Randomized dictatorship and the Kalai-Smorodinsky bargaining solution

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Abstract

"Randomized dictatorship," one of the simplest ways to solve bargaining situations, works as follows: a fair coin toss determines the "dictator"—the player to be given his first-best payoff. The two major bargaining solutions, that of Nash (1950) and that of Kalai and Smorodinsky (1975), Pareto-dominate this process. However, whereas the existing literature offers axiomatizations of the Nash solution in which this domination plays a central role (Moulin (1983), de Clippel (2007)), it does not provide an analogous result for Kalai-Smorodinsky. This paper fills in this gap: a characterization of the latter is obtained by combining the aforementioned domination with three additional axioms: Pareto optimality, individual monotonicity, and a weakened version of the Perles-Maschler (1981) super additivity axiom.

Keywords: Bargaining, Kalai-Smorodinsky solution; Randomized Dictatorship.

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1 Introduction

The classic *bargaining problem*, originated in Nash (1950), is defined as a pair (S, d), where $S \subset \mathbb{R}^2$ is the *feasible set*, representing all possible (v-N.M) utility agreements between the two players, and $d \in S$, the *disagreement point*, is a point that specifies their utilities in case they do not reach a unanimous agreement on some point of S. The following assumptions are made on (S, d):

- S is compact and convex;
- $d < x$ for some $x \in S$;[1]
- For all $x \in S$ and $y \in \mathbb{R}^2$: $d \leq y \leq x \Rightarrow y \in S$.

Denote by $\mathcal{B}$ the collection of all such pairs (S, d). A *solution* is any function $\mu\colon \mathcal{B} \to \mathbb{R}^2$ that satisfies $\mu(S, d) \in S$ for all $(S, d) \in \mathcal{B}$. Given a feasible set S, the *weak Pareto frontier* of S is $WP(S) \equiv \{x \in S : y > x \Rightarrow y \notin S\}$ and the *strict Pareto frontier* of S is $P(S) \equiv \{x \in S : y \gneqq x \Rightarrow y \notin S\}$. The best that player i can hope for in the problem (S, d), given that player j obtains at least d_j utility units, is $a_i(S, d) \equiv \max\{x_i : x \in S_d\}$, where $S_d \equiv \{x \in S : x \geq d\}$. The point $a(S, d) = (a_1(S, d), a_2(S, d))$ is the *ideal point* of the problem (S, d). The *Kalai-Smorodinsky solution*, KS, due to Kalai and Smorodinsky (1975), is defined by $KS(S, d) = P(S) \cap [d; a(S, d)]$.[2] The *Nash solution*, N, due to Nash (1950), is defined to be the unique maximizer of $(x_1 - d_1)(x_2 - d_2)$ over S_d. Nash showed that this is the unique solution that satisfies the following axioms, in the statements of which (S, d) and (T, e) are arbitrary problems.

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[1]Vector inequalities: xRy if and only if $x_i R y_i$ for both $i \in \{1, 2\}$, $R \in \{>, \geq\}$; $x \gneqq y$ if and only if $x \geq y$ & $x \neq y$.

[2]Given two vectors x and y, the segment connecting them is denoted $[x; y]$.

Weak Pareto Optimality (WPO): $\mu(S,d) \in WP(S)$.[3]

Individual Rationality (IR): $\mu_i(S,d) \geq d_i$ for all $i \in \{1,2\}$.

Let F_A denote the set of positive affine transformations from $\mathbb{R}$ to itself.[4]

Independence of Equivalent Utility Representations (IEUR): $f = (f_1, f_2) \in F_A \times F_A \Rightarrow f \circ \mu(S,d) = \mu(f \circ S, f \circ d)$.[5]

Let $\pi(a,b) \equiv (b,a)$.

Symmetry (SY): $[\pi \circ S = S]\&[\pi \circ d = d] \Rightarrow \mu_1(S,d) = \mu_2(S,d)$.

Independence of Irrelevant Alternatives (IIA): $[S \subset T]\&[d = e]\&[\mu(T,e) \in S] \Rightarrow \mu(S,d) = \mu(T,e)$.

