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Research Article

Dynamics of a Nonautonomous Stochastic SIS Epidemic Model with Double Epidemic Hypothesis

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We investigate the dynamics of a nonautonomous stochastic SIS epidemic model with nonlinear incidence rate and double epidemic hypothesis. By constructing suitable stochastic Lyapunov functions and using Has'minskii theory, we prove that there exists at least one nontrivial positive periodic solution of the system. Moreover, the sufficient conditions for extinction of the disease are obtained by using the theory of nonautonomous stochastic differential equations. Finally, numerical simulations are utilized to illustrate our theoretical analysis.

1. Introduction

The SIS (Susceptible-Infected-Susceptible) model is a basic biological mathematical model describing susceptible and infected epidemic process and is first introduced by Kermack and McKendrick [1]. The SIS model is defined in that individuals start off susceptible, at some stage catch the disease, and after a short infectious period become susceptible again [2]. Therefore, some deterministic SIS epidemic models have been studied by many authors [3–10]. Recently, the authors of [11–13] investigated the epidemic model with double epidemic hypothesis which has two epidemic diseases caused by two different viruses. For example, the deterministic SIS epidemic model with nonlinear saturated incidence rate and double epidemic hypothesis can be expressed as follows [11]:

$$\begin{aligned}\dot{S}(t) &= A - dS(t) - \frac{\beta_1 S(t) I_1(t)}{a_1 + I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{a_2 + I_2(t)} \\ &\quad + r_1 I_1(t) + r_2 I_2(t), \\ \dot{I}_1(t) &= \frac{\beta_1 S(t) I_1(t)}{a_1 + I_1(t)} - (d + \alpha_1 + r_1) I_1(t), \\ \dot{I}_2(t) &= \frac{\beta_2 S(t) I_2(t)}{a_2 + I_2(t)} - (d + \alpha_2 + r_2) I_2(t),\end{aligned} \tag{1}$$

where $S(t)$, $I_1(t)$, and $I_2(t)$ represent the number of susceptibles and infected individuals with viruses A and B at time t, respectively. The parameters in model (1) have the following meanings: A is the total input susceptible population size, d represents the natural death rate of S, I_1, and I_2, β_i represents the disease transmission coefficient between compartments S and I_i $(i = 1, 2)$, r_1 and r_2 are the recovery rates of the two diseases, and α_1 and α_2 are mortality rates due to diseases, respectively. Functions $\beta_1 S(t) I_1(t)/(a_1 + I_2(t))$ and $\beta_2 S(t) I_2(t)/(a_2 + I_2(t))$ represent two different types of saturated incidence rates for the two epidemic diseases $I_1(t)$ and $I_2(t)$. All parameter values are nonnegative.

In the real world, population systems and epidemic systems are inevitably infected by some uncertain environmental disturbances. Hence, many authors have introduced stochastic interferences into differential systems, and the stochastic dynamics of such systems were widely investigated (see [14–28]). Moreover, numerous scholars have investigated some stochastic epidemic models (see [29–34]). For example, in [11, 30] they obtained thresholds of the stochastic system which determines the extinction and persistence of the epidemic. Zhang et al. [29] proved that there is a unique ergodic stationary distribution of his model. We assume that environment fluctuations will manifest themselves mainly as fluctuations in the saturated response rate, so that $\beta_i S(t)I_i(t)/(a_i+I_i(t)) \to \beta_i S(t)I_i(t)/(a_i+I_i(t)) + (\sigma_i S(t)I_i(t)/(a_i+I_i(t)))\dot{B}_i(t)$ $(i=1,2)$, where $B(t)=(B_1(t),B_2(t))$ is a standard Brownian motion with intensity $\sigma_i>0$ $(i=1,2)$. Therefore, a stochastic model is described by [11]

$$\begin{aligned}
\mathrm{d}S(t) &= \left(A - dS(t) - \frac{\beta_1 S(t) I_1(t)}{a_1 + I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{a_2 + I_2(t)} + r_1 I_1(t) + r_2 I_2(t)\right)\mathrm{d}t - \frac{\sigma_1 S(t) I_1(t)}{a_1 + I_1(t)}\mathrm{d}B_1(t) - \frac{\sigma_2 S(t) I_2(t)}{a_2 + I_2(t)}\mathrm{d}B_2(t),\\
\mathrm{d}I_1(t) &= \left(\frac{\beta_1 S(t) I_1(t)}{a_1 + I_1(t)} - (d + \alpha_1 + r_1) I_1(t)\right)\mathrm{d}t + \frac{\sigma_1 S(t) I_1(t)}{a_1 + I_1(t)}\mathrm{d}B_1(t),\\
\mathrm{d}I_2(t) &= \left(\frac{\beta_2 S(t) I_2(t)}{a_2 + I_2(t)} - (d + \alpha_2 + r_2) I_2(t)\right)\mathrm{d}t + \frac{\sigma_2 S(t) I_2(t)}{a_2 + I_2(t)}\mathrm{d}B_2(t).
\end{aligned} \tag{2}$$

However, many infectious diseases of human fluctuate over time and often show the seasonal morbidity. Therefore, the existence of periodic solutions of some nonautonomous epidemic models was explored [35–37]. Recently, many scholars focused on nonautonomous stochastic periodic systems. With the development of stochastic differential equations and application of Has'minskii theory, the existence of stochastic periodic solution has been studied [23, 38, 39]. In [23], Zhang et al. considered a nonautonomous stochastic Lotka-Volterra predator-prey model with impulsive effects; they got thresholds for stochastic persistence and extinction of the system. Authors of [38–40] investigated periodic solution of a stochastic nonautonomous epidemic model.

Based on the discussion above, in this paper, we consider a nonautonomous stochastic SIS model with periodic coefficients

$$\begin{aligned}
\mathrm{d}S(t) &= \left(A(t) - d(t)S(t) - \frac{\beta_1(t) S(t) I_1(t)}{a_1(t) + I_1(t)} - \frac{\beta_2(t) S(t) I_2(t)}{a_2(t) + I_2(t)} + r_1(t) I_1(t) + r_2(t) I_2(t)\right)\mathrm{d}t\\
&\quad - \frac{\sigma_1(t) S(t) I_1(t)}{a_1(t) + I_1(t)}\mathrm{d}B_1(t) - \frac{\sigma_2(t) S(t) I_2(t)}{a_2(t) + I_2(t)}\mathrm{d}B_2(t),\\
\mathrm{d}I_1(t) &= \left(\frac{\beta_1(t) S(t) I_1(t)}{a_1(t) + I_1(t)} - (d(t) + \alpha_1(t) + r_1(t)) I_1(t)\right)\mathrm{d}t + \frac{\sigma_1(t) S(t) I_1(t)}{a_1(t) + I_1(t)}\mathrm{d}B_1(t),\\
\mathrm{d}I_2(t) &= \left(\frac{\beta_2(t) S(t) I_2(t)}{a_2(t) + I_2(t)} - (d(t) + \alpha_2(t) + r_2(t)) I_2(t)\right)\mathrm{d}t + \frac{\sigma_2(t) S(t) I_2(t)}{a_2(t) + I_2(t)}\mathrm{d}B_2(t),
\end{aligned} \tag{3}$$

where the parameter functions $A(t)$, $d(t)$, $\beta_i(t)$, $a_i(t)$, $r_i(t)$, $\alpha_i(t)$, $\sigma_i(t)$ $(i=1,2)$ are positive, nonconstant, and continuous periodic functions with positive period T.

To the best of our knowledge, there are only few works on research of nonautonomous stochastic epidemic models with nonlinear saturated incidence rate and double epidemic hypothesis. Therefore, based on an autonomous stochastic epidemic model, we propose a nonautonomous stochastic model and investigate the existence of stochastic periodic solution and the extinction of the two epidemic diseases.

This paper is organized as follows. In Section 2, we give some definitions and known results. In Section 3, we prove that system (3) has a unique global positive solution. In Section 4, we present sufficient condition for the existence a nontrivial positive periodic solution of system (3). In Section 5, we obtain the sufficient conditions of system (3) for extinction of the two epidemic diseases. In Section 6, we carry out a series of numerical simulations to illustrate our theoretical findings.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). The function $B_i(t)$ $(i=1,2,3,4)$ is defined on this complete probability space.

For simplicity, some notations are given first. If $f(t)$ is an integrable function defined on $[0,\infty)$, define $\langle f\rangle_t = (1/t)\int_0^t f(s)ds$, $t>0$. If $f(t)$ is a bounded function on $[0,\infty)$, define $f^l = \inf_{t\in[0,\infty)} f(t)$ and $f^u = \sup_{t\in[0,\infty)} f(t)$.

Here we present some basic theory in stochastic differential equations which are introduced in [41].

