A Very Brief Introduction to Standard Classical and Intuitionistic Dialogical Logic

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The present paper, that provides an introduction to standard dialogical logic, has mainly didactic purposes. Thus, the study of the metalogical properties has been left totally by side. However, two sections have been added for those readers willing to go through a technically more demanding material. The first one contains a technically rigorous presentation of standard dialogical logic the second one presents, some very recent work towards a logic with content, applied to modal logic, where modal logic is developed in a purely dialogical way: instead of worlds, we have contexts, constituted by hypothetical assertions and where transitions between worlds are understood as extending contexts by the means of questions.

1 Introduction:

Dialogical logic developed by Paul Lorenzen and Kuno Lorenz, was the result of a solution to some of the problems that arouse in Lorenzen' *Operative Logik* (1955).[1] We can not discuss here thoroughly the passage from the operative to the dialogical approach, though as pointed out by Peter Schroeder-Heister, the insights of Operative logic had lasting consequences in the literature on proof-theory and still deserve attention nowadays.[2] Moreover, the notion of *harmony* formulated by the antirealists and particularly by Dag Prawitz has been influenced by Lorenzen's notions of *admissibility*; *eliminability* and *inversion*. However, on my view, the dialogical tradition is rather a rupture than a continuation of the operative project and it might be confusing to start by linking conceptually both projects together.

Dialogical Logic was suggested at the end of the 1950s by Paul Lorenzen and then worked out by Kuno Lorenz. [3]Inspired by Wittgenstein's *meaning as use* the basic idea of the dialogical approach to logic is that the meaning of the logical constants is given by the norms

[1] Cf. Lorenz 2001.
[2] Schröder-Heister 2008.
[3] The main original papers are collected in Lorenzen/Lorenz (1978). For an historical overview see Lorenz (2001). Other papers have been collected more recently in Lorenz (2008, 2010a,b). A detailed account of recent developments since Rahman (1993), can be found in Rahman/Keiff (2005), Keiff (2009) and Rahman (2012). For the underlying metalogic see Clerbout (2013, 2014). For textbook presentations: Kamlah/Lorenzen (1972, 1984), Lorenzen/Schwemmer (1975), Redmond/Fontaine (2011) and Rückert (2011a). For the key role of dialogic in regaining the link between dialectics and logic, see Rahman/Keff (2010). Keiff (2004a,b, 2007) and Rahman (2009) study Modal Dialogical Logic. Fiutek et al. (2010) study the dialogical approach to belief revision. Clerbout/Gorisse/Rahman (2011) studied Jain Logic in the dialogical framework. Popek (2012) develops a dialogical reconstruction of medieval *obligationes*. Rahman/Tulenheimo (2009) study the links between GTS and Dialogical Logic. For other books see Redmond (2010) – on fiction and dialogic – Fontaine (2013) – on intentionality, fiction and dialogues – and Magnier (2013) – dynamic epistemic logic and legal reasoning in a dialogical framework.

or rules for their use. This feature of its underlying semantics quite often motivated the dialogical approach to be understood as a *pragmatist* semantics.[4]
The point is that those rules that fix meaning may be of more than one type, and that they determine the kind of reconstruction of an argumentative and/or linguistic practice that a certain kind of language games called dialogues provide. As mentioned above the dialogical approach to logic is not a logic but a semantic rule-based framework where different logics could developed, combined or compared. However, for the sake of simplicity and exemplification I will introduce only to the dialogical version of classical and intuitionist logics.

In a dialogue two parties argue about a thesis respecting certain fixed rules. The player that states the thesis is called Proponent (**P**), his rival, who puts into question the thesis is called Opponent (**O**). In its original form, dialogues were designed in such a way that each of the plays end after a finite number of moves with one player winning, while the other loses. Actions or moves in a dialogue are often understood as *utterances*[5] or as speech-acts[6]. The point is that the rules of the dialogue do not operate on expressions or sentences isolated from the act of uttering them.[7]. The rules are divided into particle rules or rules for logical constants (*Partikelregeln*) and structural rules (*Rahmenregeln*). The structural rules determine the general course of a dialogue game, whereas the particle rules regulate those moves (or utterances) that are requests (to the moves of a rival) and those moves that are answers (to the requests).

Crucial for the dialogical approach are the following points

1. The distinction between local (rules for logical constants) and global meaning (included in the structural rules)
2. The player independence of local meaning
3. The distinction between the play level (local winning or winning of a play) and the strategic level (existence of a winning strategy).
4. A notion of validity that amounts to winning strategy *independently of any model* instead of winning strategy for every model.
5. The notion of winning in a *formal play* instead of winning strategy in a model.

2. Dialogical Logic and Meaning

2.1 Local Meaning

Particle rules:
In dialogical logic, the particle rules are said to state the *local semantics*: what is at stake is only the request and the answer corresponding to the utterance of a given logical constant, rather than the whole context where the logical constant is embedded.

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[4] Quite often it has said that dialogical logics has a *pragmatic* approach to meaning. I concede that the terminology might be misleading and induce one to think that the theory of meaning involved in dialogic is not semantics at all. Helge Rückert proposes the more appropriate formulation *pragmatistische Semantik* (*pragmatist semantics*), perhaps even better*: pragmatist theory of meaning*.
[5] Cf. Rahman/Rückert 2001, 111 and Rückert 2001, chapter 1.2.
[6] Cf. Keiff 2007.
[7] Tulenheimo 2010.

- The standard terminology makes use of the terms *challenge* or *attack* and *defence*.[8] However let me point out that at the local level (the level of the particle rules) this terminology should be devoid of strategic underpinning.

- *Declarative utterances* involve the use of formulae, *interrogative utterances* do not involve the use of formulae

The following table displays the particle rules, where X and Y stand for any of the players **O** or **P**:

$\vee, \wedge, \rightarrow, \neg, \forall, \exists$	**Challenge**	**Defence**
X: $\alpha\vee\beta$	**Y**: ?-$\vee$	**X**: α *or* **X**: β (**X** chooses)
X: $\alpha\wedge\beta$	**Y**: ?$\wedge$1 or **Y**: ?$\wedge$2 (**Y** chooses)	**X**: α *respectively* **X**: β
X: $\alpha\rightarrow\beta$	**Y**: α (**Y** challenges by uttering α and requesting β)	**X**: β
X: $\neg\alpha$	**Y**: α	— (no defence available)
X: $\forall x\alpha$	**Y**: ?-$\forall x/k$ (**Y** chooses)	**X**: $\alpha\,[x/k]$
X: $\exists x\alpha$	**Y** ?$\exists$	**X**: $\alpha\,[x/k]$ (**X** chooses)

In the diagram, $\alpha[x/k]$ stands for the result of substituting the constant *k* for every occurrence of the variable *x* in the formula A.

