On the relationship between fixed points and iteration in admissible set theory without foundation

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Abstract

In this article we show how to use the result in Jäger and Probst [7] to adapt the technique of pseudo-hierarchies and its use in Avigad [1] to subsystems of set theory without foundation. We prove that the theory KPi^0 of admissible sets without foundation, extended by the principle (Σ-FP), asserting the existence of fixed points of monotone Σ operators, has the same proof-theoretic ordinal as KPi^0 extended by the principle (Σ-TR), that allows to iterate Σ operations along ordinals. By Jäger and Probst [6] we conclude that the meta-predicative Mahlo ordinal $\varphi\omega00$ is also the ordinal of $\mathsf{KPi}^0 + (\Sigma$-FP). Hence the relationship between fixed points and iteration persists in the framework of set theory without foundation.

Keywords: Admissible sets; Fixed points; Iteration; Pseudo-hierarchies

1 Introduction

In classical set theory, the relationship between fixed points and iteration is evident. Given a monotone operator Γ on the power set $\mathsf{P}(a)$ of some set a, a fixed point is obtained by iterating the operator along the ordinals: Start with an arbitrary set $I_\Gamma^0 := x \subseteq a$, apply the operator in the successor case, i.e. $I_\Gamma^{\alpha+1} := \Gamma(I_\Gamma^\alpha)$ and take the union at limit stages, i.e. $I_\Gamma^\lambda := \bigcup_{\{\alpha<\lambda\}} I_\Gamma^\alpha$. The monotonicity of Γ and a cardinality argument assure that there is an ordinal σ such that I_Γ^σ is a fixed point of the operator Γ. On the other hand, iterating an operator along a well-ordering can be seen as the iterative construction of a fixed point of a suitable monotone operator.

In (meta)-predicative subsystems of second order arithmetic things are more complex. One cannot prove the existence of well-orderings long enough for the iteration process to become stationary. Nevertheless, the existence of fixed points follows if we assume that the well-orderedness of $\prec$ implies the existence of a hierarchy H with $(H)_\alpha = I_\Gamma^\alpha$ for all α in the field of $\prec$. This principle is called arithmetical transfinite recursion, or (ATR) for short, and was shown by Avigad [1] to be equivalent over the theory $\mathsf{ACA_0}$ to the principle (FP) that claims the existence of fixed points of monotone operators defined by an arithmetical formula. To prove the existence of fixed points given (ATR), Avigad makes use of so called pseudo-hierarchies, i.e. hierarchies where the underlying ordering is only a linear ordering, not a well-ordering. Their existence follows from the fact that over $\mathsf{ACA_0}$ the Π_1^1 formula $\mathsf{WO}(\prec)$, asserting that $\prec$ is a well-ordering, is not equivalent to any Σ_1^1 formula. For the other direction, Avigad showed that a hierarchy can be obtained as a fixed point of a suitable monotone operator describing the inductive build-up of the hierarchy.

Our aim is to use pseudo-hierarchies also in the framework of Kripke-Platek set theory without foundation. The problem is, that it is now consistent to assume that there is a Σ formula $A(\prec)$ equivalent to the statement "$\prec$ is a well-ordering". Under this assumption, the existence of pseudo-hierarchies is in general not provable. Moreover, "being a well-ordering" is then a Δ predicate. Hence Π_1^1 comprehesion is available, and in particular, the ordinal of the theory $\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{TR})$, the meta-predicative Mahlo ordinal $\varphi\omega00$, becomes provable. To save the pseudo-hierarchy argument, we observe that the theory $\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{TR})$ can be consistently extended by an axiom claiming the existence of a set for which transfinite induction up to $\varphi\omega00$ fails. As follows from Jäger and Probst [7], the resulting theory $[\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{TR})]^\dagger$ still has the same proof-theoretic ordinal as $\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{TR})$, but being a well-ordering is no longer a Δ predicate. Consequently, the technique of pseudo-hierarchies can be applied to prove the existence of fixed points of monotone Σ operators, given the iteration principle $(\Sigma\text{-}\mathsf{TR})$. Hence $\varphi\omega00$ is an upper bound for $\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{FP})$. To show that $\varphi\omega00$ is also a lower bound, we argue how the well-ordering proof for the theory $\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{TR})$ given in Jäger and Probst [6] can be adjusted to the theory $\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{FP})$. This concludes the argument that extending $\mathsf{KPi^0}$ either by an axiom that allows us to iterate Σ operations along arbitrary well-orderings or an axiom that asserts the existence of fixed points of monotone Σ operators leads to theories of the same proof-theoretic strength. It that sense the relationship between fixed points and iteration persists in the framework of admissible set theory without foundation.

The plan of this paper is as follows: In the next section we introduce the theories $\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{FP})$ and $\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{TR})$. Section 3 recalls some ordinal theoretic facts that are relevant in the sequel. In section 4 we show how $[\mathsf{KPi^0} + (\Sigma\text{-}\mathsf{TR})]^\dagger$ proves the existence of fixed points of monotone Σ operators on the power set of the natural

numbers. This gives us $\varphi\omega 00$ as an upper bound. That $\varphi\omega 00$ is also a lower bound is shown in section 5, and section 6 contains some concluding remarks.

2 The theories $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ and $\mathsf{KPi}^0 + (\Sigma\text{-TR})$

We start by giving a short review of the theory KPi^0. Our version of KPi^0 is formulated with the natural numbers as *urelements*. Accordingly, we let L_1 denote a standard languages of first order arithmetic with variables $a, b, c, \ldots, u, v, w, x, y, z, \ldots$ (possibly with subscripts), a constant 0 as well as function and relation symbols for all primitive recursive functions and relations. The theory KPi^0 is formulated in the extension $\mathcal{L}^* = \mathsf{L}_1(\in, \mathsf{N}, \mathsf{S}, \mathsf{Ad})$ of L_1 by the membership relation symbol $\in$, the set constant N for the set of natural numbers and the unary relation symbols S and Ad for sets and admissible sets, respectively.

The *number terms* of $\mathcal{L}^*$ are inductively generated from the variables, the constant 0 and the symbols for the primitive recursive functions; the *terms* $r, s, t, \ldots$ of $\mathcal{L}^*$ are the number terms of L_1 plus the set constant N. The formulas $A, B, C, \ldots$ of $\mathcal{L}^*$ as well as the Δ_0, Σ, Π, Σ_n and Π_n formulas of L_1 are defined as usual. Further, equality between objects is not represented by a primitive symbol, but defined by

$$(s = t) := \begin{cases} (s \in \mathsf{N} \wedge t \in \mathsf{N} \wedge (s =_\mathsf{N} t)) \ \vee \\ (\mathsf{S}(s) \wedge \mathsf{S}(t) \wedge (\forall x \in s)(x \in t) \wedge (\forall x \in t)(x \in s)) \end{cases}$$

where $=_\mathsf{N}$ is the symbol for the primitive recursive equality on the natural numbers. The formula A^s is the result of replacing each unrestricted quantifier $\exists x(\ldots)$ and $\forall x(\ldots)$ in A by $(\exists x \in s)(\ldots)$ and $(\forall x \in s)(\ldots)$, respectively. In addition, we freely make use of all standard set-theoretic notations. So we write $\mathsf{Fun}(f)$ and $\mathsf{Tran}(x)$ for the Δ_0 formulas expressing that f is a function and that x is a transitive set. $\mathsf{Field}(x)$ is used to denote the field of an ordering x and $\mathsf{Dom}(f)$ denote the set $\{u : \exists v \ (u, v) \in f\}$, and $\mathsf{Rng}(f)$ denotes the set $\{v : \exists u \ (u, v) \in f\}$, where$(x, y)$ stands for the standard ordered pair, $\{\{x\}, \{x, y\}\}$. As usual, $x \subseteq y$ states that x is a subset of the set y, i.e. $x \subseteq y := \mathsf{S}(x) \wedge \mathsf{S}(y) \wedge (\forall u \in x)(u \in y)$.

The $\mathcal{L}^*$ theory KPi^0 is based on classical first order logic with equality. Its non-logical axioms can be divided into the following five groups.

I. Ontological axioms. We have for all terms $r, \vec{s}$ and t of L_1, all function symbols $\mathcal{H}$ and relation symbols $\mathcal{R}$ of L_1 and all axioms $A(\vec{a})$ of group III whose free variables belong to the list $\vec{a}$:

(1) $a \in \mathsf{N} \leftrightarrow \neg\mathsf{S}(a)$,

(2) $\vec{a} \in \mathsf{N} \rightarrow \mathcal{H}(\vec{a}) \in \mathsf{N}$,

(3) $\mathcal{R}(\vec{a}) \to \vec{a} \in \mathsf{N}$,

(4) $a \in b \to \mathsf{S}(b)$,

(5) $\mathsf{Ad}(a) \to (\mathsf{N} \in a \wedge \mathsf{Tran}(a))$,

(6) $\mathsf{Ad}(a) \to (\forall \vec{x} \in a) A^a(\vec{x})$,

(7) $\mathsf{Ad}(a) \wedge \mathsf{Ad}(b) \to a \in b \vee a = b \vee b \in a$.

