FORMALIZATION AND INFINITY

ANDRÉ PORTO

Department of Philosophy
Federal University of Goiás
Campus Samambaia - Caixa Postal 131
74001-970 GOIÂNIA, GO
BRAZIL

andre.s.porto@uol.com.br

Abstract: This article discusses some of Chateaubriand's views on the connections between the ideas of *formalization* and *infinity,* as presented in chapters 19 and 20 of *Logical Forms*. We basically agree with his criticisms of the standard construal of these connections, a view we named "*formal proofs as ultimate provings*", but we suggest an alternative way of picturing that connection based on some ideas of the late Wittgenstein.

Keywords: Chateaubriand. Proofs. Formalization. Infinity. Strict finitism.

FORMALIZAÇÃO E INFINITUDE

Resumo: Este artigo analisa algumas das propostas de Chateaubriand sobre a conexão entre as idéias de *formalização* e *infinitude*, da maneira com elas aparecem nos capítulos 19 e 20 de *Logical Forms*. Basicamente concordamos com suas críticas à forma usual de se entender essas conexões, uma concepção que chamamos de "*provas formais como demonstrações supremas*", mas ainda sugerimos uma concepção alternativa dessa conexão baseada em algumas idéias do Wittgenstein tardio.

Palavras chave: Chateaubriand. Provas. Formalização. Infinitude. Finitismo estrito.

1. INTRODUCTION

Ever since Gödel showed the untenability of Hilbert's program there have been theoretical attempts to widen the notion of "formal proof" so as to include more expressive devices than Hilbert's austere finitary calculus. In chapter 19 of Chateaubriand's *Logical Forms* there is one such proposal, a quite bold one at that:

> A formal proof, or deduction, is a representation of the logical form of certain proofs, or arguments, and there is no reason for these representations of logical form to be limited to finite structures. (Chateaubriand 2005, p. 292)

In this paper we try to evaluate this proposal. We will first go over Chateaubriand's main criticisms of traditional views on *formalization*. We will basically agree with most of his criticisms of the idea we called "*formal proofs as ultimate provings*" and its way of picturing the connections between the two notions (*formalization* and *finiteness*). Towards the end of our article we will suggest an alternative (and we believe a more adequate) view of that connection. To do that we will extensively use an argument extracted from Wittgenstein's *Big Typescript* concerning *recurring infinite decimals*.

2. FORMAL PROOFS AS "ULTIMATE PROVINGS"

The main point of Chateaubriand's *Proof and Logical Deduction,* chapter 19 of *Logical Forms,* is not a direct proposal of a more liberal view of formalization. In fact, the very idea of a "formal proof" enters into his argument only towards the end of that chapter and is discussed a little bit more thoroughly under the name "idealized proofs" in the following chapter 20. Chapter 19 is centrally involved in a discussion of two constraints on the idea of "proof" *in general* (not only *formal* proofs): the *finiteness* and *effectiveness* constraints. But there seems to be a wider issue involved

there. That chapter can also be seen as a devastating critique of an ordinary and rather simple minded conception of what formalization is all about: the idea that formal proofs are just a form of "ultimate provings".[1]

Let us first lay down the essential steps of this more general argument. The view we called "*formal proofs as ultimate provings*" is just the idea that, to understand what a *formal proof* is, all we have to do is to start out by properly characterizing what *provings* (in general) are, what they are *supposed to do*. The next step is to point out that provings have a central epistemological role to perform: the evaluation of mathematical research claims. And in order to properly perform that role, so the argument goes, they should be *publicly verifiable* justifications. The proposal of formal proofs as "ultimate provings" is then simply the idea that if we just *push these requirements to the limit* what we get in the end is exactly our ordinary notion of a "*formal proof*".

It is in this final section of the argument that we finally encounter the two constraints on formal proofs discussed by Chateaubriand, the constraints of *finiteness* and *effectiveness*. What we called the view of formal proofs as "ultimate provings" includes, in this last section of the argument, the proposal that we could somehow extract *both* constraints out of the requirements of *publicity* and *verifiability*. It is towards these two last proposals that Chateaubriand directs his main criticisms.

3. THE NOTION OF A "FORMAL PROOF"

Before beginning the evaluation of the view that *formal proofs* are just *ordinary provings pushed to the limit*, we need some characterization, however provisional, of the very notion of a

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[1] I'll follow Chateaubriand and use the term "provings" to refer to ordinary, non-formal proofs.