One interesting way to look at the local meaning is as rendering an abstract view (on the semantics of the logical constant) that distinguishes between the following types of actions:

a) Choice of declarative utterances (=:disjunction and conjunction).
b) Choice of interrogative utterances involving individual constants (=: quantifiers).
c) Switch of the roles of defender and challenger (=: conditional and negation). As we will discuss later on we might draw a distinction between the switches involved in the local meaning of negation and the conditional).

Let us briefly mention two crucial issues to which we will come back later on

- **Player independence:** The particle rules are symmetric in the sense that they are player independent – that is why they are formulated with the help of variables for players. Compare with the rules of tableaux or sequent calculus that are asymmetric: one set of rules for the

[8] See Keiff 2007, Rahman/Clerbout/Keiff 2009. Tero Tulenheimo pointed out that this might lead the reader to think that already at the local level there are strategic features and that this contravenes a crucial feature of the dialogical framework. Indeed, Laurent Keiff [2007] introduced the terminology *requests* and *answers.* However the dialogical vocabulary has been established with the former choice and it would be perhaps confusing to change it once more.

*true(*left)-side other set of rules for the *false(*right)-side. The symmetry of the particle rules provides, as we will see below, the means to get rid of tonk-like-operators.

- **Sub-formula property**: If the local meaning of a particle # occurring in φ involves declarative utterances, these utterances must be constituted by sub-formulae of φ.[9]

2.2 Global Meaning

Structural rules:

(SR 0) (starting rule):
The initial formula is uttered by **P** (if possible). It provides the topic of the argumentation. Moves are alternately uttered by **P** and **O**. Each move that follows the initial formula is either a request or an answer.

> **Comment**: The proviso *if possible* relates to the utterance of elementary propositione. See formal rule *(SR 2)* below.

(SR 1) (no delaying tactics rule):
Both **P** and **O** may only make moves that change the situation.

Comments: This rule should assure that plays are finite (though there might an infinite number of them). There are several formulations of it with different advantages and disadvantages. The original formulation of Lorenz made use of ranks; other devices introduced explicit restrictions on repetitions. Ranks seem to be more compatible with the general aim of the dialogical approach of distinguishing between the play level and the strategic level. Other non-repetition rules seem to presuppose the strategic level. In fact, if we take meaning to be constituted by interaction we need a way to ensure finiteness of plays because there is no such thing as an infinite interaction. That is to say, finiteness is an essential property of interaction. The potential infinity of plays required by proofs involving infinite domains is accounted for at the level of strategies (see Clerbout (2013,2014). . Let us describe here the rule that implements the use of ranks.

- After the move that sets the thesis players **O** and **P** each choose a natural number n and m respectively (termed their repetition ranks). Thereafter the players move alternately, each move being a request or an answer.

- In the course of the dialogue, **O** (**P**) may attack or defend any single (token of an) utterance at most n (or m) times.

(SR 2) (formal rule):[10]
P may not utter an elementary proposition unless **O** uttered it first. Elementary propositione can not be challenged.

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[9] This has been pointed out by Laurent Keiff and by Helge Rückert in several communications.
[10] See a discussion of this rule below in the commentaries about the dialogical notion of validity

The dialogical framework is flexible enough to define the so-called *material dialogues*, that assume that elementary propositione have a fixed truth-value:

*(SR *2) (rule for material dialogues):*
Only elementary propositione standing for true propositions may be uttered. Elementary propositione
standing for false propositions can not be uttered.

(SR 3) (winning rule):
X wins iff it is Y's turn but he cannot move (either challenge or defend).

Global meaning
These rules determine the meaning of a formula where a particle occurs as a main operator in every possible play.

(SR 4i) (intuitionist rule):[11]
In any move, each player may challenge a (complex) formula uttered by his partner or he may defend himself against the last challenge that has not yet been defended.
or
(SR 4c) (classical rule):
In any move, each player may challenge a (complex) formula uttered by his partner or he may defend himself against any challenge (including those challenges that have already been defended once).

- Notice that the dialogical framework offers a fine-grained answer to the question: Are intuitionist and classical negation the same negations? Namely: The particle rules are the same but it is the global meaning that changes.

In the dialogical approach validity is defined via the notion of *winning strategy*, where winning strategy for X means that for any choice of moves by Y, X has at least one possible move at his disposal such that he (X) wins:

Validity (definition):
A formula is valid in a certain dialogical system iff **P** has a formal winning strategy for this formula.

Thus,

- α is classically valid if there is a winning strategy for **P** in the formal dialogue Dc(α).
- α is intuitionistically valid if there is a winning strategy for **P** in the formal dialogue Dint (α).

Comments on the dialogical notion of validity:

Helge Rückert pointed out, and rightly so, that the formal rule triggers a novel notion of validity. Validity, is not being understood as being true in every model, but as *having a winning strategy independently of any model* or more generally independently of any

[11] In the standard literature on dialogues, there is an asymmetric version of the intuitionist rule, called E-rule since Felscher [1985]. For a discussion of this see appendices 1 and 2.

material grounding claim (such as truth or justification). The copy-cat strategy implicit in the formal rule is not copy cat of groundings but copy-cat of declarative utterances involving elementary propositione. Moreover one should add that there is the notion of *formal play*, that does not seem to correspond to nothing in model-theoretic approach: a formal play is not playing in a model.

In fact, one could see the formal rule as a process the first stage of which starts with what Laurent Keiff called *contentious* dialogues.[12] Contentious dialogues are dialogues where a player X utters one or more elementary propositione that are dependent upon a given ground and X is not prepared to put this ground into question – one can think of it as a claim of having some kind of ground (or a claim of truth) for it. Moreover, the antagonist is willing to concede this ground for the sake of the argument.[13] Now, if we would like to avoid to have the result that an elementary proposition is true by the only reason that the player X stated it – that is, if we want to find a way out of contentious dialogues, then there are two possible ways:

- either we accept some principle of grounding external to the dialogue itself (and thus external to the interaction of the players)
  or
- we look for a player principle of grounding that is internal to the dialogue and dependent on the interaction of the players.

The first ways leads to material dialogues the second to formal ones

If we are willing to accept something like *material truth*, then we can think that the grounds upon which the elementary propositione depend are facts of the world and a grounded elementary propositione is a way to say that it is true. However, something more general might be thought too, such as true in virtue of some players' independent ground. This is the basis on which the rule for material dialogues has been formulated

Rückert pointed out that the formal rule establishes a kind of game where one of the players must play without knowing what the antagonist's justifications of the elementary propositione are. Thus, according to this view, the passage to formal dialogues relates to the switch to some kind of games with incomplete information. Now, if the ultimate grounds of a dialogical thesis are elementary propositione and if this is implemented by the use of a formal rule, then the dialogues are in this sense necessarily asymmetric. Indeed, if both contenders were restricted by the formal rule no elementary proposition can ever be uttered. Thus, we implement the formal rule by designing one player, called the *proponent*, whose utterances of elementary propositione are, at least, at the start of the dialogue restricted by this rule.

