Immanent Reasoning or Equality in Action

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A dialogical genealogy of the notion of Equality in Constructive Type Theory

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V. Identity and Equality as a predicate from the dialogical point of view

The case of equality as a predicate s also based on the copy-cat move and this applies to both the intensional and the extensional case. We start with the former.

V.1 The intensional identity-predicate *Id*

V.1.1 The generation of the *Id*

The main point of the intensional identity predicate ***Id*** is that the equality that it expresses is based on the ontological level. The identity predicate ***Id*** in *A* expresses the fact that if *a* and *b* are definitionally the same play_objects in *A*, and *a* : *A*, then there is a play object dependent on *a* for the prop ***Id***(*A*, *a*, *b*). If it is the proponent who posits the identity, he must have posited before *a* : *A* and *a* = *b* : *A*. Since these are elementary posits, he must have overtaken them from **O**. The point is that **P** "imports" some definitional equality into the propositional level by producing an identity predicate. This yields already its formation rule:

Posit	Challenge	Defence
X ! ***Id***(A, a_i, a_j) : prop	**Y** $?_{F1}$ ***Id*** **Y**$?_{F2}$ ***Id*** **Y**$?_{F3}$ ***Id***	**X** ! *A* : *set* **X** ! *ai* : *A* **X** ! *aj* : *A*

Since ***Id***(*A*, *a*, *b*).expresses identity of *a* and *b*, the play object that makes the identity true, is a play object, expressed as $Ins^{Id\text{-}a}(a)$, the only internal structure of which is its dependence on *a*. In fact the case $Ins^{Id\text{-}a}(a)$: ***Id***(*A*, *a*,*a*) is the most basic one. We will start with it

Reflexivity. If a player stated *a* : *A*, then the challenger can ask for the predicate of identity generated by this posit. The defender must then bring forward the reflexivity of the predicate ***Id*** on <u>*a*</u> in *A*. Recall that, if it is the Proponent who brought forward the initial posit, then, this move is the result of some kind of copy-cat move. Moreover, since what it produces is reflexivity, it is a direct copy-cat move, that creates a corresponding identity-predicate. The play object of the resulting proposition is the instruction $Ins^{Id\text{-}a}(a)$ that is resolved with *a*. These yield the following rules:

The introduction of $Ins^{Id\text{-}a}(a)$

Posit	Challenge	Defence
X ! *a* : *A*	**Y** ? $_{Id\text{-}a}$	**X** ! $Ins^{Id\text{-}a}(a)$: ***Id***(*A*, *a*, *a*)

The resolution of $Ins^{Id\text{-}a}(a)$

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Posit	**Challenge**	**Defence**
X ! $\pi[Ins^{Id\text{-}a}(a)]$	**Y** / $_{Id\text{-}a}$?	**X** ! $\pi[a]$

Assume now that a player associated in his posit the play object *c* with some *prop* constituted by ***Id.*** In such a case we would like to be able to make explicit the depend-play object that *c* encodes. Since, as we will discuss below; expressions of the form ***Id***(*A*, *a*, *b*), are inhabited by only one play-object, namely $Ins^{Id\text{-}a}(a)$ we can safely lay-down the following rule
:

Posit	**Challenge**	**Defence**
X ! *c* : ***Id***(*A*, *a*, *b*)	**Y** ! *a* : *A*, *b* : *A* (provided this has been established by the formation play of ***Id***(*A*, *a*, *b*)	**X** ! $Ins^{Id\text{-}a}(a)$: ***Id***(*A*, *a*, *b*)

We can now deal with cases involving more than one expression a_i. In fact, as we will discuss below, the rule is not necessary since it results from applying the reflexivity rule and the rules of definitional identity. Nevertheless, it is practical to have it as a separate rule:

The case *Id*(*A*, *a*, *b*) .