Whereas the first four axioms are widely accepted, IIA has raised some criticism. The idea behind a typical such criticism is that the bargaining solution could, or even should, depend on the shape of the feasible set. In particular, Kalai and Smorodinsky (1975) noted that when the feasible set expands in such a way that for every feasible payoff for player 1 the maximal feasible payoff for player 2 increases, it may be the case that player 2 loses from this expansion under the Nash solution. Given $x \in S_d$, let $g_i^S(x_j)$ be the maximal possible payoff for i in S given that j's payoff is x_j, where $\{i,j\} = \{1,2\}$. What Kalai and Smorodinsky noticed, is that N violates the follow-

[3] A natural strengthening of this axiom is Pareto Optimality (PO), which requires $\mu(S,d) \in P(S)$ for all $(S,d) \in \mathcal{B}$.

[4] i.e., the set of functions f of the form $f(x) = \alpha x + \beta$, where $\alpha > 0$.

[5] If $f_i \colon \mathbb{R} \to \mathbb{R}$ for each $i = 1,2$, $x \in \mathbb{R}^2$, and $A \subset \mathbb{R}^2$, then: $(f_1, f_2) \circ x \equiv (f_1(x_1), f_2(x_2))$ and $(f_1, f_2) \circ A \equiv \{(f_1, f_2) \circ a : a \in A\}$.

ing axiom, in the statement of which (S, d) and (T, d) are arbitrary problems with a common disagreement point.

Individual Monotonicity (IM):

$$[a_j(S,d) = a_j(T,d)]\&[g_i^S(x_j) \le g_i^T(x_j)\ \forall x \in S_d \cap T_d] \Rightarrow \mu_i(S,d) \le \mu_i(T,d).$$

Furthermore, they showed that when one deletes the controversial IIA from the list of Nash's axioms and replaces it by IM, a characterization of KS obtains.

The class of solutions that satisfy the three common axioms to Nash and Kalai-Smorodinsky is large, and includes interesting and intuitive solutions. For example, it includes the *Perles-Maschler solution* (due to Perles and Maschler (1981)) and the *equal area solution* (due to Anbarci and Bigelow (1994)).[6] There is no wonder, then, that IIA is viewed as one of the most essential properties exhibited by N, and the same is true for the relation between IM and KS.

One of the simplest ways to solve bargaining problems is by a lottery: the players flip a fair coin and the winer (say, player i) gets to be the "dictator," who obtains his first-best payoff, $a_i(S, d)$, while the loser obtains d_j. It seems reasonable to demand that a "good" solution be better than this randomized dictatorship. Formally, it seem reasonable to impose the following axiom on the solution, in the statement of which (S, d) is an arbitrary problem.

Midpoint Domination (MD): $\mu(S,d) \ge \frac{1}{2}d + \frac{1}{2}a(S,d)$.

[6]The equal area solution for (S, d) is defined to be the point $u \in P(S)$ such that $[d; u]$ splits S_d into two parts with equal area; a description of the Perles-Maschler solution will be given in subsection 2.1 below.

Both N and KS satisfy this axiom.[7] In fact, Moulin (1983), in one of the simplest and most elegant axiomatizations of N, showed that it is characterized by IIA and MD alone.[8] Anbarci (1998), who considered the bargaining model with the normalization $d = \mathbf{0} \equiv (0, 0)$, showed that KS is characterized by IM and the following axiom.

Balanced Focal Point (BFP): If $S =$ conv hull$\{\mathbf{0}, (a, b), (\lambda a, 0), (0, \lambda b)\}$ for some $\lambda \in [1, 2]$, then $\mu(S, \mathbf{0}) = (a, b)$.