"*formal proof*". Fortunately we don't have to look very far to do that. There is one readily available characterization of that notion, one that could even deserve to be called "*the standard* notion of formalization", deriving as it does from the work of Frege (and maybe also from Hilbert's). The idea is that a formal proof is just an *argument* (a proving) subject to the further constraint that its *syntactical form* perfectly parallels its *logical form*. This image of a (partial) parallelism between thought and linguistic expression appears very clearly in Frege:

> As a vehicle for the expression of thoughts, language must model itself upon what happens at the level of thought. (...) Once we have come to an understanding about what happens at the linguistic level, we may find it easier to go on and apply what we have understood to what holds at the level of thought – to what is mirrored in language. (Frege 1964, p. 259)

and in Hilbert:

> The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds. Thinking, it so happens, parallels speaking and writing... (Van Heijnoort 1971, p. 475)

If we accept this very natural idea of a (partial) parallelism between language and thought, it appears that all we would have to do to reach formalization would be to add the further requirement of a *gapless rendering* of thought, i.e. of a *perfect* parallelism between logical and syntactical forms. And this is of course exactly how Frege introduces his newly constructed *Begriffsschrift*:

> To prevent anything intuitive from penetrating here unnoticed, I had to bend every effort to keep the chain of inferences free of gaps. In attempting to comply with this requirement in the strictest possible way I found the inadequacy of language to be an obstacle....

> This deficiency led me to the idea of the present ideography. (Van Heijnoort 1971, pp. 5-6)[2]

It is important to notice here that Chateaubriand seems to *accept* this classical concept of formalization. Apparently for him, just as for Frege, a formal proof *is* a (perfect) representation of its underlying logical form. According to the passage we had already cited in our introduction: "*A formal proof...is a representation of the logical form of certain (...) arguments*" (Chateaubriand 2005, p. 292). And, as the final section of that passage (about the constraint of finiteness) shows, Chateaubriand's problem does not seem to hang on this classical concept of formalization. Instead he objects to the (further) idea that we could somehow *extract* the constraints of *finiteness* and *effectiveness* out of it. This seems to be the central point of Chateaubriand proposal: *disentangling the notion of a formal proof from that of recursively characterized structures*. Before we discuss this proposal, let us quickly go over three main arguments used by him to back up his suggestions.

4. DIVERGENCE BETWEEN CONVINCIBILITY AND ACCEPTANCE

The first main argumentative line Chateaubriand employs against the idea of formal proofs as ultimate provings is not only to point out its highly idealized elements but to further accuse this view of *distortion*. This misrepresentation would come up because of its failure in distinguishing two different *positive* attitudes a subject might have regarding an *argument*: *acceptance* and *conviction*. To *accept an argument* may mean as little as finding oneself unable to point out anything *wrong* with that argument. In contrast to this rather *minimal* degree of assent, we have the much more *positive*

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[2] This same idea appears also in the *Tractatus* 3.325 (Wittgenstein 1961, p. 16).

idea of *being convinced by an argument*. In this last case, not only one cannot find any actual *false step* within the presented line of reasoning, but one does not *think these mistakes are there at all*. Or at least, one thinks that, even if any minor mistakes *were found*, they could be (easily) *repaired*.

The argument is reminiscent of Descartes[3]: mere *acceptance* should be carefully distinguished from *conviction*. In Chateaubriand's text we find:

> Although we may be able to verify algorithmically that each step conforms to some rule of inference previously recognized as valid, we may be unable to recognize how the truth of the theorem proved derives from the premises used in the proof. (Chateaubriand 2005, p. 289)

The problem here is that formal proofs could at best be a idealization of the process of *acceptance*, not of *conviction*. The paramount way to find mistakes in an argument is to break it apart into several smaller steps and check each step separately. This is exactly what a formal proof has to offer: small, easily checkable steps leading from the main premises all the way to the conclusion. But this might only indicate to us that there *seems to be nothing wrong* there. We may still find ourselves very far from anything like "(full) conviction".

