Objective Knowledge and the not Dispensability of Epistemic Subjects . Some remarks on Popper's notion of objective knowledge

Shahid Rahman, Juan Redmond, Nicolas Clerbout

Objective Knowledge and the not Dispensability of Epistemic Subjects
Some remarks on Popper's notion of *objective knowledge*[1]

Shahid Rahman*   Juan Redmond**   Nicolas Clerbout**

shahid.rahman@univ-lille3.fr , juan.redmond @uv.cl, nicolas.clerbout@ uv.cl

(* Professor for Logic and Epistemology, Univ. Lille, CNRS, UMR 8163 - STL - Savoirs Textes Langage, F-59000 Lille, France; **Professors of Philosophy, Instituto de Filosofía, Facultad de Humanidades, Universidad de Valparaíso, Chile)

Abstract:
While discussing his notion of objective knowledge Popper introduces the idea of dispensability of knowing subjects, the autonomy of knowledge and the argumentative function of language. The main claim of our paper is that, on our view, objective knowledge produced by argumentative interaction is not achieved by dispensing knowing subjects but by differentiating between the play and the strategic levels of argumentation, where a distinction should be drawn between a particular concrete knowing subject engaged in that interaction, say Karl, and an arbitrary one. Moreover, the theory of meaning deployed at the level of plays, the so-called *local meaning*, provides already some kind of objective knowledge, since it is player-independent and it does not reduce to the purely logical one of the strategy level. In fact, despite our criticism we think that Popper's discussion of the notion objective knowledge still deserves to be reflected on, particularly so in view of new results concerning the formalization of Brouwer's *Creating Subject*.

1   Introduction:

It is nowadays not difficult to raise objections against Karl's Popper's theory of knowledge based on the existence of a large literature, past and contemporary, dedicated to the criticism of falsificationism such as the penetrating recent papers of Susan Haack (2009, 2012), one of them reprinted in the present volume, and the excellent overview of those criticisms provided by Stephen Thornton (2013) in his entry on Karl Popper in the *Stanford-Encylcopedia of Philosophy*. Indeed, the devastating survey of Thornton suggests that Popper's last stand took him to retract most of the claims on which falsificationism is based. Let us quote, just as an example of the state of the art on the wide-spread criticisms, the last lines of Thornton's paper:

> *Popper's final position is that he acknowledges that it is impossible to discriminate science from non-science on the basis of the falsifiability of the scientific statements* alone*; he recognizes that scientific theories are predictive, and consequently prohibitive,* only *when taken in conjunction with auxiliary hypotheses, and he also recognizes that readjustment or modification of the latter is an integral part of scientific practice. Hence his final concern is to outline conditions which indicate when such*

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modification is genuinely scientific, and when it is merely ad hoc. *This is itself clearly a major alteration in his position, and arguably represents a substantial retraction on his part: [...].*
On the other hand, the shift in Popper's own basic position is taken by some critics as an indicator that falsificationism, for all its apparent merits, fares no better in the final analysis than verificationism (Thornton (2013), chapter 9).

Still, it is undeniable that his work had a world-wide influence in philosophy of science. Moreover, despite the harsh criticisms there are nevertheless some adamant defenders of Popper's general views[2], particularly so in relation to his socio-political writings.

In relation to his view on philosophy of science, some of the nowadays defenders focus on Popper's legacy of the notion of *objective knowledge* based in a *three-world ontology*. In fact, some recent results mentioned below and the nowadays rapid spreading of epistemic approaches motivate indeed a closer look at Popper's view on that matter and more general on the notion of objective knowledge itself.

Let us, first recall the background that originated Popper's inception of a three-world ontology in order to provide a notion of objective knowledge compatible with his general falsificationism-tenets.

The idea of objective knowledge, that Popper admits having borrowed (with some modifications) from Bolzano and Frege amounts to the slogan *Knowledge without Knowers* underlying his paper of 1968 *Epistemology Without a Knowing Subject*[3]. The paper constituted his contribution to the famous volume *Logic, Methodology and Philosophy of Science III*, of 1968 edited by Heyting/Mostowski/Robinson/Suppes and that contained papers of the finest philosophers of logic of those days involved in developing constructivism, intuitionism and epistemic logic.