Apparently, the formal rule introduces an asymmetry in relation to the commitments of **O** and **P** particularly so in the case of the utterance of the conditional. Indeed, if **O** utters a conditional, then **P**' challenge commits him to a declarative utterance that must at the end be based on atomic moves of **O**. If it is **O** that challenges a conditional no such commitment will be triggered. But it would be a mistake to draw from this fact the conclusion that the local meaning of the conditional is not symmetric. The very point of

[12] Cf. Clerbout/Keiff/Rahman 209 and in Keiff/Rahman 2010.
[13] Cf. Keiff/Rahman 2010 (156-157), where this is linked to some specific passages of Plato's *Gorgias* (472b-c).

player independence is that it is a property of the meaning of the logical particles not of the dialogue as a whole where **P** is committed to a thesis. Furthermore the asymmetry of the winning strategy is triggered by the semantic asymmetry of the formal rule. It is the possibility to isolate local meaning from validity commitments that allows dialogicians to speak of the symmetry of the logical constants and this prevents tonk-like operators from being introduced in the dialogical framework.

3 Examples

In the following examples, the outer columns indicate the numerical label of the move, the inner columns state the number of a move targeted by an attack. Expressions are not listed following the order of the moves, but writing the defence on the same line as the corresponding attack, thus showing when a round is closed. Recall, from the particle rules, that the sign "—" signalises that there is no defence against the attack on a negation.

For the sake of a simpler notation we will not record in the dialogue the rank choices but assume the uniform rank **O**: n=1 **P**: m=2:

Ex. 1: Classical and intuitionistic rules

In the following dialogue played with classical structural rules **P**' move 4 answers **O**'s challenge in move 1, since **P**, according to the classical rule, is allowed to defend (once more) himself from the challenge in move 1. **P** states his defence in move 4 though, actually **O** did not repeat his challenge – we signalise this fact by inscribing the not repeated challenge between square brackets.

O			**P**		
				$p \vee \neg p$	0
1	$?_{\vee}$	0		$\neg p$	2
3	p	2		—	
[1]	[$?_{\vee}$]	[0]		p	4

Classical rules. **P** wins.

In the dialogue displayed below about the same thesis as before, **O** wins according to the intuitionistic structural rules because, after the challenger's last attack in move 3, **t**he intuitionist structural rule forbids **P** to defend himself (once more) from the challenge in move 1.

O			**P**		
				$p \vee \neg p$	0
1	$?_{\vee}$	0		$\neg p$	2
3	p	2		—	

Intuitionist rules. **O** wins.

Ex. 2: The following example shows that P wins double negation if he plays with the classical rule but loses if it is the intuitionistic one

	O			P	
				¬¬p→p	0
1	¬¬p	0		p	4
	-		1	¬p	2
3	p	2		-	

Classical rules. **P** wins

P will not win with the intuitionistic rule since he must answer to last challenge. The defensive move 4 is then forbidden since the last challenge was move 3 and 4 answers the challenge brought forward by **O** in his move 1:

	O			P	
				¬¬p→p	0
1	¬¬p	0			
	-		1	¬p	2
3	p	2		-	

Intuitionistic rules.O wins

P cannot win since **O** challenges with move 3 and P has no more legal move at his disposal. O wins since he has the **last word.**

Ex. 3:
In the following example the Proponent can win the double negation of the third excluded, despite the fact that he is playing with intuitionistic rules. Recall – form the preceding example - that intuitionist logic double negation **is not** equivalent as the positive version of expression. Here P makes use of the repetition rank 2. That he is allowed to challenge twice the same move of his antagonist.

	O			P	
				¬¬(p∨¬p)	0
1	¬(p∨¬p)	0		-	
	-		1	p∨¬p	2
3	?-∨	2		¬p	4
5	p	4		-	
			1	p∨¬p	6
7	?-∨	6		p ☺	8

Ex. 4:

Similar applies to the following

	O			P	
				¬¬(¬¬p→p)	0
1	¬(¬¬p→p)	0		-	
	-		1	¬¬p→p	2
3	¬¬p	2			
			3	¬p	4
5	p	4		-	
			1	¬¬p→p	6
7	¬¬p	6		p ☺	8

In the following example. we separated two branches or subdialogues only because of didactic aims not because this separation is intrinsical to the development of a dialogue . Each of the branches is motivated by a possible choice of **O**. In one subdialogue he chooses to defend himself by bringing forward the left part of the disjunction (it is his choice anyway) and in the other the right one. However, he looses in both. In fact these plays, build what Clerbout (2013,2014) calls the core of the winning strategy. That is **P** wins independently of O's choices and that is why these plays build the core of the a proof of the validity of the thesis brought forward by P. Notice that **P** will win irrespectively if intuitionistic or classical rules are used.

Ex. 5:

	O			P	
				[(p∨q)∧¬p]→q	0
1	[(p∨q)∧¬p]	0			
3	¬p		1	?-$\wedge_2$	2
5	p∨q		1	?-$\wedge_1$	4
			5	?-∨	6

	O			P	
				[(p∨q)∧¬p]→q	0
1	[(p∨q)∧¬p]	0		q☺	8'
3	¬p		1	?-$\wedge_2$	2
5	p∨q		1	?-$\wedge_1$	4
7'	q		5	?-∨	6

	O			P	
				[(p∨q)∧¬p]→q	0
1	[(p∨q)∧¬p]	0			
3	¬p		1	?-$\wedge_2$	2
5	p∨q		1	?-$\wedge_1$	4

7	p		5	?-∨	6
			3	p☺	8

In the following examples we leave the reader to check if the thesis is winnable by P with both intuitionistic and classical rules

Ex. 6:

	O			P	
				∃x (Px∨Qx) →∃x (Px∨Qx)	0
1	∃x (Px∨Qx)	0		∃x (Px∨Qx)	2
3	?-∃	2		Pki∨Qki	6
5	Pki ∨Qki		1	?-∃	4

Branch 1

	O			P	
				∃x (Px∨Qx) →∃x (Px∨Qx)	0
1	∃x (Px∨Qx)	0		∃x (Px∨Qx)	2
3	?-∃	2		Pki ∨Qki	6
5	Pki∨Qki		1	?-∃	4
7	?-∨	6		Qki ☺	10
9	Qki		5	?-∨	8

Branch 2

	O			P	
				∃x(Px∨Qx) →∃x(Px∨Qx)	0
1	∃x(Px∨Qx)	0		∃x(Px∨Qx)	2
3	?-∃	2		Pki ∨Qki	6
5	Pki ∨Qki		1	?-∃	4
7	?-∨	6		Pki ☺	10
9	Pki		5	?-∨	8

