ADMISSIBLE RULES AND THE LEIBNIZ HIERARCHY

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Abstract. This paper provides a semantic analysis of admissible rules and associated completeness conditions for *arbitrary* deductive systems, using the framework of abstract algebraic logic. Algebraizability is not assumed, so the meaning and significance of the principal notions vary with the level of the Leibniz hierarchy at which they are presented. As a case study of the resulting theory, the non-algebraizable fragments of relevance logic are considered.

1. Introduction

Many researchers have considered the question: to what extent can we interpret a logic plausibly in its own meta-language? Disjunction properties are one manifestation of this concern. A problem in the reverse spirit is the derivability of admissible rules. Following Lorenzen [29], we say that a rule of inference is *admissible* in a formal system if its addition to the system produces no new theorems. A simple example is the rule of necessitation, $x \;/\; \Box x$, which is admissible (and not derivable) in quasi-normal modal logics. Less trivially, the process of cut elimination shows that underivable cut rules are admissible in suitable sequent calculi.

The *algebraizable* logics of Blok and Pigozzi [7] constitute the framework for some prominent treatments of admissibility, such as Rybakov's monograph [62]. On the other hand, the quasi-normal modal systems and the cut-free subsystems of substructural logics are not algebraizable. The present paper analyzes the semantics of admissible rules in the context of *arbitrary* deductive systems, indicating which tools of abstract algebraic logic [15, 21] are really needed at various stages of the theory, while also supplying some new results. The paper is largely self-contained, but its purpose is not to survey the now-substantial literature on admissibility in particular systems, such as intermediate, modal and fuzzy logics. The reader is referred to [62] for work of this kind done before 1997. Important subsequent developments are summarized, for instance, in [13], where ample references are given.

It is well known that certain logics possess no algebraic semantics at all. Fortunately, however, *every* deductive system $\vdash$ has a nontrivial semantics, $\mathsf{Mod}^*(\vdash)$, comprising its *reduced* matrix models [67]. For several reasons,

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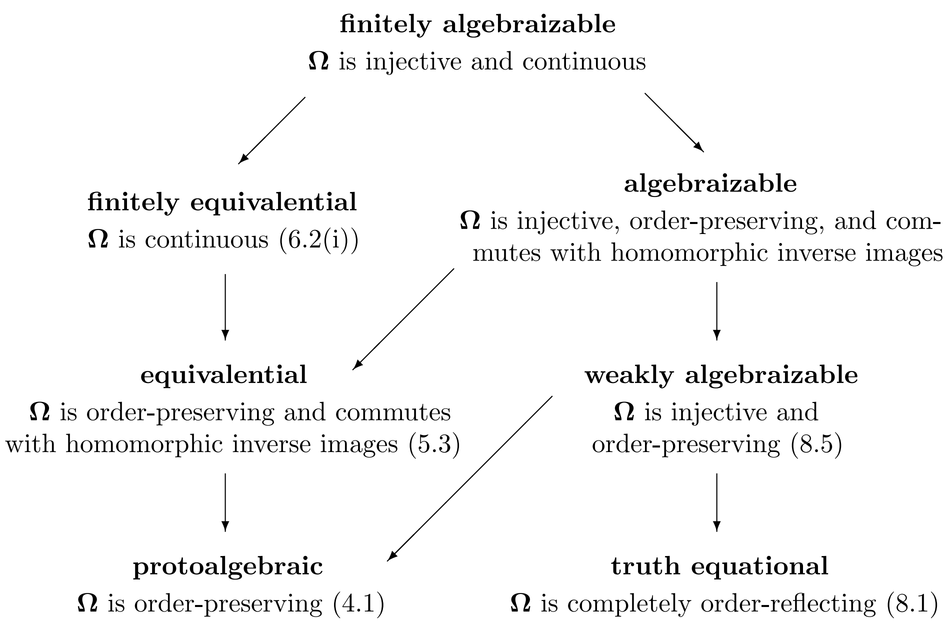
this semantics is considered canonical in abstract algebraic logic, and it will guide our analysis of syntactic notions throughout. For simplicity, we confine the present discussion to *sentential* systems, although it is not necessary to do so (see Section 11).

If $\langle \boldsymbol{A}, F\rangle$ is a matrix model of $\vdash$, then F is called a $\vdash$*–filter* of the algebra $\boldsymbol{A}$, and $\langle \boldsymbol{A}, F\rangle$ can be collapsed to a reduced matrix model by 'factoring out' the *Leibniz congruence* $\boldsymbol{\Omega}^{\boldsymbol{A}}F$. This is the largest congruence of $\boldsymbol{A}$ that turns F into a union of congruence classes. The *Leibniz operator* of $\vdash$ is the collection, taken over all $\boldsymbol{A}$, of the maps $F \mapsto \boldsymbol{\Omega}^{\boldsymbol{A}}F$ (F a $\vdash$–filter of $\boldsymbol{A}$). The action of this operator is purely algebraic—it depends only on the structure of the signature.

Because the Leibniz operator is defined for every possible $\vdash$, its behaviour serves to classify deductive systems. The outcome is the *Leibniz hierarchy*, which is depicted rather cryptically in the accompanying diagram. The numbers refer to less cryptic descriptions of the levels, recounted in the present paper (but established elsewhere). Roughly speaking, the hierarchy calibrates the degree to which a deductive system admits algebraic treatment. The arrows are implications between the indicated $\boldsymbol{\Omega}$–properties. Our aim here is to analyze admissibility for systems at the 'sub-algebraizable' levels of the hierarchy.

A portion of the Leibniz Hierarchy



Theorem 2.12 asserts that, in an arbitrary deductive system $\vdash$, a rule R is admissible iff every reduced matrix model of $\vdash$ is a homomorphic image of an R–validating subdirect product $\mathcal{B}$ of reduced matrix models of

$\vdash$. (If an element of $\mathcal{B}$ is designated, then so is its image; the converse is not imposed.) There is no guarantee that $\mathcal{B}$ itself can be chosen reduced (Fact 2.11), nor that its reduced subdirect factors will validate R. Obviously, this characterization becomes more attractive in systems where the reduced matrix models are *closed* under subdirect products. These are exactly the *protoalgebraic logics*, i.e., the ones where a rudimentary *implication* connective is definable. Thus, R is admissible in a protoalgebraic system $\vdash$ iff $\mathsf{Mod}^*(\vdash)$ actually *includes* R–validating homomorphic pre-images for all of its members. The characterization acquires a purely algebraic form in *weakly algebraizable* systems, i.e., protoalgebraic ones where the designation predicate is equationally definable over the reduced models. It takes an 'almost algebraic' form in *order algebraizable logics.* (See Theorems 4.4, 8.7 and 9.3.)

Theorem 4.7 shows that a protoalgebraic finitary system $\vdash$ will be *structurally complete*—in the sense that all of its admissible finite rules are derivable—provided that all of its finitely generated relatively subdirectly irreducible reduced matrix models are weakly projective in $\mathsf{Mod}^*(\vdash)$. In this case, moreover, $\vdash$ is *hereditarily* structurally complete, i.e., all of its finitary extensions are structurally complete as well. It is notable that no Leibniz-condition stronger than protoalgebraicity is needed here. The result applies, for instance, to the Gödel-Dummett logic **LC** (a.k.a. **G**) and to the negation-less fragment of the system $\mathbf{RM^t}$ (from relevance logic). For these two systems, hereditary structural completeness was proved directly in [20] and [45], respectively.

The *equivalential* deductive systems have a well-behaved generalized *bi-conditional* ($\leftrightarrow$), and in the *finitely equivalential* ones, this bi-conditional has a finite definition. To stipulate that all admissible rules of an equivalential system $\vdash$ are derivable (including the infinite ones) is to demand that $\mathsf{Mod}^*(\vdash)$ be the closure of a suitable *free* reduced matrix model under the combination of isomorphisms, submatrices, direct products and a fourth class operator whose meaning depends on the number of variables (Theorem 5.7). This result extends an early finding of Prucnal and Wroński [55]. A finitely equivalential finitary system $\vdash$ is structurally complete iff $\mathsf{Mod}^*(\vdash)$ is generated as a *universal Horn class* by the same free reduced model (Theorem 6.4). In that case, any two nontrivial members of $\mathsf{Mod}^*(\vdash)$ are contained, up to isomorphism, in a third member. And, in the event of hereditary structural completeness, the finitary extensions of $\vdash$ form a distributive lattice—this is implicit in Gorbunov [23].

A further consequence of structural completeness in equivalential systems is that any two nontrivial 0–generated reduced matrix models are isomorphic (Theorem 7.7). We do not need the full force of structural completeness to prove this, however. It follows from a weak variant called *overflow completeness*, isolated recently by Wroński [72]. The proof utilizes an analysis of the existential positive first order theory of $\mathsf{Mod}^*(\vdash)$, inspired by the main result of [72]. The analysis is given in Theorems 7.3 and 7.5, and it rules out overflow completeness for a large class of fuzzy and/or substructural logics (Examples 7.8, 8.10).

None of the above results presupposes algebraizability. Natural admissibility problems are abundant in non-algebraizable logics, but, to the best of the present author's knowledge, structural completeness has not yet been established for any significant non-algebraizable system. A future exception might be the implication fragment **BCIW** of the relevance logic **R**. The question of structural completeness for **BCIW** has been open for some time. A little fresh light is thrown on this problem in Section 10, where a case study of the non-algebraizable fragments of **R** is undertaken.

2. Admissible Rules

We work within a fixed but arbitrary algebraic language. Its signature and its infinite set of variables—denoted by *Var*—are assumed to be well-ordered (not necessarily countable). All algebras considered have this type, unless we say otherwise. The universe of an algebra $\boldsymbol{A}$ is denoted as A, and is assumed non-empty. Recall that (sentential) *formulas* are elements of the absolutely free algebra $\boldsymbol{Fm}$ generated by Var, and *substitutions* are endomorphisms of $\boldsymbol{Fm}$. A *rule* is a pair $\langle \Gamma, \alpha \rangle$, where $\Gamma \cup \{\alpha\} \subseteq Fm$. It is a *finite* rule if the set Γ is finite.

Throughout this paper, $\vdash$ denotes a (sentential) *deductive system*, i.e., a substitution-invariant consequence relation over formulas, cf. [15, 21, 68]. Thus, the *theorems* of $\vdash$ are the formulas α such that $\emptyset \vdash \alpha$ (briefly, $\vdash \alpha$), while the *derivable* rules of $\vdash$ are just its elements, i.e., the pairs $\langle \Gamma, \alpha \rangle$ for which $\Gamma \vdash \alpha$. Among other standard abbreviations, we signify '$\Gamma \vdash \alpha$ for all $\alpha \in \Pi$' by $\Gamma \vdash \Pi$, and '$\Gamma \vdash \Pi$ and $\Pi \vdash \Gamma$' by $\Gamma \dashv\vdash \Pi$. The *extensions* of $\vdash$ are the deductive systems in the same language that are supersets of $\vdash$. They form a set that is closed under arbitrary intersections.

Notation. (i) x, y, z (with or without indices) stand for distinct variables.
(ii) $\gamma_1, \ldots, \gamma_n \;/\; \alpha$ abbreviates a finite rule $\langle \{\gamma_1, \ldots, \gamma_n\}, \alpha \rangle$.
(iii) $T^{\vdash}$ denotes the set of all theorems of $\vdash$.
(iv) $\vdash + \langle \Gamma, \alpha \rangle$ denotes the smallest extension of $\vdash$ containing a rule $\langle \Gamma, \alpha \rangle$.

Definition 2.1. ([29]) We call $\langle \Gamma, \alpha \rangle$ an *admissible rule* of $\vdash$ if every *theorem* of $\vdash + \langle \Gamma, \alpha \rangle$ is already a theorem of $\vdash$.

Here, Γ need not be finite. Also, $\vdash$ is not assumed *finitary*, i.e., there is no guarantee that when $\Pi \vdash \varphi$, then $\Pi' \vdash \varphi$ for some finite $\Pi' \subseteq \Pi$. If $\vdash$ is finitary and Γ is finite, then $\vdash + \langle \Gamma, \alpha \rangle$ is still finitary. For in this case, $\vdash$ is *axiomatized* by some formal system $\mathbf{F}$ of axioms and *finite* inference rules, i.e., it is the natural deducibility relation $\vdash_{\mathbf{F}}$ [31]. Then, $\vdash + \langle \Gamma, \alpha \rangle$ is just $\vdash_{\mathbf{F} \cup \{\langle \Gamma, \alpha \rangle\}}$. In the sequel, we often attribute properties of $\vdash_{\mathbf{F}}$ to $\mathbf{F}$.

Note that $\vdash_{\mathbf{F}}$ remains a deductive system when we allow infinite inference rules in $\mathbf{F}$. Then, $\Gamma \vdash_{\mathbf{F}} \alpha$ means that there is a possibly infinite *well-ordered* proof of α from Γ in $\mathbf{F}$. The systems $\vdash_{\mathbf{F}} + \langle\Gamma,\alpha\rangle$ and $\vdash_{\mathbf{F}\,\cup\,\{\langle\Gamma,\alpha\rangle\}}$ still coincide. In particular, $\vdash + \langle\Gamma,\alpha\rangle$ is just $\vdash_{\vdash\,\cup\,\{\langle\Gamma,\alpha\rangle\}}$. Even in this case:

Fact 2.2. *$\langle\Gamma,\alpha\rangle$ is admissible in $\vdash$ iff every substitution that turns all the formulas in Γ into theorems of $\vdash$ also turns α into a theorem of $\vdash$.*

The argument from right to left proceeds by (possibly transfinite) induction on the length of a proof in $\vdash \cup \{\langle\Gamma,\alpha\rangle\}$. Finite induction suffices when Γ is finite and $\vdash$ is finitary.

Recall that a (sentential) *matrix* $\langle \boldsymbol{A}, F\rangle$ comprises an algebra $\boldsymbol{A}$ and a subset F of A. The *designated* elements of this matrix are the elements of F, and $\langle \boldsymbol{A}, F\rangle$ is said to *validate* a rule $\langle\Gamma,\alpha\rangle$ if $h(\alpha) \in F$ for every homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h[\Gamma] \subseteq F$. The rules validated by the matrices in a class K constitute the *consequence relation of* K. This is always a deductive system, but it is seldom finitary.

Since matrices are first order structures, we need not define their submatrices (i.e., substructures), direct and subdirect products, or ultraproducts. By Łos' Theorem, the validity of a finite rule persists in ultraproducts, while the other three constructions preserve arbitrary rules. There are two possible definitions of a homomorphism between structures, however, so we need to be explicit about this terminology:

Definition 2.3. A *matrix homomorphism* from $\langle \boldsymbol{B}, G\rangle$ into $\langle \boldsymbol{A}, F\rangle$ is an (algebraic) homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ such that $h[G] \subseteq F$, i.e., $G \subseteq h^{-1}[F]$.

We call $\langle \boldsymbol{A}, F\rangle$ a *homomorphic image* of $\langle \boldsymbol{B}, G\rangle$ if there is a matrix homomorphism h from $\langle \boldsymbol{B}, G\rangle$ into $\langle \boldsymbol{A}, F\rangle$ such that $h[B] = A$.

Clearly, for any $\alpha \in Fm$, the class of matrices validating $\langle\emptyset,\alpha\rangle$ is closed under homomorphic images. Also, if $\langle \boldsymbol{A}, F\rangle$ is a subdirect product of matrices, then each of the subdirect factors is a homomorphic image of $\langle \boldsymbol{A}, F\rangle$.

Note that we do not require $h^{-1}[F] \subseteq G$ in Definition 2.3. Throughout this paper, homomorphisms between structures preserve the indicated relations (as well as all operations) but they are not assumed to *reflect* the relations. Of course, an *isomorphism* is a bijective homomorphism whose inverse is also a homomorphism. In particular, a matrix isomorphism preserves and reflects the set of designated elements. More generally:

Definition 2.4. A matrix homomorphism h from $\langle \boldsymbol{B}, G\rangle$ into $\langle \boldsymbol{A}, F\rangle$ is said to be *strict* if $G = h^{-1}[F]$.

In this case, every rule validated by $\langle \boldsymbol{A}, F\rangle$ is validated by $\langle \boldsymbol{B}, G\rangle$ (and conversely, if $h[B] = A$).

If θ is a congruence of an algebra $\boldsymbol{A}$ and F is a union of θ–classes of $\boldsymbol{A}$, then we abbreviate $\{a/\theta : a \in F\}$ as F/θ. In this case, the natural surjection from $\langle \boldsymbol{A}, F\rangle$ to $\langle \boldsymbol{A}/\theta, F/\theta\rangle$ is a strict matrix homomorphism, for if $b \in A$ and $b/\theta \in F/\theta$, then $b \in F$.

For any matrix $\langle \boldsymbol{A}, F\rangle$, the *Leibniz congruence* $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the largest congruence of $\boldsymbol{A}$ for which F is a union of congruence classes. By Lemma 2.9

below, $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ identifies the elements of A having the same *definable* properties in the first order equality-free language of $\langle \boldsymbol{A}, F\rangle$ (hence the allusion to Leibniz, coined in [6]). In particular, $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ always exists. We omit the superscript when $\boldsymbol{A} = \boldsymbol{Fm}$. We say that $\langle \boldsymbol{A}, F\rangle$ is (Leibniz-) *reduced* if no non-identity congruence of $\boldsymbol{A}$ makes F a union of congruence classes, i.e., if $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the identity relation $\mathrm{id}_A = \{\langle a, a\rangle : a \in A\}$. This means that any strict matrix homomorphism from $\langle \boldsymbol{A}, F\rangle$ onto another matrix must be an isomorphism, and reduced matrices were originally called 'simple' (see [67]).

Notation. We abbreviate $\langle \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F,\ F/\boldsymbol{\Omega}^{\boldsymbol{A}}F\rangle$ as $\langle \boldsymbol{A}, F\rangle^*$.