Chateaubriand's example of the two alternative proofs of the formula for the sum of the first *n* natural numbers is an illustration of just this point. The *geometrical* proof is totally convincing, but it is not organized into small checkable steps. On the contrary, we seem to have a single argumentative movement that is completely satisfying, none the less. Contrasting with it, the *inductive* proof appears well segmented into logically independent steps. But that

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[3] See for example the rule III of Descartes (1999).

does not mean that if we do go ahead and check it, we end up being completely convinced by it:

> Take a beginning student who knows little mathematics...After a few runs he may accept [the inductive proof], accepting that each step is correct, but he may not be entirely convinced. (Chateaubriand 2005, p. 288)

Here we come up against another persistent theme in Chateaubriand's argumentation: there is no single *non idealized* concept of either *verifiability* or *convincibility*, a concept that could be generalized to all *individuals*:

> What is a proof for a professional mathematician may not be a proof for an undergraduate, and what is a proof for an undergraduate may not be a proof for someone who can only compare strings of symbols. (Chateaubriand 2005, p. 312)

The only group of agents which would actually resemble the idealized conception of proof checking would be, of course, *computers*. But it would certainly be unacceptable to simply *defer* all mathematical judgments to them. Somewhere somehow we would still have to do some judging, at least by checking the checkers, the machines. It would be very easy to get into an infinite regress here.[4] And we would have ended up deferring all our *conviction* as well.

5. FINITENESS AS A NON-CONSTRAINING CONSTRAINT

Let us move on to the arguments directly involving *infinity*. Chateaubriand discusses two major arguments in favor of the constraint of *finiteness* regarding formal proofs. The first one has to

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[4] Chateaubriand actually accuses Church of a very similar vicious regress. (Chateaubriand 2005, pp. 286-287)

do with the public, intersubjective, character of science and the second deals again with the notion of verification. Let us start with the public constraint. Chateaubriand is discussing a list of short arguments by Enderton, all leading to the idea of formal proofs as ultimate provings. The third point on the list suggests that to be *fully communicable* an argument has to be finite (Chateaubriand 2005, p. 281). His reply to Enderton involves a rather curious twist of the argument (my reconstruction of Chateaubriand's argument may be a little free here, but the twist is there, I think). He seems to be challenging his interlocutor to indicate precisely what (infinitary) contents are not communicable. The catch is obvious: if anyone *succeeds* in pointing out a non-communicable content, this person would, by that very success, have found a way to communicate it.

The point of his argument is to call attention to a strange idea: in *some sense* finiteness is a sort of *non constraining constraint*.[5] Again, in *some sense,* all communication is always (trivially) finite. Infinity could never provide us with an example of a *non-communicable content*: the very fact that we do in fact *meaningfully refer to* and *employ* such concepts would block that possibility.

> I can give [communicate] an infinite proof in exactly the same way in which I give an infinite set of hypotheses; namely, by describing it in an understandable way. (Chateaubriand 2005, p. 283)

6. INFINITY AND THE IDEA OF "DESCRIPTION"

Although very important, Chateaubriand's point concerning infinity is clearly non conclusive. We didn't really have to take Enderton as requiring finiteness *of the syntactical structures* of our proofs. That could well be a "non-constraining constraint". Instead,

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[5] Chateaubriand employs a very powerful passage from Hilbert to make this point. Cf. Chateaubriand (2005, Chapter 19, note 3).

we could more charitably interpret him as simply pointing out that (semantical) contents involving infinite structures may not be "directly representable" by parallel syntactical structures. We cannot, say, *syntactically list* all digits of Pi's decimal expansion. Our only way out would be a *general description* of how they could be generated. But then one could still complain that the idea of "*description*" introduces the very element which would bar it as an example of a *formal proof*. We would just have to invoke the concept of formalization as "perfect parallelism between semantical and syntactical structures". We would have certainly lost this *perfect isomorphism* simply because there would be a *cardinal gap* between the two structures.

Chateaubriand is very conscious of this second possible argumentative line regarding formal proofs. In fact, he introduces very sharply the key distinction necessary to set it up, the distinction between *describing* structures and directly *presenting* them. Regarding our common understanding of what formal proofs are, he says:

> Either one says that the linguistic sequences used in communicating the proofs *are* the proofs, or one says that they *describe* proofs, or *indicate* proofs, or *communicate* proofs. (Chateaubriand 2005, pp. 283-284)

But he does not worry about this entire replying strategy because, as we said before, he has a second argumentative line, based on the notion of verification, which would push any such attempt towards the dreadful *strict finitism*.