Note that for the particular value $\lambda = 2$, BFP becomes the requirement that MD be satisfied on triangles. It therefore seems that MD is closely related to the two main pillars of bargaining theory, N and KS. Moulin's and de Clippel's results formalize this point clearly regarding the relation between MD and N. Anbarci's result "almost" makes the analogous point regarding KS. The "almost" here is due to two reasons. First, BFP is not MD. Second, whereas the axioms MD, IM, and IIA enjoy clear economic interpretations, the intuition behind BFP is less obvious.[9] The goal of the current paper is to push forward in the direction pointed out by Anbarci and provide an MD-based axiomatization for KS. This is done in the next section. Section 3 concludes, and the Appendix collects a few technicalities.

2 The main result

When one scales a problem by a fixed factor, a new problem obtains; that is, given $(S, d) \in \mathcal{B}$ and $c > 0$, $(cS, cd) \in \mathcal{B}$.[10] When one adds two feasible sets, a new feasible

[7] That KS satisfies MD is obvious; the fact that N satisfies it was proved by Sobel (1980).

[8] A related result has been obtained by de Clippel (2007), who showed that N is characterized by MD and one other axiom—*disagreement point convexity* (see his paper for the details).

[9] The justification for BFP is that the equal areas to the north-west and south-east of the focal point (a, b) can be viewed as representing equivalent concessions.

[10] $cS \equiv \{cs : s \in S\}$.

set obtains. These basic properties have intuitive economic interpretations,[11] and they call for some discipline that a "reasonable" solution is expected to adhere to. The scaling property is expressed in the following axiom, in the statement of which (S, d) is an arbitrary problem.

Homogeneity (HOM): For all $c > 0$: $\mu(cS, cd) = c\mu(S, d)$.

Set addition is considered in the following axiom, which is due to Perles and Maschler (1981); in its statement, (S, d) and (T, d) are arbitrary problems with a common disagreement point.

Super Additivity (SA): $\mu(S, d) + \mu(T, d) \leq \mu(S + T, d)$.[12]

Any solution that satisfies HOM and SA automatically satisfies the following axiom; in its statement, (S, d) and (T, d) are arbitrary problems with a common disagreement point.

Super Additivity* (SA*): For all $\lambda \in [0, 1]$: $\lambda\mu(S, d) + (1 - \lambda)\mu(T, d) \leq \mu(\lambda S + (1 - \lambda)T, d)$.

SA* provides the players the incentives to reach early agreements in situations that involve uncertainty regarding the feasible set.[13] This axiom, however, is rather strong. The following is a substantial weakening of it; in its statement, (S, d) and (T, d) are arbitrary problems with a common disagreement point.

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[11] See Thomson (1994) for a detailed discussion of these interpretations.

[12] $S + T \equiv \{s + t : s \in S, t \in T\}$.

[13] Myerson (1981) considers essentially the same axiom, but in a slightly different model; in his model there are only feasible sets, and no disagreement points.

Restricted Super Additivity (RSA): For all $\lambda \in [0,1]$:

$$[S = S_d]\&[T = T_d]\&[a(S,d) = a(T,d)] \Rightarrow$$
$$\lambda\mu(S,d) + (1-\lambda)\mu(T,d) \leq \mu(\lambda S + (1-\lambda)T, d).$$

In words, RSA imposes the requirement of SA*, but only on pairs of problems which are "similar," in the sense that both are embedded in a common rectangle of best and worst outcomes. It is straightforward that KS satisfies this property. On the other hand, the other major solution, N, does not.[14] With this axiom at hand, we are ready to turn to the main result.

Theorem 1. *A solution μ satisfies PO, MD, IM, and RSA if and only if $\mu = KS$.*