A matrix of the form $\langle \boldsymbol{A}, F\rangle^*$ is always reduced and, by the above remarks, it validates the same rules as $\langle \boldsymbol{A}, F\rangle$. In particular:

Fact 2.5. *$\langle \Gamma, \alpha\rangle$ is admissible in $\vdash$ iff it is validated by $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$.*

This follows from Fact 2.2, which says in effect that $\langle \Gamma, \alpha\rangle$ is admissible in $\vdash$ iff it is validated by $\langle \boldsymbol{Fm}, T^{\vdash}\rangle$. Consequently, the admissible rules of $\vdash$ always form an extension of $\vdash$. Finitarity is normally lost in the passage to this extension (see Example 3.6). Moreover, when $\vdash$ has a recursive set of theorems, it may fail to have a recursive set of admissible finite rules [11, 69], even if it is finitary and finitely axiomatized in a finite signature.

For any cardinal $\mathfrak{m}$, a first order structure (e.g., a matrix) is said to be $\mathfrak{m}$*–generated* if its pure algebra reduct has a generating set with at most $\mathfrak{m}$ elements. *Finitely generated* means $\mathfrak{m}$–generated for some finite $\mathfrak{m}$. A structure is *finite* if its universe is a finite set.

When $\langle \boldsymbol{A}, F\rangle$ validates all the derivable rules of $\vdash$, it is called a *matrix model* of $\vdash$, and F is then called a $\vdash$*–filter* of $\boldsymbol{A}$. The set $Fi_{\vdash}\boldsymbol{A}$ of all $\vdash$–filters of $\boldsymbol{A}$ is closed under arbitrary intersections, hence it becomes a complete lattice $\boldsymbol{Fi}_{\vdash}\boldsymbol{A}$ when ordered by set inclusion. The elements of $Fi_{\vdash}\boldsymbol{Fm}$ are called $\vdash$*–theories*.

Definition 2.6. ([68]) A reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ is said to be *relatively subdirectly irreducible* (with respect to $\vdash$), or briefly *RSI*, provided that, whenever $\langle \boldsymbol{A}, F\rangle$ is a subdirect product of *reduced* matrix models $\langle \boldsymbol{B}_i, G_i\rangle$ ($i \in I$) of $\vdash$, then at least one of the projections $\pi_j \colon \prod_{i\in I} B_i \to B_j$ restricts to a matrix isomorphism from $\langle \boldsymbol{A}, F\rangle$ onto $\langle \boldsymbol{B}_j, G_j\rangle$.

This extends the usual notion of an *algebra* being relatively subdirectly irreducible in a class of similar algebras (to which it belongs). We need to recall the following.

Lemma 2.7. *Let $\langle \boldsymbol{A}, F\rangle$ be a reduced matrix model of $\vdash$.*

(i) *$\langle \boldsymbol{A}, F\rangle$ is RSI iff F is completely meet-irreducible in $\boldsymbol{Fi}_{\vdash}\boldsymbol{A}$.*
(ii) *If $\vdash$ is finitary or $\langle \boldsymbol{A}, F\rangle$ is finite, then $\langle \boldsymbol{A}, F\rangle$ is isomorphic to a subdirect product of RSI reduced matrix models of $\vdash$.*
(iii) *If a rule $\langle \Gamma, \alpha\rangle$ is underivable in $\vdash$, then it is invalidated by some $\mathfrak{m}$–generated reduced matrix model $\langle \boldsymbol{C}, H\rangle$ of $\vdash$, where $\mathfrak{m}$ is the number*

of variables occurring in formulas from $\Gamma \cup \{\alpha\}$. *If, in addition,* $\vdash$ *is finitary or* $\langle \boldsymbol{C}, H\rangle$ *is finite, then* $\langle \boldsymbol{C}, H\rangle$ *can be chosen RSI as well.*

Proof. The proofs of (i) and (ii) can be found in [68, Sec. 3.7].

(iii) Let J be the intersection of all $\vdash$–theories containing Γ. Then $\langle \Gamma, \alpha\rangle$ is invalidated by an obvious $\mathfrak{m}$–generated submatrix $\langle \boldsymbol{B}, G\rangle$ of $\langle \boldsymbol{Fm}, J\rangle$. Although $\langle \boldsymbol{B}, G\rangle$ need not be reduced, it validates the same rules as the $\mathfrak{m}$–generated reduced matrix $\langle \boldsymbol{B}, G\rangle^*$. Since $\langle \boldsymbol{Fm}, J\rangle$ is a matrix model of $\vdash$, so are $\langle \boldsymbol{B}, G\rangle$ and $\langle \boldsymbol{B}, G\rangle^*$. Suppose $\vdash$ is finitary or $\langle \boldsymbol{B}, G\rangle^*$ is finite. Then (ii) guarantees that $\langle \boldsymbol{B}, G\rangle^*$ is isomorphic to a subdirect product of RSI reduced matrix models $\langle \boldsymbol{B}_i, G_i\rangle$ $(i \in I)$ of $\vdash$, each of which is still $\mathfrak{m}$–generated, and $\langle \boldsymbol{B}, G\rangle^*$ validates any rule validated by all of these subdirect factors. Consequently, $\langle \boldsymbol{B}_i, G_i\rangle$ invalidates $\langle \Gamma, \alpha\rangle$ for some $i \in I$. □

The logical significance of reduced matrices comes from the following weak variant of Lemma 2.7(iii).

Theorem 2.8. ([67]) *The derivable rules of* $\vdash$ *are exactly the rules validated by the* reduced *matrix models of* $\vdash$.

In particular, the theorems *of* $\vdash$ *are just the formulas taking only designated values in every reduced matrix model of* $\vdash$.

Notation. $\mathsf{Mod}^*(\vdash)$ denotes the class of all reduced matrix models of $\vdash$.

As a semantics for $\vdash$, the class of *all* matrix models is an unexciting variant of the syntax, but $\mathsf{Mod}^*(\vdash)$ is a much more algebraically structured class in general. Theorem 2.8 yields the expected algebraic completeness theorems in all familiar cases, e.g., the reduced matrix models of classical [resp. intuitionistic] propositional logic are just the pairs $\langle \boldsymbol{A}, \{\top\}\rangle$ such that $\boldsymbol{A}$ is a Boolean [resp. Heyting] algebra with greatest element $\top$. More generally:

Lemma 2.9. *Given a matrix* $\langle \boldsymbol{A}, F\rangle$ *and* $a, b \in A$, *we have* $a \equiv_{\boldsymbol{\Omega}^{\boldsymbol{A}} F} b$ *iff the following is true: for every formula* $\alpha(x, y_1, \dots, y_n)$ *and* $\bar{c} = c_1, \dots, c_n \in A$,

$$\alpha^{\boldsymbol{A}}(a, \bar{c}) \in F \quad \textit{iff} \quad \alpha^{\boldsymbol{A}}(b, \bar{c}) \in F.$$

A restricted form of Lemma 2.9 can be found in Łoś [30]. (In its present form, it appears in [63] and [14].) The following facts are easily proved and well known; see for instance [8] or [15]. Only item (iii) relies on Lemma 2.9.

Lemma 2.10. *Let* $\langle \boldsymbol{A}, F\rangle$ *be a matrix model of* $\vdash$, *and let* $h\colon \boldsymbol{B} \to \boldsymbol{A}$ *be a homomorphism of algebras. Then*

(i) $\langle \boldsymbol{B}, h^{-1}[F]\rangle$ *is also a matrix model of* $\vdash$,
(ii) $h^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}} F] \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}} h^{-1}[F]$, *and*
(iii) *if* h *is surjective, then* $h^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}} F] = \boldsymbol{\Omega}^{\boldsymbol{B}} h^{-1}[F]$.

Admissibility and Homomorphisms.

We seek to clarify the relationship between admissible rules and surjective homomorphisms. Consider a matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and, for simplicity, assume that it is $|\mathit{Var}|$–generated. If $\langle \Gamma, \alpha \rangle$ is admissible in $\vdash$, then $\langle \boldsymbol{A}, F \rangle$ is a homomorphic image of a matrix model of $\vdash + \langle \Gamma, \alpha \rangle$, viz. $\langle \boldsymbol{Fm}, T^{\vdash} \rangle$. In this case, the reduced matrix $\langle \boldsymbol{Fm}, T^{\vdash} \rangle^*$ is also a model of $\vdash + \langle \Gamma, \alpha \rangle$, but $\langle \boldsymbol{A}, F \rangle$ need *not* be a homomorphic image of $\langle \boldsymbol{Fm}, T^{\vdash} \rangle^*$, even when $\langle \boldsymbol{A}, F \rangle$ is itself reduced. More strongly:

Fact 2.11. *There exist a finitary system $\vdash$, a finite admissible rule $\langle \Gamma, \alpha \rangle$ of $\vdash$ and a finite reduced matrix model $\langle \boldsymbol{A}, F \rangle$ of $\vdash$, such that $\langle \boldsymbol{A}, F \rangle$ is* not *a homomorphic image of any reduced matrix model of $\vdash + \langle \Gamma, \alpha \rangle$.*

Proof. In the subsignature $\Box, \Diamond, \top$ of modal logic, the axiom $\top$ and the inference rule $\Diamond x \;/\; \Box\Diamond x$ determine a finitary deductive system $\vdash$ whose set of theorems is $\{\top\}$. It is easy to see that $Fm/\boldsymbol{\Omega}\{\top\}$ has just two elements, viz. $\{\top\}$ and $Fm \setminus \{\top\}$. Let $\boldsymbol{A} = \langle \{\bot, a, \top\}, \Box, \Diamond, \top \rangle$, where $\bot, a, \top$ are distinct and $\Box$ is the identity function and $\Diamond\bot = \bot$ and $\Diamond a = \Diamond\top = \top$. Then $\langle \boldsymbol{A}, \{\top\} \rangle \in \mathsf{Mod}^*(\vdash)$. The rule $\Box x \;/\; y$ is validated by $\langle \boldsymbol{Fm}, \{\top\} \rangle^*$, but not by $\langle \boldsymbol{A}, \{\top\} \rangle$, so it is admissible and not derivable in $\vdash$.

Now suppose $\langle \boldsymbol{B}, G \rangle \in \mathsf{Mod}^*(\vdash)$ validates $\Box x \;/\; y$. We show that there is no surjective matrix homomorphism from $\langle \boldsymbol{B}, G \rangle$ to $\langle \boldsymbol{A}, \{\top\} \rangle$. Suppose, on the contrary, that h is such a homomorphism. Then $G \neq B$, because $G \subseteq h^{-1}[\{\top\}]$ and $|A| > 1$. As $\langle \boldsymbol{B}, G \rangle$ validates both $\Diamond x \;/\; \Box\Diamond x$ and $\Box x \;/\; y$, it validates $\Diamond x \;/\; y$. So, since $B \not\subseteq G$, it follows that $\Box b, \Diamond b \notin G$ for all $b \in B$. Let $b, b' \in B \setminus h^{-1}[\{\top\}]$. Considering the form of any $\alpha(x, \bar{y}) \in Fm$, we see that for any $\bar{c} \in B$, we have $\alpha^{\boldsymbol{B}}(b, \bar{c}) \in G$ iff $\alpha^{\boldsymbol{B}}(b', \bar{c}) \in G$. Thus, $\langle b, b' \rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}} G$ (by Lemma 2.9), i.e., $b = b'$ (as $\langle \boldsymbol{B}, G \rangle$ is reduced). This shows that at most one element of B is *not* mapped to $\top$ by h, contradicting surjectivity. $\Box$

Despite Fact 2.11, admissibility can be characterized in terms of reduced models and homomorphic images (and without reference to generative size). The appropriate characterization is item (iii) below.

Theorem 2.12. *The following conditions are equivalent.*

(i) *$\langle \Gamma, \alpha \rangle$ is an admissible rule of $\vdash$.*
(ii) *Every matrix model of $\vdash$ is a homomorphic image of a matrix model of $\vdash + \langle \Gamma, \alpha \rangle$.*
(iii) *Every reduced matrix model of $\vdash$ is a homomorphic image of a matrix model of $\vdash + \langle \Gamma, \alpha \rangle$ that is itself a subdirect product of reduced matrix models of $\vdash$.*

In (ii) *and* (iii), *'Every' could be replaced by 'Every finitely generated', without loss of strength (even if Γ is infinite). If $\vdash$ is finitary then, in* (iii), *we can replace 'Every' by 'Every RSI' (with or without 'finitely generated').*

Proof. (i) $\Rightarrow$ (ii): Given a matrix model $\langle \boldsymbol{A}, F \rangle$ of $\vdash$, let $\boldsymbol{U}$ be an absolutely free algebra with free generating set Y, where $|Y| = \max\{|\mathit{Var}|, |A|\}$. Then there is a surjective homomorphism $h \colon \boldsymbol{U} \to \boldsymbol{A}$. Let G be the least $\vdash$–filter of $\boldsymbol{U}$. Lemma 2.10(i) shows that $h^{-1}[F]$ is a $\vdash$–filter of $\boldsymbol{U}$, so $G \subseteq h^{-1}[F]$, whence $\langle \boldsymbol{A}, F\rangle$ is a homomorphic image of $\langle \boldsymbol{U}, G\rangle$. It remains to show that $\langle \boldsymbol{U}, G\rangle$ validates $\langle \Gamma, \alpha\rangle$. (This would follow from Fact 2.2 if $\boldsymbol{A}$ was given to be $|\,\mathit{Var}|$–generated, as $\langle \boldsymbol{U}, G\rangle$ would then be isomorphic to $\langle \boldsymbol{Fm}, T^{\vdash}\rangle$, but we must consider the possibility that $|Y| > |\,\mathit{Var}|$.) Let $k\colon \boldsymbol{Fm} \to \boldsymbol{U}$ be a homomorphism such that $k[\Gamma] \subseteq G$. We must prove that $k(\alpha) \in G$.

Since $\boldsymbol{Fm}$ is a $|\,\mathit{Var}|$–generated algebra, so is $k[\boldsymbol{Fm}]$. In the subuniverse lattice of any algebra, the finitely generated subuniverses are compact, so each element of any generating set for $k[\boldsymbol{Fm}]$ belongs to the subalgebra of $\boldsymbol{U}$ generated by a finite subset of Y. Thus, $k[\boldsymbol{Fm}]$ is contained in the subalgebra of $\boldsymbol{U}$ generated by some $X \subseteq Y$, where $|X| \leqslant |\,\mathit{Var}|$ (as Var is infinite). Choose a bijection $g\colon Z \to \mathit{Var}$, where $X \subseteq Z \subseteq Y$. Then g can be extended to a homomorphism $\widetilde{g}\colon \boldsymbol{U} \to \boldsymbol{Fm}$. Now $\widetilde{g}^{-1}[T^{\vdash}]$ is a $\vdash$–filter of $\boldsymbol{U}$, by Lemma 2.10(i), so $G \subseteq \widetilde{g}^{-1}[T^{\vdash}]$. Therefore, $\widetilde{g}k[\Gamma] \subseteq T^{\vdash}$. Since $\widetilde{g}k$ is a substitution and $\langle \Gamma, \alpha\rangle$ is admissible in $\vdash$, we infer that $\widetilde{g}k(\alpha) \in T^{\vdash}$.

It is not immediate that $k(\alpha) \in G$, as it may happen that $\widetilde{g}^{-1}[T^{\vdash}] \nsubseteq G$. Nevertheless, $k(\alpha) = \varphi^{\boldsymbol{U}}(\bar{u})$ for some $\varphi \in \mathit{Fm}$ and some $\bar{u} = u_1, \dots, u_n \in X$, where $u_1, \dots, u_n$ are distinct (see [10, Thm. II.10.3(c)] if necessary). Since $\widetilde{g}$ and g agree on X, where g is injective, the variables $g(u_1), \dots, g(u_n)$ are also distinct, and $\widetilde{g}k(\alpha)$ is $\varphi(g(u_1), \dots, g(u_n))$. Recall that this formula is a theorem of $\vdash$, so φ is a theorem as well, because $\vdash$ is substitution-invariant. Then $k(\alpha) = \varphi^{\boldsymbol{U}}(\bar{u}) \in G$, as G is a $\vdash$–filter of $\boldsymbol{U}$.

(ii) $\Rightarrow$ (iii): Let $\langle \boldsymbol{A}, F\rangle$ be a reduced matrix model of $\vdash$, so $\boldsymbol{\Omega}^{\boldsymbol{A}}F = \mathrm{id}_A$. By (ii), there is a matrix model $\langle \boldsymbol{B}, G\rangle$ of $\vdash + \langle \Gamma, \alpha\rangle$ and a surjective homomorphism $h\colon \boldsymbol{B} \to \boldsymbol{A}$ with $h[G] \subseteq F$. Then $G \subseteq h^{-1}[F] \in \mathit{Fi}_{\vdash}\boldsymbol{B}$. Let

$$\theta = \bigcap_{G \subseteq G' \in \mathit{Fi}_{\vdash}\boldsymbol{B}} \boldsymbol{\Omega}^{\boldsymbol{B}}G'.$$

Using Lemma 2.10(iii), we obtain

$$\theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}}h^{-1}[F] = h^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}}F] = h^{-1}[\mathrm{id}_A] = \ker h.$$

There is therefore a well defined homomorphism $\widetilde{h}$ from $\boldsymbol{B}/\theta$ onto $\boldsymbol{A}$, given by $\widetilde{h}\colon b/\theta \mapsto h(b)$. Observe that $\theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}}G$, i.e., G is a union of θ–classes, so $\langle \boldsymbol{B}/\theta, G/\theta\rangle$ is a matrix model of $\vdash + \langle \Gamma, \alpha\rangle$ (because $\langle \boldsymbol{B}, G\rangle$ is). Also, $\widetilde{h}[G/\theta] = h[G] \subseteq F$. Now $\langle \boldsymbol{B}/\theta, G/\theta\rangle$ is naturally isomorphic to a subdirect product of all $\langle \boldsymbol{B}, G'\rangle^*$ such that $G \subseteq G' \in \mathit{Fi}_{\vdash}\boldsymbol{B}$, and each of these subdirect factors is a reduced matrix model of $\vdash$.