Ex. 7:

	O			P	
				∀x(Ac∨Bc) → (∀xAx∨∀xBx)	0
1	∀x(Ax∨Bx)	0		(∀xAx∨∀xBx)	2
3	?-∨	2		∀xAx	4
5	?-ki	4			
7	Aki∨Bki		1	?-ki	6
			7	? ∨	8
		Branch 1			
				∀x(Ac∨Bc) → (∀xAx∨∀xBx)	0
1	∀x(Ax∨Bx)	0		(∀xAx∨∀xBx)	2
3	?-∨	2		∀xAx	4

5	?-ki	4		Aki	10
7	Aki∨Bki		1	?-ki	6
9	Aki		7	? ∨	8
			Branch 2		
				∀x(Ac∨Bc) → (∀xAx∨∀xBx)	0
1	∀x(Ax∨Bx)	0		(∀xAx∨∀xBx)	2
3	?-∨	2		∀xAx	4
5	?-ki	4			
7	Aki∨Bki		1	?-ki	6
9	Bki		7	? ∨	8
				∀xBx	10
11	?-kj	10			

Ex. 8:

	O			P	
				∀x(∀xAx→Ax)→∀xAx	0
1	∀x(∀xAx→Ax)	0		∀xAx	2
3	?-ki	2			
5	∀xAx→Aki		1	?-ki	4
			5	∀xAx	6

Option 1:

Branch 1

	O			P	
				∀x(∀xAx→Ax)→∀xAx	0
1	∀x(∀xAx→Ax)	0		∀xAx	2
3	?-ki	2		Aki☺	8
5	∀xAx→Aki		1	?-ki	4
7	Aki		5	∀xAx	6

Branch 2

	O			P	
				∀x(∀xAx→Ax)→∀xAx	0
1	∀x(∀xAx→Ax)	0		∀xAx	2
3	?-ki	2			
5	∀xAx→Aki		1	?-ki	4
			5	∀xAx	6
7	?-kj	6			
9	∀xAx→Akj		1	?-kj	8
			9	∀xAx	10
11	?-kz	10			

			1	?-kz	12
				→∞	

Option 2:

Branch 1

	O			P	
				∀x(∀xAx→Ax)→∀xAx	0
1	∀x(∀xAx→Ax)	0		∀xAx	2
3	?-ki	2		Aki☺	8
5	∀xAx→Aki		1	?-ki	4
7	Aki		5	∀xAx	6

Branch 2

	O			P	
				∀x(∀xAx→Ax) → ∀xAx	0
1	∀x(∀xAx→Ax)	0		∀xAx	2
3	?-ki	2			
5	∀xAx→Aki		1	?-ki	4
			5	∀xAx	6
7	?-ki	6			
9	∀xAx→Aki			?-ki	8
			9	∀xAx	10
11	?-ki	10			

Ex. 9:

	O			P	
				¬∀x∃xAxy	0
1	∀x∃xAxy	0			
3	∃xAxki		1	?-ki	2
			3	?-∃	4

Branch 1

	O			P	
				¬∀x∃xAxy	0
1	∀x∃xAxy	0			
3	∃xAxki		1	?-ki	2
5	Akjki		3	?- ∃	4
7	∃xAxkj		1	?-kj	6
9	Akzkj		7	?-∃	8
			1		

The rank 2 forbids P to start again with a new challenge. However it should be clear that even if the rank would be higher O will always win. The second following branch that the clever opponent can finish the affair wihout involveing higher ranks. It is sufficient for him to chose the same individual constant. Thus, unless it has been conceded that A,xy is reflexive before starting the play O wins (in this branch) by making the choice of not changing the individual constant introduced by P. .

Branch 2

	O			P	
				¬∀x∃xAxy	0
1	∀x∃xAxy	0			
3	∃xAxki		1	?-ki	2
5	Akiki		3	?-∃	4

4 A Condensed Technical Presentation of Standard Dialogical Logic[14]

Let L be a first-order language built as usual upon the propositional connectives, the quantifiers, a denumerable set of individual variables, a denumerable set of individual constants and a denumerable set of predicate symbols (each with a fixed arity).

We extend the language L with two labels **O** and **P**, standing for the players of the game, and the question mark `?'. When the identity of the player does not matter, we use variables **X** or **Y** (with **X**≠**Y**). A *move* is an expression of the form `**X**-*e*', where *e* is either a formula φ of L or the form `?[φ_1,..., φ_n]'.

We now present the rules of dialogical games. There are two distinct kinds of rules named particle (or local) rules and structural rules. We start with the particle rules.

Previous move	**X**-φ∧ψ	**X**-φ∨ψ	**X**-φ→ψ	**X**-¬φ
Challenge	**Y**-?[φ] or **Y**-?[ψ]	**Y**-?[φ,ψ]	**Y**-φ	**Y**-φ
Defence	**X**-φ resp. **X**-ψ	**X**-φ or **X**-ψ	**X**-ψ	– –

Previous move	**X**-∀xφ	**X**-∃xφ
Challenge	**Y**-?[φ(a/x)]	**Y**-?[φ(a_1/x),...,φ(a_n/x)]
Defence	**X**-φ(a/x)	**X**-φ(a_i/x) with 1≤ i ≤n

In this table, the a_is are individual constants and φ(a_i/x) denotes the formula obtained by replacing every occurrence of x in φ by a_i. When a move consists in a question of the form `?[φ_1,...,φ_n]', the other player chooses one formula among φ_1,...,φ_n and plays it. We can thus distinguish between conjunction and disjunction on the one hand, and universal and existential quantification on the other hand, in terms of which player has a choice. In the cases of conjunction and universal quantification, the challenger chooses which formula he asks for. Conversely, in the cases of disjunction and existential quantification, the defender is the one

[14] The following brief presentation of standard dialogical logic has been extracted from Nicolas Clerbout (2013, 2014).

who can choose between various formulas. Notice that there is no defence in the particle rule for negation.