7. THE THREAT OF STRICT FINITISM

We can only verify, list, completely represent, *finite structures*. It is not possible for us, finite creatures, to perform infinite tasks. And so, the argument goes on, any *fully explicit* representation

would have per force to be a *finite* representation. The finiteness constraint on formal proofs would then be here to stay. This last argument in favor of the finiteness constraint seems to be present in all discussions on the notion of formalization, albeit frequently in a non-explicit form. In fact, as Chateaubriand points out (Chateaubriand 2005, Chapter 19, note 7), we can also identify a version of that argument at the very foundations of most constructivists' attacks on classical mathematics. There seems to be a common element in both the standard constructivists attacks on classical mathematics and ordinary presentations of the notion of *formalization*.

The central strategy behind Chateaubriand's reply to this important (double) challenge is also not unheard of.[6] It involves basically *accepting the argument* but complaining that it does not go *far enough.* According to it, we would have inadvertently left some *undesirable idealized elements* in our concept of *formal proof*. The suggestion is that these further idealized elements must also go, that we should further restrict our formalism to what is *concretely feasible.* But if we did that, of course, we would end up finding ourselves in a most unwelcome position: that of *strict finitism.* Let us quickly go over the details.

If we invoke the constraints of what *we can really list*, *verify*, *etc*, we quickly realize that to block only *infinite* structures would be to require too little. Quite clearly, *infinite* lists, say, are not the only tasks that would be beyond our actual listing capabilities. We would have to require more than just *finiteness* to insure *feasibility.* Our syntactical structures would have not only to be *finite*, they would have to be "much smaller" than that:

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[6] Cf. for example Wright (1993, p. 107).

> Our description, of course, has to be less than finite; it has to be such that the person will be able to take it in, to understand it. (...) as I said before, it is less than finite, it is a very small finite, and, at best, a feasible finite. (Chateaubriand 2005, p. 283)

The problem here is that if we accept the idea of a complete and explicit parallelism between logical form and syntactical form *and also* restrict syntax to feasible syntax we end up with something entirely different from our ordinary concept of a "*formal proof*". For example, we would have to substitute the usual finite closure clauses of our inductive definitions of syntactical concepts by much more severe restrictions. These restrictions would have to take into account what "normal human beings" (or computers) *can actually perform*. And that would reintroduce the awkward problem of having to discriminate several computing groups (children, grownups, gifted mathematicians, computers, etc.).

8. *IN PRINCIPLE* AND *REAL* POSSIBILITIES

Chateaubriand is quite right. There *really is* a fundamental mistake in all arguments in favor of such notions as "*constructivity*", "*formalization*", etc. in terms of the notion of *feasibility*. One cannot employ the idea of "effectively executable operations" to support either constructivist's constraints on classical mathematics or the ordinary recursive view of formalization. The reason for this is quite direct. As we pointed out elsewhere (Pereira and Porto 2003), to appeal to notions such as "what *can be executed*" is to introduce *modality* into the discussion, the idea of what could "*possibly be performed*". The problem is then how one should construe these *modal notions*.

Two main options are available. We can stick to the more restrictive notion of "*real possibility*", of "what can *really* be executed", "*could be a fact*". Or we can be more liberal and accept

some idealization, appealing to the wider notion of "possibility *in principle*", of a (mere) *logical* possibility. The dilemma comes up because to obtain, either normal constructive mathematics (instead of strict finitism), or the ordinary recursive conception of formal languages, we would obviously need to employ the wider notion of "*in principle capability*". Otherwise we would be back to the exotic ideas of "grownup's math", "children's math", and so on. We would *need* the wider notion (to get the desired results), but in the end we would have to accept that the argument is *only viable with the more restrictive notion*. So we need to get to *abstract* constructibility, but we are firmly tied up to *concrete* executability. And we certainly cannot bug the classic mathematician because of his highly idealized conception of a "mathematical reality" and at the same time beg for his forgiveness on the equally idealized notion of "*possibility in principle*" (on the grounds that we need it to get *our* desired results).

What should we do then? What is the final outcome of all our discussion? Must we accept Chateaubriand's (rather wild) proposal of enlarging the concept of formalization so as to include "*deductions with an infinite structure, and deductions involving non-algorithmically checkable steps*" (Chateaubriand 2005, p. 295)? Should we call *these* "formal proofs"? Would there still be *non-formal proofs*, then? More than that: were there no problems, no philosophical difficulties, regarding the compatibility of *modern highly infinitary structures* vis-à-vis *concrete physical (and mental) reality*? No need of tracing some restriction to infinitary arguments?