(iii) $\Rightarrow$ (i): Let $\varphi \in \mathit{Fm}$ be a non-theorem of $\vdash$. Since φ involves only finitely many variables, $\langle \emptyset, \varphi\rangle$ is invalidated by some finitely generated reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$, which can be chosen RSI if $\vdash$ is finitary (see Lemma 2.7(iii)). Even in its restricted form, item (iii) of the present theorem implies that $\langle \boldsymbol{A}, F\rangle$ is a homomorphic image of a matrix model $\langle \boldsymbol{B}, G\rangle$ of $\vdash + \langle \Gamma, \alpha\rangle$, so $\langle \boldsymbol{B}, G\rangle$ cannot validate $\langle \emptyset, \varphi\rangle$. Therefore, φ is not a theorem of $\vdash + \langle \Gamma, \alpha\rangle$. This shows that $\langle \Gamma, \alpha\rangle$ is admissible in $\vdash$. $\square$

Fact 2.11 shows that in Theorem 2.12(iii), the pre-image of the given reduced model of $\vdash$ can't always be chosen reduced. Also, its reduced subdirect factors are not guaranteed to validate $\langle \Gamma, \alpha \rangle$. Generally, a matrix model $\langle \boldsymbol{B}, G \rangle$ of $\vdash$ won't decompose subdirectly into reduced models of $\vdash$, unless $\theta = \mathrm{id}_B$ in the proof of (ii) $\Rightarrow$ (iii). (This θ is called the *Suszko congruence of* $\langle \boldsymbol{B}, G \rangle$ *w.r.t.* $\vdash$ in [16, 57].) For systems at certain levels of the Leibniz hierarchy, however, the characterization in 2.12(iii) can be simplified—see Sections 4 and 8.

3. Derivability of Admissible Rules

The observation below goes back at least to Makinson [32].

Theorem 3.1. *The following conditions on a [finitary] deductive system* $\vdash$ *are equivalent.*

(i) *Every admissible [finite] rule of* $\vdash$ *is derivable in* $\vdash$.
(ii) *For every [finitary] deductive system* $\vdash_1$, *if* $\vdash$ *and* $\vdash_1$ *have the same language and the same theorems, then* $\vdash_1 \subseteq \vdash$.

An extension $\vdash'$ of $\vdash$ is *axiomatic* if there is a set Δ of formulas, closed under substitution, such that for any set $\Gamma \cup \{\alpha\}$ of formulas, we have $\Gamma \vdash' \alpha$ iff $\Gamma, \Delta \vdash \alpha$. Note that $\vdash$ counts as an axiomatic extension of itself. The axiomatic extensions of $\vdash_{\mathbf{F}}$ all have the form $\vdash_{\mathbf{F}'}$, where $\mathbf{F}'$ is obtained by adding suitable axioms to $\mathbf{F}$, without adding any new inference rules.

Theorem 3.2. *The following conditions on a [finitary] deductive system* $\vdash$ *are equivalent.*

(i) *For every [finitary] extension* $\vdash'$ *of* $\vdash$, *all admissible [finite] rules of* $\vdash'$ *are derivable in* $\vdash'$.
(ii) *For every* axiomatic *extension* $\vdash'$ *of* $\vdash$, *all admissible [finite] rules of* $\vdash'$ *are derivable in* $\vdash'$.
(iii) *Every [finitary] extension of* $\vdash$ *is an axiomatic extension of* $\vdash$.

Proof. The proof in the finitary case is given in [46, Thm. 2.6], and we can imitate it in the non-finitary case, with the help of Theorem 3.1. □

Definition 3.3. (Pogorzelski [49, 50]) A deductive system is said to be *structurally complete* if all of its admissible *finite* rules are derivable in it.

A *finitary* deductive system is said to be *hereditarily structurally complete* if it and all of its finitary extensions are structurally complete.

Sufficient conditions for the derivability of admissible rules are given in the next result. Partial converses will be supplied later, in Theorems 5.7 and 6.4. Item (ii) below is a variant of [55, Thm. 1], but the general notion of a reduced matrix and the connection with Lemma 2.7(ii) are not made explicit in [55].

Theorem 3.4. *Let $\vdash$ be a finitary deductive system.*

(i) *Suppose that, for each finitely generated RSI reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$, there is a strict matrix homomorphism from $\langle \boldsymbol{A}, F\rangle$ into an ultrapower of $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$. Then $\vdash$ is structurally complete.*

(ii) *Suppose that, for each $|Var|$–generated RSI reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$, there is a strict matrix homomorphism from $\langle \boldsymbol{A}, F\rangle$ into $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$. Then every admissible (finite or infinite) rule of $\vdash$ is derivable in $\vdash$.*

Proof. (i) Let $\langle \Gamma, \alpha\rangle$ be underivable in $\vdash$, where Γ is finite. Lemma 2.7(iii) shows that $\langle \Gamma, \alpha\rangle$ is invalidated by some finitely generated RSI reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$. By assumption, $\langle \boldsymbol{A}, F\rangle$ is mapped into an ultrapower of $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$ by some strict matrix homomorphism g. Since g is strict, $\langle \Gamma, \alpha\rangle$ is not validated by the ultrapower. Consequently, it is not validated by $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$, because Γ is finite. Then, by Fact 2.5, $\langle \Gamma, \alpha\rangle$ is not admissible in $\vdash$, and so $\vdash$ is structurally complete.

(ii) can be proved similarly, because every underivable rule of $\vdash$ is invalidated in some $|Var|$–generated RSI reduced matrix model of $\vdash$, and no ultrapower is involved in the statement of (ii). □

Recall that $\vdash$ is said to be *tabular* if it has a finite matrix model that invalidates $\langle \emptyset, \alpha\rangle$ whenever α is not a theorem of $\vdash$. We say that $\vdash$ is *strongly finite* if it is the consequence relation of some finite set of finite matrices.

Theorem 3.5. ([67]) *Every strongly finite deductive system is finitary.*

A strongly finite system must be tabular, because a set of matrices and its direct product validate the same rules of the form $\langle \emptyset, \alpha\rangle$. As a partial converse, if a finitary tabular system has a *deduction-detachment theorem* (DDT) in the sense of [9, 15], then it is strongly finite. This follows from Corollary 2.5.20 and Theorem 2.6.2 in [15]. For our purposes, a *fragment* of a deductive system $\vdash$ is the set of all derivable *rules* of $\vdash$ in some restricted signature; it is obviously a deductive system in its own right. The following example will be needed in subsequent arguments.

Example 3.6. The *intermediate implicational logics* are the finitary extensions of the $\to$ fragment of intuitionistic logic. All of these systems are structurally complete [51] (hence hereditarily so), but only the tabular logics among them can derive all of their own admissible *infinite* rules [54]. Thus, every *non*-tabular logic in this class is a finitary system whose system of admissible rules is non-finitary. There are $2^{\aleph_0}$ non-tabular logics of this kind [70]. In view of Theorem 3.2(iii), the intermediate implicational logics are *axiomatic* extensions of the $\to$ fragment of intuitionistic logic, so they inherit the standard DDT, viz. $\Gamma, \alpha \vdash \beta$ iff $\Gamma \vdash \alpha \to \beta$. There is therefore no difference between tabularity and strong finiteness for these systems. Also, all claims made in this example remain true if we add conjunction to the signature [53].

Medvedev's logic of finite problems is an example of a finitary system that is structurally complete, but not hereditarily so [52]. It seems to be the only such sentential logic currently known, although an equational system with similar features is identified in [4, Ex. 2.14.4]. Medvedev's system is not finitely axiomatizable [33].

4. Protoalgebraic Systems

The protoalgebraic deductive systems are the ones where a rudimentary conditional ($\to$) can be simulated by binary formulas. More exactly:

Theorem 4.1. ([6]) *The following conditions on* $\vdash$ *are equivalent.*

(i) *There is a set* $\boldsymbol{\rho}$ *of binary formulas* $\rho(x,y)$ *of* $\vdash$ *such that* $\vdash \boldsymbol{\rho}(x,x)$ *and* $x,\, \boldsymbol{\rho}(x,y) \vdash y$.
(ii) *Whenever* F *and* G *are* $\vdash$*–filters of an algebra* $\boldsymbol{A}$, *with* $F \subseteq G$, *then* $\boldsymbol{\Omega}^{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G$.
(iii) $\mathsf{Mod}^*(\vdash)$ *is closed under subdirect products.*

In this case, if $\vdash$ *is finitary, then the set* $\boldsymbol{\rho}$ *can be chosen finite in* (i).

Definition 4.2. We say that $\vdash$ is *protoalgebraic* if it satisfies the equivalent conditions in Theorem 4.1.

Note that an extension of a protoalgebraic system is itself protoalgebraic.

For present purposes, a first order structure is said to be *trivial* if its universe has just one element and all of its indicated relations are non-empty. Thus, a trivial *matrix* validates all rules in its language. A *reduced* matrix $\langle \boldsymbol{A}, F\rangle$ is nontrivial iff $F \neq A$ (since $\boldsymbol{\Omega}^{\boldsymbol{A}} A = A \times A = \boldsymbol{\Omega}^{\boldsymbol{A}} \emptyset$). In particular, if $\vdash$ is a *consistent* deductive system (i.e., $T^{\vdash} \neq Fm$), then $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$ is nontrivial. For in this case, $T^{\vdash} = \emptyset$ or $\boldsymbol{\Omega} T^{\vdash} \neq Fm \times Fm$.

Notation. For any first order language $\mathcal{L}$, and any class K of $\mathcal{L}$–structures, we use $\mathrm{H}(\mathsf{K})$, $\mathrm{I}(\mathsf{K})$, $\mathrm{S}(\mathsf{K})$, $\mathrm{P}(\mathsf{K})$, $\mathrm{P_S}(\mathsf{K})$ and $\mathrm{P_U}(\mathsf{K})$ to denote the respective closures of K under homomorphic and isomorphic images, substructures, direct and subdirect products, and ultraproducts. We interpret the direct product (and any ultraproduct) of the empty family of $\mathcal{L}$–structures as the trivial $\mathcal{L}$–structure with universe $\{\emptyset\}$. Therefore, if K is closed under P (or $\mathrm{P_S}$ or $\mathrm{P_U}$), then K contains a trivial structure.

Let $\mathcal{L}$ be a first order language with equality. Recall that the *atomic* $\mathcal{L}$*–formulas* are either formal equations $\alpha = \beta$ between $\mathcal{L}$–terms, or expressions $R(\alpha_1, \dots, \alpha_m)$, where R is a relation symbol of $\mathcal{L}$, having (finite positive) rank m, and $\alpha_1, \dots, \alpha_m$ are $\mathcal{L}$–terms. *Atomic sentences* are the universal closures $\forall \bar{x}\, \Phi$ of atomic formulas Φ. An *atomic class* is a class of structures axiomatized by a set of atomic sentences. In the absence of relation symbols, these are just varieties of algebras.

The *atomic closure* of a class K of $\mathcal{L}$–structures is the smallest atomic class containing K. It is equal to $\mathrm{HSP}(\mathsf{K})$ [35], which coincides with $\mathrm{HP_S}(\mathsf{K})$

[25]. Consequently, K is itself an atomic class iff it is closed under H, S and P, or equivalently, under H and $\mathrm{P_S}$. (Proofs of these generalized Birkhoff-Kogalevskiĭ theorems are accessible in [24, pp. 64, 82–3] as well.) In particular, if K is closed under $\mathrm{P_S}$, then $\mathrm{H}(\mathsf{K})$ is the atomic closure of K. Applying this to Theorem 4.1(iii), we obtain:

Theorem 4.3. *If* $\vdash$ *is protoalgebraic, then* $\mathrm{H}(\mathsf{Mod}^*(\vdash))$ *is the atomic closure of* $\mathsf{Mod}^*(\vdash)$.

It follows that $\mathrm{S}(\mathsf{Mod}^*(\vdash)) \subseteq \mathrm{H}(\mathsf{Mod}^*(\vdash))$ whenever $\vdash$ is protoalgebraic, although the matrices in $\mathrm{S}(\mathsf{Mod}^*(\vdash))$ need not be reduced, and the ones in $\mathrm{H}(\mathsf{Mod}^*(\vdash))$ need not be models of $\vdash$.

Theorem 4.4. *Suppose* $\vdash$ *is protoalgebraic. Then the following conditions are equivalent.*

(i) $\langle \Gamma, \alpha \rangle$ *is an admissible rule of* $\vdash$.
(ii) *Every reduced matrix model of* $\vdash$ *is a homomorphic image of a reduced matrix model of* $\vdash + \langle \Gamma, \alpha \rangle$.
(iii) $\mathsf{Mod}^*(\vdash)$ *and* $\mathsf{Mod}^*(\vdash + \langle \Gamma, \alpha \rangle)$ *have the same atomic closure.*

The last two assertions of Theorem 2.12 *apply equally here in* (ii).

Proof. Combine Theorems 2.12(iii), 4.1(iii) and 4.3. □

Example 4.5. The modal system $\mathbf{S4}^{\mathrm{MP}}$ has the theorems of **S4** as its axioms, and $x,\, y \vee \neg\, x \;/\; y$ (*modus ponens*) as its sole inference rule. It is not algebraizable [15, Ex. 4.8.3] (nor even weakly algebraizable in the sense of Section 8 below), but it is obviously protoalgebraic, with $\{y \vee \neg\, x\}$ in the role of $\boldsymbol{\rho}$ in Theorem 4.1(i).

If we add the rule of *necessitation*, $x \;/\; \Box x$, to $\mathbf{S4}^{\mathrm{MP}}$, we get a familiar system for **S4**, whose reduced matrix models are just the pairs $\langle \boldsymbol{A}, \{\top\} \rangle$ where $\boldsymbol{A}$ is an interior algebra with greatest element $\top$. The reduced matrix models of $\mathbf{S4}^{\mathrm{MP}}$ itself are the pairs $\langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an interior algebra and F a lattice filter of $\boldsymbol{A}$ containing no $\Box$–closed lattice filter other than $\{\top\}$. Thus, the identity map $a \mapsto a$ makes $\langle \boldsymbol{A}, F \rangle$ a homomorphic image of $\langle \boldsymbol{A}, \{\top\} \rangle$, witnessing Theorem 4.4(ii)'s criterion for admissibility of the necessitation rule in an extremely simple way.

A matrix isomorphism from $\langle \boldsymbol{B}, G \rangle$ onto a submatrix of $\langle \boldsymbol{A}, F \rangle$ is called an *embedding* of $\langle \boldsymbol{B}, G \rangle$ into $\langle \boldsymbol{A}, F \rangle$. An injective (i.e., one-to-one) matrix homomorphism is an embedding iff it is strict. Thus, some injective matrix homomorphisms are *not* embeddings.

Definition 4.6. A reduced matrix model $\langle \boldsymbol{A}, F \rangle$ of $\vdash$ is said to be *weakly projective* (with respect to $\vdash$) provided that, whenever $\langle \boldsymbol{A}, F \rangle$ is a homomorphic image of a *reduced* matrix model $\langle \boldsymbol{B}, G \rangle$ of $\vdash$, then there is an embedding from $\langle \boldsymbol{A}, F \rangle$ into $\langle \boldsymbol{B}, G \rangle$.

This extends a common notion of weak projectivity in classes of algebras (where the concepts of embedding and injective homomorphism coincide).

Theorem 4.7. *Suppose $\vdash$ is protoalgebraic and finitary. If every finitely generated RSI reduced matrix model of $\vdash$ is weakly projective, then $\vdash$ is hereditarily structurally complete.*

Proof. Consider an axiomatic extension $\vdash'$ of $\vdash$. By Theorem 3.2, it is enough to show that $\vdash'$ is structurally complete.

Let $\langle \boldsymbol{A}, F\rangle$ be a reduced matrix model of $\vdash'$. Then $Fi_{\vdash'}\boldsymbol{A}$ is an interval of the lattice $\boldsymbol{Fi}_{\vdash}\boldsymbol{A}$, because $\vdash'$ is axiomatic over $\vdash$ (see [15, Prop. 0.8.3] if necessary). Therefore, F is completely meet-irreducible in $\boldsymbol{Fi}_{\vdash'}\boldsymbol{A}$ iff it is completely meet-irreducible in $\boldsymbol{Fi}_{\vdash}\boldsymbol{A}$. So, $\langle \boldsymbol{A}, F\rangle$ is RSI with respect to $\vdash'$ iff it is RSI with respect to $\vdash$, by Lemma 2.7(i). Moreover, if $\langle \boldsymbol{A}, F\rangle$ is weakly projective with respect to $\vdash$, then it is clearly weakly projective with respect to $\vdash'$. This means that all the assumptions of the present theorem persist in $\vdash'$, so it suffices to show that $\vdash$ is structurally complete.

By Theorem 4.4, an admissible finite rule $\langle \Gamma, \alpha\rangle$ of $\vdash$ is validated by a homomorphic pre-image of each finitely generated RSI matrix $\langle \boldsymbol{A}, F\rangle$ in $\mathsf{Mod}^*(\vdash)$, and the pre-image can also be chosen reduced. By the weak projectivity assumption, every such $\langle \boldsymbol{A}, F\rangle$ embeds into its pre-image, whence $\langle \boldsymbol{A}, F\rangle$ itself validates $\langle \Gamma, \alpha\rangle$. Thus, by Lemma 2.7(iii), $\langle \Gamma, \alpha\rangle$ is derivable in $\vdash$, and so $\vdash$ is structurally complete. □

An infinitary analogue of this result could be proved in the same way: if every $|\mathit{Var}|$–generated RSI reduced matrix model of a protoalgebraic finitary system $\vdash$ is weakly projective, then every admissible (possibly infinite) rule of an extension of $\vdash$ is derivable in the extension. But the assumptions in this result are very strong, and the only obvious applications are to systems where every RSI reduced matrix model is finite. In contrast, Theorem 4.7 has nontrivial applications (see Example 8.9) and a partial converse (Theorem 6.14). In the proof of Theorem 4.7, the appeal to Theorem 4.4 could be replaced by an appeal to the following result.