In general, consider the l-dimensional stochastic differential equation

$$\mathrm{d}X(t) = f(X(t),t)\,\mathrm{d}t + g(X(t),t)\,dB(t), \quad t \ge t_0, \tag{4}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^l$. $B(t)$ stands for a l-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Denote by $C^{2,1}(\mathbb{R}^l \times [t_0, \infty]; \mathbb{R}_+)$ the family of all nonnegative functions $V(X,t)$ defined on $\mathbb{R}^l \times [t_0,\infty]$ such that they are continuously twice differentiable in X and once in t. The differential operator L of (4) is defined by [41]

$$L = \frac{\partial}{\partial t} + \sum f_i (X,t) \frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{l} \left[g^T (X,t)\, g(X,t)\right]_{ij} \frac{\partial^2}{\partial X_i \partial X_j}. \tag{5}$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^l \times [t_0, \infty]; \mathbb{R}_+)$, then

$$LV(X,t) = V_t(X,t) + V_X(X,t) f(X,t) + \frac{1}{2}\operatorname{trace}\left[g^T(X,t) V_{XX}(X,t)\, g(X,t)\right], \tag{6}$$

where $V_t = \partial V/\partial t$, $V_X = (\partial V/\partial x_1, \ldots, \partial V/\partial x_l)$, and $V_{XX} = (\partial^2 V/\partial X_i \partial X_j)_{l\times l}$. In view of Itô's formula, if $X(t) \in \mathbb{R}^l$, then

$$\mathrm{d}V(X(t),t) = LV(x(t),t)\,\mathrm{d}t + V_X(X(t),t)\, g(X(t),t)\, dB(t). \tag{7}$$

Definition 1 (see [42]). A stochastic process $\xi(t) = \xi(t,\omega)$ $(-\infty < t < +\infty)$ is said to be periodic with period T if for every finite sequence of numbers $t_1, t_2, \ldots, t_n$ the joint distribution of random variables $\xi(t_1+h), \xi(t_2+h), \ldots, \xi(t_n+h)$ is independent of h, where $h = kT$, $k = \pm 1, \pm 2, \ldots$.

It is shown in [42] that a Markov process $x(t)$ is T-periodic if and only if its transition probability function is T-periodic and the function $\mathbb{P}_0(t,A) = \mathbb{P}\{x(t) \in A\}$ satisfies the equation

$$\mathbb{P}_0(s,A) = \int_{\mathbb{R}^l} \mathbb{P}_0(s,dx)\, \mathbb{P}(s,x,s+T,A) = \mathbb{P}_0(s+T,A). \tag{8}$$

Consider the following equation:

$$X(t) = X(t_0) + \int_{t_0}^{t} b(s,X(s))\, ds + \sum_{r=1}^{k}\int_{t_0}^{t} \sigma_r(s,X(s))\, dB_r(s), \quad X \in \mathbb{R}^l. \tag{9}$$

Lemma 2 (see [42]). *Suppose that coefficients of (9) are T-periodic in t and satisfy the condition*

$$\left|b(s,x) - b(s,y)\right| + \sum_{r=1}^{k} \left|\sigma_r(s,x) - \sigma_r(s,y)\right| \leq P\left|x-y\right|, \tag{10}$$

$$|b(s,x)| + \sum_{r=1}^{k} \left|\sigma_r(s,x)\right| \leq P(1+|x|),$$

in every cylinder $I \times U$, where P is a constant. And suppose further that there exists a function $V(t,x) \in C^2$ in $\mathbb{R}^d$ which is T-periodic in t and satisfies the following conditions:

(A_1) $\inf_{|x|\geq \mathbb{R}} V(t,x) \to \infty$ *as* $\mathbb{R} \to \infty$.

(A_2) $LV(t,x) \leq -1$ *outside some compact set, where the operator L is given by*

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{l} b_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{l} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}, \quad a_{ij} = \sum_{r=1}^{k} \sigma_r^i(t,x)\, \sigma_r^j(t,x). \tag{11}$$

Then there exists a solution of (9) which is a T-periodic Markov process.

Lemma 3 (see [2], strong law of large numbers). *Let $M = \{M_t\}_t \geq 0$ be a real-valued continuous local martingale vanishing at $t = 0$.*

Then

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty \quad a.s. \Longrightarrow \lim_{t\to\infty} \frac{M_t}{\langle M,M\rangle_t} = 0 \quad a.s. \tag{12}$$

and also

$$\limsup_{t\to\infty} \frac{\langle M,M\rangle_t}{t} < \infty \quad a.s. \Longrightarrow \lim_{t\to\infty} \frac{M_t}{t} = 0 \quad a.s. \tag{13}$$

3. Existence and Uniqueness of the Global Positive Solution

In this section, we prove that system (3) has a unique global positive solution.

Theorem 4. *For any initial value $(S(0), I_1(0), I_2(0)) \in R_+^3$, there is a unique positive solution $(S(t), I_1(t), I_2(t))$ of (3) on $t \geq 0$ and the solution will remain in R_+^3 with probability one.*

Proof. From system (3), we can get

$$\begin{aligned}\frac{\mathrm{d}(S(t)+I_1(t)+I_2(t))}{\mathrm{d}t} &= A(t) - d(t)(S(t)+I_1(t)+I_2(t)) - (\alpha_1(t)I_1(t) + \alpha_2(t)I_2(t)) \\ &\leq A(t) - d(t)(S(t)+I_1(t)+I_2(t)) \\ &\leq A^u - d^l(S(t)+I_1(t)+I_2(t)).\end{aligned} \tag{14}$$

Then

$$\lim_{t\to+\infty}(S(t)+I_1(t)+I_2(t)) \leq \frac{A^u}{d^l}; \tag{15}$$

obviously, we have

$$\limsup_{t\to+\infty} S(t) \le \frac{A^u}{d^l},\quad \limsup_{t\to+\infty} I_1(t) \le \frac{A^u}{d^l},\quad \limsup_{t\to+\infty} I_2(t) \le \frac{A^u}{d^l}. \tag{16}$$

Since the coefficients of system (3) satisfy the local Lipschitz conditions, then for any given initial value $(S(0), I_1(0), I_2(0)) \in R_+^3$, there is a unique local solution $(S(t), I_1(t), I_2(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To demonstrate that this solution is global, we only need to prove that $\tau_e = \infty$ a.s.

Let $k_0 > 0$ be sufficiently large for any initial value $S(0), I_1(0)$, and $I_2(0)$ lying within the interval $[1/k_0, k]$. For each integer $k \ge k_0$, define the following stopping time:

$$\tau_k = \inf\left\{t \in [0,\tau_e) : \min\{S(t), I_1(t), I_2(t)\} \le \frac{1}{k} \text{ or } \max\{S(t), I_1(t), I_2(t)\} \ge k\right\}, \tag{17}$$

where we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty}\tau_k$; hence $\tau_\infty \le \tau_k$ a.s. Next, we only need to verify $\tau_\infty = \infty$ a.s. If this statement is false, then there exist two constants $T > 0$ and $\epsilon \in (0,1)$ such that

$$\mathbb{P}\{\tau_\infty \le T\} > \epsilon. \tag{18}$$

Thus there is an integer $k_1 \ge k_0$ such that

$$\mathbb{P}\{\tau_k \le T\} \ge \epsilon, \quad k \ge k_1. \tag{19}$$

Define a C^2-function $V: \mathbb{R}_+^3 \to \mathbb{R}_+$ as follows:

$$V(S, I_1, I_2) = S - 1 - \ln S + I_1 - 1 - \ln I_1 + I_2 - 1 - \ln I_2; \tag{20}$$

the nonnegativity of this function can be obtained from

$$u - 1 - \ln u \ge 0, \quad u > 0. \tag{21}$$

Applying Itô's formula yields

$$\begin{aligned} dV(S, I_1, I_2) &= LV(S, I_1, I_2)\,dt \\ &\quad + (2SI_1 - S - I_1)\frac{\sigma_1(t)}{a_1(t) + I_1} dB_1(t) \\ &\quad + (2SI_2 - S - I_2)\frac{\sigma_2(t)}{a_2(t) + I_2} dB_2(t), \end{aligned} \tag{22}$$