Particle rules provide an abstract description of how the game can proceed locally: they specify the way a formula can be challenged and defended according to its main logical constant. In this way we say that these rules govern the local level of meaning. Strictly speaking, the expressions occurring in the table above are not actual moves because they feature formulas schemata and the players are not specified. Moreover, these rules are indifferent to any particular situations that might occur during the game. For these reasons we say that the description provided by the particle rules is abstract. The words "challenge" and "defence" are convenient to name certain moves according to their relationship with other moves. Such relationships can be precisely defined in the following way. Let Σ be a sequence of moves. The function p_Σ assigns a position to each move in Σ, starting with 0. The function F_Σ assigns a pair $[m,Z]$ to certain moves N in Σ, where m denotes a position smaller than $p_\Sigma(N)$ and Z is either C or D, standing respectively for "challenge" and "defence". That is, the function F_Σ keeps track of the relations of challenge and defence as they are given by the particle rules. A *play* (or dialogue) is a legal sequence of moves, i.e., a sequence of moves which observes the game rules. The rules of the second kind that we mentioned, the structural rules, give the precise conditions under which a given sentence is a play. The *dialogical game* for φ, written $\mathsf{D}(\varphi)$, is the set of all plays with φ as the thesis (see the Starting rule below). The structural rules are the following:

SR0 (Starting rule) Let φ be a complex formula of L. For every $\pi\in\mathsf{D}(\varphi)$ we have:

- $p_\pi(\mathbf{P}\text{-}\varphi)=0$,
- $p_\pi(\mathbf{O}\text{-n}:=i)=1$,
- $p_\pi(\mathbf{P}\text{-m}:=j)=2$

In other words, any play π in $\mathsf{D}(\varphi)$ starts with $\mathbf{P}$-φ. We call φ the *thesis* of the play and of the dialogical game. After that, the Opponent and the Proponent successively choose a positive integer called *repetition rank*. The role of these integers is to ensure that every play ends after finitely many moves, in a way specified by the next structural rule.

SR1 (Classical game-playing rule)

- Let $\pi\in\mathsf{D}(\varphi)$. For every M in π with $p_\pi(M)>2$ we have $F_\pi(M)=[m',Z]$ with $m'<p_\pi(M)$ and $Z\in\{C,D\}$
- Let r be the repetition rank of player $\mathbf{X}$ and $\pi\in\mathsf{D}(\varphi)$ such that
  - the last member of π is a $\mathbf{Y}$ move,
    - M_0 is a $\mathbf{Y}$ move of position m_0 in π,
    - $M_1,\ldots,M_n$ are $\mathbf{X}$ moves in π such that $F_\pi(M_1)=\ldots=F_\pi(M_n)=[m_0,Z]$.

  Consider the sequence[15] $\pi'=\pi*N$ where N is an $\mathbf{X}$ move such that $F_{\pi'}(N)=[m_0,Z]$. We have $\pi'\in\mathsf{D}(\varphi)$ only if $n < r$.

The first part of the rule states that every move after the choice of repetition ranks is either a challenge or a defence. The second part ensures finiteness of plays by setting the player's repetition rank as the maximum number of times he can challenge or defend against a given move of the other player.

15 We use $\pi*N$ to denote the sequence obtained by adding move N to the play π.

SR2 (Formal rule) Let ψ be an elementary sentence, N be the move **P**- ψ and M be the move **O**-ψ . A sequence π of moves is a play only if we have: if $N\in\pi$ then $M\in\pi$ and $p_\pi(M)<p_\pi(N)$.

A play is called *terminal* when it cannot be extended by further moves in compliance with the rules. We say it is **X** terminal when the last move in the play is an **X** move.

SR3 (Winning rule) Player **X** wins the play π only if it is **X** terminal.

Consider for example the following sequences of moves: **P**-Qa→Qa, **O**-n:=1, **P**-m:=12, **O**-Qa, **P**-Qa

. We often use a convenient table notation for plays. For example, we can write this play as follows:

	O			P	
				Qa→Qa	0
1	n:=1			m:=12	2
3	Qa	(0)		Qa	4

The numbers in the external columns are the positions of the moves in the play. When a move is a challenge, the position of the challenged move is indicated in the internal columns, as with move 3 in this example. Notice that such tables carry the information given by the functions *p* and *F* in addition to represent the play itself.

However, when we want to consider several plays together – or example when building a strategy - such tables are not that perspicuous. So we do not use them to deal with dialogical games for which we prefer another perspective. The *extensive form* of the dialogical game $\mathsf{D}(\varphi)$ is simply the tree representation of it, also often called the game-tree. More precisely, the extensive form E_φ of $\mathsf{D}(\varphi)$ is the tree (T,l,S) such that:

i) Every node t in T is labelled with a move occurring in $\mathsf{D}(\varphi)$

ii) $l: T\rightarrow \mathrm{N}$

iii) $S\subseteq T^2$ with:

- There is a unique t_0 (the root) in T such that $l(t_0)=0$, and t_0 is labelled with the thesis of the game.
- For every $t\neq t_0$ there is a unique t' such that $t'St$.
- For every t and t' in T, if tSt' then $l(t')=l(t)+1$.
- Given a play π in $\mathsf{D}(\varphi)$ such that $p_\pi(M')=p_\pi(M)+1$ and t, t' respectively labelled with M and M', then tSt'.

A *strategy* for Player **X** in $\mathsf{D}(\varphi)$ is a function which assigns an **X** move M to every non terminal play π with a **Y** move as last member such that extending π with M results in a play. An **X** strategy is *winning* if playing according to it leads to **X**'s victory no matter how **Y** plays.

A strategy can be considered from the viewpoint of extensive forms: the extensive form of an **X** strategy σ in $\mathsf{D}(\varphi)$ is the tree-fragment $\mathsf{E}_{\varphi,\sigma}=(T_\sigma, l_\sigma, S_\sigma)$ of E_φ such that:

i) The root of $\mathsf{E}_{\varphi,\sigma}$ is the root of E_φ.

ii) Given a node t in E_φ labelled with an **X** move, we have that $tS_\sigma t'$ whenever tSt'.

iii) Given a node t in E_φ labelled with a **Y** move and with at least one t' such that tSt', then there is a unique $\sigma(t)$ in T_σ where $tS_\sigma\sigma(t)$ and $\sigma(t)$ is labelled with the **X** move prescribed by σ.

Here are some examples of results which pertain to the level of strategies.[16]

- Winning **P** strategies and leaves. *Let* w *be a winning **P** strategy in* $\mathsf{D}(\varphi)$. *Then every leaf in* $\mathsf{E}_{\varphi,\mathrm{w}}$ *is labelled with a **P** signed atomic sentence.*
- Determinacy. *There is a winning **X** strategy in* $\mathsf{D}(\varphi)$ *if and only if there is no winning **Y** strategy in* $\mathsf{D}(\varphi)$.
- Soundness and Completeness of Tableaux. *Consider first-order tableaux and first-order dialogical games. There is a tableau proof for* φ *if and only if there is a winning **P** strategy in* $\mathsf{D}(\varphi)$.

  By soundness and completeness of the tableau method with respect to model-theoretical semantics, it follows that existence of a winning **P** strategy coincides with validity: *There is a winning **P** strategy in* $\mathsf{D}(\varphi)$ *if and only if* φ *is valid.*

EXAMPLES OF EXTENSIVE FORMS

Extensive forms of dialogical games and of strategies are infinitely generated trees (trees with infinitely many branches). Thus it is not possible to actually write them down. But an illustration remains helpful, so we add Figures 1 and 2 below.