Theorem 4.8. *Let $\vdash$ be protoalgebraic. Then every $|\mathit{Var}|$–generated reduced matrix model of $\vdash$ is a homomorphic image of $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$.*

Proof. Let $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\vdash)$ be $|\mathit{Var}|$–generated. Then there is a function from Var onto a generating set for $\boldsymbol{A}$, and it extends to a homomorphism h from $\boldsymbol{Fm}$ onto $\boldsymbol{A}$. Now $h^{-1}[F]$ is a $\vdash$–theory and $T^{\vdash} \subseteq h^{-1}[F]$. Since $\vdash$ is protoalgebraic and h is surjective and $\langle \boldsymbol{A}, F\rangle$ is reduced, it follows from Theorem 4.1(ii) and Lemma 2.10(iii) that

$$\boldsymbol{\Omega} T^{\vdash} \subseteq \boldsymbol{\Omega} h^{-1}[F] = h^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}} F] = h^{-1}[\mathrm{id}_A] = \ker h,$$

so the function $\widetilde{h}\colon \alpha/\boldsymbol{\Omega} T^{\vdash} \mapsto h(\alpha)$ ($\alpha \in \mathit{Fm}$) is a well defined homomorphism from $\boldsymbol{Fm}/\boldsymbol{\Omega} T^{\vdash}$ onto $\boldsymbol{A}$. Clearly, $\widetilde{h}[T^{\vdash}/\boldsymbol{\Omega} T^{\vdash}] \subseteq F$. □

Fact 2.11 shows that Theorem 4.8 would fail if we dropped the assumption that $\vdash$ is protoalgebraic.

5. Equivalential Systems

The equivalential deductive systems are the ones whose Leibniz operators are atomically definable. More precisely:

Definition 5.1. A set $\boldsymbol{\rho}$ of binary formulas $\rho(x,y)$ is called a set of *equivalence formulas* for $\vdash$ if, for every matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and all $a,b \in A$,

$$a \equiv_{\boldsymbol{\Omega}^{\boldsymbol{A}} F} b \;\text{ iff }\; \boldsymbol{\rho}^{\boldsymbol{A}}(a,b) \subseteq F.$$

We say that $\vdash$ is *equivalential* if it has a set of equivalence formulas.

It follows from Theorem 2.8 that a deductive system has at most one set of equivalence formulas, up to inter-derivability. Clearly, if $\vdash$ is equivalential, then so are its extensions. Equivalential systems originate in [55], where a definition resembling the next lemma was given.

Lemma 5.2. ([68, pp. 222–3]) *A set $\boldsymbol{\rho}$ of binary formulas is a set of equivalence formulas for $\vdash$ iff*

$$\vdash \boldsymbol{\rho}(x,x),$$
$$x,\, \boldsymbol{\rho}(x,y) \vdash y, \text{ and}$$
$$\boldsymbol{\rho}(x_1,y_1),\dots,\boldsymbol{\rho}(x_n,y_n) \vdash \boldsymbol{\rho}(\sigma(x_1,\dots,x_n),\, \sigma(y_1,\dots,y_n))$$

for every connective σ in the signature of $\vdash$, where n is the rank of σ.

Thus, equivalence formulas function as a generalized bi-conditional ($\leftrightarrow$), and the Lindenbaum-Tarski construction can be carried out in a recognizable fashion in any equivalential system.

Theorem 5.3. (cf. [8, 15, 27]) *The following conditions on $\vdash$ are equivalent.*

(i) *$\vdash$ is equivalential.*
(ii) *$\vdash$ is protoalgebraic and for every matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and every algebraic homomorphism $h\colon \boldsymbol{B} \to \boldsymbol{A}$, we have*
$$h^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}} F] = \boldsymbol{\Omega}^{\boldsymbol{B}} h^{-1}[F]$$
(even if h is not surjective).
(iii) *$\vdash$ is protoalgebraic and, whenever $\langle \boldsymbol{B}, G\rangle$ is a submatrix of a matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$, then $\boldsymbol{\Omega}^{\boldsymbol{B}} G = (B\times B) \cap \boldsymbol{\Omega}^{\boldsymbol{A}} F$.*
(iv) *$\mathsf{Mod}^*(\vdash)$ is closed under submatrices and direct products.*

It is well known that if $\vdash$ is equivalential, then $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$ is freely generated by $\{x/\boldsymbol{\Omega}T^{\vdash} : x \in \mathit{Var}\}$ in the concrete category $\mathsf{Mod}^*(\vdash)$ (equipped with all matrix homomorphisms). Indeed, for each $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\vdash)$, any function from $\{x/\boldsymbol{\Omega}T^{\vdash} : x \in \mathit{Var}\}$ into A can be extended to a matrix homomorphism $\widetilde{h}\colon \langle \boldsymbol{Fm}, T^{\vdash}\rangle^* \to \langle \boldsymbol{A}, F\rangle$, as in the proof of Theorem 4.8. The difference is that we rely on Theorem 5.3(ii) instead of Lemma 2.10(iii) when showing that $\widetilde{h}$ is well defined, because the homomorphism in the role of h is no longer guaranteed to be surjective. The map $x \mapsto x/\boldsymbol{\Omega}T^{\vdash}$ is injective on Var whenever $\vdash$ is protoalgebraic and *strongly consistent*—in the sense that $\alpha \nvdash \beta$ for some $\alpha, \beta \in Fm$. For then, in Theorem 4.1(i), we must have $\boldsymbol{\rho}(x,y) \nsubseteq T^{\vdash}$, whence $\boldsymbol{\rho} \neq \emptyset$ and $x \not\equiv_{\boldsymbol{\Omega} T^{\vdash}} y$ (because $\boldsymbol{\rho}(x,x) \subseteq T^{\vdash}$).

Lemma 5.4. *Let $h\colon \langle \boldsymbol{B}, G\rangle \to \langle \boldsymbol{A}, F\rangle$ be a matrix homomorphism between matrix models of $\vdash$, where $\langle \boldsymbol{B}, G\rangle$ is reduced. If $\vdash$ is equivalential and h is strict, then h is injective, and therefore an embedding.*

Proof. Let $\boldsymbol{\rho}$ be a set of equivalence formulas for $\vdash$, and let $b, b' \in B$ with $h(b) = h(b')$. Then $h[\boldsymbol{\rho}^{\boldsymbol{B}}(b,b')] = \boldsymbol{\rho}^{\boldsymbol{A}}(h(b), h(b')) \subseteq F$, so $\boldsymbol{\rho}^{\boldsymbol{B}}(b,b') \subseteq G$, as h is strict. Consequently, $b = b'$, because $\langle \boldsymbol{B}, G\rangle$ is reduced. □

Notation. For any class K of similar first order structures, we define

$$\mathrm{U}(\mathsf{K}) \;=\; \{\mathcal{A} :\ \text{every } |\mathit{Var}|\text{–generated substructure of } \mathcal{A} \text{ belongs to } \mathsf{K}\}.$$

Lemma 5.5. $\mathrm{U}(\mathsf{Mod}^*(\vdash)) \subseteq \mathsf{Mod}^*(\vdash)$, *for every deductive system* $\vdash$.

Proof. Clearly, if all $|\mathit{Var}|$–generated substructures of $\langle \boldsymbol{A}, F\rangle$ are matrix models of $\vdash$, then so is $\langle \boldsymbol{A}, F\rangle$ itself. Also, if $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ and $\boldsymbol{B}$ is the subalgebra of $\boldsymbol{A}$ generated by $\{a, b\}$, then $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}}(F \cap B)$. This follows from Lemma 2.9, and it shows that a matrix will be reduced whenever all of its 2–generated submatrices are reduced. □

A class K of similar structures is called a *UISP–class* if it is closed under the class operators U, I, S and P. The smallest such class containing K is UISP(K).

Theorem 5.6. *If $\vdash$ is equivalential, then the map $\vdash' \mapsto \mathsf{Mod}^*(\vdash')$ is a bijection from the extensions of $\vdash$ to the UISP–subclasses of $\mathsf{Mod}^*(\vdash)$. Its inverse sends a UISP–class $\mathsf{K} \subseteq \mathsf{Mod}^*(\vdash)$ to the consequence relation of K.*

Proof. Let $\boldsymbol{\rho}$ be a set of equivalence formulas for $\vdash$ (and hence for its extensions). Regardless of equivalentiality, when $\vdash'$ and $\vdash''$ extend $\vdash$, then

$$\vdash' \;\subseteq\; \vdash'' \quad \text{iff} \quad \mathsf{Mod}^*(\vdash'') \subseteq \mathsf{Mod}^*(\vdash'), \tag{1}$$

by Theorem 2.8. In particular, the map $\vdash' \mapsto \mathsf{Mod}^*(\vdash')$ is injective on the extensions of $\vdash$. Equivalentiality ensures that each $\mathsf{Mod}^*(\vdash')$ is indeed a UISP–class: see Theorem 5.3(iv) and Lemma 5.5. To prove surjectivity, consider a UISP–class $\mathsf{K} \subseteq \mathsf{Mod}^*(\vdash)$, and let $\vdash'$ be the consequence relation of K. Then $\vdash'$ is a deductive system extending $\vdash$, and $\mathsf{K} \subseteq \mathsf{Mod}^*(\vdash')$. For the reverse inclusion, let $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\vdash')$. We must show that $\langle \boldsymbol{A}, F\rangle \in \mathsf{K}$.

Because $\mathsf{Mod}^*(\vdash')$ is closed under submatrices and K is closed under U, we may assume that $\boldsymbol{A}$ is $|\mathit{Var}|$–generated. So, there is a surjective homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$. Note that $h^{-1}[F]$ is a $\vdash$–theory, by Lemma 2.10(i). Consequently, for each $\alpha \in Fm \backslash h^{-1}[F]$, the rule $\langle h^{-1}[F], \alpha\rangle$ is not derivable in $\vdash'$, i.e., there exist $\langle \boldsymbol{B}_\alpha, G_\alpha\rangle \in \mathsf{K}$ and a homomorphism $g_\alpha\colon \boldsymbol{Fm} \to \boldsymbol{B}_\alpha$

such that $g_\alpha[h^{-1}[F]] \subseteq G_\alpha$ but $g_\alpha(\alpha) \notin G_\alpha$ (by definition of $\vdash'$). Let $g\colon \boldsymbol{Fm} \to \prod_\alpha \boldsymbol{B}_\alpha$ be the homomorphism induced by all of the g_α. Then

$$h^{-1}[F] = g^{-1}\left[\prod\nolimits_\alpha G_\alpha\right]. \tag{2}$$

Observe that $\langle \prod_\alpha \boldsymbol{B}_\alpha, \prod_\alpha G_\alpha \rangle$ is a reduced matrix model of $\vdash'$, because K is closed under P and contained in $\mathsf{Mod}^*(\vdash')$. Now

$$\ker h = \ker g, \tag{3}$$

by (2), because the law

$$x = y \iff \boldsymbol{\rho}(x,y) \text{ consists of designated elements}$$

is valid throughout $\mathsf{Mod}^*(\vdash')$. It follows from (2) and (3) that the map $h(\alpha) \mapsto g(\alpha)$ is a well defined isomorphism from $\langle \boldsymbol{A}, F \rangle$ onto a submatrix of $\langle \prod_\alpha \boldsymbol{B}_\alpha, \prod_\alpha G_\alpha \rangle$. Therefore, $\langle \boldsymbol{A}, F \rangle \in \mathsf{K}$, because K is closed under I, S and P. □

Because the connectives and variables of a deductive system are assumed to form sets, the extensions of the system also constitute a set. So, although $\mathsf{Mod}^*(\vdash)$ is a proper class, Theorem 5.6 allows us to treat its collection of UISP–subclasses as a set—actually a lattice, ordered by $\subseteq$, provided that $\vdash$ is equivalential. Then the bijection $\vdash' \mapsto \mathsf{Mod}^*(\vdash')$ is a lattice anti-isomorphism, by (1).

In the next result, the equivalence of conditions (i) and (iii) in the finitary case is essentially due to Prucnal and Wroński (see [55, Thm. 2]).

Theorem 5.7. *Let $\vdash$ be equivalential. Then the following two conditions are equivalent.*

(i) *Every admissible (finite or infinite) rule of $\vdash$ is derivable in $\vdash$.*
(ii) $\mathsf{Mod}^*(\vdash) = \mathrm{UISP}\,\langle \boldsymbol{Fm}, T^{\vdash} \rangle^*$.

Moreover, these conditions imply the next one.

(iii) *Every $|Var|$–generated RSI reduced matrix model of $\vdash$ can be embedded into $\langle \boldsymbol{Fm}, T^{\vdash} \rangle^*$.*

If $\vdash$ is finitary, then all three of the above conditions are equivalent.

Proof. (i) ⇔ (ii): By Fact 2.5 and Theorem 2.8, the admissible rules of $\vdash$ are the rules validated by $\langle \boldsymbol{Fm}, T^{\vdash} \rangle^*$, i.e., by $\mathrm{UISP}\,\langle \boldsymbol{Fm}, T^{\vdash} \rangle^*$, while the derivable rules are the ones validated by $\mathsf{Mod}^*(\vdash)$. Thus, (i) holds iff $\mathrm{UISP}\,\langle \boldsymbol{Fm}, T^{\vdash} \rangle^*$ and $\mathsf{Mod}^*(\vdash)$ validate the same rules. But both are UISP–classes, so (i) and (ii) are equivalent, in view of Theorem 5.6.

(ii) ⇒ (iii) is a consequence of the definitions, because $\mathrm{SP}(\mathsf{K}) \subseteq \mathrm{P_S S}(\mathsf{K})$ for any class K of similar structures, and $\mathsf{Mod}^*(\vdash)$ is closed under submatrices.

If $\vdash$ is finitary, then (iii) ⇒ (i) instantiates Theorem 3.4(ii), because matrix embeddings are strict. □

Combining Theorems 4.3, 4.4 and 5.6, we obtain:

Theorem 5.8. *Suppose* $\vdash$ *is equivalential. Then every admissible rule of* $\vdash$ *is derivable in* $\vdash$ *iff, for any UISP–class* K, *if* $\mathsf{K} \subsetneqq \mathsf{Mod}^*(\vdash)$ *then* $\mathrm{H}(\mathsf{K}) \subsetneqq \mathrm{H}(\mathsf{Mod}^*(\vdash))$, *i.e.,* $\mathsf{Mod}^*(\vdash) \not\subseteq \mathrm{H}(\mathsf{K})$.

Given classes $\mathsf{K}_1 \subseteq \mathsf{K}_2$, both closed under I, S and P, we call K_1 a *relative atomic subclass* of K_2 if $\mathsf{K}_1 = \mathsf{K}_2 \cap \mathsf{C}$ for some atomic class C. This amounts to asking that K_1 can be axiomatized, relative to K_2, by some set of atomic sentences. Since $\mathsf{K}_2 \cap \mathrm{H}(\mathsf{K}_1)$ is the smallest relative atomic subclass of K_2 containing K_1, Theorem 5.8 readily implies the corollary below.

Corollary 5.9. *Suppose* $\vdash$ *is equivalential. Then the following conditions are equivalent.*

(i) *Every extension of* $\vdash$ *has the property that each of its admissible rules is derivable.*
(ii) *For any UISP–classes* $\mathsf{K}_1, \mathsf{K}_2 \subseteq \mathsf{Mod}^*(\vdash)$, *if* $\mathsf{K}_1 \subsetneqq \mathsf{K}_2$ *then* $\mathrm{H}(\mathsf{K}_1) \subsetneqq \mathrm{H}(\mathsf{K}_2)$.
(iii) *Every UISP–subclass* K *of* $\mathsf{Mod}^*(\vdash)$ *is a relative atomic subclass of* $\mathsf{Mod}^*(\vdash)$, *i.e.,* $\mathsf{Mod}^*(\vdash) \cap \mathrm{H}(\mathsf{K}) = \mathsf{K}$.

6. Finitely Equivalential Systems

Let $\mathcal{L}$ be a first order language with equality. The (strict) *universal Horn sentences* of $\mathcal{L}$ are the first order $\mathcal{L}$–sentences of the form

$$\forall \bar{x}\ \big((\&_{i<n} \Phi_i) \implies \Psi \big),$$

where $n \in \omega$ and $\Phi_0, \ldots, \Phi_{n-1}, \Psi$ are atomic $\mathcal{L}$–formulas. (If these atomic formulas are variable-free, then the quantifier is not required, i.e., $\bar{x}$ may be empty.) Let K be a class of $\mathcal{L}$–structures. We call K a (strict) *universal Horn class* if it can be axiomatized by a set of universal Horn $\mathcal{L}$–sentences. The smallest such class containing K is $\mathrm{ISPP_U}(\mathsf{K})$. This is a refinement, by Grätzer and Lakser [26], of a result of Maltsev [35]. Thus, K is itself a universal Horn class iff it is closed under I, S, P and $\mathrm{P_U}$. (Russian and Polish authors often follow Maltsev in referring to universal Horn classes as 'quasivarieties', even if they do not consist of pure algebras.)