where

$$\begin{aligned} LV &= \left(1 - \frac{1}{S}\right)\left(A(t) - d(t)S - \frac{\beta_1(t)SI_1}{a_1(t)+I_1} - \frac{\beta_2(t)SI_2}{a_2(t)+I_2} + r_1(t)I_1 + r_2(t)I_2\right) + \frac{1}{2} \\ &\quad \cdot \frac{\sigma_1^2(t)I_1^2}{(a_1(t)+I_1)^2} + \frac{1}{2}\frac{\sigma_2^2(t)I_2^2}{(a_2(t)+I_2)^2} + \left(1 - \frac{1}{I_1}\right) \\ &\quad \cdot \left(\frac{\beta_1(t)SI_1}{a_1(t)+I_1} - (d(t)+\alpha_1(t)+r_1(t))I_1\right) + \frac{1}{2} \\ &\quad \cdot \frac{\sigma_1^2(t)S^2}{(a_1(t)+I_1)^2} + \left(1 - \frac{1}{I_2}\right)\left(\frac{\beta_2(t)SI_2}{a_2(t)+I_2} - (d(t)+\alpha_2(t)+r_2(t))I_2\right) + \frac{1}{2}\frac{\sigma_2^2(t)S^2}{(a_2(t)+I_2)^2} \\ &= A(t) - d(t)S - \frac{A(t)}{S} + d(t) + \frac{\beta_1(t)I_1}{a_1(t)+I_1} + \frac{\beta_2(t)I_2}{a_2(t)+I_2} - \frac{r_1(t)I_1}{S} - \frac{r_2(t)I_2}{S} \\ &\quad + \frac{\sigma_1^2(t)I_1^2}{2(a_1(t)+I_1)^2} + \frac{\sigma_2^2(t)I_2^2}{2(a_2(t)+I_2)^2} - (d(t) + \alpha_1(t))I_1 - \frac{\beta_1(t)S}{a_1(t)+I_1} + d(t) + \alpha_1(t) + r_1(t) \\ &\quad + \frac{\sigma_1^2(t)S^2}{2(a_1(t)+I_1)^2} - (d(t)+\alpha_2(t))I_2 - \frac{\beta_2(t)S}{a_2(t)+I_2} + d(t) + \alpha_2(t) + r_2(t) + \frac{\sigma_2^2(t)S^2}{2(a_2(t)+I_2)^2} \le A^u \\ &\quad + 3d^u + \alpha_1^u + \alpha_2^u + r_1^u + r_2^u + \beta_1^u + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} + \frac{\sigma_1^{2u}A^{2u}}{2a_1^{2l}d^{2l}} + \frac{\sigma_2^{2u}A^{2u}}{2a_2^{2l}d^{2l}} := K, \end{aligned} \tag{23}$$

where K is a positive constant.

So we have

$$\begin{aligned} dV(S, I_1, I_2) &\le K\,dt + (2SI_1 - S - I_1)\frac{\sigma_1(t)}{a_1(t)+I_1} dB_1(t) \\ &\quad + (2SI_2 - S - I_2)\frac{\sigma_2(t)}{a_2(t)+I_2} dB_2(t). \end{aligned} \tag{24}$$

Integrating (24) from 0 to $\tau_k \wedge T$ and taking expectations on both sides yield

$$\begin{aligned} &\mathbb{E}V(S(\tau_k \wedge T), I_1(\tau_k \wedge T), I_2(\tau_k \wedge T)) \\ &\quad \le V(S(0), I_1(0), I_2(0)) + KT. \end{aligned} \tag{25}$$

Let $\Omega_k = \{\tau_k \le T\}$; from inequality (25) we can see that $P(\Omega_k) \ge \epsilon$. We have

$$\begin{aligned} &V(S(\tau_k \wedge T), I_1(\tau_k \wedge T), I_2(\tau_k \wedge T)) \\ &\quad \ge (k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right). \end{aligned} \tag{26}$$

By (25) and (26), one has

$$V(S(0), I_1(0), I_2(0)) + KT \geq \mathbb{E}[1\Omega_k(\omega) \cdot V(S(\tau_k \wedge T), I_1(\tau_k \wedge T), I_2(\tau_k \wedge T))] \geq \epsilon (k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 - \ln \frac{1}{k}\right), \quad (27)$$

where $1\Omega_k$ is the indicator function of Ω_k.

Let $k \to \infty$; we have

$$\infty > V(S(0), I_1(0), I_2(0)) + KT = \infty. \quad (28)$$

So we obtain $\tau_\infty = \infty$. The proof is completed. □

4. Existence of Nontrivial T-Periodic Solution

In this section, we verify that system (3) admits at least one nontrivial positive T-periodic solution. Define

$$\mathfrak{R} = \sum_{i=1}^{2} \frac{\langle A\alpha_i \beta_i \rangle_T}{\langle d + \alpha_i + r_i + \sigma_i^2 A^{2u}/2a_i^2 d^{2l} \rangle_T \langle d + \sigma_1^2/2 + \sigma_2^2/2 \rangle_T \langle A + a_1\alpha_1 + a_2\alpha_2 \rangle_T}. \quad (29)$$

Theorem 5. *When $(a_1(t)r_1(t)/(\beta_1(t) - a_1(t)d(t)))^l > A^u/d^l$ and $(a_2(t)r_2(t)/(\beta_2(t) - a_2(t)d(t)))^l > A^u/d^l$ hold, if $\mathfrak{R} > 1$, then there exists a nontrivial positive T-periodic solution of system (3).*

Proof. Define a C^2-function $\overline{V} : [0, \infty) \times \mathbb{R}_+^3 \to \mathbb{R}_+$:

$$\overline{V}(t, S, I_1, I_2) = M(-(e_1 + e_2)\ln S - c_1 \ln I_1 - c_2 \ln I_2 + S + I_1 + I_2 + \omega(t)) - \ln S - I_1 - I_2 + \frac{1}{\theta + 1}(S + I_1 + I_2)^{\theta+1} = M(V_1 + \omega(t)) + V_2 + V_3 + V_4 + V_5, \quad (30)$$

where

$$c_1 = \frac{\langle A\alpha_1\beta_1 \rangle_T}{\langle d + \alpha_1 + r_1 + \sigma_1^2 A^{2u}/2a_1^2 d^{2l} \rangle_T^2 \langle d + \sigma_1^2/2 + \sigma_2^2/2 \rangle_T},$$

$$e_1 = \frac{\langle A\alpha_1\beta_1 \rangle_T}{\langle d + \alpha_1 + r_1 + \sigma_1^2 A^{2u}/2a_1^2 d^{2l} \rangle_T \langle d + \sigma_1^2/2 + \sigma_2^2/2 \rangle_T^2},$$

$$c_2 = \frac{\langle A\alpha_2\beta_2 \rangle_T}{\langle d + \alpha_2 + r_2 + \sigma_2^2 A^{2u}/2a_2^2 d^{2l} \rangle_T^2 \langle d + \sigma_1^2/2 + \sigma_2^2/2 \rangle_T},$$

$$e_2 = \frac{\langle A\alpha_2\beta_2 \rangle_T}{\langle d + \alpha_2 + r_2 + \sigma_2^2 A^{2u}/2a_2^2 d^{2l} \rangle_T \langle d + \sigma_1^2/2 + \sigma_2^2/2 \rangle_T^2}, \quad (31)$$

and $0 < \theta < \min\{1, d^l/(\sigma_1^{2u} + \sigma_2^{2u})\}$; M is a sufficiently large positive constant and satisfies the following conditions:

$$d^l - \theta(\sigma_1^{2u} + \sigma_2^{2u}) > 0,$$
$$-M\lambda + \max\{D, E\} \leq -2, \quad (32)$$

where

$$\lambda = \langle A + a_1\alpha_1 + a_2\alpha_2 \rangle_T (\mathfrak{R} - 1),$$

$$D = \sup_{(S, I_1, I_2) \in R_+^3} \left\{ (d^u + \alpha_2^u + r_2^u) I_2 - \frac{1}{2}(d^l - \theta(\sigma_1^{2u} + \sigma_2^{2u})) I_2^{\theta+1} + O + d^u + \beta_1^u + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \right\},$$

$$E = \sup_{(S, I_1, I_2) \in R_+^3} \left\{ (d^u + \alpha_1^u + r_1^u) I_1 - \frac{1}{2}(d^l - \theta(\sigma_1^{2u} + \sigma_2^{2u})) I_1^{\theta+1} + O + d^u + \beta_1^u + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \right\}, \quad (33)$$

where

$$O = \sup_{(S, I_1, I_2) \in \mathbb{R}_+^3} \left\{ A^u (S + I_1 + I_2)^\theta - \frac{1}{2}(d^l - \theta(\sigma_1^{2u} + \sigma_2^{2u}))(S + I_1 + I_2)^{\theta+1} \right\} < \infty. \quad (34)$$

Next we prove that condition (A_1) in Lemma 2 holds. It is easy to check that $\overline{V}(t, S, I_1, I_2)$ is a T-periodic function in t and satisfies

$$\liminf_{K \to \infty, (S, I_1, I_2) \in \mathbb{R}_+^3 \setminus \mathbb{U}_k} \overline{V}(t, S, I_1, I_2) = \infty, \quad (35)$$

where $\mathbb{U}_k = (1/k, k) \times (1/k, k) \times (1/k, k)$ and $k > 1$ is a sufficiently large number.