Proof. It is easy to check that KS satisfies the axioms. Let μ be an arbitrary solution that satisfies them. Let (S,d) be an arbitrary problem. Wlog, we can assume that $d = \mathbf{0}$. Let $a \equiv a(S,d)$ and let $k \equiv KS(S,d)$. The midpoint of the aforementioned problem is $m \equiv \frac{1}{2}a$. Let $\theta \in (0,1)$ be the number that satisfies $(1-\theta)m + \theta a = k$. Let $\Delta \equiv$ conv hull$\{\mathbf{0}, (a_1, 0), (0, a_2)\}$, let $R \equiv \{x \in \mathbb{R}^2 : \mathbf{0} \leq x \leq a\}$ and let $Q \equiv (1-\theta)\Delta + \theta R$. By MD, $\mu(\Delta, d) = m$. By PO, $\mu(R,d) = a$. By RSA, $\mu(Q,d) = k$. Now, let $T \equiv$ conv hull$(\Delta \cup \{k\})$. The combination of PO, IM, and the fact that $\mu(Q,d) = k$ implies $\mu(T,d) = k$. By IM, $\mu(S,d) = k$. □

2.1 Independence of the axioms

The *midpoint solution*, $m(S,d) \equiv \frac{1}{2}d + \frac{1}{2}a(S,d)$, satisfies all of the axioms from Theorem 1 but PO.[15] The *equal loss solution*, $EL(S,d) \equiv a(S,d) - (l,l)$, where l is the minimal number such that the aforementioned expression is in S, satisfies all of them

[14] See the Appendix.

[15] Consider the following modification of $m(S,d)$: $m^*(S,d) \equiv m(S,d) + (e,e)$, where e is the maximal number such that the aforementioned expression is in S. It is easy to check that this solution satisfies WPO, MD, IM, and RSA; it shows that PO cannot be weakened to WPO without rendering the conclusion of Theorem 1 false.

but MD (this solution is due to Chun (1988)). The following solution satisfies all the axioms but RSA. Given an arbitrary (S, d), let us denote, for short, its ideal point and midpoint by a and m, respectively. Consider the following *piecewise linear monotone path solution*: it assigns to each (S, d) the point $P(S) \cap \{[m; (\frac{1}{2}a_1 + \frac{1}{2}m_1, m_2)] \cup [(\frac{1}{2}a_1 + \frac{1}{2}m_1, m_2); a]\}$. It is easy to see that this solution satisfies PO, MD, and IM.

As an example of a solution that satisfies all the axioms but IM, the following bargaining solution, the *Perles-Maschler solution*, PM, presents itself as a candidate. This solution is defined on $\mathcal{B}_o$: the class of problems $(S, d) \in \mathcal{B}$ where $d = \mathbf{0}$, $S = S_d$, and $WP(S) = P(S)$.[16] Given (S, d) as above, $PM(S, d)$ is the unique point $u \in P(S)$ that satisfies:

$$\int_{(0,a_2)}^{u} \sqrt{-dxdy} = \int_{u}^{(a_1,0)} \sqrt{-dxdy}, \tag{1}$$

where the integrals are taken along the arcs of $\partial S = P(S)$.[17] It is well-known that this solution satisfies PO and RSA (in fact: SA and SA*), and that it violates IM. What is left to prove is that it satisfies MD. To this end, the following lemma will be useful.

Lemma 1. *Let $a, b > 0$, let $l \in (0, a]$ and let $h \in (0, b]$. Let $S \equiv$ conv hull$\{\boldsymbol{0}, (a + l, 0), (0, b + h), (a, b)\}$. Then $MP(S, \boldsymbol{0}) \geq \frac{1}{2}(a + l, b + h)$.*

Proof. Make the assumptions of the lemma. Let $S_1 \equiv \{x \in S : x_2 \geq b\}$ and $S_2 \equiv \{x \in S : x_1 \geq a\}$. Let $T_1 \equiv S_1 - (0, b)$ and $T_2 \equiv S_2 - (a, 0)$. By Lemma 3.2 in Perles and Maschler (1981), $S = T_1 + T_2$. Therefore, since MP is supper additive, $MP(S, \mathbf{0}) \geq$

[16] The choice $d \equiv \mathbf{0}$ is a mere normalization; equivalently, one can consider the collection of all (S, d) with a common d and where $S = S_d$, which is simply a d-translation of the Perles-Maschler setting.

[17] The solution MP can be defined also for problems for which $WP(S) \neq P(S)$, but then the expression (1) needs to be amended in order to account for the possibility that the Pareto boundary contains a segment parallel to an axis. This is only a technically that I will ignore for the sake of the ease of presentation.