II. **Number-theoretic axioms.** We have for all axioms $A(\vec{a})$ of Peano arithmetic PA which are not instances of the schema of complete induction and whose free variables belong to the list $\vec{a}$:

(Number theory) $\vec{a} \in \mathsf{N} \to A^{\mathsf{N}}(\vec{a})$.

III. **Kripke Platek axioms.** For all Δ_0 formulas $A(u)$ and $B(u,v)$ of $\mathcal{L}^*$:

(Pair) $\exists x (a \in x \wedge b \in x)$,

(Tran) $\exists x (a \subseteq x \wedge \mathsf{Tran}(x))$,

(Δ_0-Sep) $\exists y (\mathsf{S}(y) \wedge y = \{x \in a : A(x)\})$,

(Δ_0-Col) $(\forall x \in a) \exists y B(x,y) \to \exists z (\forall x \in a)(\exists y \in z) B(x,y)$.

IV. **Limit axiom.** It is used to formalize that each set is element of an admissible set, hence we claim:

(Lim) $\exists x (a \in x \wedge \mathsf{Ad}(x))$.

V. **Complete induction on N.** The only induction principle included in the axioms of KPi^0 is the following axiom of complete induction on the natural numbers for sets:

(S-$\mathsf{I_N}$) $0 \in a \wedge (\forall x \in \mathsf{N})(x \in a \to x + 1 \in a) \to \mathsf{N} \subseteq a$.

The monograph Barwise [2] provides an excellent introduction into general admissible set theory. Theories of admissible sets without foundation, on the other hand, have been studied, in particular, in Jäger [3, 4]. It is shown there, among other things, that the proof-theoretic ordinal of KPi^0 is Γ_0.

In this article, however, we are mainly interested in the relationship between the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{FP})$ that extends KPi^0 by the axiom schema $(\Sigma\text{-}\mathsf{FP})$, claiming the existence of fixed points of monotone Σ operators on the power set of the natural

numbers, and the theory $\mathsf{KPi}^0 + (\Sigma\text{-TR})$, allowing to iterate Σ operators along ordinals. In order to give a compact formulation of our principle $(\Sigma\text{-FP})$ we introduce the following abbreviations: For all formulas $A(\vec{u}, v, w)$ of $\mathcal{L}^*$ we set

$$\begin{aligned}
\mathsf{Op}_A^{\mathsf{N}}(\vec{a}) &:= (\forall x \subseteq \mathsf{N})(\exists! y \subseteq \mathsf{N}) A(\vec{a}, x, y), \\
\mathsf{Mon}_A^{\mathsf{N}}(\vec{a}) &:= (\forall u, v, x, y \subseteq \mathsf{N})[A(\vec{a}, u, x) \wedge A(\vec{a}, v, y) \wedge u \subseteq v \to x \subseteq y].
\end{aligned}$$

$\mathsf{Op}_A^{\mathsf{N}}(\vec{a})$ expresses that for the parameters $\vec{a}$ the formula $A(\vec{a}, x, y)$ defines a functional relation on $\mathsf{P}(\mathsf{N})$. If in addition $\mathsf{Mon}_A^{\mathsf{N}}(\vec{a})$ holds, this functional relation is monotone in the usual sense and we say that A defines a monotone operator for the parameters $\vec{a}$. The principle $(\Sigma\text{-FP})$ takes now the following form: For each Σ formula $A(\vec{u}, v, w)$ of $\mathcal{L}^*$ with at most the variables $\vec{u}, v, w$ free we have

$(\Sigma\text{-FP})$ $\qquad\qquad \mathsf{Op}_A^{\mathsf{N}}(\vec{a}) \wedge \mathsf{Mon}_A^{\mathsf{N}}(\vec{a}) \to (\exists x \subseteq \mathsf{N}) A(\vec{a}, x, x).$

Remark 1 *The purpose of the restriction to operators on $\mathsf{P}(\mathsf{N})$ is to keep the fixed point axiom persistent. If instead, we add to KPi^0 a fixed point principle that claims arbitrary monotone Σ operators to have fixed points, we do not increase the proof-theoretic strength. In this case however, a further strengthening, e.g. by foundation leads to an inconsistent theory. For a further discussion, see section 6.*

The theory $\mathsf{KPi}^0 + (\Sigma\text{-TR})$ is the extension of the theory KPi^0 by the axiom schema $(\Sigma\text{-TR})$ that allows to iterate Σ operations along ordinals. In the current context, ordinals are hereditarily transitive well-founded sets:

$$\begin{aligned}
\mathsf{Wf}(a, \in) &:= \forall x(x \subseteq a \wedge x \neq \emptyset \to (\exists y \in x)(\forall z \in y)(z \notin x)), \\
\mathsf{Ord}(a) &:= \mathsf{Tran}(a) \wedge (\forall x \in a)\mathsf{Tran}(x) \wedge \mathsf{Wf}(a, \in).
\end{aligned}$$

To formulate the iteration principle, we introduce for each Σ formula $D(\vec{u}, x, y, z)$ of $\mathcal{L}^*$, with at most the variables $\vec{u}, x, y, z$ free, the formula

$$\mathsf{Hier}_D^{\mathbb{O}}(\vec{a}, b, f) := \left\{ \begin{array}{l} \mathsf{Ord}(b) \wedge \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = b \wedge \\ (\forall x \in b) D(\vec{a}, x, f{\restriction}x, f(x)). \end{array} \right.$$

The iteration principle now takes the form

$(\Sigma\text{-TR})$ $\qquad \mathsf{Ord}(b) \wedge (\forall x \in b)\forall y \exists! z D(\vec{a}, x, y, z) \to \exists f \mathsf{Hier}_D^{\mathbb{O}}(\vec{a}, b, f)$

for all Σ formulas $D(\vec{u}, x, y, z)$ of $\mathcal{L}^*$ with at most the variables $\vec{u}, x, y, z$ free.

As shown in Jäger and Probst [6] the theory $\mathsf{KPi}^0 + (\Sigma\text{-TR})$ enables us also to iterate Σ operations along arbitrary well-orderings. Given a set a and a binary relation $b \subseteq a \times a$, we write $\mathsf{Lin}(a, b)$ if b is a strict linear ordering on a. A linear

ordering is a well-ordering if any non-empty subset of its domain has a least element with respect to this ordering,

$$\mathsf{Wo}(a, b) := \mathsf{Lin}(a, b) \wedge \forall x (x \subseteq a \wedge x \neq \emptyset \rightarrow (\exists y \in x)(\forall z \in x)((z, y) \notin b)).$$

We define for each Σ formula $D(\vec{u}, x, y, z)$ of $\mathcal{L}^*$ the formula

$$\mathsf{Hier}_D(\vec{a}, b, c, f) := \left\{ \begin{array}{l} \mathsf{Wo}(b, c) \wedge \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = b \wedge \\ (\forall x \in b) D(\vec{a}, x, \{(y, f(y)) : (y, x) \in c\}, f(x)), \end{array} \right.$$

in order to state the theorem below.

Theorem 2 *For all Σ formulas $D(\vec{u}, x, y, z)$ of $\mathcal{L}^*$ that contain at most the variables $\vec{u}, x, y, z$ free, $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ proves*

$$\mathsf{Wo}(b, c) \wedge (\forall x \in b) \forall y \exists! z D(\vec{a}, x, y, z) \rightarrow \exists f \mathsf{Hier}_D(\vec{a}, b, c, f).$$

In [6] it is also shown that the proof-theoretic ordinal of $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ is $\varphi\omega 00$. In the sequel we will show that the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{FP})$ has the same proof-theoretic ordinal, and hence unfolds yet another aspect of metapredicative Mahlo. The Mahlo axiom schema for admissible set theory without foundation claims Π_2 reflection on admissible sets,

$$(\Pi_2\text{-}\mathsf{Ref}^{\mathsf{Ad}}) \qquad\qquad A(\vec{a}) \rightarrow \exists x (\vec{a} \in x \wedge \mathsf{Ad}(x) \wedge A^x(\vec{a}))$$

for all Π_2 formulas $A(\vec{u})$ of $\mathcal{L}^*$ with at most the variables $\vec{u}$ free. The theory $\mathsf{KPi}^0 + (\Pi_2\text{-}\mathsf{Ref}^{\mathsf{Ad}})$ is denoted by KPm^0 and is analyzed in detail in Jäger and Strahm [8] and Strahm [13].

We conclude the section by mentioning a result about KPi^0 that plays an important role in section 5 where we derive a lower bound for $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{FP})$. This result is due to Jäger and a detailed proof is given e.g. in [6].