We will devote the last sections of our article to the presentation of an *alternative argumentative route* to the problem of setting up some restriction on the notion of *infinity*. This alternative route, due to Wittgenstein, was no accidental result within that author's philosophy. In our opinion, more than any other philosopher, Wittgenstein was centrally aware of the challenges presented by arguments such as Chateaubriand's. Of course we

won't revise Wittgenstein's complete philosophical proposals here. Instead, we will concentrate on one single argument presented by him in the context of his discussion of *infinite recurring decimals*.

9. WITTGENSTEIN'S DISCUSSIONS ON INFINITE RECURRING DECIMALS

As we said, Wittgenstein proposes an alternative route to what we might call *the problem of tracing the finiteness constraint* without resorting to *modal notions* (and its associated specter of *strict finitism*). As we shall see, instead of being based on the notion of "*capacity*", his constraint has to do with *semantics*, with a differentiation between *general* and *singular terms*.

Wittgenstein's alternative proposal comes up within some rather obstinate discussions of some of the most elementary infinitary statements in mathematics, equations such as:

$$1 \div 7 = 0,142 \ldots$$

Surprisingly, the philosopher finds these rather plain mathematical statements *objectionable*. Fortunately his qualms are not hard to understand.

In a nut-shell, Wittgenstein deems such statements not to be *well formed*. The reasoning is quite direct and is entirely due to Frege. To be well formed, each side of an identity statement should contain a *singular term*. Wittgenstein doesn't have any problems with the *left complex term*:

$$\div (1, 7)$$

but he does find the *right term*

0,142 ...

unacceptable. Again, his point is quite direct. If "0,142 ..." *were a singular term*, if "1 ÷ 7 = 0,142 ..." *were a (true) equation*, what should we say about all the following *equations*:

$$\frac{1421}{9999} = 0{,}142...$$

$$\frac{1422}{9999} = 0{,}142...$$

$$\frac{1423}{9999} = 0{,}142...?\,[7]$$

Should we also call them "*true*"? Because of the transitivity of identity, that would obviously not be an option.

The problem, Wittgenstein says, is that "0,142 ..." *is not a singular term!* (Wittgenstein 2005a) It is *a general term*. Something like "*a number with a decimal expansion beginning by '0', '1', '4' and '2'*". As (Kripke 1972, 18) points out, the difficulty here is that "*an indefinite number of rules* [i.e., of different numbers] ... *are compatible with any such finite initial segment*". In other words, if "0,142 ..." were a *singular* term, it would not satisfy *unicity*. And if it were a *general* term, what is it doing on one of the sides of an identity statement? That's Wittgenstein's point.

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[7] Following (Kripke 1972), we will say that these equations involve "non-standard interpretations".

10. EXTRACTING INFINITE EXPANSIONS OUT OF FINITE SEGMENTS

It is important to realize the correct scope of Wittgenstein's argument. In contrast with *strict finitism*, he does not intend to rule out any mathematical expressions involving infinite concepts. For example, according to him

$$1 \div 7 = 0,\dot{1}4285\dot{7}$$

would be a perfectly well formed statement. In fregean terms, the problem is with the *mode of presentation* employed by each expression to single out its referent.

The phrase "$0,\dot{1}4285\dot{7}$" offers us a *syntactical base*, the string "0^,"[8] and a *recursive operation*, the operation of "filling n decimal places with the digits "142857". So we could rewrite the second equation maybe as:

$$\forall n\big(Dividing_{(n)}(1, 7) = 0\char`^,\char`^Filling[142857]_{(n-1)}\big)$$

or even as:

$$[\lambda n.\ Dividing_{(n)}(1, 7)] = [\lambda n.\ 0\char`^,\char`^Filling[142857]_{(n-1)}]$$

The important point is that, taken together, the base and the recursive operation *do single out a number*, in such a way that the expression

$$\imath n \big(0\char`^,\char`^Filling[142857]_{(n-1)}\big)$$

does succeed in referring to an object.

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[8] We employ the symbol "^" for concatenation.