By Itô's formula, we obtain

$$
\begin{aligned}
LV_1 &= -\frac{e_1+e_2}{S}\left(A(t) - d(t)S - \frac{\beta_1(t)SI_1}{a_1(t)+I_1} - \frac{\beta_2(t)SI_2}{a_2(t)+I_2} + r_1(t)I_1 + r_2(t)I_2\right) \\
&\quad - \frac{c_1}{I_1}\left(\frac{\beta_1(t)SI_1}{a_1(t)+\mathrm{I}_1} - (d(t)+\alpha_1(t)+r_1(t))I_1\right) + \frac{e_1+e_2}{2}\frac{\sigma_1^2(t)I_1^2}{(a_1(t)+I_1)^2} - \frac{c_2}{I_2}\left(\frac{\beta_2(t)SI_2}{a_2(t)+I_2}\right. \\
&\quad \left. - (d(t)+\alpha_2(t)+r_2(t))I_2\right) + \frac{e_1+e_2}{2} \cdot \frac{\sigma_2^2(t)I_2^2}{(a_2(t)+I_2)^2} + \frac{c_1}{2}\frac{\sigma_1^2(t)S^2}{(a_1(t)+I_1)^2} + \frac{c_2}{2} \\
&\quad \cdot \frac{\sigma_2^2(t)S^2}{(a_2(t)+I_2)^2} + A(t) - d(t)(S+I_1+I_2) - \alpha_1(t) \cdot I_1 - \alpha_2(t)I_2 \le -e_1\frac{A(t)}{S} - \frac{c_1\beta_1(t)S}{a_1(t)+I_1} - \alpha_1(t) \\
&\quad \cdot (a_1(t)+I_1) - e_2\frac{A(t)}{S} - \frac{c_2\beta_2(t)S}{a_2(t)+I_2} - \alpha_2(t) \cdot (a_2(t)+I_2) + c_1\left(d(t)+\alpha_1(t)+r_1(t)\right. \\
&\quad \left. + \frac{\sigma_1^2(t)A^{2u}}{2a_1^2(t)d^{2l}}\right) + e_1\left(d(t) + \frac{\sigma_1^2(t)}{2} + \frac{\sigma_2^2(t)}{2}\right) + c_2\left(d(t)+\alpha_2(t)+r_2(t)+\frac{\sigma_2^2(t)A^{2u}}{2a_2^2(t)d^{2l}}\right) \\
&\quad + e_2\left(d(t)+\frac{\sigma_1^2(t)}{2}+\frac{\sigma_2^2(t)}{2}\right) - (e_1+e_2)\left(\frac{r_1(t)}{S} + d(t) - \frac{\beta_1(t)}{a_1(t)}\right)I_1 - (e_1+e_2)\left(\frac{r_2(t)}{S}+d(t)\right. \\
&\quad \left. - \frac{\beta_2(t)}{a_2(t)}\right)I_2 + A(t) + a_1(t)\alpha_1(t) + a_2(t)\alpha_2(t) \\
&\le -3\left(A(t)\alpha_1(t)\beta_1(t)c_1e_1\right)^{1/3} - 3\left(A(t)\alpha_2(t)\beta_2(t)c_2e_2\right)^{1/3} + e_1\left(d(t)+\frac{\sigma_1^2(t)}{2}\right. \\
&\quad \left. + \frac{\sigma_2^2(t)}{2}\right) + c_1\left(d(t)+\alpha_1(t)+r_1(t) + \frac{\sigma_1^2(t)A^{2u}}{2a_1^2(t)d^{2l}}\right) + c_2\left(d(t)+\alpha_2(t)+r_2(t)\right. \\
&\quad \left. + \frac{\sigma_2^2(t)A^{2u}}{2a_2^2(t)d^{2l}}\right) + e_2\left(d(t)+\frac{\sigma_1^2(t)}{2}+\frac{\sigma_2^2(t)}{2}\right) + A(t) + a_1(t)\alpha_1(t) + a_2(t)\alpha_2(t) - (e_1+e_2) \\
&\quad \cdot \left(\frac{r_1(t)}{S} + d(t) - \frac{\beta_1(t)}{a_1(t)}\right)I_1 - (e_1+e_2)\left(\frac{r_2(t)}{S} + d(t) - \frac{\beta_2(t)}{a_2(t)}\right)I_2 \coloneqq \mathfrak{R}_0(t) - (e_1+e_2)\left(\frac{r_1(t)}{S}\right. \\
&\quad \left. + d(t) - \frac{\beta_1(t)}{a_1(t)}\right)I_1 - (e_1+e_2)\left(\frac{r_2(t)}{S}+d(t) - \frac{\beta_2(t)}{a_2(t)}\right)I_2,
\end{aligned} \tag{36}
$$

where

$$
\begin{aligned}
\mathfrak{R}_0(t) = 3\left(A(t)\alpha_1(t)\beta_1(t)c_1e_1\right)^{1/3} &+ c_1\left(d(t)+\alpha_1(t)+r_1(t)+\frac{\sigma_1^2(t)A^{2u}}{2a_1^2(t)d^{2l}}\right) + e_1\left(d(t)+\frac{\sigma_1^2(t)}{2}+\frac{\sigma_2^2(t)}{2}\right) \\
&- 3\left(A(t)\alpha_2(t)\beta_2(t)c_2e_2\right)^{1/3} + c_2\left(d(t)+\alpha_2(t)+r_2(t)+\frac{\sigma_2^2(t)A^{2u}}{2a_2^2(t)d^{2l}}\right) \\
&+ e_2\left(d(t)+\frac{\sigma_1^2(t)}{2}+\frac{\sigma_2^2(t)}{2}\right) + A(t) + a_1(t)\alpha_1(t) + a_2(t)\alpha_2(t).
\end{aligned} \tag{37}
$$

Note that $(a_1(t)r_1(t)/(\beta_1(t) - a_1(t)d(t)))^l > A^u/d^l$ and $(a_2(t)r_2(t)/(\beta_2(t) - a_2(t)d(t)))^l > A^u/d^l$ hold; then

$$L(V_1) \le \mathfrak{R}_0(t). \tag{38}$$

Define the T-periodic function $\omega(t)$ satisfying

$$\omega'(t) = \langle\mathfrak{R}_0\rangle_T - \mathfrak{R}_0(t). \tag{39}$$

So

$$
\begin{aligned}
L(V_1+\omega(t)) &\le \langle\mathfrak{R}_0\rangle_T \\
&= -\langle A + a_1\alpha_1 + a_2\alpha_2\rangle_T(\mathfrak{R}-1) \\
&\coloneqq -\lambda.
\end{aligned} \tag{40}
$$

Applying Itô's formula, we can also have

$$LV_2 = -\frac{A(t)}{S} + d(t) + \frac{\beta_1(t) I_1}{a_1(t) + I_1} + \frac{\beta_2(t) I_2}{a_2(t) + I_2} - \frac{r_1(t) I_1}{S} - \frac{r_2(t) I_2}{S} + \frac{\sigma_1^2(t) I_1^2}{2(a_1(t) + I_1)^2} + \frac{\sigma_2^2(t) I_2^2}{2(a_2(t) + I_2)^2} \le -\frac{A^l}{S} + d^u + \beta_1^u + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2},$$

$$LV_3 = -\frac{\beta_1(t) S I_1}{a_1(t) + I_1} + (d(t) + \alpha_1(t) + r_1(t)) I_1 \le (d^u + \alpha_1^u + r_1^u) I_1,$$

$$LV_4 = -\frac{\beta_2(t) S I_2}{a_2(t) + I_2} + (d(t) + \alpha_2(t) + r_2(t)) I_2 \le (d^u + \alpha_2^u + r_2^u) I_2,$$

$$\begin{aligned} LV_5 &= (S + I_1 + I_2)^\theta (A(t) - d(t) S - d(t) I_1 - d(t) I_2 - \alpha_1(t) I_1 - \alpha_2(t) I_2) + \theta (S + I_1 + I_2)^{\theta-1} \cdot \left(\frac{\sigma_1^2(t) S^2 I_1^2}{(a_1(t) + I_1)^2} + \frac{\sigma_2^2(t) S^2 I_2^2}{(a_2(t) + I_2)^2} \right) \le A^u (S + I_1 + I_2)^\theta - d^l (S + I_1 + I_2)^{\theta+1} + \theta (S + I_1 + I_2)^{\theta-1} (\sigma_1^{2u} + \sigma_2^{2u}) S^2 \\ &\le A^u (S + I_1 + I_2)^\theta - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) (S + I_1 + I_2)^{\theta+1} - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) (S + I_1 + I_2)^{\theta+1} \le O - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) S^{\theta+1} - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) I_1^{\theta+1} - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) I_2^{\theta+1}, \end{aligned} \tag{41}$$

where

$$O = \sup_{(S, I_1, I_2) \in \mathbb{R}_+^3} \left\{ A^u (S + I_1 + I_2)^\theta - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) (S + I_1 + I_2)^{\theta+1} \right\} < \infty. \tag{42}$$