Posit	**Challenge**	**Defence**
X ! *a* : *A* … **X** ! *a* = *b* : *A*	**Y** ? $_{Id}$	**X** ! $Ins^{Id\text{-}a}(a)$: ***Id***(*A*, *a*, *b*)

Notice, once again, that the play-objects for ***Id***(*A*, *a*, *a*) and for ***Id***(*A*, *a*, *b*), are the same, namely *r*(*a*). The point is that the predicate establishes the identity of *a* and *b*, so that, to use a Fregean terminology, they "denote", the same play-object.

V.1.2 The substitution rule for *Id*

Let us start by considering the dialogical use of a general form of substitution that should provide the play-level correspondent of the general rule we presented in the chapter on the CTT-notion of the intensional equality-predicate.

Assume that player X made use of the equality-predicate in order to establish the equality between two terms, say t_1 and t_2. Assume too that X has posited Ct_1. ¨Player Y can now posit Ct_2, by taking that the predicative equality between both terms allows him to posit the elementary proposition Ct_2 (given the posit Ct_1 of his antagonist). In the case that X is the opponent, this triggers a kind of indirect copy-cat: **P** does not copy exactly the same term, but he posits an elementary expression that is equivalent to one of the **O** modulo-the equality of the terms involved. The play object for the resulting proposition is $Ins^{Id\text{-}ab}(c, d)$, and the components *c* and *d*, allow tracing back the play-objects for the propositions that lead to the substitution, namely the play-object for the identity and the play object for the proposition on which the substitution is carried out. In fact, if wish to achieve the same degree of generality than the one in CTT we need to include cases where C includes the play-object $r(t_i)$.

Posit	***Id*-Substitution**
X ! $a : A$ **X** ! $b : A$ **…** **X** ! c : ***Id***(A, a, b) **…**	
X ! $d : \varphi[a, a, Ins^{Id\text{-}a}(a)]$	**Y** ! $e : \varphi[a', b', c']$
X ? ***Id****^{e}*	**Y** ! $Ins^{Id\text{-}ab}(c, d) : \varphi[a', b', c']$
After Y's answer X challenges the play-object *e*	
X ? A; a' /, **X** ? A, b' / **X** ? A, c' /	**Y** ! $Ins^{Id\text{-}ab}(c, d) : \varphi[a'/a, b', c'], a' : A$ **Y** ! $Ins^{Id\text{-}ab}(c, d) : \varphi[a', b'/a, c'], b' : A$ **Y** ! $Ins^{Id\text{-}ab}(c, d) : \varphi[a', b', c'/ Ins^{Id\text{-}a}(a)], a' : A$
After **Y**'s answer X challenges the components of φ	**Y** makes the substitutions explicit that support his previous posit After the answer only a challenge on the instruction in this expression is allowed. After the instruction has been resolved , no further challenge on that expression is possible.

The resolution of the instruction $Ins^{Id\text{-}ab}(c, d)$ gives back d. The idea is that, since a an b are identical, the substitution yields a proposition that is the same modulo the identity-predicate, and share therefore the same play-object
:

Posit	**Challenge**	**Defence**
X ! $\pi[Ins^{Id\text{-}ab}((c, d))]$	**Y** / ***Id-ab***?	**X** ! $\pi[d]$

The dialogical rules for the identity-predicate are closer to the formulation of Thompson (1999, pp. 109-113) than the ones described in our overview of CTT where we presented the developments of Nordström et al. (1990).
Leibnz's Law can be formulated as a special case. But let us allow a more liberal version:

Posit	***Id*-Leibniz-substitution**
X ! $t_1 : A$ **X** ! $t_2 : A$ **…** **X** ! c : ***Id***(A, t_1, t_2) **…**	
X ! $d : \varphi[t_1]$	**Y** ! $e : \varphi[t_j]$ (where t_j is either t_1 or t_2)
X ? ***Id****^{e}*	**Y !** $Lbz^{Id\text{-}t1t2}(c, d) : \varphi[t_j]$

After Y's answer X challenges the play-object *e* **Y !** / $Lbz^{Id\text{-}t1t2}$ After the answer only a challenge on the instruction in this expression is allowed.	 **Y !** $d : \varphi[t_j]$ After the instruction has been resolved , no further challenge on that expression is possible.