$MP(T_1, \mathbf{0}) + (T_2, \mathbf{0}) = MP(S_1, \mathbf{0}) + (S_2, \mathbf{0}) - (a, b) = (\frac{a}{2}, b + \frac{h}{2}) + (a + \frac{l}{2}, \frac{b}{2}) - (a, b)$. The last equality follows from the fact that the combination of SY and IEUR implies that MD is satisfied on triangles, and MP indeed satisfies the aforementioned axioms. $\square$

Proposition 1. *For every $(S, d) \in \mathcal{B}_o$ and $i \in \{1, 2\}$: $MP_i(S, d) \geq \frac{1}{2} a_i(S, d)$.*

Proof. Since MP is continuous on $\mathcal{B}_o$,[18] it is enough to prove the proposition for the case where $S = S_d$ is a polygon. Let n denote the number of sides of ∂S. I will prove the proposition by induction on n. The statement of the proposition is obviously true for $n = 1$, and, by Lemma 1, it is also true for $n = 2$. I will therefore assume correctness for $n - 1$ and prove for n, where $n \geq 3$.

Let $(S, d) \in \mathcal{B}_o$ be such that $S = S_d$ is a polygon, and let $n \geq 3$ be the number of sides of ∂S. Select two points, $q^1, q^2 \in \partial S$, such that:

1. $S_1 \equiv \{x \in S : x_2 \geq q^1_2\}$ and $S_2 \equiv \{x \in S : x_1 \geq q^2_1\}$ are triangles,
2. at least one of q^1 and q^2 is a vertex of S, and
3. S_1 and S_2 have the same area.

By 2, $S_2 \equiv \{x \in S : x_1 \geq q^1_1, x_2 \geq q^2_2\}$ is a polygon and ∂S_2 has fewer than n sides (specifically, $n-1$ or $n-2$). Let T_i be the translate of S_i to the origin. By Lemma 3.3 in Perles and Maschler (1981), $S = (T_1 + T_3) + T_2$. By Lemma 1, $MP(T_1 + T_3, \mathbf{0}) \geq \frac{1}{2} a(T_1 + T_3, \mathbf{0})$ and by the induction's hypothesis $MP(T_2, \mathbf{0}) \geq \frac{1}{2} a(T_2, \mathbf{0})$. Since MP is supper additive, $MP(S) = MP((T_1 + T_3) + T_2) \geq MP(T_1 + T_3) + MP(T_2)$, and therefore $MP(S) \geq \frac{1}{2}[a(T_1 + T_3, \mathbf{0}) + a(T_2, \mathbf{0})]$. The observation that $a(T_1 + T_3, \mathbf{0}) + a(T_2, \mathbf{0}) = a((T_1 + T_3) + T_2, \mathbf{0})$ completes the proof. $\square$

[18] i.e., if $(S_n, \mathbf{0}) \subset \mathcal{B}_o$ is such that S_n converges to S in the Hausdorff metric and $(S, \mathbf{0}) \in \mathcal{B}_o$, then $MP(S_n, \mathbf{0})$ converges to $MP(S, \mathbf{0})$.

3 Conclusion

MD expresses the demand that a bargaining solution should be unambiguously better than "randomized dictatorship." RSA provides, at least under some restrictions on the underlying uncertainty, incentives for early agreements. PO is obvious. Finally, IM is, informally speaking, the essence of the Kalai-Smorodinsky solution. In this paper I have shown that in the 2-person bargaining problem, this solution is uniquely pinned down by these axioms. The axioms are independent. In the process of proving this independence it was shown that the Perles-Maschler solution, like the Kalai-Smorodinsky and Nash solutions, satisfies MD. This result is of interest in its own right.