Nothing of the sort is true of "0,142...". The mode of presentation used here is just to offer us an initial *finite segment* of an expansion and with it aspire to single out one particular *infinite* expansion. As Kripke (1972, p. 18) points out, despite intelligence tests, "*mathematical and philosophical sophisticates*" know that this cannot be done. No *finite initial segment* alone, be it as long as we please, can succeed in completely picking out one single infinite expansion. I take this to be one of Wittgenstein's central lessons in his famous example of the continuation of the series "2, 4, 6, 8,..".[9]

11. INFINITARY LOGIC

Let us now take Chateaubriand's discussion of his infinitary set of hypothesis *A* (Chateaubriand 2005, p. 285):

$$\forall x \forall y \forall z \, ((Rxy \,\&\, Ryz) \rightarrow Rxz)$$
$$\forall x \, \neg Rxx$$
$$Ra_1a_2$$
$$Ra_2a_3$$
$$Ra_3a_4$$
$$\vdots$$

He claims that *A* could only have an infinite set as a domain for the relation *R* (Chateaubriand 2005, p. 285). Now, is this really *true*? Aside from the obviously intended interpretation (in which *R* would indeed have an infinite domain), what would prevent us from using the following alternative "non standard" continuations?

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[9] Cf. Wittgenstein (2005b § 185 and § 208).

$$Ra_1a_2 \qquad\qquad Ra_1a_2$$
$$Ra_2a_3 \qquad\qquad Ra_2a_3$$
$$Ra_3a_4 \qquad\qquad Ra_3a_4$$
$$Ra_1a_2 \quad\text{or}\quad Ra_4a_1$$
$$Ra_2a_3 \qquad\qquad Ra_1a_2$$
$$Ra_3a_4 \qquad\qquad Ra_2a_3$$
$$\vdots \qquad\qquad Ra_3a_4$$
$$Ra_4a_1$$
$$\vdots$$

Of course, if we picked, say, our second non standard interpretation, *R* would not have an infinite domain anymore. *R* would have *no model* and *A* would be *contradictory*!

The problem here is with the very notation that distinguishes *infinitary logic* from ordinary *first order logic*: its prodigal employment of *elision dots* (or of similar notations such as "and so on...") to denote *infinite expansions*. According to what we saw before, this "elision dots notation" would be just another instance of a generally (flawed) idea: the suggestion that one could extract a single infinite expansion out of *finite initial segments* such as:

$$Ra_1a_2$$
$$Ra_2a_3$$
$$Ra_3a_4$$

12. SOME FINAL REMARKS

It is my opinion that many of Chateaubriand's challenges to widely accepted views on the foundations of logic and mathematics are both correct and profound. I find particularly stimulating his independent and daring approaches to these issues. His discussion of the issue of Henkin vs. "absolute" interpretations of higher order

logics is especially interesting and would demand careful evaluation. As to the issues reviewed in this article, I don't intend to have settled any of the questions and problems raised by its discussions. It is quite clear that the difficulties involved are very delicate and run deep. My idea was just to suggest some alternative arguments and thus restore some connection between *formalization* and *finiteness*.

REFERENCES

CHATEAUBRIAND, O. *Logical Forms. Part I: Truth and Descriptions.* Campinas: Unicamp, Centro de Lógica, Epistemologia e História da Ciência, 2001. (Coleção CLE, v. 34)

————. *Logical Forms. Part II: Logic, Language, and Knowledge.* Campinas: Unicamp, Centro de Lógica, Epistemologia e História da Ciência, 2005. (Coleção CLE, v. 42)

DESCARTES, R. *Regras para a Orientação do Espírito.* São Paulo: Martins Fontes, 1999.

FREGE, G. *Posthumous Writings.* Berkeley: California University Press, 1964.

KRIPKE, S. *Wittgenstein on Rules and Private Language.* Oxford: Basil Blackwell, 1972.

PEREIRA, L.C., PORTO, A. "Algumas Considerações sobre a Noção Construtiva de Verdade." *O Que Nos Faz Pensar*, 17, pp. 107-123, 2003.

PORTO, A. "Wittgenstein on Mathematical Identities". *Forthcoming.*

VAN HEIJENOORT, J. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931.* Cambridge: Harvard University Press, 1971.

WITTGENSTEIN, L. *Tractatus Logico-Philosophicus.* London: Routledge & Kegan Paul, 1961.

_____. *The Big Typescript.* Oxford: Basil Blackwell, 2005a.

_____. *Philosophical Investigations.* Oxford: Basil Blackwell, 2005b.

WRIGHT, C. *Realism, Meaning and Truth.* Cambridge: Basil Blackwell, 1993.