In the context of equivalential deductive systems, $\mathsf{Mod}^*(\vdash)$ is a universal Horn class iff it is *elementary* (i.e., axiomatizable by a set of first order sentences), iff it is closed under ultraproducts. This follows from Łos' Theorem and Theorem 5.3(iv). In general, if $\mathsf{Mod}^*(\vdash)$ is closed under ultraproducts, then $\vdash$ is finitary: see [15, Cor. 0.4.6].

Definition 6.1. A deductive system is said to be *finitely equivalential* if it has a finite set of equivalence formulas (cf. Definition 5.1).

Theorem 6.2.

(i) ([8, 27]) $\vdash$ *is finitely equivalential iff* $\boldsymbol{\Omega}^{\boldsymbol{A}} \bigcup_{i \in I} F_i = \bigcup_{i \in I} \boldsymbol{\Omega}^{\boldsymbol{A}} F_i$ *whenever* $\{F_i : i \in I\}$ *is a* $\subseteq$*–directed set of* $\vdash$*–filters of an algebra* $\boldsymbol{A}$ *such that* $\bigcup_{i \in I} F_i$ *is still a* $\vdash$*–filter (as it will be, if* $\vdash$ *is finitary).*

(ii) ([14, 8]) $\vdash$ *is finitary and finitely equivalential iff* $\mathsf{Mod}^*(\vdash)$ *is a universal Horn class.*

In (ii), if Dx formalizes 'x is designated', and if $\boldsymbol{\rho}$ is a finite set of equivalence formulas for $\vdash$, then $\mathsf{Mod}^*(\vdash)$ is axiomatized by

$$\forall x\,\forall y\;\big(x = y \iff \&_{\rho\in\boldsymbol{\rho}}\, D\rho(x,y)\big)$$

as well as all

$$\forall \bar{x}\;\big(\big(\&_{\gamma\in\Gamma}\, D\gamma\big) \implies D\alpha\big) \tag{4}$$

such that $\langle \Gamma, \alpha\rangle$ is a derivable finite rule of $\vdash$. If $\vdash$ is the deducibility relation of a finitary formal system $\mathbf{F}$, then we may restrict (4) to the inference rules $\langle \Gamma, \alpha\rangle$ of $\mathbf{F}$, including the axioms (considered as pairs $\langle \emptyset, \alpha\rangle$). Now Theorem 5.6 specializes as follows.

Theorem 6.3. (cf. [15, p. 190]) *If* $\vdash$ *is finitely equivalential and finitary, then* $\vdash' \,\mapsto \mathsf{Mod}^*(\vdash')$ *is a lattice anti-isomorphism from the finitary extensions of* $\vdash$ *to the universal Horn subclasses of* $\mathsf{Mod}^*(\vdash)$. *Its inverse sends a universal Horn class* $\mathsf{K} \subseteq \mathsf{Mod}^*(\vdash)$ *to the consequence relation of* K.

Theorem 6.4. *Let* $\vdash$ *be finitary and finitely equivalential. Then the following conditions are equivalent.*

(i) $\vdash$ *is structurally complete.*
(ii) $\mathsf{Mod}^*(\vdash) = \mathbb{ISPP}_{\mathrm{U}}\,\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$.
(iii) *Every finitely generated RSI reduced matrix model of* $\vdash$ *can be embedded into an ultrapower of* $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$.

The proof is similar to that of Theorem 5.7, but we exploit Theorems 6.3 and 3.4(i), rather than 5.6 and 3.4(ii).

Corollary 6.5. *Let* $\vdash$ *be finitely equivalential and finitary.*

If $\vdash$ *is structurally complete, then* $\mathsf{Mod}^*(\vdash)$ *has the* joint embedding property, *i.e., whenever* $\langle \boldsymbol{A}, F\rangle$ *and* $\langle \boldsymbol{B}, G\rangle$ *are nontrivial reduced matrix models of* $\vdash$, *then there exists* $\langle \boldsymbol{C}, H\rangle \in \mathsf{Mod}^*(\vdash)$ *such that both* $\langle \boldsymbol{A}, F\rangle$ *and* $\langle \boldsymbol{B}, G\rangle$ *can be embedded into* $\langle \boldsymbol{C}, H\rangle$.

Proof. A universal Horn class has the joint embedding property iff it is generated by a single structure (see [36, p. 288] or [24, Prop. 2.1.19]), so the result follows from Theorem 6.4(ii). □

Recall that an $\mathcal{L}$–structure $\mathcal{A}$ is said to be *locally embeddable into* a class K of $\mathcal{L}$–structures if every finite subset B of the universe of $\mathcal{A}$ can be extended to an isomorphic copy of a structure from K, in such a way that the $\mathcal{A}$–induced relations and partial operations on elements of B are preserved. In this case, $\mathcal{A}$ itself can be embedded into an ultraproduct of a non-empty subfamily of K (see [24, Thm. 1.2.8]). The converse holds when the signature is finite, because the tables of relations and partial operations on a finite subset of $\mathcal{A}$ are then embodied in a first order (existential) sentence, whose negation must persist in ultraproducts of non-empty families. For a single structure $\mathcal{C}$, we take 'locally embeddable into $\mathcal{C}$' to mean locally embeddable into $\{\mathcal{C}\}$. In conjunction with Theorem 3.4(i) and the implication (i) $\Rightarrow$ (iii) from Theorem 6.4, these remarks yield:

Corollary 6.6. *If every finitely generated RSI reduced matrix model of a finitary system* $\vdash$ *is locally embeddable into* $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$, *then* $\vdash$ *is structurally complete. The converse holds if* $\vdash$ *is also finitely equivalential, with a finite signature.*

For finitely equivalential finitary systems, any generating class for the universal Horn class $\mathsf{Mod}^*(\vdash)$ can play the role of the finitely generated RSI reduced matrix models in the sufficient condition for structural completeness given by Corollary 6.6. (This follows from the implication (ii) $\Rightarrow$ (i) in Theorem 6.4.) A purely algebraic specialization of this last claim appears in [12, Thm. 3.3].

Definition 6.7. We say that $\vdash$ has the *strong finite model property* if every finite rule that is underivable in $\vdash$ is invalidated by some finite matrix model of $\vdash$. (The model can be chosen reduced and RSI, by Lemma 2.7(iii).)

Theorem 6.8. *Let* $\vdash$ *be a finitely equivalential finitary deductive system with the strong finite model property, having a finite signature.*

Then $\vdash$ *is structurally complete iff every finite RSI reduced matrix model of* $\vdash$ *can be embedded into* $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$.

Proof. ($\Rightarrow$) This follows from Corollary 6.6, because a *finite* structure is locally embeddable into a structure $\mathcal{C}$ iff it is embeddable into $\mathcal{C}$.

($\Leftarrow$) Let $\langle \Gamma, \alpha\rangle$ be an admissible finite rule of $\vdash$. By Fact 2.5, $\langle \Gamma, \alpha\rangle$ is validated by $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$. So, by assumption, $\langle \Gamma, \alpha\rangle$ is validated by all finite RSI reduced matrix models of $\vdash$, and is therefore derivable in $\vdash$, by the strong finite model property. □

If an equivalential system is tabular (e.g., if it is strongly finite), then it is finitely equivalential, because there are only finitely many binary operations on a finite set.

Theorem 6.9. *If* $\vdash$ *is equivalential and strongly finite, then each of its RSI reduced matrix models is finite.*

Proof. Let M be a finite set of finite reduced matrices whose consequence relation $\vdash$ is equivalential. Then $\vdash$ is finitely equivalential and finitary (see Theorem 3.5), and since it is the consequence relation of M, it is also the consequence relation of the universal Horn subclass $\mathrm{ISPP_U}(\mathsf{M})$ of $\mathsf{Mod}^*(\vdash)$, whence $\mathsf{Mod}^*(\vdash) = \mathrm{ISPP_U}(\mathsf{M})$, by Theorem 6.3. The latter is really $\mathrm{ISP}(\mathsf{M})$, because the isomorphic closure of a finite set of finite matrices is closed under ultraproducts. So, $\mathsf{Mod}^*(\vdash) = \mathrm{IP_S S}(\mathsf{M})$. In particular, every RSI matrix in $\mathsf{Mod}^*(\vdash)$ embeds into a member of M, and is therefore finite. □

Theorem 6.10. *Let $\vdash$ be strongly finite and equivalential, with a finite signature. If $\vdash$ is structurally complete, then each of its admissible* infinite *rules is derivable in $\vdash$.*

Proof. As above, $\vdash$ is finitely equivalential and finitary. Since $\vdash$ is strongly finite, it has the strong finite model property. If $\vdash$ is also structurally complete then, by Theorems 6.8 and 6.9, every RSI reduced matrix model of $\vdash$ can be embedded into $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$. In this case, by Theorem 5.7, every admissible (possibly infinite) rule of $\vdash$ is derivable in $\vdash$. $\square$

Definition 6.11. If $\boldsymbol{\rho}$ is a set of equivalence formulas for $\vdash$, and if $\boldsymbol{\rho}(\alpha, \beta)$ consists of theorems of $\vdash$, then α and β are said to be *logically equivalent* in $\vdash$. (This makes sense, because equivalence formulas are essentially unique.) An equivalential system is *locally tabular* if it has only finitely many logically inequivalent formulas in n fixed variables, for every finite n.

If an equivalential system is tabular, then it is locally tabular, and if it is locally tabular then it is finitely equivalential. All of the intermediate implicational logics are locally tabular, by Diego's Theorem [18, 38]. So, by the results cited in Example 3.6, we cannot weaken strong finiteness to local tabularity in Theorem 6.10. On the other hand, Theorem 6.9 can be generalized as follows: if a locally tabular equivalential system has, up to isomorphism, only finitely many finite RSI reduced matrix models, then it has no infinite RSI reduced matrix model. The proof adapts that of Quackenbush's Theorem [10, Thm. V.3.8] and uses Lemma 2.7(ii).

A universal Horn class K is said to be *primitive* if every universal Horn subclass of K is a relative atomic subclass of K. Theorem 5.8 and Corollary 5.9 finitize as follows, via Theorem 6.3.

Theorem 6.12. *Let $\vdash$ be finitely equivalential and finitary.*

(i) *$\vdash$ is structurally complete iff, for any universal Horn class K, if $\mathsf{K} \subsetneqq \mathsf{Mod}^*(\vdash)$ then $\mathsf{Mod}^*(\vdash) \not\subseteq \mathrm{H}(\mathsf{K})$.*
(ii) *$\vdash$ is hereditarily structurally complete iff $\mathsf{Mod}^*(\vdash)$ is primitive.*

Gorbunov proved that, for any primitive universal Horn class K, the lattice of universal Horn subclasses of K is distributive (see [23] or [24, Prop. 5.1.22]). Combining this with Theorems 6.3 and 6.12(ii), we obtain:

Theorem 6.13. *If a finitely equivalential finitary deductive system is hereditarily structurally complete, then its finitary extensions form a distributive lattice.*

(The finitary extensions are axiomatic in this case, by Theorem 3.2.)

A universal Horn class K is said to be *locally finite* if every finitely generated member of K is finite. An equivalential deductive system $\vdash$ is locally tabular iff $\mathsf{Mod}^*(\vdash)$ is locally finite. Indeed, an n–element subset of *Var*, factored by $\boldsymbol{\Omega} T^{\vdash}$, generates a submatrix $\langle \boldsymbol{A}_n, F_n\rangle$ of $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$ that is reduced (by Theorem 5.3(iv)), and the homomorphic images of $\langle \boldsymbol{A}_n, F_n\rangle$ include all n–generated reduced matrix models of $\vdash$ (by an obvious adaptation of Theorem 4.8). Now $\vdash$ is locally tabular iff $\langle \boldsymbol{A}_n, F_n\rangle$ is finite for all finite n, iff $\mathsf{Mod}^*(\vdash)$ is locally finite.

A further finding of Gorbunov is that a locally finite universal Horn class K is primitive iff every finite relatively subdirectly irreducible member of K is weakly projective in K (see [23] or [24, Prop. 5.1.24]). This yields the following result, which is a partial converse for Theorem 4.7.

Theorem 6.14. *Let $\vdash$ be equivalential, locally tabular and finitary. Then $\vdash$ is hereditarily structurally complete iff every finite RSI reduced matrix model of $\vdash$ is weakly projective.*

7. Overflow Rules

Again, let $\mathcal{L}$ be a first order language with equality. Recall that, up to logical equivalence, an existential positive $\mathcal{L}$–sentence is a sentence of the form $\exists\bar{x}\,\Phi$, where Φ is a disjunction of one or more $\mathcal{L}$–formulas, each of which is a conjunction of one or more atomic $\mathcal{L}$–formulas. (If no variable occurs in Φ, then no quantifiers are required.)

In Bergman [4], a quasivariety K of algebras is said to be *structurally complete* if every proper subquasivariety of K generates a proper subvariety of the variety $\mathbb{H}(\mathsf{K})$. By [4, Thm. 2.7], every existential positive first order sentence over a structurally complete *variety* K is either true throughout K or false in all nontrivial members of K. This is a one-way implication, but Wroński [72] isolates a weak form of structural completeness that exactly characterizes Bergman's condition on existential positive sentences, while demanding only that K be a *quasivariety* of algebras. Wroński's characterization asks that K should satisfy every (finite) quasi-equation

$$\big(\&_{i<n}\,\alpha_i = \beta_i\big) \implies x = y \tag{5}$$

such that (i) x, y are distinct variables absent from the equations on the left of $\Longrightarrow$, and (ii) for every substitution h, if K satisfies $h(\alpha_i) = h(\beta_i)$ for all $i < n$, then K satisfies $h(x) = h(y)$ [72, Fact 2]. A natural phrasing of (ii) is '(5) is admissible in the equational consequence relation of K.' Theorems 7.3 and 7.5 below are inspired by these insights. (It is possible to unify the present account with the framework of [4, 72], by considering Gentzen systems—see Section 11.)

In [72], the quasi-equation (5) is called an 'overflow rule' if (i) holds. In our context, the following definition is appropriate.

Definition 7.1. If Γ is a set of formulas of $\vdash$, none of which contains an occurrence of the variable y, then $\langle \Gamma, y\rangle$ is called an *overflow rule* of $\vdash$.

For the rest of this section, $\mathcal{L}$ denotes the first order language, with equality, of $\mathsf{Mod}^*(\vdash)$, and *Var* (the set of variables of $\vdash$) also serves as the set of variables of $\mathcal{L}$. Recall that the unary designation predicate D belongs to $\mathcal{L}$. By an *existential positive $\mathcal{L}$–condition*, we shall mean a formal expression

$$\exists \bar{x} \textstyle\bigvee_{i \in I} \&_{j \in J_i} \Phi_{ij}, \tag{6}$$

where I and all of the J_i are non-empty possibly infinite sets, every Φ_{ij} is an atomic $\mathcal{L}$–formula, and $\bar{x}$ is a possibly infinite (and possibly empty) sequence of variables, including all that occur in (6).

Lemma 7.2. *Let $\langle \boldsymbol{A}, F\rangle$ be a nontrivial reduced matrix model of $\vdash$, let $\langle \Gamma, y\rangle$ be an overflow rule of $\vdash$, with $\Gamma \neq \emptyset$, and let $\bar{x}$ be the sequence of variables occurring in Γ (taken in any order).*

Then $\exists \bar{x} \&_{\gamma \in \Gamma} D\gamma$ is true in $\langle \boldsymbol{A}, F\rangle$ iff $\langle \boldsymbol{A}, F\rangle$ does not validate $\langle \Gamma, y\rangle$.

The proof is easy, because a nontrivial reduced matrix has at least one non-designated element and, for the purpose of assigning values to variables, y is independent of the variables in Γ.

Theorem 7.3. *If every equality-free existential positive $\mathcal{L}$–condition is true either in every member of $\mathsf{Mod}^*(\vdash)$ or in no nontrivial member of $\mathsf{Mod}^*(\vdash)$, then every admissible overflow rule of $\vdash$ is derivable in $\vdash$.*

The converse holds if $\vdash$ is equivalential, in which case it applies to all *existential positive $\mathcal{L}$–conditions, not only the equality-free ones.*

Proof. We may assume without loss of generality that $\vdash$ is strongly consistent, so the matrix $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$ is nontrivial.

($\Rightarrow$) Let $\langle \Gamma, y\rangle$ be an underivable overflow rule of $\vdash$. We need to show that $\langle \Gamma, y\rangle$ is inadmissible in $\vdash$, so we may assume that $\Gamma \neq \emptyset$. By Theorem 2.8, $\langle \Gamma, y\rangle$ is invalidated by some reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$, which must be nontrivial, as the trivial matrices validate all rules. Now $\exists \bar{x} \&_{\gamma \in \Gamma} D\gamma$ is true in $\langle \boldsymbol{A}, F\rangle$, by Lemma 7.2, and it is an equality-free existential positive $\mathcal{L}$–condition, so it is true in all reduced matrix models of $\vdash$, by assumption. In particular, it is true in $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$. By Lemma 7.2 again, $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$ does not validate $\langle \Gamma, y\rangle$, so $\langle \Gamma, y\rangle$ is inadmissible in $\vdash$, by Fact 2.5.

($\Leftarrow$) Consider an existential positive $\mathcal{L}$–condition $\exists \bar{x}\, \Phi$ that is true in some nontrivial reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$, where Φ is a formal disjunction of expressions Φ_i, $i \in I$, each of which is a formal conjunction of atomic $\mathcal{L}$–formulas. Then $\exists \bar{x}\, \Phi_i$ is true in $\langle \boldsymbol{A}, F\rangle$ for some $i \in I$. It suffices to show that $\exists \bar{x}\, \Phi_i$ is true in every reduced matrix model of $\vdash$.