Therefore

$$\begin{aligned} L\overline{V}(t, S, I_1, I_2) &\le -M \langle A + a_1 \alpha_1 + a_2 \alpha_2 \rangle_T (\mathfrak{R} - 1) + (d^u + \alpha_1^u + r_1^u) I_1 + (d^u + \alpha_2^u + r_2^u) I_2 - \frac{A^l}{S} - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) S^{\theta+1} \\ &\quad - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) I_1^{\theta+1} - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) I_2^{\theta+1} + O + d^u + \beta_1^u + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \\ &= -M\lambda - \frac{A^l}{S} + (d^u + \alpha_1^u + r_1^u) I_1 + (d^u + \alpha_2^u + r_2^u) I_2 - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) S^{\theta+1} - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) I_1^{\theta+1} \\ &\quad - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) I_2^{\theta+1} + O + d^u + \beta_1^u + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2}. \end{aligned} \tag{43}$$

Now, we are in the position to construct a compact subset U such that A_2 in Lemma 2 holds. Define the following bounded closed set:

$$U = \left\{ (S, I_1, I_2) \in \mathbb{R}_+^3 : \epsilon \le S \le \frac{1}{\epsilon},\ \epsilon \le I_1 \le \frac{1}{\epsilon},\ \epsilon \le I_2 \le \frac{1}{\epsilon} \right\}, \tag{44}$$

where $\epsilon > 0$ is a sufficiently small number. In the set $\mathbb{R}_+^3 \backslash U$, we can choose ϵ sufficiently small such that

$$-M\lambda - \frac{A^l}{\epsilon} + C \le -1, \tag{45}$$

$$-M\lambda + (d^u + \alpha_1^u + r_1^u) \epsilon + D \le -1, \tag{46}$$

$$-M\lambda + (d^u + \alpha_2^u + r_2^u) \epsilon + E \le -1, \tag{47}$$

$$-M\lambda - \frac{1}{2} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) \frac{1}{\epsilon^{\theta+1}} + C \le -1, \tag{48}$$

$$-M\lambda - \frac{1}{4} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) \frac{1}{\epsilon^{\theta+1}} + G \le -1, \tag{49}$$

$$-M\lambda - \frac{1}{4} (d^l - \theta (\sigma_1^{2u} + \sigma_2^{2u})) \frac{1}{\epsilon^{\theta+1}} + H \le -1, \tag{50}$$

where C, D, E, G, and H are positive constants which can be found from the following inequations (52), (54), (56), (59), and (61), respectively. For the sake of convenience, we divide into six domains,

$$U_1 = \left\{ (S, I_1, I_2) \in \mathbb{R}^3_+,\ 0 < S < \epsilon \right\},$$
$$U_2 = \left\{ (S, I_1, I_2) \in \mathbb{R}^3_+,\ 0 < I_1 < \epsilon \right\},$$
$$U_3 = \left\{ (S, I_1, I_2) \in \mathbb{R}^3_+,\ 0 < I_2 < \epsilon \right\},$$
$$U_4 = \left\{ (S, I_1, I_2) \in \mathbb{R}^3_+,\ S > \frac{1}{\epsilon} \right\}, \quad (51)$$
$$U_5 = \left\{ (S, I_1, I_2) \in \mathbb{R}^3_+,\ I_1 > \frac{1}{\epsilon} \right\},$$
$$U_6 = \left\{ (S, I_1, I_2) \in \mathbb{R}^3_+,\ I_2 > \frac{1}{\epsilon} \right\}.$$

Next we will prove that $L\overline{V}(S, I_1, I_2) \le -1$ on $\mathbb{R}^3_+ \backslash U$, which is equivalent to proving it on the above six domains.

Case 1. If $(S, I_1, I_2) \in U_1$, one can see that

$$\begin{aligned} L\overline{V} \le & -M\lambda - \frac{A^l}{S} + \left(d^u + \alpha_1^u + r_1^u\right) I_1 \\ & + \left(d^u + \alpha_2^u + r_2^u\right) I_2 \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + O + d^u + \beta_1^u \\ & + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \le -M\lambda - \frac{A^l}{S} + C \\ \le & -M\lambda - \frac{A^l}{\epsilon} + C, \end{aligned} \quad (52)$$

where

$$\begin{aligned} C = \sup_{(S,I_1,I_2)\in R^3_+} & \left\{ \left(d^u + \alpha_1^u + r_1^u\right) I_1 + \left(d^u + \alpha_2^u + r_2^u\right) I_2 \right. \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + O + d^u + \beta_1^u + \beta_2^u \\ & \left. + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \right\}. \end{aligned} \quad (53)$$

According to (45), we have $L\overline{V} \le -1$ for all $(S, I_1, I_2) \in U_1$.

Case 2. If $(S, I_1, I_2) \in U_2$, one can get that

$$\begin{aligned} L\overline{V} \le & -M\lambda + \left(d^u + \alpha_1^u + r_1^u\right) I_1 + \left(d^u + \alpha_2^u + r_2^u\right) I_2 \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + O + d^u + \beta_1^u \\ & + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \\ \le & -M\lambda + \left(d^u + \alpha_1^u + r_1^u\right) I_1 + D \\ \le & -M\lambda + \left(d^u + \alpha_1^u + r_1^u\right) \epsilon + D, \end{aligned} \quad (54)$$

where

$$\begin{aligned} D = \sup_{(S,I_1,I_2)\in R^3_+} & \left\{ \left(d^u + \alpha_2^u + r_2^u\right) I_2 \right. \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + O + d^u + \beta_1^u + \beta_2^u \\ & \left. + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \right\}. \end{aligned} \quad (55)$$

In view of (46), we can obtain that $L\overline{V} \le -1$ for all $(S, I_1, I_2) \in U_2$.

Case 3. If $(S, I_1, I_2) \in U_3$, we have

$$\begin{aligned} L\overline{V} \le & -M\lambda + \left(d^u + \alpha_2^u + r_2^u\right) I_2 + \left(d^u + \alpha_1^u + r_1^u\right) I_1 \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} + O + d^u + \beta_1^u \\ & + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \\ \le & -M\lambda + \left(d^u + \alpha_2^u + r_2^u\right) I_2 + E \\ \le & -M\lambda + \left(d^u + \alpha_2^u + r_2^u\right) \epsilon + E, \end{aligned} \quad (56)$$

where

$$\begin{aligned} E = \sup_{(S,I_1,I_2)\in R^3_+} & \left\{ \left(d^u + \alpha_1^u + r_1^u\right) I_1 \right. \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} + O + d^u + \beta_1^u + \beta_2^u \\ & \left. + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \right\}. \end{aligned} \quad (57)$$

By (47), we can conclude that $L\overline{V} \le -1$ for all $(S, I_1, I_2) \in U_3$.

Case 4. If $(S, I_1, I_2) \in U_4$, one can derive that

$$\begin{aligned} L\overline{V} \le & -M\lambda - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) S^{\theta+1} \\ & + \left(d^u + \alpha_1^u + r_1^u\right) I_1 \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} \\ & + \left(d^u + \alpha_2^u + r_2^u\right) I_2 \\ & - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + O + d^u + \beta_1^u \\ & + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \\ \le & -M\lambda - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) S^{\theta+1} + C \\ \le & -M\lambda - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) \frac{1}{\epsilon^{\theta+1}} + C. \end{aligned} \quad (58)$$

Together with (48), we can deduce that $L\overline{V} \le -1$ for all $(S, I_1, I_2) \in U_4$.

Case 5. If $(S, I_1, I_2) \in U_5$, it follows that

$$\begin{aligned} L\overline{V} &\le -M\lambda - \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} \\ &\quad + \left(d^u + \alpha_1^u + r_1^u\right) I_1 \\ &\quad - \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} \\ &\quad + \left(d^u + \alpha_2^u + r_2^u\right) I_2 \\ &\quad - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + O + d^u + \beta_1^u \\ &\quad + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \\ &\le -M\lambda - \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} + G \\ &\le -M\lambda - \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) \frac{1}{\epsilon^{\theta+1}} + G, \end{aligned} \tag{59}$$

where

$$\begin{aligned} G = \sup_{(S,I_1,I_2)\in R_+^3} &\left\{\left(d^u + \alpha_1^u + r_1^u\right) I_1\right. \\ &- \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} + \left(d^u + \alpha_2^u + r_2^u\right) I_2 \\ &- \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + O + d^u + \beta_1^u + \beta_2^u \\ &\left.+ \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2}\right\}. \end{aligned} \tag{60}$$

By virtue of (49), we can deduce that $L\overline{V} \le -1$ for all $(S, I_1, I_2) \in U_5$.