Lemma 3 *Let $D(\vec{u}, v)$ be a Δ_0 formula of $\mathcal{L}^*$ with at most the variables $\vec{u}, v$ free. Then KPi^0 proves that*

$$\exists x (D(\vec{a}, x) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x)) \rightarrow$$
$$\exists y (y = \bigcap \{x : D(\vec{a}, x) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x)\} \wedge D(\vec{a}, y) \wedge \vec{a} \in y \wedge \mathsf{Ad}(y)).$$

This lemma implies for example that in KPi^0 for any set x the intersection x^+ of all admissibles containing x is an admissible itself,

$$x^+ := \bigcap \{y : x \in y \wedge \mathsf{Ad}(y)\}.$$

3 Ordinal-theoretic preliminaries

The aim of this section is to review some ordinal-theoretic facts that will be relevant in the sequel. As in, for example, Jäger and Strahm [8] we work with the ternary Veblen function, that is obtained from the binary φ function as follows:

(1) $\varphi 0\beta\gamma$ is $\varphi\beta\gamma$,

(2) if $\alpha > 0$, then $\varphi\alpha 0\gamma$ denotes the γth ordinal which is strongly critical with respect to all functions $\lambda\xi.\lambda\eta.\varphi\delta\xi\eta$ for $\delta < \alpha$,

(3) if $\alpha > 0$ and $\beta > 0$, then $\varphi\alpha\beta\gamma$ denotes the γth common fixed point of the functions $\lambda\eta.\varphi\alpha\delta\eta$ for $\delta < \eta$.

Let Ξ_0 be the least ordinal greater than 0 which is closed under addition and the ternary φ function. In the following we will work with a standard primitive recursive notation system $(\mathsf{OT}, \lhd)$ for all ordinals less than Ξ_0. All required definitions are straightforward generalizations of those used for building a notation system for Γ_0 (cf. [9, 11]) and are omitted. In the reminder of this article we let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$ range other the set OT and the terms $\hat{0}, \hat{1}, \hat{2}, \dots$ act as codes for the finite ordinals. Moreover, ordinal constants such as $0, 1, \omega$ and $\varphi\omega 00$ are often used instead of the corresponding ordinal notations.

We set for all primitive recursive relations $\prec$ and all number terms s:

$$
\begin{aligned}
\mathsf{Prog}(\prec, x) &:= (\forall u \in \mathsf{Field}(\prec))[\forall v(v \prec u \to v \in x) \to u \in x], \\
\mathsf{TI}(\prec, x) &:= \mathsf{Prog}(\prec, x) \to \mathsf{Field}(\prec) \subseteq x. \\
\mathsf{TI}(\prec, x, s) &:= \mathsf{Prog}(\prec, x) \to (\forall n \prec s)(n \in x).
\end{aligned}
$$

To be consistent with the notation in [6] we often write $\mathsf{TI}(\mathfrak{a}, x)$ instead of $\mathsf{TI}(\lhd, \mathfrak{a}, x)$, where $\lhd$ is the special well-ordering of our notation system. An ordinal α is called provable in the theory T - formulated in $\mathcal{L}^*$ or a similar language - if there exists a primitive recursive well-ordering $\prec$ of order-type α such that

$$\mathsf{T} \vdash \forall x \mathsf{TI}(\prec, x).$$

The least ordinal which is not provable in T is called the *proof-theoretic ordinal* of T and denoted by $|\mathsf{T}|$.

4 Upper bound

To determine an upper bound for the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{FP})$, we show that the fixed point axiom $(\Sigma\text{-}\mathsf{FP})$ is provable in a theory denoted by $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ which is a

conservative extension of $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ in the sense that its proof-theoretic ordinal is still $\varphi\omega00$. The theory $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ is designed to prove the existence of so called pseudo-hierarchies. This allows us to adapt Avigad's argument in [1] to show that $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ proves the existence of fixed points of monotone Σ operators.

As shown in [6], $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ has proof-theoretic ordinal $\varphi\omega00$. Therefore the sentence

$$\forall x \mathsf{TI}(x, \varphi\omega00)$$

is not provable in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ and hence the theory $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ that extends the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ by the axiom

$$\exists x \neg \mathsf{TI}(x, \varphi\omega00)$$

is consistent. Moreover, this extension still has the same proof-theoretic ordinal.

Lemma 4
$$|\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})| = |[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger|.$$

Proof: This result is a consequence of results in Jäger and Probst [7]. For the reader's convenience we summarize the main steps. Assume that there is a primitive recursive well-ordering $\prec$ such that

$$[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger \vdash \forall x \mathsf{TI}(\prec, x).$$

Our aim is to show that the ordertype of $\prec$ is less than $\varphi\omega00$. Since KPm^0 proves the iteration principle $(\Sigma\text{-}\mathsf{TR})$ (cf. [6]) we have that

$$\mathsf{KPm}^0 \vdash \exists x \neg \mathsf{TI}(x, \varphi\omega00) \rightarrow \forall y \mathsf{TI}(\prec, y).$$

It follows from the proof-theoretic analysis of KPm^0 given in Jäger and Strahm [8] that a standard semi-formal system PA^∞ for second order arithmetic proves

$$(*) \qquad\qquad \mathsf{PA}^\infty \vdash^{\alpha_0}_0 \mathsf{TI}(\varphi\omega00, X), \mathsf{TI}(\prec, Y)$$

for an $\alpha_0 < \varphi\omega00$.
Inspecting the proof of the boundedness lemma (cf. e.g. Pohlers [9]) we observe that also the following slightly more general statement holds:

Lemma 5 (Boundedness Lemma) *Suppose that $\prec_1$ and $\prec_2$ are primitive recursive well-orderings, Γ is a finite set of U-positive formulas of PA^∞, Δ is a finite set of V-positive formulas of PA^∞, V does not occur in Γ and U does not occur in Δ. Further, let $\Lambda_{\vec{s}} := \{s_1 \notin U, \ldots, s_k \notin U\}$ and $\Lambda_{\vec{t}} := \{t_1 \notin V, \ldots, t_l \notin V\}$. If*

$$\mathsf{PA}^\infty \vdash^{\alpha}_0 \neg\mathsf{Prog}(\prec_1, U), \Lambda_{\vec{s}}, \Gamma, \neg\mathsf{Prog}(\prec_2, V), \Lambda_{\vec{t}}, \Delta$$

then it follows that

$$\mathbb{N} \models \bigvee\{F[(\prec_1)_{\gamma_1}/U] : F \in \Gamma\} \quad or \quad \mathbb{N} \models \bigvee\{F[(\prec_2)_{\gamma_2}/V] : F \in \Delta\},$$

where we have $\gamma_i = \beta_i + 2^\alpha$ *and* $\beta_1 = \mathsf{max}\{|s_1^{\mathbb{N}}|_{\prec_1}, \ldots, |s_k^{\mathbb{N}}|_{\prec_1}\}$ *and* $\beta_2 = \mathsf{max}\{|t_1^{\mathbb{N}}|_{\prec_2}, \ldots, |t_l^{\mathbb{N}}|_{\prec_2}\}.$

As in [9], if $\prec$ is a well-ordering we assign to each natural number in the field of $\prec$ an ordinal by the recursion $|n|_{\prec} := \{|m|_{\prec} : m \prec n\}$, and $(\prec)_\sigma$ is the set $\{n \in \mathbb{N} : |n|_{\prec} \in \sigma\}$. We write $t^{\mathbb{N}}$ for the value of the closed term t and $\mathbb{N} \models F[S/U]$ to indicate that F holds in the standard model provided the relation symbol U is interpreted by the set S.

Applying the above lemma to $(*)$ yields that

$$\mathbb{N} \models \forall x \in \mathsf{Field}(\varphi\omega00)(|x|_{\lhd} \leq \omega^{\alpha_0}) \text{ or } \mathbb{N} \models \forall x \in \mathsf{Field}(\prec)(|x|_{\prec} \leq \omega^{\alpha_0}).$$

The first statement is obviously wrong and hence the ordertype of $\prec$ is less than $\varphi\omega00$. □

Pseudo-hierarchies are hierarchies where the underlying ordering is not a well-ordering. If $D(\vec{u}, x, y, z)$ is a formula of $\mathcal{L}^*$ we write

$$\mathsf{PSH}_D(\vec{a}, b, c, f) := \left\{ \begin{array}{l} \mathsf{Lin}(b, c) \wedge \neg\mathsf{Wo}(b, c) \wedge \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = b \wedge \\ (\forall x \in b)D(\vec{a}, x, \{(y, f(y)) : (y, x) \in c\}, f(x)) \end{array} \right.$$

to express that f is a pseudo-hierarchy for $D(\vec{u}, x, y, z)$ with parameters $\vec{a}$ along the ordering c on b.

The next lemma is a key step in showing that $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ proves the existence of pseudo-hierarchies for Σ formulas of $\mathcal{L}^*$. It states that there is no Σ formula $A(u, v)$ that is equivalent to the assertion that b is a well-ordering. The proof of the lemma exhibits the role of the axiom $\exists x \neg\mathsf{TI}(\lhd, x, \varphi\omega00)$: It is to exclude that $\mathsf{Wo}(x, y)$ is a Δ predicate.