The model can be formulated for the n-person case in an analogous manner to the 2-person description from above; the definitions and axioms have straightforward (and well-known) multi-person counterparts. Since Theorem 1 extends to the n-person case and its proof is essentially unchanged relatively to the 2-person case, a full-fledged re-formulation of the model in n dimension is omitted. However, some remarks regarding the independence of the axioms are in place. The midpoint solution and the piecewise linear monotone path solution from above have counterparts in the multi-person case. The equal-loss solution, on the other hand, may fail to exist when there are more than two players. Nevertheless, it is well-defined in the (n-person) model in which the assumption of compact feasible sets is replaced by unboundedness from below, or *free disposal*.[19] Finally, the existence of a multi-person solution that satisfies all the axioms but IM is also nontrivial. In the Appendix I describe a rich domain of 3-person problems on which such a solution exists.

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[19]In such a model, RSA needs to be amended so that "$S = S_d$" ("$T = T_d$") is replaced by "S = comprehensive hull of S_d" ("T = comprehensive hull of T_d").

Appendix

The following example shows that N fails to satisfy RSA.

Example 1: Let $S = \text{conv hull}\{\mathbf{0}, (1,0), (0,2)\}$ and let $T \equiv \text{conv hull}(S \cup \{(1, 1+\epsilon)\})$, for some small $\epsilon > 0$. Let $Q \equiv \frac{\epsilon}{1+\epsilon} S + \frac{1}{1+\epsilon} T$. We have that $N(S, \mathbf{0}) = (\frac{1}{2}, 1)$, $N(T, \mathbf{0}) = (1, 1+\epsilon)$, and $N(Q, \mathbf{0}) = (1, 1)$. The requirement of RSA fails for player 2.

Bellow is a construction of a 3-person solution that satisfies all the axioms but IM. The generalization to any $n \geq 3$ is straightforward.

Example 2: Given a 3-person bargaining problem (S, d), define the set $X(S, d)$ as follows:

$$X((S, d)) \equiv \{(a, b) : (a, b, m_3(S, d)) \in S\}.$$

Consider the domain of *smooth* 3-person problems: those 3-person (S, d) such that $WP(S) = P(S)$, and where $P(S)$ does not contain hyperplanes; that is, for all distinct $x, y \in P(S)$ and $\alpha \in (0, 1)$, the point $\alpha x + (1 - \alpha) y$ is not in $P(S)$. Define the solution μ^* on this domain by:

$$\mu^*(S, d) \equiv (MP_1(X((S, d)), (d_1, d_2)), MP_2(X((S, d)), (d_1, d_2)), m_3(S, d)).$$

It is easy to see that this solution satisfies (the multi-person counterparts of) PO, RSA, and MD.

References

Anbarci, N. (1998), Simple characterizations of the Nash and Kalai/Smorodinsky solutions, *Theory and Decision*, **45**, 255-261.

Anbarci, N. and J. Bigelow (1994), The area monotonic solution to the cooperative bargaining problem, *Mathematical Social Sciences*, **28**, 133-142.

Chun, Y. (1988), The equal-loss principle for bargaining problems, *Economic Letters*, **26**, 103-106.

de Clippel, G. (2007), An axiomatization of the Nash bargaining solution, *Social Choice and Welfare*, **29**, 201-210.

Kalai, E. and Smorodinsky, M. (1975), Other solutions to Nash's bargaining problem, *Econometrica*, **43**, 513-518.

Moulin, H. (1983), Le choix social utilitariste. Ecole Polytechnique Discussion Paper.

Myerson, R.B. (1981), Utilitarianism, Egalitarianism, and the timing effect in social choice problems, *Econometrica*, **49**, 883-897.

Nash, J. F. (1950), The bargaining problem, *Econometrica*, **18**, 155-162.

Perles, M.A. and Maschler, M. (1981), The super-additive solution for the Nash bargaining game, *International Journal of Game Theory*, **10**, 163-193.

Sobel, J. (1980), Distortion of utilities and the bargaining problem, *Econometrica*, **49**, 597-620.

Thomson, W. (1994), "Cooperative models of bargaining." In: Aumann R.J., Hart S. (eds) Handbook of game theory, vol 2. North-Holland, Amsterdam, 1237-1284.