Let $\boldsymbol{\rho}$ be a set of equivalence formulas for $\vdash$. Then $\boldsymbol{\rho} \neq \emptyset$, because $\vdash$ is strongly consistent. Every equational subformula $\alpha = \beta$ of Φ_i can be replaced in Φ_i by $\&_{\rho \in \boldsymbol{\rho}} D\rho(\alpha, \beta)$, without affecting the truth of $\exists \bar{x}\, \Phi_i$ in any reduced matrix model of $\vdash$. We may therefore assume that Φ_i has the form $\&_{\gamma \in \Gamma} D\gamma$ for some non-empty $\Gamma \subseteq Fm$. Since *Var* is an infinite set, we may also assume that some $y \in Var$ does not occur in any member of Γ (otherwise, by standard cardinality arguments, the set of apparent variables of Γ can be replaced by a $|Var|$–element proper subset of itself, without affecting the truth of $\exists \bar{x}\, \Phi_i$ in any $\mathcal{L}$–structure). Then, because $\exists \bar{x}\, \Phi_i$ is true in $\langle \boldsymbol{A}, F\rangle$, which is nontrivial, Lemma 7.2 shows that $\langle \boldsymbol{A}, F\rangle$ does not validate the overflow rule $\langle \Gamma, y\rangle$. Consequently, $\langle \Gamma, y\rangle$ is not derivable in $\vdash$. So, by assumption, $\langle \Gamma, y\rangle$ is not admissible in $\vdash$. This means that $\langle \Gamma, y\rangle$ is not validated by $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$, by Fact 2.5. It follows from Lemma 7.2 that $\exists \bar{x}\, \Phi_i$ is true in $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$.

It is easy to see that the truth of $\exists \bar{x}\, \Phi_i$ persists in homomorphic images and in superstructures. But every $|\,Var|$–generated reduced matrix model of $\vdash$ is a homomorphic image of $\langle \boldsymbol{Fm}, T^{\vdash}\rangle^*$, by Theorem 4.8, and every reduced matrix model of $\vdash$ has $|\,Var|$–generated submatrices, all of which still belong to $\mathsf{Mod}^*(\vdash)$, by Theorem 5.3(iv). So, $\exists \bar{x}\, \Phi_i$ is true in every reduced matrix model of $\vdash$, as required. ☐

Definition 7.4. We shall say that $\vdash$ is *overflow complete* if every admissible *finite* overflow rule of $\vdash$ is derivable in $\vdash$.

Finitizing Theorem 7.3 and its proof, we obtain:

Theorem 7.5. *Let $\vdash$ be finitely equivalential. Then $\vdash$ is overflow complete iff every existential positive $\mathcal{L}$–sentence holds either in all of the nontrivial reduced matrix models of $\vdash$, or in none of them.*

Remark 7.6. If $\vdash$ is merely equivalential and overflow complete, then the equality-free existential positive (finite) $\mathcal{L}$–sentences still hold either in all or in none of the nontrivial members of $\mathsf{Mod}^*(\vdash)$. This is established by the proof of Theorem 7.3.

Note that a matrix is 0–generated only if its signature includes a constant symbol (because we exclude empty structures from consideration).

Theorem 7.7. *Let $\vdash$ be equivalential. If $\vdash$ is overflow complete, then any two nontrivial 0–generated reduced matrix models of $\vdash$ are isomorphic.*

Proof. Let $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\vdash)$ be 0–generated and nontrivial, so $\vdash$ has a constant symbol, c say. The map $x \mapsto c^{\boldsymbol{A}}$ ($x \in Var$) extends to a homomorphism $g\colon \boldsymbol{Fm} \to \boldsymbol{A}$, and g must be surjective, because $\boldsymbol{A}$ is 0–generated. Since $T^{\vdash} \subseteq g^{-1}[F]$ and $\vdash$ is equivalential (hence protoalgebraic), Theorem 4.1(ii) shows that $\boldsymbol{\Omega} T^{\vdash} \subseteq \boldsymbol{\Omega} g^{-1}[F]$.

Because $\vdash$ has a constant symbol, its variable-free formulas constitute a subalgebra $\boldsymbol{B}$ of $\boldsymbol{Fm}$. Let $G = T^{\vdash} \cap B$, so $\langle \boldsymbol{B}, G\rangle^* \in \mathsf{Mod}^*(\vdash)$. By Theorem 5.3(iii) and Lemma 2.10(iii),

$$\boldsymbol{\Omega}^{\boldsymbol{B}} G = (B \times B) \cap \boldsymbol{\Omega} T^{\vdash} \subseteq \boldsymbol{\Omega} g^{-1}[F] = g^{-1}[\boldsymbol{\Omega}^{\boldsymbol{A}} F] = \ker g,$$

as $\langle \boldsymbol{A}, F\rangle$ is reduced. Thus, $\widetilde{g}\colon \alpha/\boldsymbol{\Omega}^{\boldsymbol{B}} G \mapsto g(\alpha)$ ($\alpha \in B$) is a well defined matrix homomorphism from $\langle \boldsymbol{B}, G\rangle^*$ to $\langle \boldsymbol{A}, F\rangle$, and $\widetilde{g}$ is surjective, again because $\langle \boldsymbol{A}, F\rangle$ is 0–generated.

We show that $\widetilde{g}$ is strict. For each $\alpha \in B$, the expression $D\alpha$ is an existential positive $\mathcal{L}$–sentence, because α is a variable-free formula of $\vdash$. For the same reason, if $\alpha \in B$ and $\widetilde{g}(\alpha/\boldsymbol{\Omega}^{\boldsymbol{B}} G) \in F$, then $D\alpha$ is true in

$\langle \boldsymbol{A}, F\rangle$. In this case, since $\langle \boldsymbol{A}, F\rangle$ is nontrivial and reduced, and since $\vdash$ is overflow complete, it follows from Remark 7.6 that $D\alpha$ is true in all members of $\mathsf{Mod}^*(\vdash)$. Then $\alpha \in T^{\vdash}$, by Theorem 2.8, whence $\alpha \in G$, i.e., $\alpha/\boldsymbol{\Omega}^{\boldsymbol{B}}G \in G/\boldsymbol{\Omega}^{\boldsymbol{B}}G$. This confirms that $\widetilde{g}$ is strict. Consequently, $\widetilde{g}$ is an embedding, by Lemma 5.4, and so $\widetilde{g}\colon \langle \boldsymbol{B}, G\rangle^* \cong \langle \boldsymbol{A}, F\rangle$. But $\langle \boldsymbol{B}, G\rangle^*$ is fixed, so the proof is complete. □

Example 7.8. Substructural logics that lack the *weakening axiom*

$$x \to (y \to x)$$

are often formulated with an inferential negation, $\neg\, x = x \to \mathbf{f}$, where $\mathbf{f}$ is a constant symbol. In these systems, $\{x \to y,\, y \to x\}$ is a set of equivalence formulas. In a reduced matrix model, the cardinality of the submatrix generated by $\{\mathbf{f}\}$ may vary with the choice of model, even if we restrict the signature to $\to$. For example, the 4–element algebra in the proof of Theorem 10.10 is $\to$ generated by $\{\mathbf{f}\}$, and so is the 2–element Boolean algebra (where $\mathbf{f}$ is the lower element). So, when $\mathbf{f}$ and $\to$ are both definable, these algebras become 0–generated. Since they are not isomorphic, Theorem 7.7 rules out overflow completeness (and thereby structural completeness) for countless substructural logics with $\to, \mathbf{f}$, and without weakening. Not all of these systems are algebraizable.

It is easy to see that a deductive system is overflow complete iff, for each of its underivable finite rules, there is a substitution turning all of the rule's premises into theorems. Recently, Cintula and Metcalfe [12] have studied this condition under the name *passive structural completeness.*

8. Truth Equational and Weakly Algebraizable Systems

A deductive system is truth equational if its unary designation predicate is equationally definable over its reduced matrix models. To be precise, the theorem below was proved in [57] (and more directly in [59, Thm. 37]).

Theorem 8.1. *The following conditions on $\vdash$ are equivalent.*

(i) *There is a set $\boldsymbol{\tau}$ of pairs $\tau = \langle \tau_\ell(x), \tau_r(x)\rangle$ of unary formulas such that, for every reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and every $a \in A$,*

$$a \in F \;\textit{ iff }\; \big(\tau_\ell^{\boldsymbol{A}}(a) = \tau_r^{\boldsymbol{A}}(a) \textit{ for all } \tau \in \boldsymbol{\tau}\big).$$

(ii) *Whenever F_i $(i \in I)$ and G are $\vdash$–filters of an algebra $\boldsymbol{A}$, such that $\bigcap_{i\in I} \boldsymbol{\Omega}^{\boldsymbol{A}} F_i \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G$, then $\bigcap_{i\in I} F_i \subseteq G$.*

For example, in the reduced matrix models of classical or intuitionistic propositional logic, the displayed condition in (i) is realized as '$a \in F$ iff $a = \top$'. In substructural logics without weakening, this is no longer true, but instead, (i) is witnessed by '$a \in F$ iff $a = a \vee (a \to a)$'.

Definition 8.2. We say that $\vdash$ is *truth equational* if it satisfies the equivalent conditions of Theorem 8.1.

Observe that this demand persists in extensions. The reduced matrix models of a truth equational system $\vdash$ are evidently determined by their algebra reducts, i.e., whenever $\langle \boldsymbol{A}, F\rangle$, $\langle \boldsymbol{A}, G\rangle \in \mathsf{Mod}^*(\vdash)$, then $F = G$. In fact, this remains true for subdirect products of reduced matrix models.

Notation. We denote by $\mathsf{Alg}^*(\vdash)$ the class of all algebra reducts $\boldsymbol{A}$ of reduced matrix models $\langle \boldsymbol{A}, F\rangle$ of $\vdash$. The *algebraic counterpart* $\mathsf{Alg}(\vdash)$ of $\vdash$ is defined as $\mathbb{IP}_{\mathbb{S}}(\mathsf{Alg}^*(\vdash))$, the closure of $\mathsf{Alg}^*(\vdash)$ under subdirect products (and isomorphisms).

Remark 8.3. If $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ are both as in Theorem 8.1(i), then

$$\big(\&_{\tau\in\boldsymbol{\tau}}\, \tau_\ell(x) = \tau_r(x)\big) \iff \big(\&_{\tau\in\boldsymbol{\tau}'}\, \tau_\ell(x) = \tau_r(x)\big)$$

is clearly valid in $\mathsf{Alg}^*(\vdash)$, and therefore in $\mathsf{Alg}(\vdash)$.

Note that $\mathsf{Alg}(\vdash) = \mathsf{Alg}^*(\vdash)$ if $\vdash$ is protoalgebraic, by Theorem 4.1(iii). Even if $\vdash$ is not protoalgebraic, truth equationality permits a slight relaxation of the admissibility criterion in Theorem 2.12(iii). This follows from the first item in the next lemma.

Lemma 8.4. *Let $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, G\rangle$ be matrix models of a truth equational system $\vdash$, where $\langle \boldsymbol{A}, F\rangle$ is reduced, and let $h\colon \boldsymbol{B} \to \boldsymbol{A}$ be an algebraic homomorphism.*

(i) *If $\langle \boldsymbol{B}, G\rangle$ is a subdirect product of reduced matrix models of $\vdash$ (in particular, if $\langle \boldsymbol{B}, G\rangle$ is itself reduced), then h is a matrix homomorphism from $\langle \boldsymbol{B}, G\rangle$ into $\langle \boldsymbol{A}, F\rangle$.*

(ii) *If h is a matrix homomorphism from $\langle \boldsymbol{B}, G\rangle$ into $\langle \boldsymbol{A}, F\rangle$, and G is a union of* $(\ker h)$*–classes, then h is strict.*

(iii) *Every injective matrix homomorphism from $\langle \boldsymbol{B}, G\rangle$ into $\langle \boldsymbol{A}, F\rangle$ is an embedding.*

Proof. Let $\boldsymbol{\tau}$ be as in Theorem 8.1(i).

(i) Let $\langle \boldsymbol{B}, G\rangle$ be a subdirect product of reduced matrix models $\langle \boldsymbol{B}_i, G_i\rangle$ $(i \in I)$ of $\vdash$, and let $b \in G$. Then, for each $i \in I$, we have $b(i) \in G_i$, because $\langle \boldsymbol{B}_i, G_i\rangle$ is a homomorphic image of $\langle \boldsymbol{B}, G\rangle$. Thus, for all $\tau \in \boldsymbol{\tau}$ and $i \in I$, we have $\tau_\ell^{\boldsymbol{B}_i}(b(i)) = \tau_r^{\boldsymbol{B}_i}(b(i))$, because $\langle \boldsymbol{B}_i, G_i\rangle$ is reduced, and so $\tau_\ell^{\boldsymbol{B}}(b) = \tau_r^{\boldsymbol{B}}(b)$. Then $\tau_\ell^{\boldsymbol{A}}(h(b)) = h(\tau_\ell^{\boldsymbol{B}}(b)) = h(\tau_r^{\boldsymbol{B}}(b)) = \tau_r^{\boldsymbol{A}}(h(b))$ for all $\tau \in \boldsymbol{\tau}$. Since $\langle \boldsymbol{A}, F\rangle$ is reduced, this implies that $h(b) \in F$, as required.

(ii) Let $h(b) \in F$, where $b \in B$. We must show that $b \in G$. For each $\tau \in \boldsymbol{\tau}$, we have $h(\tau_\ell^{\boldsymbol{B}}(b)) = \tau_\ell^{\boldsymbol{A}}(h(b)) = \tau_r^{\boldsymbol{A}}(h(b)) = h(\tau_r^{\boldsymbol{B}}(b))$, as $\langle \boldsymbol{A}, F\rangle$ is reduced. Thus, $\ker h$ identifies $\tau_\ell^{\boldsymbol{B}}(b)$ with $\tau_r^{\boldsymbol{B}}(b)$. But $\ker h \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}} G$, as G is a union of $(\ker h)$–classes, so $\tau_\ell^{\boldsymbol{B}}(b) \equiv_{\boldsymbol{\Omega}^{\boldsymbol{B}} G} \tau_r^{\boldsymbol{B}}(b)$ for all $\tau \in \boldsymbol{\tau}$. Therefore, $b/\boldsymbol{\Omega}^{\boldsymbol{B}} G \in G/\boldsymbol{\Omega}^{\boldsymbol{B}} G$, as $\langle \boldsymbol{B}, G\rangle^* \in \mathsf{Mod}^*(\vdash)$, and so $b \in G$.

(iii) is an instance of (ii), because G is a union of id_B–classes. ☐

Theorem 8.5. ([17]) *The following conditions on $\vdash$ are equivalent.*

(i) *$\vdash$ is both protoalgebraic and truth equational.*

(ii) *For every algebra* $\boldsymbol{A}$, *the map* $F \mapsto \boldsymbol{\Omega}^{\boldsymbol{A}} F$ *is injective and order-preserving (with respect to* $\subseteq$*) on the* $\vdash$*–filters of* $\boldsymbol{A}$.

(iii) *For every algebra* $\boldsymbol{A}$, *the map* $F \mapsto \boldsymbol{\Omega}^{\boldsymbol{A}} F$ *defines a lattice isomorphism from the* $\vdash$*–filters of* $\boldsymbol{A}$ *onto the* $\mathsf{Alg}(\vdash)$*–congruences of* $\boldsymbol{A}$, *i.e., the congruences* θ *such that* $\boldsymbol{A}/\theta \in \mathsf{Alg}(\vdash)$.

Definition 8.6. ([17]) We say that $\vdash$ is *weakly algebraizable* if it satisfies the equivalent conditions of Theorem 8.5.

Admissibility in weakly algebraizable systems can be characterized in terms of pure algebras, rather than matrices, as follows.

Theorem 8.7. *Let* $\vdash$ *be weakly algebraizable. Then the following conditions are equivalent.*

(i) $\langle \Gamma, \alpha \rangle$ *is an admissible rule of* $\vdash$.

(ii) *Every algebra in* $\mathsf{Alg}(\vdash)$ *is a homomorphic image of an algebra belonging to* $\mathsf{Alg}(\vdash + \langle \Gamma, \alpha \rangle)$.

(iii) *Every algebra in* $\mathsf{Alg}(\vdash)$ *is a homomorphic image of one in which*

$$\big(\&_{\tau \in \boldsymbol{\tau},\, \gamma \in \Gamma}\, \tau_\ell(\gamma) = \tau_r(\gamma)\big) \implies \big(\&_{\tau \in \boldsymbol{\tau}}\, \tau_\ell(\alpha) = \tau_r(\alpha)\big)$$

is valid, where $\boldsymbol{\tau}$ *is as in Theorem* 8.1(i).

Proof. Since $\vdash$ and its extensions are protoalgebraic and truth equational, Theorem 4.4 and Lemma 8.4(i) combine to prove the equivalence of conditions (i) and (ii) of the present theorem. The meaning of (iii) is independent of the choice of $\boldsymbol{\tau}$, by Remark 8.3, and the equivalence of (ii) and (iii) is just a consequence of the definitions. ☐

Theorem 8.7 generalizes [46, Thm. 7.11], which dealt only with *algebraizable* systems; the present proof is also simpler. Algebraizability was introduced in [7] and is discussed in detail in [9, 15, 21, 48]. For present purposes, it suffices to note that

> a deductive system $\vdash$ is [finitely] algebraizable iff it is both truth equational and [finitely] equivalential.

The usual definition of algebraizability asks that $\vdash$ be *equivalent*—in a suitable sense—to the equational consequence relation of a class C of pure algebras. (In this case, we can choose $\mathsf{C} = \mathsf{Alg}(\vdash) = \mathsf{Alg}^*(\vdash)$.) The pertinent notion of equivalence is discussed in several recent papers, particularly [5], but we shall not need to use it here. *Orthologic* is an example of a weakly algebraizable system that is not algebraizable, see [17, 34]. In this example, $\{\langle x, \top \rangle\}$ can play the role of $\boldsymbol{\tau}$ in Theorem 8.1(i).

We do not need the full force of algebraizability in order to prove the next result. It follows from Theorem 4.7, via Lemma 8.4(i),(iii).