Let us apply first the ***Id***-predicate-copy-cat-rule in order to obtain a play for the symmetry of ***Id***[1]

Symmetry

	O			**P**	
	A1: c : **Id**(A, a, b) A2: $a : A$ A3 $b : A$			e : **Id**(A, b, a)	0
1	m:= 1			n:= 2	2
3	? $\boldsymbol{Id}^{e}$			$Ins^{Id\text{-}ab}(c, Ins^{Id\text{-}a}(a))$: **Id**(A, b, a)	6
5	$Ins^{Id\text{-}a}(a)$: **Id**(A, a, a)		A2	**Y** ? $_{Id\text{-}a}$	4
7	? b / , ?a /	6		$Ins^{Id\text{-}ab}(c, Ins^{Id\text{-}a}(a))$: **Id**$(A, b/a, a/a)$; a , $b : A$	8
9	/ ***Id-ab***?	6		$Ins^{Id\text{-}a}(a)$: **Id**(A, b, a)	10
11	/ ***Id-a***?	10	5	a : **Id**(A, b, a)	12

Remark: We could have split move 7 (and 8) in two; but for the sake of simplicity we carried out both possible challenges on the same line.

The following example deploys the use of Leibniz's substitution for the case of transitivity:[2]

Transitivity

	O			**P**	
	A1 - ! v : **Id**(A, a, b) A2 - ! w : **Id**(A, b, c) A3 - ! $a, b, c : A$,			! e : **Id**(A, a, c)	0
1	m:= 1			n:= 2	2
3	? $\boldsymbol{Id}^{e}$			! $Lbz^{Id\text{-}ab}$(v, w) : **Id**(A, a, c), $a, c : A$,	4
5	? / $Lbz^{Id\text{-}ab}$;	4		! w : **Id**(A, a, c)	6

V.2 The extensional identity-predicate *Eq*

V.2.1 The generation of *Eq*

The dialogical process that yields the extensional predicate ***Eq*** is simpler than the other forms of equality. Once **O** introduced a definitional equality between, **P** is allowed to introduce a predicative version, in such a way that the play-object for the resulting proposition is the play-object *eq*, that does not depend upon the play-objects involved in the definitional equality that generated ***Eq***. Hence; from eq one cannot trace back the play-objects on the basis

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[1] In fact it is the most relevant play for the building of the winning strategy

[2] Again, the play is not sufficient for the development of a winning strategy but it in fact the most relevant play for doing so.

of which the predicate ***Eq*** has been generated. Accordingly, the resulting play-object for ***Eq*** can-not be challenged and every play object c for ***Eq***(A, a, b) is definitionally equal to ***eq***.

Let us start with the formation-rule

Posit	**Challenge**	**Defence**
X ! ***Eq***(A, a_i, a_j) : prop	**Y** $?_{F1}$ ***Eq*** **Y**$?_{F2}$ ***Eq*** **Y**$?_{F3}$ ***Eq***	**X** ! A : *set* **X** ! $ai : A$ **X** ! $aj : A$

The introduction of *Ins*$^{Id\text{-}a}$(*a*)

Posit	**Challenge**	**Defence**
X ! $a_i = a_j$: A	**Y** ? $_{ai = aj}$	**X** ! eq : ***Eq***(A, a_i, a_j)

The play-object *eq*

Posit	**Challenge**	**Defence**
X ! c : ***Eq***(A, a_i, a_j)	**Y** ! $a_i : A$, $a_j : A$	**X** ! eq : ***Eq***(A, a_i, a_j)

V.2.2 From *Eq* to definitional equality

X	**Y**
X ! $a_1 : A$ **X** ! $a_2 : A$ **…** **X** ! c : ***Eq*** (A, a_1, a_2)	
X ! ***eq*** : ***Eq*** (A, a_1, a_2)	**Y ? c**^{Eq}
	Y ! $a_1 = a_2 : A$
	After **X**'s answer on challeng on c **Y** can posit the definitional equality, and this cannot be challenged