Lemma 6 *For all* Σ *formulas* $A(\vec{u}, v, w)$ *of* $\mathcal{L}^*$ *the theory* $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ *proves:*

$$\forall \vec{x} \neg (\forall y, z[A(\vec{x}, y, z) \leftrightarrow \mathsf{Wo}(y, z)]).$$

Proof: Assume that there are sets $\vec{a}$ and a Σ formula $A(\vec{u}, v, w)$ such that

$(*)$ $\qquad\qquad\qquad\qquad \forall y, z[A(\vec{a}, y, z) \leftrightarrow \mathsf{Wo}(y, z)]$

holds. We show that under this assumption we could embed $\Pi_1^1\text{-}\mathsf{CA}_0$ into the theory $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$. Thus the theory $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ would prove $(\forall x \subseteq \mathbb{N})\mathsf{TI}(x, \varphi\omega00)$. Since our ordinal notations are elements of $\mathbb{N}$ and $\Pi_1^1\text{-}\mathsf{CA}_0$ proves the ordinal $\varphi\omega00$, this contradicts $\exists x \neg\mathsf{TI}(x, \varphi\omega00)$.

Let $\Psi(u, \vec{v}, \vec{V})$ be a Π_1^1 formula of L_2 . As shown e.g. in [12], ACA_0 proves that for all natural numbers $\vec{m}$ and all subsets $\vec{Y}$ of N there is a set T dependent on Ψ and the parameters $\vec{m}$ and $\vec{Y}$ such that for all x,

(1) $(T)_x$ is a tree,

(2) $(T)_x$ is recursive in $\vec{Y}$,

(3) $\Psi(x, \vec{m}, \vec{Y}) \leftrightarrow \mathsf{WO}(\mathsf{KB}((T)_x))$,

where $\mathsf{KB}(S)$ denotes the restriction of the Kleene/Brower ordering to the tree S, $\mathsf{WO}(X)$ is the L_2 formula saying that $\{(x, y) : \langle x, y \rangle \in X\}$ is a linear ordering that is well-founded and $(X)_t := \{x : \langle x, t \rangle \in X\}$, where $\langle \cdot, \cdot \rangle$ is a standard primitive recursive pairing function. To simplify the notation we also use capital letters $X, Y, Z, \ldots$ in $\mathcal{L}^*$ formulas to denote subsets of N.
If we write $\mathsf{Wo}(z)$ for $\mathsf{Wo}(\mathsf{Field}(z), z)$ and let Ψ^* denote the translation of Ψ to the language $\mathcal{L}^*$ then $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ proves that

$$(\forall x \in \mathsf{N})[\Psi^*(x, \vec{m}, \vec{Y}) \leftrightarrow \mathsf{Wo}(\{(u, v) : \langle u, v \rangle \in \mathsf{KB}((T)_x)\})].$$

Now the assumption $(*)$ puts us in position to apply Δ separation, i.e. there is a set $Z \subseteq \mathsf{N}$ with
$$\forall x[x \in Z \leftrightarrow x \in \mathsf{N} \wedge \Psi^*(x, \vec{m}, \vec{Y})].$$

But this means that we could embed $\Pi_1^1\text{-}\mathsf{CA}_0$ into $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$. $\square$

The ordering underlying a hierarchy is by definition a well-ordering. This fact allows us to prove properties of the hierarchy by transfinite induction along this well-ordering. For pseudo-hierarchies we can not apply this method of proof. Fortunately this does not matter, since $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ proves the existence of pseudo-hierarchies with "nice" properties:

Lemma 7 (Existence of pseudo-hierarchies) *Assume that $D(\vec{u}, x, y, z)$ and $B(\vec{u}, x, y, z)$ are Σ formulas of $\mathcal{L}^*$ and let A be an arbitrary formula of $\mathcal{L}^*$ such that $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ proves*

$$A \to \forall b, c\ [\mathsf{Wo}(b, c) \to \exists f(\mathsf{Hier}_D(\vec{a}, b, c, f) \wedge B(\vec{a}, b, c, f))].$$

Then $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ also proves

$$A \to \exists d, e, g\ [\mathsf{PSH}_D(\vec{a}, d, e, g) \wedge B(\vec{a}, d, e, g)].$$

Proof: Let A and $\vec{a}$ such that $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ proves

$(*)$ $\qquad A \to \forall b, c[\mathsf{Wo}(b, c) \to \exists f(\mathsf{Hier}_D(\vec{a}, b, c, f) \wedge B(\vec{a}, b, c, f))].$

Now we define

$$\mathsf{Hier}'_D(\vec{a}, b, c, f) : = \begin{cases} \mathsf{Lin}(b,c) \wedge \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = b \wedge \\ (\forall x \in b) D(\vec{a}, x, \{(y, f(y)) : (y, x) \in c\}, f(x)) \end{cases}$$

$$E(\vec{a}, b, c) : = \exists f[\mathsf{Hier}'_D(\vec{a}, b, c, f) \wedge B(\vec{a}, b, c, f)].$$

Note that $E(\vec{a}, b, c)$ is a Σ formula of $\mathcal{L}^*$. Hence lemma 6 yields that

$$\neg \forall x, y[E(\vec{a}, x, y) \leftrightarrow \mathsf{Wo}(x, y)].$$

So there are sets d and e such that

$$[\neg \mathsf{Wo}(d, e) \wedge E(\vec{a}, d, e)] \vee [\mathsf{Wo}(d, e) \wedge \neg E(\vec{a}, d, e)].$$

Assuming A, the second disjunct contradicts $(*)$, and the first disjunct is equivalent to

$$\exists g[\mathsf{PSH}_D(\vec{a}, d, e, g) \wedge B(\vec{a}, d, e, g)].$$

$\square$

To prove the existence of a fixed point of a given monotone Σ operation on the powerset of N, we first show that there is a pseudo-hierarchy such that the Σ operation specifies the transition from one stage of the hierarchy to the next. Then this pseudo-hierarchy is divided into an upper part that contains no least stage, and the corresponding lower part. The careful definition of the pseudo-hierarchy then implies that the union of the stages in the lower part equals the intersection of the stages in the upper part and is also a fixed point of the given operation. The details of this argument are carried out in the proof of the following theorem.

Theorem 8 (Existence of fixed points) *Let $F(\vec{u}, x, y)$ be a Σ formula of $\mathcal{L}^*$. Then $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ proves*

$$\mathsf{Op}^{\mathsf{N}}_F(\vec{a}) \wedge \mathsf{Mon}^{\mathsf{N}}_F(\vec{a}) \rightarrow (\exists x \subseteq \mathsf{N}) F(\vec{a}, x, x).$$

Proof: Suppose that $\mathsf{Op}^{\mathsf{N}}_F(\vec{a})$ and $\mathsf{Mon}^{\mathsf{N}}_F(\vec{a})$ hold. Next we set

$$D(\vec{u}, x, y, z) := \begin{cases} (\bigcup \mathsf{Rng}(y) \not\subseteq \mathsf{N} \wedge z = \bigcup \mathsf{Rng}(y)) \vee \\ (\bigcup \mathsf{Rng}(y) \subseteq \mathsf{N} \wedge x = x \wedge F(\vec{u}, \bigcup \mathsf{Rng}(y), z)). \end{cases}$$

Now assume that $\mathsf{Wo}(b, c)$. Since $\mathsf{Op}^{\mathsf{N}}_F(\vec{a})$ and $\mathsf{Mon}^{\mathsf{N}}_F(\vec{a})$ it follows immediately from the definition of D that

$$(\forall x \in b) \forall y \exists! z D(\vec{a}, x, y, z).$$

Hence Theorem 2 yields that there is a hierarchy f such that $\mathsf{Hier}_D(\vec{a}, b, c, f)$ holds. Moreover we can prove that f has the following properties expressed by the Σ formula $B(\vec{a}, b, c, f)$ that is the conjunction of the three formulas below:

(1) $(\exists x \in b)(\forall y \in b)((y, x) \notin c)$,

(2) $(\forall x, y \in b)[(x, y) \in c \to f(x) \subseteq f(y) \subseteq \mathsf{N}]$,

(3) $(\forall x \in b)(\forall n \in \mathsf{N})[n \in f(x) \to (\exists y \in b)(x = y \vee (y, x) \in c) \wedge (n \in f(y) \wedge (\forall z \in b)(z, y) \in c \to (n \notin f(z)))]$.

Thus $B(\vec{a}, b, c, f)$ states that the set b has a least element with respect to the ordering c, that the function f is monotone with $\mathsf{Rng}(f) \subseteq \mathsf{P}(\mathsf{N})$ and that if n is an element of $f(x)$ for some $x \in b$, then there is a least y with respect to the ordering c such that $n \in f(y)$.

The first and the third property follow directly from that fact that c is a well-ordering on b. The second property is shown by transfinite induction along c on b: We set

$$C(v) := (\forall x, y \in b)[(x, y) \in c \wedge (y, v) \in c \to f(x) \subseteq f(y) \subseteq \mathsf{N}]$$

and assume that $w \in b$ and that for all $v \in b$ with $(v, w) \in c$ we have $C(v)$. We aim to show that $C(w)$. So assume that $x, y \in b$ with $(x, y) \in c$ and $(y, w) \in c$. $\mathsf{Hier}_D(\vec{a}, b, c, f)$ implies that $D(\vec{a}, y, \{(z, f(z)) : (z, y) \in c\}, f(y))$. Now $C(y)$ implies that $\bigcup\{f(z) : (z, y) \in c\} \subseteq \mathsf{N}$. Hence the definition of D and $\mathsf{Mon}_F^\mathsf{N}(\vec{a})$ yield $f(x) \subseteq \bigcup\{f(z) : (z, y) \in c\} \subseteq f(y) \subseteq \mathsf{N}$. But this is $C(w)$.