Figure 1 partially represents the extensive form of the dialogical game for the formula $\forall x(Q(x) \rightarrow Q(x))$. Every play in this game is represented as a branch in the extensive form: we have given an example where the leftmost branch represents one of the simplest and shortest plays in the game. The root of the extensive form is labelled with the thesis. After that, the Opponent has infinitely many possible choices for her repetition rank: this is represented by the root's having infinitely many immediate successors in the extensive form. The same goes for the Proponent's repetition rank, and for every time a player is to choose an individual constant.

[16] These results are proven, together with others, in Clerbout (2013).

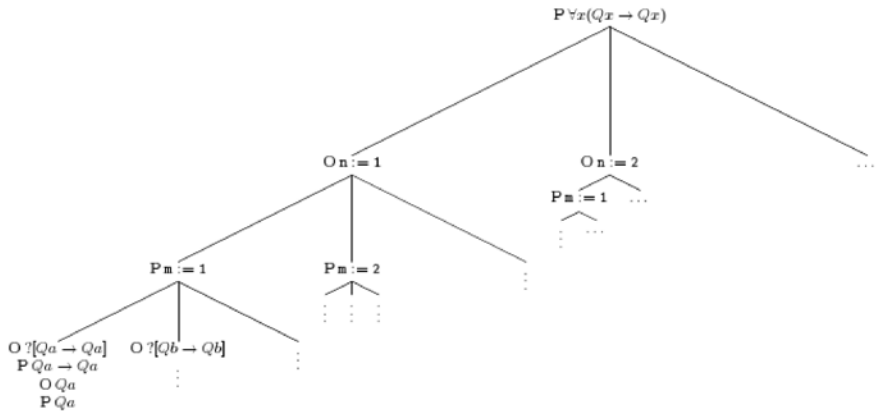

Figure 1.

Figure 2 partially represents the extensive form of a strategy for the Proponent in this game. It is a fragment of the tree of Figure 1 where each node labelled with an **O**-move has at most one successor. We do not keep track of all the possible choices for **P** any more: every time the Proponent has a choice in the game, the strategy selects exactly one of the possible moves. But since all the possible ways for the Opponent to play must be taken into account by a strategy, the other ramifications are kept. In our example, the strategy prescribes choosing the same repetition rank as the Opponent. Of course there are infinitely many other strategies available for **P**.

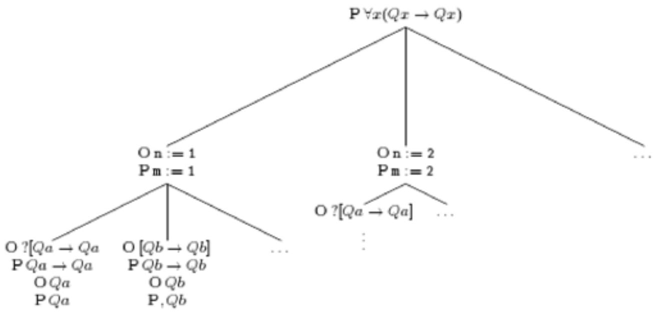

Figure 2.

5 A dialogical approach to Modal Logic: Worlds as hypotheses[17]

In fact there is already some work on dialogical modal logic (such as Rahman/Rückert (2001), Redmond/Fontaine (2011), Clerbout (2013a)). However, this approach that makes use of labels is too close to the model-theoretical approach of Kripke's modal logic: in such a setting, labels are names of model-theoretic worlds after all. What we need is a conception where worlds are introduced at the object-language level and have genuine dialogical rather than metaphysical nature. Fortunately, A. Ranta (1991, 1994) provided one half of the task by developing a CTT version of possible world-semantics. What we need now is to combine it with the dialogical approach to CTT recently developed (see Appendix).

Let us start with the CTT approach to possible worlds as developed by Ranta (1994). The main idea of Ranta is that an assertion relativized to a possible world W amounts to a hypothetical where that assertion is brought forward provided the hypothesis W and this is expressed at the level of the object-language. In other words, the modal assertions are reduced to hypotheticals. Thus, one might say that, in some way, Leibniz's (metaphysical) notion of possible world is placed into Kant's conception of hypotheticals. Thus, what we need is not

[17] Extracted From Rahman/Redmond (2014).

labels for worlds but assertions that are brought forward during a play provided some open assumptions. Accordingly, hypotheses take the place of worlds and arbitrary elements (variables) of the hypotheses the proposition is dependent on, take the place of world-labels.

More generally, and independently of the dialogical setting, this yields the following correspondences - pointed out by Ranta (1991, 1983):

A : set in W	means	A(x) : set (x : W)
A = B : set in W	means	A(x) = B(x) : set (x : W)
a : A in W	means	a(x) : A(x) (x : W)
a = b: A in W	means	a(x) = b(x): A(x) (x : W)

The CTT take on hypotheses is that when making the hypothesis A, we consider that there is an arbitrary proof of A. We use a variable *x* for such a proof in the same way as we make use of variable when considering an arbitrary element of a set. In the case of the dialogical approach to CTT, proof-objects only occur at the level of strategies, however, at the play level, play-objects provide the ontology suitable for that level. More precisely, while bare play-objects provide the ontology of categorical moves, functions provide the play-objects of hypotheticals.

Let us now make use of the translation proposed by Ranta in order to formulate in the context of dialogical logic the local meaning of quantifiers brought forward by hypothetical moves – and so doing, introduce its modal structure at the level of the object-language:

In order to avoid a heavy notation we will not write the world variable in proposition but we will assume that it occurs free in in it; for instance the expression $b(y) : (\exists x : A)\varphi\ (y : W)$ should be read as $b(y) : [(\exists x : A)\varphi](y)\ (y : W)$:

Posit	Challenge	Defence
	Y $?_{prop}$	**X** $(\exists x : A)\varphi$: prop, A, W : set
X $b(y) : (\exists x : A)\varphi\ (y : W)$	**Y** $?_L$ Or **Y** $?_R$ **[the challenger has the choice]**	**X** $L^{\exists}(b(y)) : A\ (y : W)$ Respectively **X** $R^{\exists}(b(y)) : \varphi(L(b(y)))\ (y : W)$
	Y $?_{prop}$	**X** $(\forall x : A)\varphi$: prop, A, W : set
X $b(y) : (\forall x : A)\varphi\ (y : W)$	**Y** $L^{\forall}(b(y)) : A\ \ (y : W)$	**X** $R^{\forall}(b(y)) : \varphi(L(p(y))))\varphi\ (y : W)$