Case 6. If $(S, I_1, I_2) \in U_6$, we obtain

$$\begin{aligned} L\overline{V} &\le -M\lambda - \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} \\ &\quad + \left(d^u + \alpha_1^u + r_1^u\right) I_1 \\ &\quad - \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} \\ &\quad + \left(d^u + \alpha_2^u + r_2^u\right) I_2 \\ &\quad - \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + O + d^u + \beta_1^u \\ &\quad + \beta_2^u + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2} \\ &\le -M\lambda - \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + H \\ &\le -M\lambda - \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) \frac{1}{\epsilon^{\theta+1}} + H, \end{aligned} \tag{61}$$

where

$$\begin{aligned} H = \sup_{(S,I_1,I_2)\in R_+^3} &\left\{\left(d^u + \alpha_1^u + r_1^u\right) I_1\right. \\ &- \frac{1}{2}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_1^{\theta+1} + \left(d^u + \alpha_2^u + r_2^u\right) I_2 \\ &- \frac{1}{4}\left(d^l - \theta\left(\sigma_1^{2u} + \sigma_2^{2u}\right)\right) I_2^{\theta+1} + O + d^u + \beta_1^u + \beta_2^u \\ &\left.+ \frac{\sigma_1^{2u}}{2} + \frac{\sigma_2^{2u}}{2}\right\}. \end{aligned} \tag{62}$$

It follows from (50) that $L\overline{V} \le -1$ for all $(S, I_1, I_2) \in U_6$.

Clearly, one can see from (52), (54), (56), (58), (59), and (61) that, for a sufficiently small ϵ,

$$\begin{aligned} &L\overline{V}\left(S, I_1, I_2\right) \le -1, \\ &\left(S, I_1, I_2\right) \in \mathbb{R}_+^3 \backslash U. \end{aligned} \tag{63}$$

Hence A_2 in Lemma 2 is satisfied. This completes the proof of Theorem 5. □

5. Extinction

In this section, we investigate the conditions for the extinction of the two infectious diseases of system (3).

Let

$$\begin{aligned} \mathscr{R}_1 &= \frac{A^u \langle\beta_1\rangle_T}{d^l \langle a_1\rangle_T \langle d + \alpha_1 + r_1\rangle_T} \\ &\quad - \frac{A^{2u}\left\langle\sigma_1^2\right\rangle_T}{2d^{2l}\langle a_1^2\rangle_T \langle d + \alpha_1 + r_1\rangle_T}, \\ \mathscr{R}_2 &= \frac{A^u \langle\beta_2\rangle_T}{d^l \langle a_2\rangle_T \langle d + \alpha_2 + r_2\rangle_T} \\ &\quad - \frac{A^{2u}\left\langle\sigma_2^2\right\rangle_T}{2d^{2l}\langle a_2^2\rangle_T \langle d + \alpha_2 + r_2\rangle_T}. \end{aligned} \tag{64}$$

Theorem 6. *Let $(S(t), I_1(t), I_2(t))$ be a solution of system (3) with initial value $(S(0), I_1(0), I_2(0)) \in R_+^3$.*

Then if

$$\text{(i):}\quad \left\langle\sigma_i^2\right\rangle_T > \frac{\left\langle\beta_i^2\right\rangle_T}{2\left\langle d^l + \alpha_i^l + r_i^l\right\rangle_T}, \quad i = 1, 2, \tag{65}$$

or

$$\begin{aligned} &\text{(ii):}\quad \mathscr{R}_i < 1, \\ &\left\langle\sigma_i^2\right\rangle_T \le \frac{d^l \left\langle a_i^2\right\rangle_T \langle\beta_i\rangle_T}{A^u \langle a_i\rangle_T}, \\ &\qquad i = 1, 2, \end{aligned} \tag{66}$$

hold, the two infectious diseases of system (3) go to extinction a.s.; that is,

$$\lim_{t\to+\infty} I_i(t) = 0, \quad i = 1, 2. \tag{67}$$

Proof. Applying Itô's formula to system (3), we have

$$\begin{aligned} d\ln I_i(t) = &\left(\frac{\beta_i(t) S(t)}{a_i(t)+I_i(t)} - \left(d(t)+\alpha_i(t)+r_i(t)\right) \right. \\ &\left. - \frac{\sigma_i^2(t) S^2(t)}{2\left(a_i(t)+I_i(t)\right)^2} \right) \mathrm{d}t + \frac{\sigma_i(t) S(t)}{a_i(t)+I_i(t)} \mathrm{d}B_i(t), \\ & i=1,2. \end{aligned} \tag{68}$$

Case (i). Integrating (68) from 0 to t and dividing t on both sides, we obtain

$$\begin{aligned} \frac{\ln I_i(t)}{t} &= -\frac{\sigma_i^2(t)}{2t} \int_0^t \left(\frac{S(\tau)}{a_i(\tau)+I_i(\tau)} - \frac{\beta_i(\tau)}{\sigma_i^2(\tau)} \right)^2 \mathrm{d}\tau \\ &\quad - \frac{1}{t} \int_0^t \left(d(\tau)+\alpha_i(\tau)+r_i(\tau)\right) \mathrm{d}\tau \\ &\quad + \frac{1}{t} \int_0^t \frac{\beta_i^2(\tau)}{2\sigma_i^2(\tau)} \mathrm{d}\tau + \frac{M_i(t)}{t} + \frac{\ln I_i(0)}{t} \\ &\le -\left\langle d+\alpha_i+r_i - \frac{\beta_i^2}{2\sigma_i^2} \right\rangle_t + \frac{M_i(t)}{t} \\ &\quad + \frac{\ln I_i(0)}{t}. \end{aligned} \tag{69}$$

Case (ii). Integrating (68) from 0 to t first and then dividing by t on both sides yield

$$\begin{aligned} \frac{\ln I_i(t)}{t} &= \frac{1}{t} \int_0^t \left(\frac{\beta_i(\tau) S(\tau)}{a_i(\tau)+I_i(\tau)} \right. \\ &\quad - \left(d(\tau)+\alpha_i(\tau)+r_i(\tau)\right) \\ &\quad \left. - \frac{\sigma_i^2(\tau) S^2(\tau)}{2\left(a_i(\tau)+I_i(\tau)\right)^2} \right) \mathrm{d}\tau + \frac{M_i(t)}{t} + \frac{\ln I_i(0)}{t} \\ &\le \left\langle \frac{\beta_i A^u}{a_i d^l} - \left(d+\alpha_i+r_i\right) - \frac{\sigma_i^2 A^{2u}}{2a_i^2 d^{2l}} \right\rangle_t + \frac{M_i(t)}{t} \\ &\quad + \frac{\ln I_i(0)}{t} \le \left\langle d+\alpha_i+r_i \right\rangle_T \left(\mathscr{R}_i - 1\right) + \frac{M_i(t)}{t} \\ &\quad + \frac{\ln I_i(0)}{t}, \end{aligned} \tag{70}$$

where $M_i(t) = \int_0^t \sigma_i(\tau)S(\tau)/(a_i(\tau)+I_i(\tau))\mathrm{d}B_i(\tau)$, $i=1,2$, which is a local continuous martingale with $M_i(0)=0$. By Lemma 3, we have

$$\lim_{t\to+\infty} \frac{M_i(t)}{t} = 0, \quad i=1,2. \tag{71}$$

Taking the limit superior of both sides of (69) leads to

$$\limsup_{t\to+\infty} \frac{\ln I_i(t)}{t} \le -\left\langle d+\alpha_i+r_i - \frac{\beta_i^2}{2\sigma_i^2} \right\rangle_T < 0, \tag{72}$$

which implies $\lim_{t\to+\infty} I_i(t) = 0$.

Taking the superior limit of both sides of (70) leads to

$$\limsup_{t\to+\infty} \frac{\ln I_i(t)}{t} \le \left\langle d+\alpha_i+r_i \right\rangle_T \left(\mathscr{R}_i - 1\right) < 0, \tag{73}$$

which implies $\lim_{t\to+\infty} I_i(t) = 0$, $i=1,2$. This completes the proof. □

Remark 7. Theorem 6 shows that the two diseases will die out if the white noise disturbance is large or the white noise disturbance is not large and $\mathscr{R}_i < 1$. When $\langle \sigma_i^2 \rangle_T > \langle \beta_i^2 \rangle_T / 2\langle d^l + \alpha_i^l + r_i^l \rangle_T$, the two infectious diseases of system (3) die out almost surely; that is to say, large white noise stochastic disturbance can lead to the two epidemics being extinct.

6. Numerical Simulations

Now we introduce some numerical simulations examples which illustrate our theoretical results.

Example 8. In model (3), let

$$\begin{aligned} A(t) &= 0.5 + 0.1\sin \pi t, \\ a_1(t) &= 0.3 + 0.1\sin \pi t, \\ a_2(t) &= 0.31 + 0.1\sin \pi t, \\ \beta_1(t) &= 0.62 + 0.1\sin \pi t, \\ \beta_2(t) &= 0.62 + 0.1\sin \pi t, \\ r_1(t) &= 0.2 + 0.1\sin \pi t, \\ r_2(t) &= 0.35 + 0.1\sin \pi t, \\ d(t) &= 0.2 + 0.1\sin \pi t, \\ \alpha_1(t) &= 0.2 + 0.1\sin \pi t, \\ \alpha_2(t) &= 0.25 + 0.1\sin \pi t, \\ \sigma_1(t) &= 0.1 + 0.05\sin \pi t, \\ \sigma_2(t) &= 0.1 + 0.05\sin \pi t. \end{aligned} \tag{74}$$

Note that $\mathfrak{R} > 1$, $a_1(t)r_1(t)/(\beta_1(t) - a_1(t)d(t)) > A^u/d^l$, and $a_2(t)r_2(t)/(\beta_2(t) - a_2(t)d(t)) > A^u/d^l$ hold; that is, the conditions of Theorem 5 hold. Hence, system (3) has a positive periodic solution with $T=1$. Figure 1(a) shows the periodicity of the nonautonomous stochastic model (3) with $\sigma_1 = 0$ and $\sigma_2 = 0$. Figure 1(b) shows that solution of the nonautonomous stochastic model (3) with the initial value $(S(t), I_1(t), I_2(t)) = (0.3, 0.15, 0.15)$ tends to a periodic orbit in the sense of joint distribution.