Theorem 8.8. *Suppose that* $\vdash$ *is finitary and weakly algebraizable. If every finitely generated relatively subdirectly irreducible algebra in* $\mathsf{Alg}(\vdash)$ *is weakly projective in* $\mathsf{Alg}(\vdash)$, *then* $\vdash$ *is hereditarily structurally complete.*

As $\mathsf{Alg}(\vdash)$ is closed under subdirect products, it is a variety iff it is closed under homomorphic images of the algebraic kind. In this case, an algebra in $\mathsf{Alg}(\vdash)$ will be relatively subdirectly irreducible in $\mathsf{Alg}(\vdash)$ iff it is subdirectly irreducible in the absolute sense.

Example 8.9. $\mathbf{RM^t}$ denotes the extension of relevance logic by the *mingle* axiom $x \to (x \to x)$. Here, relevance logic is formulated with the Ackermann truth constant $\mathbf{t}$ (see Section 10 for more details). Although $\mathbf{RM^t}$ is not structurally complete, its negation-less fragment $\vdash$ (i.e., its $\to, \cdot, \wedge, \vee, \mathbf{t}$ fragment) is hereditarily structurally complete. For reasons explained in [45, 46], this cannot be proved by generalizing the syntactic method known as 'Prucnal's trick' (deriving from [51]). But $\vdash$ is algebraizable and the algebraic criterion of Theorem 8.8 can be applied. Indeed, $\mathsf{Alg}(\vdash)$ is the locally finite variety of *positive Sugihara monoids* (PSMs), and it is proved in [45] that every finite subdirectly irreducible PSM is projective (hence weakly projective) in this variety.

For an algebraizable finitary system $\vdash$, if the class $\mathsf{Alg}(\vdash)$ is elementary, then it is a quasivariety. In this case, $\vdash$ is structurally complete iff every proper subquasivariety of $\mathsf{Alg}(\vdash)$ generates a proper subvariety of $\mathrm{H}(\mathsf{Alg}(\vdash))$—cf. Bergman's definition in Section 7.

Example 8.10. $\mathbf{FL_{ew}}$ denotes intuitionistic affine linear logic without exponentials (sometimes called 'BCK–logic'). It is algebraizable, and $\mathsf{Alg}(\mathbf{FL_{ew}})$ is the variety of all bounded integral commutative residuated lattices, see for instance [22]. Let $\vdash$ be a consistent axiomatic extension of the S–fragment of $\mathbf{FL_{ew}}$, where S includes at least $\to$ and $\bot$. Then $\mathsf{Alg}(\vdash)$ is a quasivariety, but it need not be a variety [71]. We define $x \to^0 y = y$ and $x \to^{n+1} y = x \to (x \to^n y)$ for $n \in \omega$. A member of $\mathsf{Alg}(\vdash)$ satisfying $x \to^n y = x \to^{n+1} y$ is said to be n*–contractive*, and every finite algebra in $\mathsf{Alg}(\vdash)$ is n–contractive for some finite n. If $\vdash$ is overflow complete, then $\mathsf{Alg}(\vdash)$ contains no simple algebra on more than two elements that is n–contractive for a finite n—in particular, $\mathsf{Alg}(\vdash)$ contains no *finite* simple algebra other than the 2–element Boolean algebra. The proof uses Theorem 7.5 and the existential positive sentence

$$\exists x\, (x^n = \bot \;\;\&\;\; \neg\, x \leqslant x),$$

which can be written in terms of $\to, \bot$ as

$$\exists x\, (x \to^n \bot = \bot \to \bot \;\;\&\;\; (x \to \bot) \to x = \bot \to \bot).$$

This sentence is false in the unique 2–element member of $\mathsf{Alg}(\vdash)$, but it would be true in any simple n–contractive member having more than two elements. The proof details can be found in [46, Prop. 10.5], but the present account is a slight improvement, as we do not assume here that $\mathsf{Alg}(\vdash)$ is a variety. This rules out overflow completeness for a large class of fuzzy logics—e.g., the finite MV–chains on three or more elements are simple algebras, so they cannot belong to $\mathsf{Alg}(\vdash)$ if $\vdash$ is overflow complete.

Since the appearance of [46], a somewhat different explanation of Example 8.10 has been given in [12, Thm. 5.16].

9. Order Algebraizable Systems

Several prominent non-algebraizable systems $\vdash$ are still order algebraizable in the sense of [60] (see Section 10 for examples). The definition asks that $\vdash$ be equivalent, in the sense of [5], to the *inequational* consequence relation of a class of partially ordered algebras. Here, however, it is convenient to work with the following characterization, whose correctness follows immediately from [60, Thm. 7.1, Cor. 6.7].

Characterization 9.1. $\vdash$ is *order algebraizable* iff its language includes a set $\boldsymbol{\rho}$ of binary formulas $\rho(x, y)$ such that, for every reduced matrix model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$, the set A is partially ordered by the relation

$$a \leqslant_F b \ \text{ iff } \ \boldsymbol{\rho}^{\boldsymbol{A}}(a, b) \subseteq F$$

and, moreover,

$$x \dashv\vdash \bigcup\{\boldsymbol{\rho}(\tau_\ell(x),\, \tau_r(x)) : \tau \in \boldsymbol{\tau}\} \tag{7}$$

for a suitable set $\boldsymbol{\tau}$ of pairs of unary formulas $\tau = \langle \tau_\ell(x), \tau_r(x)\rangle$.

In this case, we say that $\vdash$ is *$\boldsymbol{\rho}$–order algebraizable* and, by its *$\boldsymbol{\rho}$–ordered algebras*, we mean the structures $\langle \boldsymbol{A}, \leqslant_F\rangle$ arising as above from all of its reduced matrix models $\langle \boldsymbol{A}, F\rangle$.

Under these assumptions, for any $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\vdash)$ and $a \in A$, we have

$$a \in F \ \text{ iff } \ (\tau_\ell(a) \leqslant_F \tau_r(a) \ \text{ for all } \ \tau \in \boldsymbol{\tau})\,, \tag{8}$$

by (7) and the definition of $\leqslant_F$. Consequently, the map sending F to $\leqslant_F$ is injective on the $\vdash$–filters of any $\boldsymbol{A} \in \mathsf{Alg}^*(\vdash)$. There is no difference here between $\mathsf{Alg}^*(\vdash)$ and $\mathsf{Alg}(\vdash)$, because every $\boldsymbol{\rho}$–order algebraizable system is protoalgebraic—in fact equivalential, with equivalence formulas $\boldsymbol{\rho}(x, y) \cup \boldsymbol{\rho}(y, x)$ [60].

The order algebraizable systems do not appear to constitute a level of the Leibniz hierarchy, as they seem to have no simple $\boldsymbol{\Omega}$–characterization, but they are a mathematically natural subclass of the equivalential systems. Clearly, an extension of a $\boldsymbol{\rho}$–order algebraizable system $\vdash$ is itself $\boldsymbol{\rho}$–order algebraizable, and if $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ both satisfy the demands of 9.1, then

$$\big(\&_{\tau\in\boldsymbol{\tau}}\, \tau_\ell(x) \leqslant \tau_r(x)\big) \iff \big(\&_{\tau\in\boldsymbol{\tau}'}\, \tau_\ell(x) \leqslant \tau_r(x)\big)$$

is valid in the $\boldsymbol{\rho}$–ordered algebras of $\vdash$.

Remark 9.2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras, and let $\leqslant$ and $\leqslant'$ be binary relations on A and B, respectively. The conventions of Sections 2 and 4 dictate that we call $\langle \boldsymbol{A}, \leqslant\rangle$ a *homomorphic image* of $\langle \boldsymbol{B}, \leqslant'\rangle$ iff there is a surjective (algebraic) homomorphism $h\colon \boldsymbol{B} \to \boldsymbol{A}$ such that, whenever $b_1, b_2 \in B$ with $b_1 \leqslant' b_2$, then $h(b_1) \leqslant h(b_2)$.

Theorem 9.3. *Let $\vdash$ be $\boldsymbol{\rho}$–order algebraizable. Then the following conditions are equivalent.*

(i) $\langle \Gamma, \alpha \rangle$ *is an admissible rule of* $\vdash$.
(ii) *Every $\boldsymbol{\rho}$–ordered algebra of $\vdash$ is a homomorphic image of a $\boldsymbol{\rho}$–ordered algebra of $\vdash + \langle \Gamma, \alpha \rangle$ (in the sense of Remark* 9.2).
(iii) *Every $\boldsymbol{\rho}$–ordered algebra of $\vdash$ is a homomorphic image of one in which*

$$\big(\&_{\gamma \in \Gamma,\, \tau \in \boldsymbol{\tau}}\ \tau_\ell(\gamma) \leqslant \tau_r(\gamma)\big) \implies \big(\&_{\tau \in \boldsymbol{\tau}}\ \tau_\ell(\alpha) \leqslant \tau_r(\alpha)\big)$$

is valid, where $\boldsymbol{\tau}$ is as in Characterization 9.1.

Proof. (i) $\Leftrightarrow$ (ii): Since order algebraizable systems are protoalgebraic, it suffices to observe that the criterion in Theorem 4.4(ii) is equivalent, for $\vdash$, to the one in 9.3(ii). Indeed, given reduced matrix models $\langle \boldsymbol{A}, F \rangle$ and $\langle \boldsymbol{B}, G \rangle$ of $\vdash$ and a surjective homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{A}$, we have $h[G] \subseteq F$ iff h preserves order when considered as a map from $\langle \boldsymbol{B}, \leqslant_G \rangle$ to $\langle \boldsymbol{A}, \leqslant_F \rangle$. This follows from (8) and the definitions of $\leqslant_G$ and $\leqslant_F$, because h preserves the formulas occurring in $\boldsymbol{\rho}$ and in $\boldsymbol{\tau}$.

(ii) $\Leftrightarrow$ (iii) follows from the definitions, using (8). $\square$

Because a $\boldsymbol{\rho}$–order algebraizable system $\vdash$ is equivalential, its $\boldsymbol{\rho}$–ordered algebras constitute a UISP–class of $\mathcal{L}$–structures, where $\mathcal{L}$ is the first order language with equality having one (binary) relation symbol $\leqslant$ and the connectives of $\vdash$ as function symbols. We denote this UISP–class by $\mathsf{OAlg}_{\boldsymbol{\rho}}(\vdash)$. If it is elementary (and thus a universal Horn class) for a suitable $\boldsymbol{\rho}$, we say that $\vdash$ is *elementarily order algebraizable.* In that case, $\boldsymbol{\rho}$ can be chosen *finite* (whence $\vdash$ is finitely equivalential), and $\mathsf{OAlg}_{\boldsymbol{\rho}}(\vdash)$ is axiomatized by the anti-symmetry law $\forall x\, \forall y\, ((x \leqslant y \ \&\ y \leqslant x) \implies x = y)$ and suitable sentences all of the form

$$\forall \bar{x}\ \big(\big(\&_{i<n}\, \alpha_i(\bar{x}) \leqslant \beta_i(\bar{x})\big) \implies \alpha(\bar{x}) \leqslant \beta(\bar{x})\big),$$

with $n \in \omega$. This does not force $\mathsf{Mod}^*(\vdash)$ to be a universal Horn class, however, and $\vdash$ need not be finitary [58].

A *partially ordered algebra* $\langle \boldsymbol{A}, \leqslant \rangle$ comprises an algebra $\boldsymbol{A}$ and a partial order $\leqslant$ of its universe A. When $\vdash$ is elementarily $\boldsymbol{\rho}$–order algebraizable, then any universal Horn subclass K of $\mathsf{OAlg}_{\boldsymbol{\rho}}(\vdash)$ consists of partially ordered algebras, by definition. Nevertheless, the atomic class $\mathrm{H}(\mathsf{K})$ may include structures $\langle \boldsymbol{A}, \leqslant \rangle$ where $\leqslant$ is not a partial order, because both anti-symmetry and transitivity may be lost in the formation of homomorphic images. It is therefore preferable to work with $\mathsf{OAlg}_{\boldsymbol{\rho}}(\vdash) \cap \mathrm{H}(\mathsf{K})$, the relative atomic subclass of $\mathsf{OAlg}_{\boldsymbol{\rho}}(\vdash)$ generated by K. From Theorem 6.12(i), we obtain:

Theorem 9.4. *Suppose $\vdash$ is elementarily $\boldsymbol{\rho}$–order algebraizable and finitary. Then $\vdash$ is structurally complete iff every proper universal Horn subclass of $\mathsf{OAlg}_{\boldsymbol{\rho}}(\vdash)$ generates a proper relative atomic subclass of $\mathsf{OAlg}_{\boldsymbol{\rho}}(\vdash)$.*

Every algebraizable system is order algebraizable, because the identity relation is a partial order. Structural completeness has been established for few (if any) significant *non*-algebraizable logics, but there is at least one interesting conjecture of this kind in the literature. That is Problem 10.6 below, and Theorems 6.8, 9.3 and 9.4 are potentially relevant to it.

10. Fragments of Relevance Logic: A Case Study

The most natural examples of order algebraizable systems (apart from algebraizable ones) are the intensional fragments of relevance logic, linear logic, and other substructural logics without weakening. In exponential-free linear logic, no fragment with implication is structurally complete [46], but the contraction axiom turns relevance logic into a more complex case study, with some open problems.

Relevance logic is traditionally identified with the theorems of a formal system $\mathbf{R}$ (sometimes called $\mathbf{R}^{\mathbf{t}}$), whose signature is $\wedge, \vee, \cdot, \to, \neg, \mathbf{t}$. For recent surveys, see [19, 37, 61]. The postulates of $\mathbf{R}$ in any restricted signature S constitute a formal system $\mathbf{R}_S$. In particular, $\mathbf{R}_{\cdot,\to,\mathbf{t}}$ is

(B)	$(x \to y) \to ((z \to x) \to (z \to y))$	(prefixing)
(C)	$(x \to (y \to z)) \to (y \to (x \to z))$	(exchange)
(I)	$x \to x$	(identity)
(W)	$(x \to (x \to y)) \to (x \to y)$	(contraction)
	$x \to (y \to (y \cdot x))$	
	$(x \to (y \to z)) \to ((y \cdot x) \to z)$	
	$\mathbf{t}$	
	$\mathbf{t} \to (x \to x)$	
(MP)	$x,\, x \to y \ /\ y$	(modus ponens).

Whenever $\{\to\} \subseteq S \subseteq \{\cdot, \to, \mathbf{t}\}$, then $\mathbf{R}_S$ axiomatizes the S–fragment of $\vdash_{\mathbf{R}}$ [40]. Because of this, we shall not bother to distinguish notationally between $\mathbf{R}_S$ and $\vdash_{\mathbf{R}_S}$, and we refer to $\mathbf{R}_S$ itself as the S–fragment of $\mathbf{R}$. (In [40], fragments are considered as sets of theorems, rather than as subsets of a deducibility relation, but the above axiomatization is separative even for rules. This point is discussed in more detail in [28, 66].)

If $\{\to\} \subseteq S \subseteq \{\cdot, \to, \mathbf{t}\}$, then $\mathbf{R}_S$ is not (even weakly) algebraizable [7], but it is elementarily $\boldsymbol{\rho}$–order algebraizable with witness $\boldsymbol{\tau}$, where

$$\boldsymbol{\rho}(x, y) = \{x \to y\} \quad \text{and} \quad \boldsymbol{\tau}(x) = \{\langle x \to x,\, x\rangle\}.$$

We can replace $\boldsymbol{\tau}(x)$ by $\{\langle \mathbf{t}, x\rangle\}$ when $\mathbf{t}$ belongs to S. The $\{x \to y\}$–ordered algebras of $\mathbf{R}_{\cdot,\to,\mathbf{t}}$ are the Church monoids of [40], defined below.

Definition 10.1. A *Church monoid* $\langle \boldsymbol{A}, \leqslant\rangle$ comprises an algebra $\boldsymbol{A} = \langle A; \cdot, \to, \mathbf{t}\rangle$ and a partial order $\leqslant$ of A, where

(i) $\langle A; \cdot, \mathbf{t}\rangle$ is a commutative monoid (i.e., $\mathbf{t} \in A$ and $\cdot$ is a commutative and associative binary operation on A, with $a \cdot \mathbf{t} = a$ for all $a \in A$),

(ii) for all $a, b, c \in A$, if $a \leqslant b$ then $a \cdot c \leqslant b \cdot c$,
(iii) for all $a, b \in A$, $\max_{\leqslant}\{c \in A : a \cdot c \leqslant b\}$ exists and is equal to $a \to b$, and
(iv) $\langle A; \cdot, \leqslant \rangle$ is *square increasing*, i.e., $a \leqslant a \cdot a$ for all $a \in A$.

The joint content of (ii) and (iii) could be put more succinctly as follows:

(ii)′ for all $a, b, c \in A$, we have $c \leqslant a \to b$ iff $a \cdot c \leqslant b$.

The binary operation $\to$ is called *residuation.* In any Church monoid, $\to$ is completely determined by $\cdot$ and $\leqslant$, and it follows from (ii)′ that

$$a \leqslant b \text{ iff } \mathbf{t} \leqslant a \to b. \tag{9}$$

Because $\mathbf{R}_{\cdot,\to,\mathbf{t}}$ is order algebraized by Church monoids, with $\boldsymbol{\tau} = \{\langle \mathbf{t}, x \rangle\}$, the following well known fact is a manifestation of (8).

Fact 10.2. *For any set* $\Gamma \cup \{\alpha\}$ *of formulas of* $\mathbf{R}_{\cdot,\to,\mathbf{t}}$, *we have* $\Gamma \vdash_{\mathbf{R}} \alpha$ *iff every Church monoid satisfies* $\forall \bar{x}\ ((\&_{\gamma \in \Gamma}\ \mathbf{t} \leqslant \gamma) \implies \mathbf{t} \leqslant \alpha)$.