The quantification that results from these rules corresponds to what in standard modal logic is known as *actualist quantification* – i.e., the scope of the quantifiers is circumscribed by the objects that "inhabit" W. In our setting this means that the players must choose for their moves the suitable hypothetical objects, i.e. those functions defined on the hypothesis W. This yields a new approach to free logic in general and of dialogical free logic in particular: One can infer *Something is a vampire* from *Nosferatu is a vampire* only if the existential is relativized to the same world W, upon which *Nosferatu* is dependent. We cannot, for example, infer $b(y) : (\exists x : A)\ Bx(y)\ \ (y : W)$, neither from the hypothetical $a(z) : Bk(z)\ (z : V)$ nor from the categorical expression a : Bk. From the dialogical perspective; the point is that because of the formal rule, in order to win **P** can only choose those play objects that **O** has chosen before. Thus, since during the plays involving our examples, **O** brings forward $a(z) : Bk(z)\ (z : V)$ (in the first example) and a : Bk, in the second, **P** will not be able to make a suitable choice that yields a copy-cat of those moves of **O** – the closest move that P can obtain is $a(y) : Bk(y)\ (y : W)$ – that will yield a **P**-win for the first example iff V=W and a win in the second examples iff there is no (more) hypothesis. Moreover, the following tautological hypothetical implications are valid (there is winning strategy for P) : $b(y) : Bk \rightarrow (\exists x : A)\ Bx\ \ (y : W)$ and $b(y) : (\forall x : A)Bx\ \ \rightarrow Bk\ (y : W)$ only if the formation rules for Bk : prop presuppose a set such that this set is dependent upon W.

However we have not still tackled the following questions:

What corresponds to the accessibility relation of standard modal logic?

What is a set W?

How do we understand fictional objects as *Holme*s? The answer to these questions takes us to the next sections.

Epistemic alternatives, Accessibility and the Dialogical perspective. [18]

One way to see the relation between a world W_1 and world W_2 is to see it as an epistemic alternative, where W_2 is an extension of W_1 in the sense that W_2 adds information in such a way that every proposition that is true under the hypothesis W_1 is also true under the hypothesis W_2. More generally we express this situation in the following way: d(y) : W_1 (y : W_2). Thus, if W_2 is accessible from W_1, then there is a function **f** from W_2 to W_1 - see Ranta (1994, p. 147). But certainly there might many such functions, that express not only that W_2 from W_1 but also that, say, V and U are accessible too from W_1, though W_2, V, U are not accessible between them. The whole yields a tree structure with W_1 as root. In the dialogical frame the point is that if it is the case that a(b(y)) : A (b(y) : W (y : V)), and a(x)) : A (x : W), then players can bring forward b(y) : A (y : V). Interesting is the fact that one way to understand a(b(y)) is the variant of the object a(x) when 'transferred' to the world V. If the language contains modal operators, the relation between W and V, can also be brought dynamically into the play by the choices of the players as usual in dialogical modal logic – see Rahman/Rückert (2001), however, for the purpose of the study of hypothetical objects such operators are not needed.

Worlds as Contexts

Still we do not know exactly what the sets W are and in what sense they can be seen as expressing the idea of "possible world". Recall that in this context each world W, V, U is a set -, and this set is a hypothesis. What we need to elucidate is what are the elements of this set and in what sense we can say that is captures the idea of *possible*. Let us start with the latter. From the epistemic point of view possible means, different alternatives of adding knowledge provided full knowledge is not achieved. A. Ranta (1991, p. 78) links this notion of possibility with Husserl's (Cart. Med., p. 62) conception of different ways of completing what I know. Here we are then, in such a frame, possible means an alternative ways of completing not yet achieved full knowledge. Moreover, this means that possible is always an approximation to full knowledge: if the approximation were to end, then possibility will not be any more there, but full knowledge. Possible is that what always can be completed. But how to express this notion formally and how to link it with the dialogical approach? Formally speaking, a possible world is itself a set constituted by a sequence of hypothetical assertions with a dependence defined between them (this structure is called a *context*). Let us call Γ the sequence that approximates a world, then we have

a : A in Γ means $a(x_1, \ldots, x_n) : A(x_1, \ldots, x_n)\ (x_1 : A_1, \ldots, x_n : A_n(x_1 \ldots, x_{n-1}))$

and similar applies to: A : set in Γ, A = B : set in Γ, and a = b : A in Γ

As mentioned above, if contexts should capture the notion of possible world, it is important that the approximations never end. Thus, as pointed out by Ranta (1991, p. 93) worlds are a kind of **limits** of sequences of hypothetical assertions further and further specified – without ever reaching full specification. In fact there are two ways to extend a context by adding information to it, and one by reducing uncertainty into it, namely:

[18] For the CTT approach to the notion of possible worlds in the present and the following sections we follow Ranta (1991, 1994).

(1) By adding a further hypothesis. It may happen that the new proposition, say, $A(x_1, \ldots, x_n)$ be potentially true in Γ (that is, when the new proposition can be inferred from the original context – then there in growth only on actual knowledge), but it could also add a new information

(2) By introducing a definition for some variables and achieving in this way a reduction of uncertainty. For instance, the context

$\Gamma = (x_1 : A_1, \ldots x_n : A_n)$ is extended to the extension

$\Gamma, x_k = c : A_k$

So that in the new context every occurrence of x_k is substituted by *a*. The new context is obtained from *Γ* by removing the hypothesis $x_k : A_k$ by $c(x_1 \ldots x_n)$. Thus the new context is shorter than the original. Still, this operation furnishes not only the knowledge of the original context but the value of the one variable reduces the uncertainty within the context.

(3) By presenting a new context Δ such that there is function f from Δ to Γ as explained in II.2.1a

As observed by Ranta (1994, p. 146-47) 1) and 2) can be seen as special cases of 3). Thus (3) provides the most general case of extension and, at the same time deploys the constructive meaning of the notion of accessibility. Dialogically speaking, extensions of contexts are to be understood as answers to questions of specification – recall that we are dealing with a fully interpreted language. Assume that one player brings forward the hypothetical that there is a play object *for A(y), provided x is a living being, y is a human(x).* Then the first kind of extension will the triggered by a question such as is *(s)he European or Asian?*. The second kind of extension will be triggered by a *wh-question* (that is a *who*, *what* or *when* question. The third kind of extension can thought as asking the defender to establish a link between the variables between the first and the new context. Assume for instance that the initial context Γ contains the disjunction $A \vee B$ the hypothetical play object of which is the variable x. Assume further on that the new context Δ contains y : A. In such a case the player that claims that Δ is an extension of Γ must produce the definition $L^{\vee}(x) = y : A \vee B$, and similar must happen in relation to every component of Γ. Summing up, from the dialogical perspective modality can be seen as a dialogue where moves involved questions and answers in relation to underlying contexts.