So we have shown that $\mathsf{Op}_F^\mathsf{N}(\vec{a})$ and $\mathsf{Mon}_F^\mathsf{N}(\vec{a})$ imply

$$\forall b, c[\mathsf{Wo}(b, c) \to \exists f(\mathsf{Hier}_D(\vec{a}, b, c, f) \wedge B(\vec{a}, b, c, f))].$$

By lemma 7 we conclude that there exist sets d, e and a function g such that

$$\mathsf{PSH}_D(\vec{a}, d, e, g) \wedge B(\vec{a}, d, e, g).$$

Next, we divide d in an upper part u and a lower part l. Because $\neg\mathsf{Wo}(d, e)$ there is a set $u \subseteq d$ that has no least element with respect to the linear ordering e. Further we can assume that the set u is upwards closed, i.e. if $x \in u$ and if $(x, y) \in e$ then also $y \in u$. The lower part $l \subseteq d$ is now defined as $l := \{x \in d : (\forall y \in u)(x, y) \in e\}$. Since d has a least element with respect to the ordering e, l is not empty.

Looking for a candidate for a fixed point of the operator F, we set

$$w := \bigcup_{x \in l} g(x),$$
$$w' := \bigcap_{x \in u} g(x).$$

The monotonicity of g implies $w \subseteq w'$, but also $w' \subseteq w$ holds: If $y \in w'$ there is a least element $x_0 \in d$ with respect to the ordering e such that $y \in g(x_0)$. Because u

has no least element with respect to the ordering e, x_0 belongs already to l, yielding $y \in w$. This means that $w = w'$. Next we argue that w is indeed a fixed point of F.

First, we fix a set z such that $F(\vec{a}, w, z)$ holds, and then we choose an arbitrary $x \in l$. We have that

$$F(\vec{a}, \bigcup \{g(y) : (y, x) \in e\}, g(x)).$$

Further, $\mathsf{Mon}_F^{\mathsf{N}}(\vec{a})$ and the definition of w imply $g(x) \subseteq z$. This means that for all $x \in l$ we have $g(x) \subseteq z$, hence $w \subseteq z$.

Similarly, for an arbitrary $x \in u$,

$$F(\vec{a}, \bigcup \{g(y) : (y, x) \in e\}, g(x))$$

holds. This time we conclude that $z \subseteq g(x)$. So for all $x \in u$ we have $z \subseteq g(x)$, hence $z \subseteq w'$. But since $w = w'$ we have $z = w$ and therefore $F(\vec{a}, w, w)$. □

This concludes the argument that $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ proves the principle $(\Sigma\text{-}\mathsf{FP})$. Due to lemma 4, the theories $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ and $[\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})]^\dagger$ have the same proof-theoretic ordinal.

Theorem 9

$$|\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{FP})| \leq \varphi\omega00.$$

5 Lower bound

The purpose of this section is to prove that $\varphi\omega00$ is also a lower bound for the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{FP})$. Our strategy is to adapt the well-ordering proof for $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ given in [6] to the present context. This well-ordering proof is based on the well-ordering proof in Jäger, Kahle, Setzer and Strahm [5]. Also Rüede [10] adopts this proof for the treatment of Σ_1^1 transfinite dependence choice.

In the well-ordering proof in [6], one defines for each natural number n, a Δ_0 formula $\mathcal{K}_{n+1}(x)$ of $\mathcal{L}^*$ that satisfies the following property:

$$\mathcal{K}_{n+1}(y) \rightarrow (\forall x \in y)\mathsf{TI}(x, \varphi\hat{n}00).$$

Then one proves the so called Main Lemma, claiming that for every set x, there exists a set y, such that $x \in y$ and $\mathcal{K}_{n+1}(y)$. As an immediate consequence, the ordinal $\varphi\hat{n}00$ is provable for every natural number n.

We start by repeating the definitions of the Δ_0 formulas $\mathcal{K}_n(y)$. They express that y respects certain closure properties regarding hierarchies. So $\mathcal{K}_0(a)$ says that a is an admissible, and $\mathcal{K}_{n+1}(a)$ states that a is a limit of admissibles that contains with a set x also a $\mathcal{K}_n$-hierarchy f above x, whose domain is the set of notations $\mathfrak{a} \in \mathsf{OT}$ that look like a well-ordering in a. The formal definitions are given below.

Definition 10

$$\overline{\mathsf{Ad}}(a) \quad := \quad \mathsf{Ad}(a) \vee (\exists u(u \in a) \wedge (\forall x \in a)(\exists y \in a)(x \in y \wedge \mathsf{Ad}(y)))$$

$$\mathcal{T}(f) \quad := \quad \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = \mathsf{OT},$$

$$\mathcal{K}_1(a) \quad := \quad \mathsf{Ad}(a),$$

$$\mathcal{K}_{n+1}(a) \quad := \quad \overline{\mathsf{Ad}}(a) \wedge [\forall x \exists f(\mathcal{T}(f) \wedge \forall \mathfrak{a}(\mathsf{Wo}(\mathfrak{a}) \to \mathcal{H}_n(\mathfrak{a}, x, f)))]^a,$$

$$\mathcal{H}_n(\mathfrak{a}, u, f) \quad := \quad \mathcal{T}(f) \wedge (\forall \mathfrak{b} \prec \mathfrak{a})(f{\restriction}\mathfrak{b} \in f(\mathfrak{b}) \wedge u \in f(\mathfrak{b}) \wedge \mathcal{K}_n(f(\mathfrak{b}))).$$

The closure properties expressed by the Δ_0 formulas $\mathcal{K}_n(x)$ are tailored such that the following lemma holds. For a proof see lemma 11 in [6].

Lemma 11 KPi^0 *proves for each natural number n that*

$$\mathcal{K}_{n+1}(y) \to (\forall x \in y)\mathsf{TI}(x, \varphi \hat{n}00).$$

An inspection of the proof of the Main Lemma in [6] yields that one needs to iterate Σ operations that map elements of the class $\overline{\mathsf{Ad}}$ on elements of $\overline{\mathsf{Ad}}$ along (primitive) recursive well-orderings on N. Since our axiom (Σ-FP) only claims the existence of fixed points of monotone Σ operators on the powerset of N, we only can iterate Σ operations that map subsets of N to subsets of N (cf. lemma 15). To adapt the well-ordering proof, we code elements of a suitable subclass of $\overline{\mathsf{Ad}}$ as subsets of natural numbers. Thereto we draw upon the additional assumption that there exists a set for which transfinite induction up to $\varphi \omega 00$ fails. Hence our Main Lemma takes the following form.

Lemma 12 (Main Lemma) *For every natural number n greater than 0 we have that* $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ *proves:*

$$\exists x \neg \mathsf{TI}(x, \varphi \omega 00) \to \forall x \exists y(x \in y \wedge \mathcal{K}_n(y)).$$

To remove the additional assumption, observe that we have for all natural numbers n:

(1) $\mathsf{KPi}^0 + (\Sigma\text{-FP}) \vdash \forall x \mathsf{TI}(x, \varphi \omega 00) \to \forall x \mathsf{TI}(x, \varphi \hat{n}00)$,

(2) $\mathsf{KPi}^0 + (\Sigma\text{-FP}) \vdash \exists x \neg \mathsf{TI}(x, \varphi \omega 00) \to \forall x \mathsf{TI}(x, \varphi \hat{n}00)$.

(1) is a trivial observation and (2) follows from lemma 11 and lemma 12. Together, they imply that $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ proves the ordinal $\varphi \hat{n}00$ for each natural number n.

Next we introduce the class $\mathsf{Ad}_\circ$ of *good* admissibles, i.e. admissibles that contain a set for which transfinite induction up to $\varphi \omega 00$ fails,

$$\mathsf{Ad}_\circ(a) :\Leftrightarrow \mathsf{Ad}(a) \wedge (\exists x \in a)\neg \mathsf{TI}(x, \varphi \omega 00).$$

Of course, good admissibles are also linearly ordered and there is always a next good admissible. It is clear that the existence of good admissibles is an immediate consequence of the assumption $\exists x \neg \mathsf{TI}(x, \varphi\omega 00)$. Below we show how to represent good admissibles as subsets of the natural numbers. For that purpose, we assign to an admissible a the set $\mathsf{p}(a) \subseteq \mathsf{N}$ given by

$$\{e \in \mathsf{N} : (\forall x \in \mathsf{N})\{e\}(x) \in \{0,1\} \wedge \mathsf{Wo}^a(\{(m,n) \in \mathsf{N} : \{e\}(\langle m,n \rangle) = 0\})\}.$$

So $\mathsf{p}(a)$ is the set of all indices of characteristic functions whose associated relations are well-orderings in a. As usual $\{\cdot\}$ denotes Kleene brackets.