Example 9. Choose the parameters in model (3) as follows:

$$\begin{aligned} A(t) &= 0.2 + 0.2\sin \pi t, \\ a_1(t) &= 0.23 + 0.2\sin \pi t, \\ a_2(t) &= 0.25 + 0.2\sin \pi t, \\ \beta_1(t) &= 0.2 + 0.2\sin \pi t, \end{aligned}$$

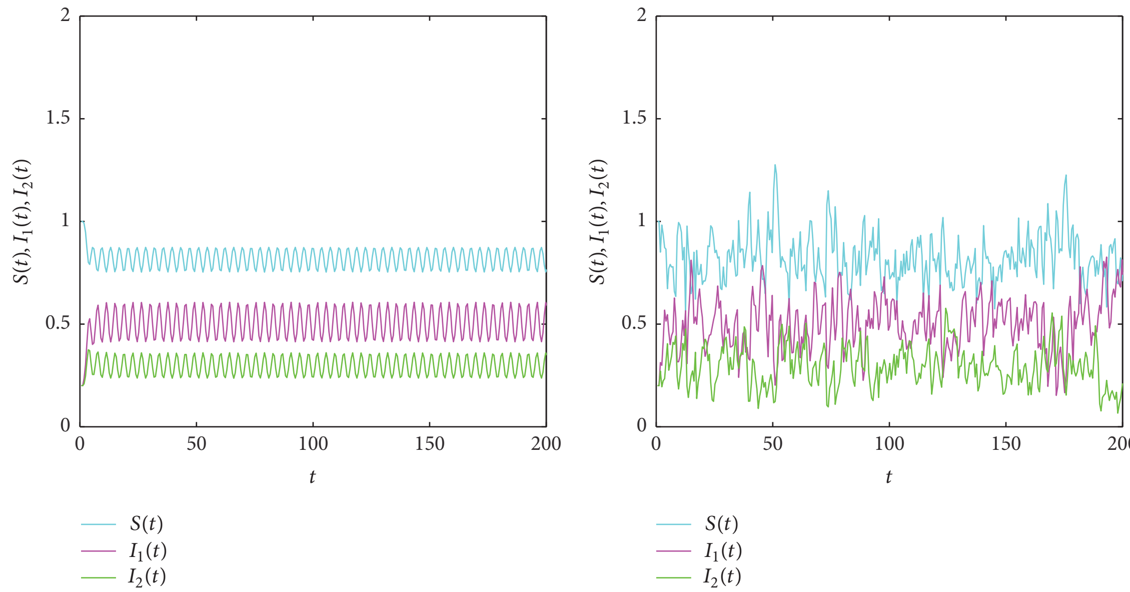

FIGURE 1: The solution $(S(t), I_1(t), I_2(t)) = (1, 0.2, 0.2)$ to the nonautonomous stochastic model (3).

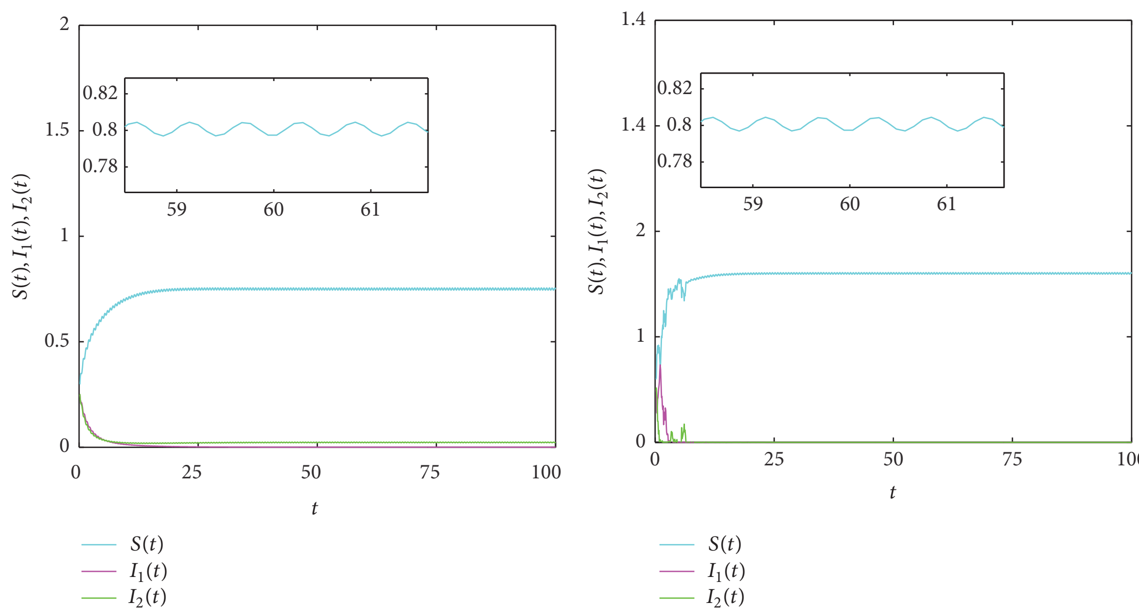

FIGURE 2: The solution $(S(t), I_1(t), I_2(t)) = (0.3, 0.25, 0.25)$ to the nonautonomous stochastic model (3).

$$\begin{aligned}
\beta_2(t) &= 0.3 + 0.2\sin \pi t,\\
r_1(t) &= 0.2 + 0.2\sin \pi t,\\
r_2(t) &= 0.3 + 0.2\sin \pi t,\\
d(t) &= 0.25 + 0.2\sin \pi t,\\
\alpha_1(t) &= 0.2 + 0.2\sin \pi t,\\
\alpha_2(t) &= 0.3 + 0.2\sin \pi t,\\
\sigma_1(t) &= 0.4 + 0.2\sin \pi t,\\
\sigma_2(t) &= 0.5 + 0.2\sin \pi t.
\end{aligned} \tag{75}$$

Note that $\langle\sigma_i^2\rangle_T > \langle\beta_i^2\rangle_T/2\langle d^l+\alpha_i^l+r_i^l\rangle_T$. Therefore, conditions (i) of Theorem 6 hold. Then the two infectious diseases will go to extinction. Figure 2(a) shows that one of two diseases in the deterministic SIS epidemic model is extinct and the other is persistent. Figure 2(b) shows that the two diseases will die out under the large white noise disturbance of model (3).

Example 10. Choose the parameters in model (3) as follows:

$$\begin{aligned}
A(t) &= 0.2 + 0.2\sin \pi t,\\
a_1(t) &= 0.23 + 0.2\sin \pi t,
\end{aligned}$$

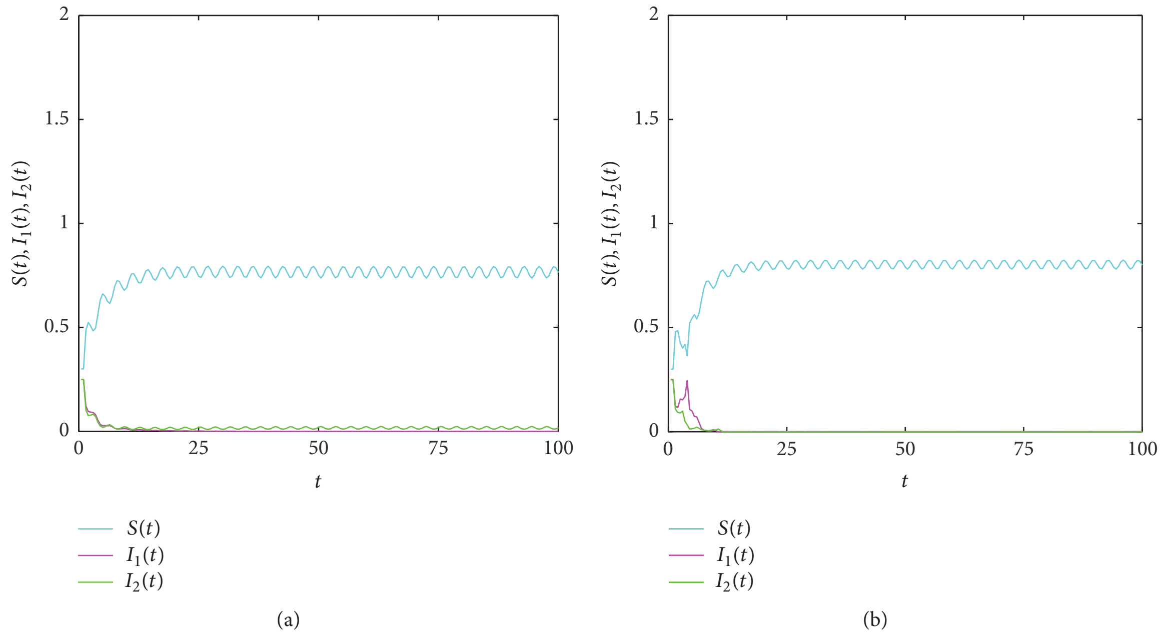

FIGURE 3: The solution $(S(t), I_1(t), I_2(t)) = (0.3, 0.25, 0.25)$ to the nonautonomous stochastic model (3).