Theorem 10.3. ([46]) *The rule* $x \to \mathbf{t},\ (x \to \mathbf{t}) \to \mathbf{t}\ /\ x$ *is admissible in* $\mathbf{R}_{\cdot,\to,\mathbf{t}}$, *and therefore in* $\mathbf{R}_{\to,\mathbf{t}}$. *Consequently,* $\mathbf{R}_{\cdot,\to,\mathbf{t}}$ *and* $\mathbf{R}_{\to,\mathbf{t}}$ *are not structurally complete.*

The proof in [46] relies on a characterization of admissibility that was confined to algebraizable systems. Thus, it detours through an algebraizable conservative extension of $\mathbf{R}_{\cdot,\to,\mathbf{t}}$. The detour can be eliminated, however, because Theorem 9.3 prescribes nothing more than order algebraizability. The argument in [46] shows that every Church monoid is a homomorphic image of one that satisfies

$$\forall x\ (x \to \mathbf{t} = \mathbf{t} \implies x = \mathbf{t})\,. \tag{10}$$

Note that (10) amounts to

$$\forall x\ ((\mathbf{t} \leqslant x \to \mathbf{t}\ \ \&\ \ \mathbf{t} \leqslant (x \to \mathbf{t}) \to \mathbf{t}) \implies \mathbf{t} \leqslant x)\,,$$

in view of (9). Thus, $x \to \mathbf{t},\ (x \to \mathbf{t}) \to \mathbf{t}\ /\ x$ is admissible in $\mathbf{R}_{\cdot,\to,\mathbf{t}}$, and it remains admissible in $\mathbf{R}_{\to,\mathbf{t}}$, because $\cdot$ does not occur in it. It is underivable in these systems, as it is underivable even in the stronger system of classical logic (where $\mathbf{t}$ is logically equivalent to $y \to y$).

Theorem 10.3 does not settle the problem of structural completeness for $\mathbf{R}_{\cdot,\to}$, but this question is rather easily disposed of by syntactic arguments, as follows.

Theorem 10.4. *The rule* $x \cdot y\ /\ x$ *is admissible in* $\mathbf{R}_{\cdot,\to,\mathbf{t}}$, *and therefore in* $\mathbf{R}_{\cdot,\to}$. *Consequently,* $\mathbf{R}_{\cdot,\to}$ *is not structurally complete.*

Proof. We use a single-conclusion sequent calculus $\mathbf{G}$ such that, for any formula α of $\mathbf{R}_{\cdot,\to,\mathbf{t}}$, the sequent $\rhd \alpha$ is provable in $\mathbf{G}$ iff α is a theorem of $\mathbf{R}$. We require, as usual, that $\mathbf{G}$ has the cut elimination property and the subformula property. Various calculi of this sort are available—see for instance [47, 65]. In these systems, no axiom has the form $\rhd \alpha \cdot \beta$. The inference rule schemata are such that any cut-free proof of $\rhd \alpha \cdot \beta$ in $\mathbf{G}$, involving no connective other than $\cdot, \to, \mathbf{t}$, must end with an execution of

$$\frac{\Delta \ \rhd \ \alpha \qquad \Sigma \ \rhd \ \beta}{\Delta, \Sigma \ \rhd \ \alpha \cdot \beta} \ (\rhd \cdot)$$

in which Δ and Σ are empty. Thus, by the cut-elimination and subformula properties, if $\alpha \cdot \beta$ is a theorem of $\mathbf{R}_{\cdot, \to, \mathbf{t}}$, then $\rhd \alpha$ is provable in $\mathbf{G}$, i.e., $\vdash_{\mathbf{R}} \alpha$. This establishes that $x \cdot y \,/\, x$ is admissible in $\mathbf{R}_{\cdot, \to, \mathbf{t}}$, and it remains admissible in $\mathbf{R}_{\cdot, \to}$, because it does not involve $\mathbf{t}$. It is underivable in $\mathbf{R}$, however, by Fact 10.2, because the implication $\mathbf{t} \leqslant x \cdot y \implies \mathbf{t} \leqslant x$ is not valid in every Church monoid. Indeed, consider the Church monoid with identity 1 on the chain $-2 < -1 < 1 < 2$, where $a \cdot b$ is the element of $\{a, b\}$ with the greater absolute value when $|a| \neq |b|$, and is otherwise the minimum of $\{a, b\}$. To invalidate the implication, set $x = -1$ and $y = 2$. □

Combining Theorems 10.4 and 9.3, we obtain a fact about residuated structures that is not obvious on algebraic grounds:

Corollary 10.5. *Every Church monoid is a homomorphic image of one that satisfies* $\forall x \, \forall y \ (\mathbf{t} \leqslant x \cdot y \implies \mathbf{t} \leqslant x)$.

The above results say nothing about the pure implication fragment $\mathbf{R}_{\to}$ of $\mathbf{R}$. This fragment is better known as **BCIW**, because it is axiomatized by (B), (C), (I), (W) and modus ponens.

Problem 10.6. ([64]) *Is* **BCIW** *structurally complete?*

In [64], Slaney and Meyer gave a syntactic proof that the $\wedge, \to$ fragment of $\mathbf{R}$ is structurally complete. They expressed hopes for a similar theorem in the case of **BCIW**, but predicted a need to resort to algebraic methods. In fact, *hereditary* structural completeness for $\mathbf{R}_{\wedge, \to}$ can be inferred from the arguments in [64] (see [46] for a generalization of this result). On the other hand, **BCIW** is *not* hereditarily structurally complete—see Remark 10.11.

The theory in Section 9 was motivated in part by the remark about algebraic methods in [64] (and the fact that **BCIW** is order algebraizable but not algebraizable). The $\{x \to y\}$–ordered algebras of **BCIW** are the $\to, \leqslant$ subreducts of Church monoids. They are finitely axiomatized structures, and **BCIW** has the strong finite model property (see [41, 44, 66]). Nevertheless, Problem 10.6 remains open, and even the following special cases seem difficult:

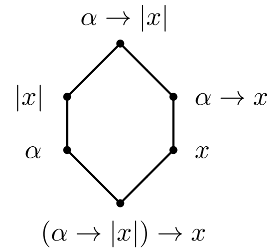
Problem 10.7. *Is the rule* $(x \to (x \to x)) \to x \,/\, x$ *admissible in* **BCIW**?

Problem 10.8. *If a rule involving only one variable is admissible in* **BCIW**, *must it be derivable in* **BCIW**?

Because of the interest in Problem 10.6, we include here an observation (Theorem 10.10) that connects these three problems together. We exploit the following result of Meyer, in which we set

$$|x| := x \to x \quad \text{and} \quad \alpha := x \to |x| \, .$$

Theorem 10.9. ([39]) *Up to logical equivalence, the one-variable formulas of* **BCIW** *are exactly the following six, where the Hasse diagram puts* β *below* γ *iff* $\beta \to \gamma$ *is a theorem of* **BCIW**.



This is the order reduct of the $\{x \to y\}$–ordered algebra of **BCIW** that comes from $\langle \boldsymbol{Fm}_1, T_1\rangle^*$, where $\boldsymbol{Fm}_1$ is the free $\to$ groupoid on one generator x, and T_1 is its intersection with the theorems of **BCIW**. In the diagram, each formula β abbreviates its own equivalence class modulo logical equivalence (i.e., modulo $\boldsymbol{\Omega} T_1$). The 6×6 Cayley table for $\to$ is given in [39]. Of the six displayed formulas, only $|x|$ and $\alpha \to |x|$ are theorems of **BCIW**.

Theorem 10.10. *If the rule*

(11) $$(x \to (x \to x)) \to x \;/\; x$$

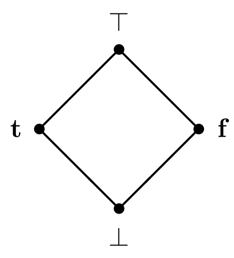
is admissible in **BCIW**, *then* **BCIW** *is not structurally complete.*

If (11) *is not admissible in* **BCIW**, *then every admissible one-variable rule of* **BCIW** *is derivable in* **BCIW**.

Proof. For the first assertion, we need only note that (11) is underivable in **BCIW**. This follows from Fact 10.2, because the implication

$$\mathbf{t} \leqslant (x \to |x|) \to x \implies \mathbf{t} \leqslant x$$

is not valid in the Church monoid $\langle A; \cdot, \to, \mathbf{t}, \leqslant\rangle$ with the following Hasse diagram, where $\bot \cdot a = a$ for all $a \in A$, and $\mathbf{f} \cdot \mathbf{f} = \top = a \cdot \top$ whenever $\bot \neq a \in A$. Indeed, $(x \to |x|) \to x$ takes the value $\top$ when we set $x = \mathbf{f}$.



For the second assertion, suppose that there exists a one-variable rule

(12) $$\alpha_1, \ldots, \alpha_n \;/\; \beta$$

that is admissible but underivable in **BCIW**, and choose (12) so that n is as small as possible. We need to show that (11) is then admissible in **BCIW**. Since (12) is underivable, β is not a theorem of **BCIW**, hence $n > 0$. Any

α_i that is a theorem of **BCIW** could be omitted from (12), contradicting the minimality of n, so no α_i is a theorem. Similarly, $\alpha_i \to \alpha_j$ cannot be a theorem unless $i = j$, for otherwise we could omit α_j from (12). This means that $\{\alpha_1, \ldots, \alpha_n\}$ is an anti-chain in the Hasse diagram of Theorem 10.9, hence $n \leqslant 2$. Finally, because (12) is underivable, there is no i for which $\alpha_i \to \beta$ is a theorem, i.e., we must have $\alpha_i \nleqslant \beta$ in the Hasse diagram, for all i. So (12) must be (11) *or* one of the following.

(i) $\alpha,\, \alpha \to x \;/\; x$
(ii) $\alpha,\, x \;/\; (\alpha \to |x|) \to x$
(iii) $\alpha,\, \alpha \to x \;/\; (\alpha \to |x|) \to x$
(iv) $\alpha \;/\; x$
(v) $\alpha \;/\; \alpha \to x$
(vi) $\alpha \;/\; (\alpha \to |x|) \to x$
(vii) $x \;/\; \alpha$
(viii) $x \;/\; (\alpha \to |x|) \to x$
(ix) $\alpha \to x \;/\; \alpha$
(x) $\alpha \to x \;/\; (\alpha \to |x|) \to x$

We show, however, that each of (i)–(x) is either derivable or inadmissible in **BCIW**, thus completing the proof.

Obviously, (i) is derivable. To see that (ii) is derivable, observe that the theorem $(\alpha \to |x|) \to (\alpha \to |x|)$ is logically equivalent in **BCIW** to

$$\alpha \to (x \to ((\alpha \to |x|) \to x)),$$

thanks to several applications of (C). Modus ponens does the rest. And, because (i) and (ii) are derivable, so is (iii).

We claim that none of (iv)–(x) is admissible in **BCIW**.

To see that (iv) is not admissible, substitute $x \to |x|$ for x. The premise of (iv) becomes $(x \to |x|) \to ((x \to |x|) \to (x \to |x|))$, which is a theorem of **BCIW**, because both

$$(x \to |x|) \to (x \to x) \quad \text{and} \quad (x \to x) \to ((x \to |x|) \to (x \to |x|))$$

are theorems (use (W) and (B)). But the conclusion of (iv) becomes $x \to |x|$, which is not a theorem.

If (v) were admissible, then the same would be true of (iv), by modus ponens. So (v) is not admissible. Similarly, the inadmissibility of (vi) follows from that of (iv), because $\alpha \to |x|$ is a theorem.

To see that (vii) is inadmissible, substitute $(x \to (x \to y)) \to (x \to y)$ for x, so the premise of (vii) becomes the theorem (W). This substitution turns α into a formula, δ say, and it suffices to show that δ is not a theorem of **BCIW**, i.e., that it is not a theorem of **R**. The set

$$A \,=\, \{0\} \cup \{2^n : n \in \omega\} \cup \{\infty\}$$

can be made into a Church monoid $\langle A; \cdot, \to, 1, \leqslant \rangle$, where $\leqslant$ is the conventional total order and $\cdot$ is ordinary multiplication on the finite elements of

A, while $0 \cdot \infty = 0$ and $a \cdot \infty = \infty$ whenever $0 \neq a \in A$. In this structure, $0 \to a = \infty = a \to \infty$ for all $a \in A$, and $\infty \to a = 0$ unless $a = \infty$, while $a \to 0 = 0$ unless $a = 0$. For finite nonzero $a, b \in A$, the value of $a \to b$ is b/a if a divides b; otherwise it is 0. Substituting 2 for x and 8 for y, we find that the value of $(x \to (x \to y)) \to (x \to y)$ is $(2 \to 4) \to 4 = 2 \to 4 = 2$. So the corresponding value of δ is $2 \to (2 \to 2) = 2 \to 1 = 0$. Since $\mathbf{t} = 1 \not\leqslant 0$, it follows that δ is not a theorem of $\mathbf{R}$, and hence that (vii) is inadmissible in **BCIW**. Moreover, this argument can be extended to show that (viii) is inadmissible, because $(0 \to |2|) \to 2 = \infty \to 2 = 0$.

Finally, note that $x \to (\alpha \to x)$ is a theorem of **BCIW** (apply (C) to (W)). Therefore, the inadmissibility of (ix) and (x) follows from that of (vii) and (viii), using modus ponens. □

Remark 10.11. A problem of Avron [2] asks whether the rule

$$x,\, (x \to (y \to y)) \to (x \to y) \;/\; y$$

is admissible in **BCIW**. As Avron observes, it is admissible but not derivable in the $\to$ fragment of $\vdash_{\mathbf{RM}^{\mathbf{t}}}$ (see Example 8.9), which is stronger than **BCIW**. This explains why **BCIW** is not *hereditarily* structurally complete.

In the literature, the most prominent admissible rule of relevance logic is the underivable *disjunctive syllogism* $x,\, y \vee \neg\, x \;/\; y$, known as (γ). The admissibility of (γ) in $\mathbf{R}$ was proved in [42]. $\mathbf{R}$ is algebraizable, and $\mathsf{Alg}(\mathbf{R})$ is the variety of *De Morgan monoids* [1, 7]. These are distributive lattice-ordered Church monoids with an involution. In [43], there is a construction showing (in effect) that every subdirectly irreducible De Morgan monoid is a homomorphic image of a De Morgan monoid satisfying $(\mathbf{t} \leqslant x \;\&\; \mathbf{t} \leqslant y \vee \neg\, x) \implies \mathbf{t} \leqslant y$.

By Theorem 7.5, a fragment of relevance logic with negation cannot be overflow complete, because the existential positive sentence $\exists x\,(x = \neg\, x)$ holds in the 3–element De Morgan monoid and fails in the 2–element De Morgan monoid. On the other hand, $\mathbf{R}_{\cdot,\to,\mathbf{t}}$ and its fragments with $\to$ are vacuously overflow complete, as they have no admissible overflow rules. Indeed, Church monoids satisfy $|\mathbf{t}| = \mathbf{t} = \mathbf{t} \cdot \mathbf{t}$ and $||x|| = |x| = |x| \cdot |x|$, so all formulas in $\cdot, \to, \mathbf{t}$ [resp. $\cdot, \to$] become theorems of $\mathbf{R}$ under the substitution that sends all variables to $\mathbf{t}$ [resp. to $|x|$ for a fixed variable x].

11. Sequent Systems

Gentzen systems may be regarded as generalized sentential deductive systems: in the role of sentential formulas, we have suitably shaped sequents of formulas $\alpha_1, \dots, \alpha_m \rhd \beta_1, \dots, \beta_n$, with the understanding that such a sequent is sent by any substitution h to

$$h(\alpha_1), \dots, h(\alpha_m) \rhd h(\beta_1), \dots, h(\beta_n).$$

Sentential systems may then be identified with the Gentzen systems in which all permissible sequents have the shape $\rhd \varphi$. The [in]equational consequence relations of classes of [ordered] algebras are special Gentzen systems.

The Leibniz classification of sentential logics can be extended to Gentzen systems $\vdash$, provided that we generalize the matrix theory appropriately. The *designated* elements of a matrix $\langle \boldsymbol{A}, F\rangle$ (i.e., the elements of F) are formal sequents of elements of A, whose shapes are among those permitted by $\vdash$. The *Leibniz congruence* $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the largest congruence θ of $\boldsymbol{A}$ such that, whenever $a_1, \dots, a_m \rhd a_{m+1}, \dots, a_n \in F$ and $a_i \equiv_\theta b_i$ for $i = 1, \dots, n$, then $b_1, \dots, b_m \rhd b_{m+1}, \dots, b_n \in F$. Again, a matrix $\langle \boldsymbol{A}, F\rangle$ is *reduced* if $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the identity relation. Theorem 2.12 remains true in this setting. The $\boldsymbol{\Omega}$–characterizations of protoalgebraicity, truth equationality and [finite] equivalentiality can be retained as definitions; for order algebraizability, see [60]. The available model-theoretic characterizations remain valid, and the syntactic characterizations are modified in natural ways—see [56] and its references. Modulo these changes, the main results of Sections 4–9 remain true as well, because they make no essential use of special syntax, and rely mostly on $\boldsymbol{\Omega}$–properties instead. Since equational consequence relations are Gentzen systems, results about structurally complete classes of algebras (such as those in [4]) are encompassed in this unified setting.

Substructural Gentzen systems that enjoy cut elimination are typically at least order algebraizable, but their cut-free subsystems cannot even be assumed protoalgebraic. So the results of Sections 4–9 cannot be used to explain cut elimination, although Theorem 2.12 is still applicable. The reduced matrix models of cut-free systems are not easily isolated, however, and it seems difficult to extract the criteria of Theorem 2.12(iii) directly from algebraic proofs of cut elimination (such as the one in [3]).

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