Lemma 13 $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ *proves:*

$$\mathsf{Ad}_\circ(a) \rightarrow \mathsf{p}(a) \notin a.$$

Proof: Suppose that a is a good admissible. We argue that $\mathsf{p}(a) \in a$ implies that a is a model of $\Pi^1_1\text{-CA}^-_0$, i.e. the theory that extends ACA_0 by comprehension for Π^1_1 formulas that do not contain set parameters. However this contradicts the definition of $\mathsf{Ad}_\circ$, namely that $(\exists x \in a)\neg\mathsf{TI}(x, \varphi\omega 00)$.

Let $\Psi(u, \vec{v})$ be a Π^1_1 formula of L_2 without set parameters. As in the proof of lemma 6, ACA_0 proves that for all natural numbers $\vec{m}$ there is a set T dependent on Ψ and the parameters $\vec{m}$ such that for all x:

(1) $(T)_x$ is a tree,

(2) $(T)_x$ is recursive ,

(3) $\Psi(x, \vec{m}) \leftrightarrow \mathsf{WO}(\mathsf{KB}((T)_x))$.

Since a is a model of ACA_0, $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ proves that

$$(\forall x \in \mathsf{N})[(\Psi^*)^a(x, \vec{m}) \leftrightarrow \mathsf{Wo}^a(\{(u,v) : \langle u,v \rangle \in \mathsf{KB}((T)_x)\})].$$

On the other hand, the definition of $\mathsf{p}(a)$ yields that for any recursive ordering $S \subseteq N$, we have that $\mathsf{Wo}^a(\{(u,v) : \langle u,v \rangle \in S\})$ is equivalent to

$$(\exists e \in \mathsf{p}(a))(\forall u,v \in \mathsf{N})[\{e\}(\langle u,v \rangle) = 1 \leftrightarrow \langle u,v \rangle \in S].$$

For all $x \in \mathsf{N}$ the set $\mathsf{KB}((T)_x)$ is recursive. Since a is also a model of $(\Delta_0\text{-Sep})$ and $\mathsf{p}(a) \in a$, there exists a set $Y \in a$ such that

$$\forall x[x \in Y \leftrightarrow x \in \mathsf{N} \wedge (\Psi^*)^a(x, \vec{m})].$$

This means that a is a model of $\Pi^1_1\text{-CA}^-_0$. $\qquad\qquad\square$

So if a is a good admissible, $\mathsf{p}(a) \notin a$ holds. Since $\mathsf{p}(a) \in a^+$, the linearity of admissibles yields that $\mathsf{p}(a)^+ = a^+$. This allows us to regain the admissible a from the set $\mathsf{p}(a)$. We define

$$\mathsf{q}(x) := \begin{cases} \emptyset^+ & : x \in \emptyset^+ \\ \text{the unique } y \text{ with } \mathsf{Ad}(y) \wedge y^+ = x^+ & : \text{otherwise} \end{cases}$$

and obtain the following corollary.

Corollary 14 *The following is provable in* $\mathsf{KPi}^0 + (\Sigma\text{-FP})$:

$$\mathsf{Ad}_\circ(a) \to a = \mathsf{q}(\mathsf{p}(a)).$$

Next we show that $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ enables us to iterate Σ operations that map subsets of natural numbers to subsets of natural numbers along well-orderings on subsets of N. First we introduce some auxiliary notions.

$$\begin{aligned} \mathsf{Fun_N}(f) &:= f \subseteq \mathsf{N} \wedge (\forall x \in f)(\exists y, z)(x = \langle y, z \rangle), \\ \mathsf{Support_N}(f) &:= \{x \in \mathsf{N} : (\exists y \in \mathsf{N})(\langle x, y \rangle \in f)\}. \end{aligned}$$

$\mathsf{Fun_N}(f)$ expresses that f is a subset of N consisting of codes of ordered pairs. Such an f can be seen to represent a function $g : \mathsf{N} \to \mathsf{P}(\mathsf{N})$,

$$g := \{(x, y) : x \in \mathsf{N} \wedge y = \{n \in \mathsf{N} : \langle x, n \rangle \in f\}\}.$$

If $n \notin \mathsf{Support_N}(f)$ then $g(n) = \emptyset$. Further, we say that f is an N-hierarchy for the $\mathcal{L}^*$ formula $D(\vec{u}, x, y, z)$ with parameters $\vec{a}$ along the well-ordering c on b, if $\mathsf{Hier}_D^\mathsf{N}(\vec{a}, b, c, f)$, where $\mathsf{Hier}_D^\mathsf{N}(\vec{a}, b, c, f)$ is the formula

$$\begin{aligned} &\mathsf{Fun_N}(f) \wedge \mathsf{Support_N}(f) \subseteq b \wedge \\ &(\forall x \in b)D(\vec{a}, x, \{\langle v, w \rangle \in f : (v, x) \in c\}, \{w : \langle x, w \rangle \in f\}). \end{aligned}$$

Lemma 15 *For all Σ formulas $D(\vec{u}, x, y, z)$ of $\mathcal{L}^*$, $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ proves: If $b \subseteq \mathsf{N}$ and $\mathsf{Wo}(b, c)$ then*

$$(\forall x \in b)(\forall y \subseteq \mathsf{N})(\exists! z \subseteq \mathsf{N})D(\vec{a}, x, y, z) \to \exists f \mathsf{Hier}_D^\mathsf{N}(\vec{a}, b, c, f).$$

Proof: We adapt Avigad's argument from [1], which shows that $\mathsf{ATR_0}$ proves the existence of fixed points of monotone operators given by an arithmetical formula, to the present context.

Assume that we have sets $\vec{a}$, $b \subseteq \mathsf{N}$ and c satisfying the left hand side of the implication. We aim to find an $f \subseteq \mathsf{N}$ such that $\mathsf{Hier}_D^\mathsf{N}(\vec{a}, b, c, f)$. Thereto, a monotone operator is defined, whose fixed point $g \subseteq \mathsf{N}$ codes the characteristic function of f.

First, we introduce two auxiliary operations mapping $\mathsf{P}(\mathsf{N}) \times \mathsf{P}(\mathsf{N}) \to \mathsf{P}(\mathsf{N})$, monotone in each component. Thereby, if $x \subseteq \mathsf{N}$, we write $\langle x \rangle$ for the set $\{\langle m, n \rangle : \langle m, n \rangle \in x\}$ and $\overline{x}$ for the set $\{n \in \mathsf{N} : n \notin x\}$.

$$B_1(\vec{u}, w, x, y, z) \quad := \quad \begin{cases} (z = \mathsf{N} \wedge \langle x \rangle \cup \langle y \rangle = \langle \mathsf{N} \rangle \wedge \langle x \rangle \cap \langle y \rangle \neq \emptyset) \vee \\ (D(\vec{u}, w, x, z) \wedge \langle x \rangle \cup \langle y \rangle = \langle \mathsf{N} \rangle \wedge \langle x \rangle \cap \langle y \rangle = \emptyset) \vee \\ (z = \emptyset \wedge \langle x \rangle \cup \langle y \rangle \neq \langle \mathsf{N} \rangle), \end{cases}$$

$$B_2(\vec{u}, w, x, y, z) \quad := \quad \begin{cases} (z = \mathsf{N} \wedge \langle x \rangle \cup \langle y \rangle = \langle \mathsf{N} \rangle \wedge \langle x \rangle \cap \langle y \rangle \neq \emptyset) \vee \\ (D(\vec{u}, w, x, \overline{z}) \wedge \langle x \rangle \cup \langle y \rangle = \langle \mathsf{N} \rangle \wedge \langle x \rangle \cap \langle y \rangle = \emptyset) \vee \\ (z = \emptyset \wedge \langle x \rangle \cup \langle y \rangle \neq \langle \mathsf{N} \rangle). \end{cases}$$

So if x contains exactly those pairs not in y and the operation defined by D assigns z to x, then the operations given by B_1 and B_2 assign to x and y the sets z and $\overline{z}$, respectively. The other cases are defined to assure monotonicity in both components.