$$
\begin{aligned}
a_2(t) &= 0.25 + 0.2 \sin \pi t, \\
\beta_1(t) &= 0.2 + 0.2 \sin \pi t, \\
\beta_2(t) &= 0.3 + 0.2 \sin \pi t, \\
r_1(t) &= 0.2 + 0.2 \sin \pi t, \\
r_2(t) &= 0.3 + 0.2 \sin \pi t, \\
d(t) &= 0.25 + 0.2 \sin \pi t, \\
\alpha_1(t) &= 0.2 + 0.2 \sin \pi t, \\
\alpha_2(t) &= 0.3 + 0.2 \sin \pi t, \\
\sigma_1(t) &= 0.2 + 0.2 \sin \pi t, \\
\sigma_2(t) &= 0.2 + 0.2 \sin \pi t.
\end{aligned}
\tag{76}
$$

Note that $\mathscr{R}_1 < 1$, $\mathscr{R}_2 < 1$, and $\langle \sigma_i^2 \rangle_T \leq d^l \langle a_i^2 \rangle_T \langle \beta_i^2 \rangle_T / A^u \langle a_i \rangle_T$. That is, conditions (ii) of Theorem 6 hold. Then the two infectious diseases will go to extinction. Figure 3(a) shows that one of two diseases in the deterministic SIS epidemic model is extinct and the other is persistent without the white noises. Figure 3(b) shows that the two diseases will die out under a small white noise disturbance of model (3).

7. Discussion and Conclusions

This paper explores the existence of nontrivial positive T-periodic solution of a nonautonomous stochastic SIS epidemic model with nonlinear growth rate and double epidemic hypothesis. By constructing a suitable stochastic Lyapunov function, we establish sufficient conditions for the existence of nontrivial positive T-periodic solution of system (3). Furthermore, the sufficient conditions for the extinction of the two diseases are obtained. Our results are given as follows:

(1) If $a_1(t)r_1(t)/(\beta_1(t) - a_1(t)d(t)) > A^u/d^l$, $a_2(t)r_2(t)/(\beta_2(t) - a_2(t)d(t)) > A^u/d^l$, and $\mathfrak{R} > 1$ hold, the SIS model has at least one nontrivial positive T-periodic solution, where

$$
\mathfrak{R} = \sum_{i=1}^{2} \frac{\langle A\alpha_i \beta_i \rangle_T}{\langle d + \alpha_i + r_i + \sigma_i^2 A^{2u}/2a_i^2 d^{2l} \rangle_T \langle d + \sigma_1^2/2 + \sigma_2^2/2 \rangle_T \langle A + a_1\alpha_1 + a_2\alpha_2 \rangle_T}. \tag{77}
$$

(2) If $\langle \sigma_i^2 \rangle_T > \langle \beta_i^2 \rangle_T / 2\langle d^l + \alpha_i^l + r_i^l \rangle_T$, the two infectious diseases go extinct.

(3) If $\langle \sigma_i^2 \rangle_T \leq \dfrac{d^l \langle a_i^2 \rangle_T \langle \beta_i^2 \rangle_T}{A^u \langle a_i \rangle_T}$ and $\mathscr{R}_i < 1$, the two infectious diseases also go extinct, where

$$
\mathscr{R}_i = \frac{A^u \langle \beta_i \rangle_T}{d^l \langle a_i \rangle_T \langle d + \alpha_i + r_i \rangle_T} - \frac{A^{2u} \langle \sigma_i^2 \rangle_T}{2d^{2l} \langle a_i^2 \rangle_T \langle d + \alpha_i + r_i \rangle_T}. \tag{78}
$$

Some interesting questions deserve further investigation. On the one hand, we may explore some realistic but complex models, such as considering the effects of impulsive or delay perturbations on system (3). On the other hand, we can concern the dynamics of a nonautonomous stochastic SIS epidemic model with two infectious diseases driven by Lévy jumps. What is more, we can also investigate the nonautonomous stochastic SIS epidemic model with two infectious diseases by a continuous time Markov chain. We will investigate these cases in our future work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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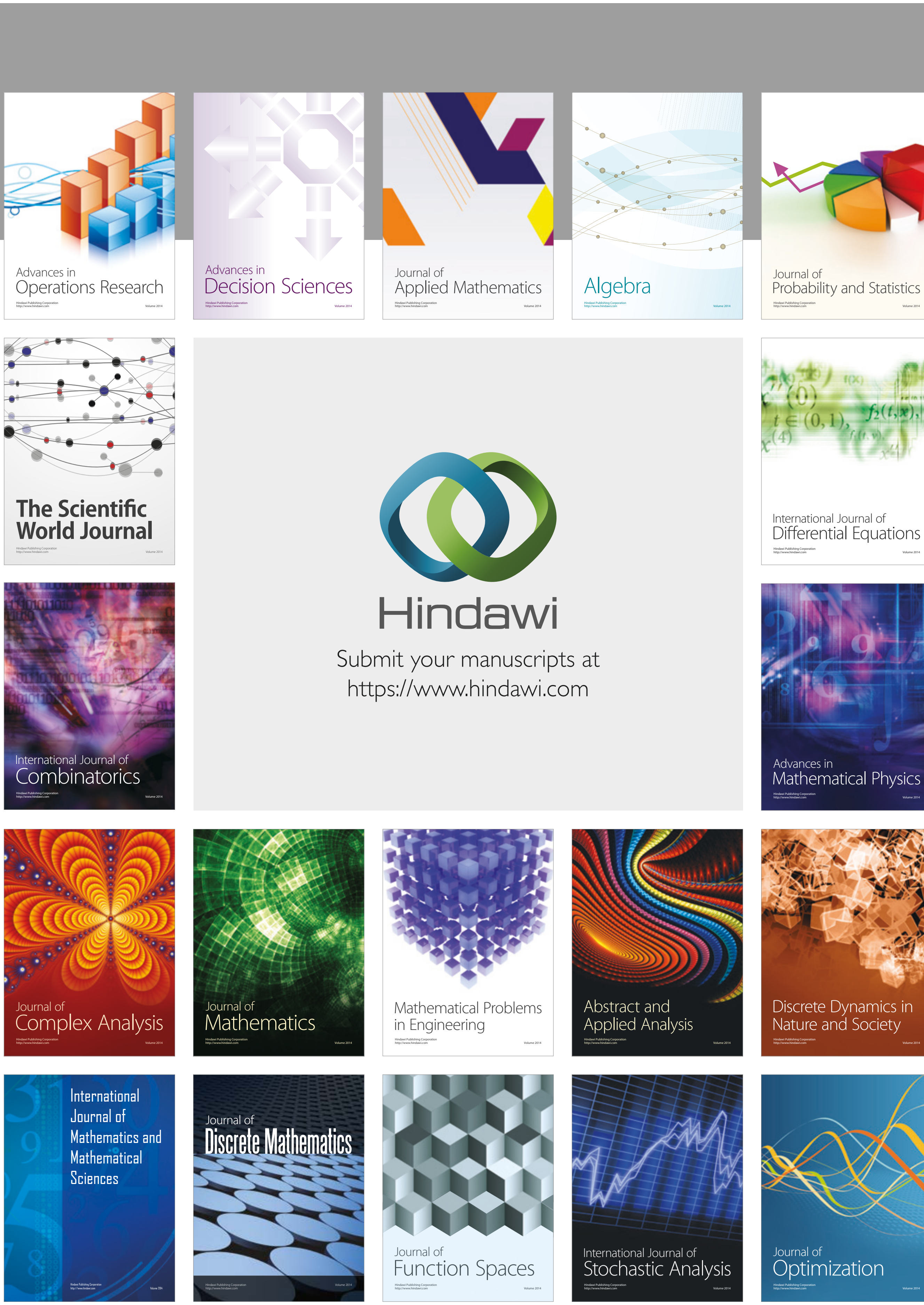
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