Appendix III The Dialogical Approach to CTT[19]

The Formation of Propositions

Before delving into the details about play-objects, let us first discuss the issue of the formation of expressions, and in particular of propositions, in the context of dialogical logic. In standard dialogical systems there is a presupposition that the players use well-formed formulas (wff's). One can check well-formedness at will, but only via the usual metareasoning by which one checks that the formula indeed observes the definition of wff. The first addendum we want to make is to allow players to question the status of expressions, in particular to question the status of something as actually standing for a proposition. Thus we start with rules giving a dialogical explanation of the *formation* of propositions. These are local rules added to the particle rules which give the local meaning of logical constants (see next section).

Let us make a remark before displaying the formation rules. Because the dialogical theory of meaning is based on argumentative interaction, dialogues feature expressions which are not posits or sentences. They also feature requests used as challenges, as illustrated by the formation rules below and the particle rules in the next section. Now, by the *No entity without type* principle, the type of these actions, which type we shall write “formation-

[19] The present overview on the dialogical approach to CTT is based on Rahman/Clerbout (2013, 2014).

request”, should be specified during a dialogue

Posit	**Challenge** [when different challenges are possible, the challenger chooses]	**Defence**
X ! Γ : set	**Y** $?_c\Gamma$, or	**X** ! $a_1 : \Gamma$, **X** ! $a_2 : \Gamma$, … **X** gives the canonical elements of Γ; provides a generation method $a_i : \Gamma \Rightarrow a_j : \Gamma$, provides the equality rules
X ! φ ∨ ψ : prop (similar applies for the rest of the propostional connectives)	**Y** $?_{F\vee 1}$ or **Y** $?_{F\vee 2}$	**X** ! φ : prop **X** ! ψ : prop
X ! (∀*x* : *A*) φ(*x*) : prop	**Y** $?_{F\forall 1}$ or **Y** $?_{F\forall 2}$	**X** ! *A* : set **X** ! φ(*x*) : prop (*x* : *A*)
X ! (∃*x* : *A*) φ(*x*) : prop	**Y** $?_{F\exists 1}$ or **Y** $?_{F\exists \square}$	**X** ! *A* : set **X** ! φ(*x*) : prop (*x* : *A*)

By definition the *falsum* symbol ⊥ is of type prop. Therefore the formation of a posit of the form ⊥ cannot be challenged.
The next rule is not a formation rule *per se* but rather a substitution rule.[20]
Posit-substitution

There are two cases in which **Y** can ask **X** to make a substitution in the context $x_i : A_i$. The first one is when in a standard play a variable (or a list of variables) occurs in a posit with a proviso. Then the challenger posits an instantiation of the proviso:

Posit	**Challenge**	**Defence**
X ! $\pi(x_1, \ldots, x_n)$ $(x_i : A_i)$	**Y** ! $\tau_1 : A_1, \ldots, \tau_n : A_n$	**X** ! $\pi(\tau_1 \ldots \tau_n)$

The second case is in a formation-play. In such a play the challenger simply posits the whole assumption as in Move 7 of the example below:

Posit	**Challenge**	**Defence**
X ! $\pi(\tau_1, \ldots, \tau_n)$ $(\tau_i : A_i)$	**Y** ! $\tau_1 : A_1, \ldots, \tau_n : A_n$	**X** ! $\pi(\tau_1, \ldots, \tau_n)$

Play objects/ The idea is now to design dialogical games in which the players' posits are of the form “*p* : φ” and acquire their meaning in the way they are used in the game – i.e., how they are challenged and defended. This requires, among others, to analyse the form of a given play-object *p*, which depends on φ, and how a play-object can be obtained from other, simpler, play-objects. The standard dialogical semantics for logical constants gives us the needed information for this purpose. The main logical constant of the expression at stake provides the basic information as to what a play-object for that expression consists of:

[20] It is an application of the original rule from CTT given in Ranta (1994, p. 30).

A play for **X** $\varphi\vee\psi$ is obtained from two plays p_1 and p_2, where p_1 is a play for **X** φ and p_2 is a play for **X** ψ. According to the particle rule for disjunction, it is the player **X** who can switch from p_1 to p_2 and vice-versa.

A play for **X** $\varphi\wedge\psi$ is obtained similarly, except that it is the player **Y** who can switch from p_1 to p_2.

A play for **X** $\varphi\rightarrow\psi$ is obtained from two plays p_1 and p_2, where p_1 is a play for **Y** φ and p_2 is a play for **X** ψ. It is the player **X** who can switch from p_1 to p_2.

The standard dialogical particle rule for negation rests on the interpretation of $\neg\varphi$ as an abbreviation for $\varphi\rightarrow\bot$, although it is usually left implicit. It follows that a play for **X** $\neg\varphi$ is also of the form of a material implication, where p_1 is a play for **Y** φ and p_2 is a play for **X** $\bot$, and where **X** can switch from p_1 to p_2.

As for quantifiers, we are dealing with quantifiers for which the type of the bound variable is always specified. We thus consider expressions of the form $(Qx : A)\varphi$, where Q is a quantifier symbol.

Posit	**Challenge**	**Defence**
X φ (where no play-object has been specified for φ)	**Y** ? play-object	X p : φ
X p : φ∨ψ	**Y** $?_{prop}$	**X** φ∨ψ : prop
	Y ?[φ,ψ]	**X** $L^{\vee}$(p) : φ Or **X** $R^{\vee}$(p) : ψ **[the defender has the choice]**
X p : φ∧ψ	**Y** $?_{prop}$	**X** φ∧ψ : prop
	Y ?[φ] Or **Y** ?[ψ] **[the challenger has the choice]**	**X** $L^{\wedge}$(p) : φ respectively **X** $R^{\wedge}$(p) : ψ
X p : φ→ψ	**Y** $?_{prop}$	**X** φ→ψ : prop
	Y $L^{\rightarrow}$(p) : φ	**X** $R^{\rightarrow}$(p) : ψ
X p : ¬φ	**Y** $?_{prop}$	**X** ¬φ : prop
	Y $L^{\bot}$(p) : φ	**X** $R^{\bot}$(p) : ⊥
X p : (∃x : A)φ	**Y** $?_{prop}$	**X** (∃x : A)φ : prop
	Y $?_L$ Or **Y** $?_R$ **[the challenger has the choice]**	**X** $L^{\exists}$(p) : A Respectively **X** $R^{\exists}$(p) : φ(L(p))
X p : {x : A | φ}	**Y** $?_L$ Or **Y** $?_R$ **[the challenger has the choice]**	**X** $L^{\{\ldots\}}$(p) : A Respectively **X** $R^{\{\ldots\}}$(p) : φ(L(p))
	Y $?_{prop}$	**X** (∀x : A)φ : prop

X p : $(\forall x : A)\varphi$	**Y** $L^{\forall}(p) : A$	**X** $R^{\forall}(p) : \varphi(L(p))$
X p : B(k) (for atomic B)	**Y** $?_{prop}$	**X** B(k) : prop
	Y ?	**X** sic (n) (X indicates that Y posited it at move n)

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