For $s \subseteq \mathsf{N}$ and $w \in b$, we further set

$$\begin{aligned} \mathsf{lev}(s, w, c) \quad &:= \quad \{\langle v, n \rangle : \langle \langle v, n \rangle, 0 \rangle \in s \wedge (v, w) \in c\}, \\ \overline{\mathsf{lev}}(s, w, c) \quad &:= \quad \{\langle v, n \rangle : \langle \langle v, n \rangle, 1 \rangle \in s \vee (v, w) \notin c\}. \end{aligned}$$

Under the assumption that s codes the characteristic function of a hierarchy and $w \in b$, then $\mathsf{lev}(s, w, c)$ is the disjoint union of all levels below w, and $\overline{\mathsf{lev}}(s, w, c)$ is its complement w.r.t. pairs. Now we choose $C(\vec{u}, v, w, x, y)$ to be a Σ formula expressing that y is the union of the two sets

$$\{\langle \langle v, n \rangle, 0 \rangle : v \in b \wedge n \in \mathsf{N} \wedge \exists z [n \in z \wedge B_1(\vec{a}, v, \mathsf{lev}(x, v, c), \overline{\mathsf{lev}}(x, v, c), z)]\},$$

$$\{\langle \langle v, n \rangle, 1 \rangle : v \in b \wedge n \in \mathsf{N} \wedge \exists z [n \in z \wedge B_2(\vec{a}, v, \mathsf{lev}(x, v, c), \overline{\mathsf{lev}}(x, v, c), z)]\}.$$

It is not hard to see that $\mathsf{Op}_C^\mathsf{N}(\vec{a}, b, c)$ and $\mathsf{Mon}_C^\mathsf{N}(\vec{a}, b, c)$, which yields a g with $C(\vec{a}, b, c, g, g)$. By transfinite induction we show that for all $v \in b$,

$$(*) \qquad\qquad (\forall n \in \mathsf{N})(\langle \langle v, n \rangle, 0 \rangle \in g \leftrightarrow \langle \langle v, n \rangle, 1 \rangle \notin g).$$

Obviously, if 0 is the c-least element of b, then $\mathsf{lev}(g, 0, c) = \emptyset$ and $\overline{\mathsf{lev}}(g, 0, c) = \langle \mathsf{N} \rangle$. Thus the claim follows from the definition of C. And if $(*)$ holds for all $v' \in b$ with $(v', v) \in c$, then $\mathsf{lev}(g, v, c) \cup \overline{\mathsf{lev}}(g, v, c) = \langle \mathsf{N} \rangle$ and $\mathsf{lev}(g, v, c) \cap \overline{\mathsf{lev}}(g, v, c) = \emptyset$, which yields the induction step.

Thus, if we let $f := \{\langle v, n \rangle : \langle \langle v, n \rangle, 0 \rangle \in g\}$ and $v \in b$, then the set $z := \{n : \langle v, n \rangle \in f\}$ satisfies $D(\vec{a}, v, \mathsf{lev}(g, v, c), z)$, therefore $\mathsf{Hier}_D^\mathsf{N}(\vec{a}, b, c, f)$. $\qquad\square$

Next we show that we can iterate Σ operations, whose range consists of good admissibles only, along well-orderings on subsets of N.

Lemma 16 *If $D(\vec{u}, x, y, z)$ is a Σ formula of $\mathcal{L}^*$, then $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ proves that the following two assertions*

(1) $b \subseteq \mathsf{N} \wedge \mathsf{Wo}(b, c) \wedge (\forall x \in b)\forall y \exists! z D(\vec{a}, x, y, z)$,

(2) $(\forall x \in b)\forall y, z D(\vec{a}, x, y, z) \rightarrow \mathsf{Ad}_\circ(z)$,

imply

(3) $\exists f \mathsf{Hier}_D(\vec{a}, b, c, f)$.

Proof: We assume that (1) and (2) hold, and set

$$u^* := \{(x, z) : x \in \mathsf{Support}_\mathsf{N}(u) \wedge z = \mathsf{q}(\{y : \langle x, y \rangle \in u\})\}.$$

The idea behind this definitions is the following: If $h \subseteq \mathsf{N}$ and $\mathsf{Fun}_\mathsf{N}(h)$, then h^* is a function that maps elements of $\mathsf{Support}_\mathsf{N}(h)$ to admissibles. Now let

$$D'(\vec{u}, x, y, z) := \exists r (D(\vec{u}, x, y^*, r) \wedge z = \mathsf{p}(r)).$$

By lemma 15 there exists a $g \subseteq \mathsf{N}$ such that $\mathsf{Hier}^\mathsf{N}_{D'}(\vec{a}, b, c, g)$. Then the function

$$f := \{(x, y) : x \in b \wedge y = \{z : \langle x, z \rangle \in g\}\}.$$

satisfies $\mathsf{Hier}_D(\vec{a}, b, c, f)$: That $\mathsf{Fun}(f)$ and $\mathsf{Dom}(f) = b$ is obvious. It remains to show that

$$(\forall x \in b) D(\vec{a}, x, \{(y, f(y)) : (y, x) \in c\}, f(x)).$$

We fix an $x \in b$. $\mathsf{Hier}^\mathsf{N}_{D'}(\vec{a}, b, c, g)$ implies that

$$D'(\vec{a}, x, \{\langle v, w \rangle : (v, x) \in c\}, \{w : \langle x, w \rangle \in g\}).$$

By the definition of D' there is a set r such that

$$D(\vec{a}, x, \{\langle v, w \rangle : (v, x) \in c\}^*, r) \wedge \{w : \langle x, w \rangle \in g\} = \mathsf{p}(r).$$

The definition of f implies that $f(x) = \mathsf{q}(\{w : \langle x, w \rangle \in g\}) = r$ and that

$$\{\langle v, w \rangle : (v, x) \in c\}^* = \{(y, f(y)) : (y, x) \in c\}.$$

Therefore, we have for all $x \in b$,

$$D(\vec{a}, x, \{(y, f(y)) : (y, x) \in c\}, f(x)),$$

which is $\mathsf{Hier}_D(\vec{a}, b, c, f)$. $\qquad\square$

In his well-ordering proof for second order arithmetic with Σ_1^1 transfinite dependent choice, Rüede makes use of Π_2^1 reflection on ω-models of ACA_0 to prove the Main Lemma. In our present context, this part is taken over by Π_2 reflection on a suitable subclass of $\overline{\mathsf{Ad}}$. This suitable subclass of $\overline{\mathsf{Ad}}$ comprises those elements of $\overline{\mathsf{Ad}}$ that are the limits of increasing ω-sequences of good admissibles,

$$\mathsf{Seq}_{\mathsf{Ad}_\circ}(f) := \left\{ \begin{array}{l} \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = \mathsf{N} \wedge \\ (\forall n \in \mathsf{N})[\mathsf{Ad}_\circ(f(n)) \wedge f(n) \in f(n+1)]. \end{array} \right.$$

Thus a sequence f in $\mathsf{Seq}_{\mathsf{Ad}_\circ}$ represents the limit $\bigcup \mathsf{Rng}(f)$, which in turn is coded by the following set of natural numbers, $\{\langle n, x \rangle : x \in \mathsf{p}(f(n))\}$. It is not hard to obtain the following generalization of the previous lemma.

Lemma 17 *If $D(\vec{u}, x, y, z)$ is a Σ formula of $\mathcal{L}^*$, then $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ proves that the following two assertions*

(1) $b \subseteq \mathsf{N} \wedge \mathsf{Wo}(b, c) \wedge (\forall x \in b) \forall y \exists! z D(\vec{a}, x, y, z)$,

(2) $(\forall x \in b) \forall y, z D(\vec{a}, x, y, z) \rightarrow \mathsf{Seq}_{\mathsf{Ad}_\circ}(z)$,

imply

(3) $\exists f \mathsf{Hier}_D(\vec{a}, b, c, f)$.

By means of lemma 16, we get the following uniform variant of Π_2 reflection on $\overline{\mathsf{Ad}}$.

Lemma 18 (Π_2 reflection on $\overline{\mathsf{Ad}}$) *For any Σ formula $A(\vec{u}, v, w)$ of $\mathcal{L}^*$ with at most the displayed variables $\vec{u}, v, w$ free, there exists a Σ formula $A^\sharp(\vec{u}, v)$ of $\mathcal{L}^*$ with at most the variables $\vec{u}, v$ free, so that the following two assertions can be proved in $\mathsf{KPi}^0 + (\Sigma\text{-FP}) + \exists z \neg \mathsf{TI}(z, \varphi\omega 00)$:*

(1) $\forall x \exists y A(\vec{a}, x, y) \rightarrow \exists! f A^\sharp(\vec{a}, f)$,

(2) $\forall x \exists y A(\vec{a}, x, y) \rightarrow$
 $\quad \forall f (A^\sharp(\vec{a}, f) \rightarrow \mathsf{Seq}_{\mathsf{Ad}_\circ}(f) \wedge \vec{a} \in \bigcup \mathsf{Rng}(f) \wedge [\forall x \exists y A(\vec{a}, x, y)]^{\bigcup \mathsf{Rng}(f)}$.

Proof: Using lemma 16, the proof of lemma 7 in [6] easily translates to the present context. □

Now the stage is set to prove the Main Lemma. Actually we show a bit more, so that we can prove the lemma by meta-induction on the natural numbers.

Lemma 19 *For every natural number n greater than 0 there exists a Σ formula $F_n(u, v)$ of $\mathcal{L}^*$ such that $\mathsf{KPi}^0 + (\Sigma\text{-FP}) + \neg \mathsf{TI}(x, \varphi\omega 00)$ proves:*

(1) $\forall x \exists! f F_n(x, f)$,

(2) $\forall x \forall f[F_n(x, f) \rightarrow \mathsf{Seq}_{\mathsf{Ad}_\circ}(f) \wedge x \in \bigcup \mathsf{Rng}(f) \wedge \mathcal{K}_n(\bigcup \mathsf{Rng}(f))]$.

Proof: Again, we refer to the proof of the corresponding Main Lemma in [6]. Replace their theorem 6 by our lemma 17. □

From the discussion at the beginning of this section we obtain the lower bound.

Theorem 20
$$|\mathsf{KPi}^0 + (\Sigma\text{-FP})| \geq \varphi\omega 00.$$

Further we can conclude from theorem 9 and results of Jäger and Strahm in [8] that the theories $\mathsf{KPi}^0 + (\Sigma\text{-TR})$, $\mathsf{KPi}^0 + (\Sigma\text{-FP})$ and KPm^0 all have the same proof-theoretic ordinal.

Corollary 21
$$|\mathsf{KPi}^0 + (\Sigma\text{-TR})| = |\mathsf{KPi}^0 + (\Sigma\text{-FP})| = |\mathsf{KPm}^0| = \varphi\omega 00.$$

6 Concluding remarks

As mentioned earlier, the restriction in the formulation of the axiom $(\Sigma\text{-FP})$ to operators on the powerset of the natural numbers is to keep the theory persistent. We will elaborate on this point a bit further.

This time, we set for all formulas $A(\vec{u}, v, w)$ of $\mathcal{L}^*$,

$$\mathsf{Op}_A(\vec{a}) \; := \; \forall x \exists! y[\mathsf{S}(x) \rightarrow \mathsf{S}(y) \wedge A(\vec{a}, x, y)],$$
$$\mathsf{Mon}_A(\vec{a}) \; := \; \forall u, v, x, y[A(\vec{a}, u, x) \wedge A(\vec{a}, v, y) \wedge u \subseteq v \rightarrow x \subseteq y].$$

The principle $(\Sigma\text{-FP}')$, claiming the existence of fixed points of monotone Σ operators on the whole universe, takes now the following form: For each Σ formula $A(\vec{u}, v, w)$ of $\mathcal{L}^*$ with at most the variables $\vec{u}, v, w$ free, we have

$(\Sigma\text{-FP}')$ $\qquad\qquad \mathsf{Op}_A(\vec{a}) \wedge \mathsf{Mon}_A(\vec{a}) \rightarrow \exists x[\mathsf{S}(x) \wedge A(\vec{a}, x, x)]$.

From the proof of theorem 8 it follows that $[\mathsf{KPi}^0 + (\Sigma\text{-TR})]^\dagger$ proves also all instances of the principle $(\Sigma\text{-FP}')$. However, if we further strengthen the theory $\mathsf{KPi}^0 + (\Sigma\text{-FP}')$ by an axiom claiming foundation

(Foundation) $\qquad\qquad \forall x(\exists u(u \in x) \rightarrow (\exists y \in x)(y \cap x = \emptyset))$,

it becomes inconsistent.

To see this, we consider the operation

$$\mathsf{o}(x) := \bigcup\{\alpha + 1 : \alpha \in x \wedge \mathsf{Ord}(\alpha)\}.$$

Note that $\mathsf{o}(x)$ is an ordinal and that $\mathsf{o}(\alpha) = \alpha$ for all ordinals α. The theory KPi^0 also proves that ordinals are linearly ordered by the elementhood relation $\in$.

Lemma 22 KPi^0 *proves that the operation* $x \mapsto \mathsf{o}(x) \cup \{\mathsf{o}(x)\}$ *is monotone but has no fixed point.*

Proof: Ordinals are linarly ordered by the $\in$ relation, hence the operation is monotone. Since ordinals are well-founded by $\in$, it has no fixed point. $\qquad\Box$

In the presence of foundation, being an ordinal is a Δ predicate, i.e. $\mathsf{Ord}(\alpha)$ is equivalent to $\mathsf{Tran}(\alpha) \wedge (\forall x \in \alpha)\mathsf{Tran}(x)$. Thus, the aforementioned operation becomes definable by a Σ formula of $\mathcal{L}^*$, which implies the inconsistency of $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{FP}') + (\mathsf{Foundation})$. Moreover, the operation $x \mapsto \mathsf{o}(x) \cup \{\mathsf{o}(x)\}$ is definable by a Σ formula of $\mathcal{L}^*$ in any extension of KPi^0 where $\mathsf{Wo}(u,v)$ is a Δ formula, i.e. where there exists a Σ formula $A(\vec{u}, v, w)$ of $\mathcal{L}^*$ such that

$$\exists \vec{x} \forall y, z[\mathsf{Wo}(y,z) \leftrightarrow A(\vec{x}, y, z)]$$

is provable. Due to the previous lemma, such an extension cannot consistently be further extended by the principle $(\Sigma\text{-}\mathsf{FP}')$.

We conclude this section by relating the consistency of $\mathsf{T} + (\Sigma\text{-}\mathsf{FP}')$ to the consistency of $\mathsf{T} + (\Sigma\text{-}\mathsf{TR})$ for theories T comprising KPi^0.

Lemma 23 *There is a Σ formula $A(u,v)$ of $\mathcal{L}^*$ such that KPi^0 proves*

$$A(x,y) \rightarrow y = \bigcap\{z : x \subseteq z \wedge \mathsf{Ad}(z)\}.$$

This justifies the notation x° for the set $\bigcap\{z : x \subseteq z \wedge \mathsf{Ad}(z)\}$.

Proof: There is a Σ formula $A(u,v)$ of $\mathcal{L}^*$ such that $A(x,y)$ implies that

$$y = \bigcap\{z \in (x^+)^+ : x \subseteq z \wedge \mathsf{Ad}(z)\}.$$

Since admissibles are linearly ordered by $\in$, an admissible z that is not an element of $(x^+)^+$ satisfies already $x \subseteq x^+ \subseteq z$. Thus, $A(x,y)$ implies $y = x^\circ$. $\mathsf{Op}_A^{\mathsf{N}}$ and $\mathsf{Mon}_A^{\mathsf{N}}$ are now obvious. $\qquad\Box$

Lemma 24 $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{FP}')$ *proves each instance of* $(\Sigma\text{-}\mathsf{TR})$.

Proof: Suppose that $A(\vec{u}, v, w)$ is a Σ formula of $\mathcal{L}^*$ and $\vec{a}$ are sets, such that $\mathsf{Op}_A^{\mathsf{N}}(\vec{a})$ and $\mathsf{Mon}_A^{\mathsf{N}}(\vec{a})$ hold. Further, assume $\mathsf{Ord}(b)$. We show that $\exists f \mathsf{Hier}_A(\vec{a}, b, f)$.

By the previous lemma there is a Σ formula B with $\mathsf{Op}_B^{\mathsf{N}}(\vec{a}, b)$ and $\mathsf{Mon}_B^{\mathsf{N}}(\vec{a}, b)$, such that $B(\vec{a}, b, x, y)$ implies

$$y = (\{v : A(\vec{a}, u, v) \wedge u \in x\} \cup \{b\})^\circ.$$

Now (Σ-FP$'$) yields a set z satisfying $B(\vec{a}, b, z, z)$. By transfinite induction along b we show that $(\forall w \in b)(\exists! f \in z)\mathsf{Hier}_A(\vec{a}, w, f)$. So suppose that $w \in b$ and for all $v \in w$, there is exactly one $g \in z$ satisfying $\mathsf{Hier}_A(\vec{a}, v, g)$. By Σ replacement, there is a unique function h with domain w, such that for $v \in w$, $h(v) \in z$ and $\mathsf{Hier}_A(\vec{a}, v, h(v))$. Moreover, h is an element of each admissible that is a superset of z, thus $h \in z$. For the induction step, let $s := \{(v', c) \in h(v) : v' \in v \in w\}$ and f the set with $A(\vec{a}, s, f)$. This yields $\mathsf{Hier}_A(\vec{a}, w, f)$. Similarly, we obtain $\exists f \mathsf{Hier}_A(\vec{a}, b, f)$. $\square$

Hence one can consistently extend a theory T by the principle (Σ-FP$'$) if and only if it is consistent to assume that $\mathsf{Wo}(u, v)$ is not a Δ formula of $\mathsf{T} + (\Sigma$-TR$)$.

Theorem 25 *Let T be a theory that comprises KPi^0. Then $\mathsf{T} + (\Sigma$-FP$')$ is consistent if and only if there is no Σ formula $A(\vec{u}, v, w)$ of $\mathcal{L}^*$ for which $\mathsf{T} + (\Sigma$-TR$)$ proves $\exists \vec{x} \forall y, z[A(\vec{x}, y, z) \leftrightarrow \mathsf{Wo}(y, z)]$.*

Proof: Suppose that $\mathsf{T} + (\Sigma$-FP$')$ is consistent. Lemma 24 yields (Σ-TR). If Wo were a Δ predicatae, the operation of lemma 22 would be definable by a Σ formula of $\mathcal{L}^*$, which contradicts (Σ-FP$'$).

If we assume the right hand side, we can consistently extend $\mathsf{T} + (\Sigma$-TR$)$ by a principle that claims that we have for every Σ formula $A(\vec{u}, v, w)$ of $\mathcal{L}^*$ that

$$\forall \vec{x} \neg \left(\forall y, z[A(\vec{x}, y, z) \leftrightarrow \mathsf{Wo}(y, z)] \right).$$

Now we can use the proof of theorem 8 to show (Σ-FP$'$). $\square$

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