Now, knowledge without a knower seems to be a *contradictio in adjecto*: The act of knowing is the product of an intentional act and as such requires of an agent performing the act. In fact, from the point of view of Frege, the third realm is inhabited by *propositions that are either true or false*: For Frege, objective knowledge amounts to the *assertion* of true propositions: what is independent of a knower is, on Frege's view, *truth* rather than *knowledge*, since knowledge is in principle an epistemic notion rather than ontological: objective knowledge is supervenient on objective truth. In other words, according to Frege, objective knowledge is assured by the possibility of an agent to access to (at least some) true propositions – though it is part of Frege's realism that there might be some inaccessible truths. Compare with the following words of Popper that leads to his thesis of the (partial) *autonomy* of the third realm:

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[2] Cf. Miller (1974a, b, 1994, 2006), Mulkay/Gilbert (1981), Nguimbi (2012), Niiniluoto (1987).

[3] Further developed in publications such as in his book *Objective Knowledge* of 1972 and his paper *Three Worlds* of 1979.

> *We can thus say that there is a kind of Platonic (or Bolzanoesque) third world of books in themselves, theories in themselves, problems in themselves, problem situations in themselves, arguments in themselves, and so on. And I assert that even though this third world is a human product, there are many theories in themselves and arguments in themselves and problem situations in themselves which have never been produced or understood and may never be produced or understood by men.* Popper (1968, p. 342).

However, Popper's falsificationism is not very much at ease with Frege's realm of eternal truths. So, Popper prefers instead to make use of Tarki's terminology and thus (letting by side some unfortunate formulations where he seems to mix up the syntactical language of derivability with the semantic language of validity and his rather confusing remarks on Göedel's incompleteness theorems), he proposes to understand *objective knowledge* as closely related to deductive closure. In order to make compatible his fallibilism with objective knowledge and in order to meet the apparent *contradictio in adjecto* of an intentional act of knowledge without knowing subject, Popper makes use of the distinction between the second and the third world. While the propositions of the third realm might have been originated in the second by some epistemic agents, its objective content is constituted by the logical consequences of these propositions, independently now of the epistemic agents that originated them. So, like in Frege, it is logic, that provides objectivity to the propositions of the third realm. Moreover, the logical features of his notion of objective knowledge should make it possible to revise or falsify a proposition at the third world, and that has been originated in the second. In fact in some earlier work, that he unfortunately abandoned, Popper (1947a,b, 1949) seems to come close to what now is called *proof-theoretical semantics*, where meaning is given by inferential rules (see Schröder-Heister (1984, 2005)). It is unfortunate since this might have provided him what he was looking for: a notion of objective knowledge entirely based on logical terms. However, on the times of his *Epistemology Without Knowing Subject*, influenced by Alfred Tarski's own rejection of proof-theoretical approaches, Popper abhorred of the idea of approaching the notion of meaning by the deployment of an epistemic framework. In fact, the construction of a third autonomous realm allowed Popper to challenge the, in those days, incipient *explicit epistemic logic* developed by Jaakko Hintikka − recall that Hintikka's famous *Knowledge and Belief* was published in the same year of the publication of Popper's paper under consideration. Since on the three-worlds-account, objective knowledge is independent of its actors, there is no problem in accepting that there are truths, that might be inaccessible to any epistemic agent: and so the problem of *logical omniscience,* that jeopardized Hintikka's modal approach to the epistemic conception of logic, disappears. Let us recall that the problem of logical omniscience relates to the implausibility of the following axiom of modal approaches to epistemic logic *If an agent knows A, he knows all of its logical consequences*.

So far, it looks as if that there is no novelty in relation to Frege's attack on subjectivism. Moreover, since Frege distinguishes truth and the access to truth − while the first concerns

ontology, the second concerns epistemology: the failure of logical omniscience amounts accepting that there are truths that might never been known, and this is not problem at all for such kind of approaches. However, Frege's distinction is not in principle available for Popper: if instead of an inaccessible truth for an epistemic subject we identify such kind of truths with objective knowledge, as Popper does, then we are committed to the awkward formulation that objective knowledge might not be epistemologically accessible. Moreover, the notion of objective knowledge Popper is aiming at, does not really involve logical truths (that is, valid propositions) but rather general falsifiable truths of empirical sciences and the logical closure of them. In order to avoid such an awkward outcome and in order to meet his own fallibilistic view on scientific knowledge, Popper introduces the, what he calls, two high-level functions of language: the *descriptive function* and the *argumentative function*. The descriptive function provides the truth of a given proposition by establishing a (metalogical) relation between the sentence (expressing that proposition) and a fact – here he meets Tarski's formal semantics. The argumentative function cares of testing if the relation established by the description is or not the adequate one. But wait a minute argumentation is about interaction between epistemic agents, so the knowing subject seems to come back. What Popper needs now is to have a logical notion of argument: sound arguments are what he seems to have in mind, that is, arguments the soundness of which is independent of the argumentative agents that bring them forward –they are autonomous, and they might be true in a model. The testing of the arguments might lead us back to the second level, where new propositions will be created and tested again at the third world-level – within a some new model:

> *The process can be described by the following somewhat oversimplified schema*
> $$P_1 + TT + EE + P_2$$
> *That is, we start from some problem PI, proceed, to a tentative solution or tentative theory **TT**, which may be (partly or wholly) mistaken; in any case it will be subject to error elimination **EE** which may consist of critical discussion or experimental tests; at any rate, new problems Pz arise from our own creative activity; and these new problems are not in general intentionally created by us, they emerge autonomously from the field of new relationships which we cannot help bringing into existence with every action, however little we intend to do so. The autonomy of the third world, and the feed-back of the third world upon the second and even the first, are among the most important facts of the growth of knowledge.* Popper (1968, p. 345)

Furthermore, if Popper's notion of objective knowledge is understood as the general rejection of epistemic approaches to knowledge based on agentivity and more precisely on assertion-conditions, then such a rejection faces the following challenges:

1. According to Popper, scientific objective knowledge is, different to Frege, hypothetical in nature. Furthermore, the argumentative function can either falsify or provide the means to continue supporting or not the plausibility of given scientific hypotheses. Thus, it looks as Popper's objective knowledge is either very close to justified (or not yet falsified) belief after all or to the antirealist notion of belief:

hypotheticals are in fact one way to express the contents of a belief! – at least from the point of antirealist point of view that Popper targets with his criticisms. Indeed, on the constructivist account hypotheticals are those expressions, where we assume (believe) that we have some way to justify the proposition involved, though we cannot yet bring forward the precise piece of evidence – the so-called *proof-object* – that justifies that proposition and that makes it irrevocably true. In fact, hypotheticals involve those propositions for which we will never have a definitive justification – see Ranta (1994, pp. 145-157) and Dango (2015). Certainly, Popperians might point out that the constructivist notion of hypothetical involve epistemic subjects and this makes them not first-class denizens of Popper's third world. However, on one hand the constructivist distinction between actual and potential knowledge (1994, pp. 91-92;) offers a straightforward response to this kind of Popperian remark, on the other the difference between actual and potential knowledge can be captured by our own proposal of distinguishing the play- from the strategy-level.

2. How do we come to realize that an argument is objective? Where do we test the arguments of the third level? Not at the second level, since this is the level of subjectivity, not a third level either, since there is no knowing subject. Since there are no knowing subjects at the third level one might wonder how the third level can come back to the second level. Notice that Tarski's notion of truth in a model, is not argumentative in nature. In the best case truth in a model might set the conditions of an argumentative process (such as in Hintikka's (1973) game theoretical semantics). But then the interaction of epistemic agents triggered by their assertions play a fundamental role. Moreover, the fallibilistic features of Popper's project seems to be closer to the dynamic feature of arguments based on *epistemic assumptions* (to be discussed in section 3), than to the static properties of truth in a model.[4]

3. Paul Humphrey (2009) stresses the crucial role that computer-simulations in contemporary science play, and challenges philosophers of science to move away from a purely anthropocentric perspective on the development of sciences. Humphrey (2009, p. 616) suggests that Popper's third world might be seen as anticipating this move, though he rapidly adds that Popper's notion is too abstract to be helpful. In fact, the Curry-Howard (see Howard (1969)) results and Per Martin-Löf's Constructive Type Theory (1984) show that the automatic running of computer programs does not per se provide an argument in favor of Popper's notion of third world. Indeed, the results of Curry-Howard and Martin-Löf show that the following isomorphism holds: *a proof is a program, the expression it proves is a*

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[4] Let us point out that by the time of Popper's talk, Paul Lorenzen's first version of dialogical logic has been already been published. Indeed, the first paper on dialogical logic *Logik und Agon* was published in 1958 and re-printed in 1978. From the very start of his work on constructive mathematics and logic Lorenzen showed that such kind of approaches do not require the subjectivist assumptions of Leo Brower's intuitionism.

type for the program, and this type is a proposition as understood by intuitionistic logic. Thus, whatever Popper's notion of assertion-free entities of the third world might mean, it cannot be identified with the automatic running of computer-programs, since the underlying logic of the running of these programs is precisely the kind of logic, that Popper rejects because it is based on assertions. More precisely, if *b* is a program and *B* the type of that program, *b* : *B* expresses the judgment that *b* is a proof of the proposition *B*. In other words *b* : *B* expresses the assertion that the proposition *B* is made true by the piece of evidence (its proof-object) *b*. Thus, the logic underlying functional programming has epistemic foundations after all.

4. A deep new result of Göran Sundholm (2014), shows that the axiomatization of Brower's theory of a creating subject (CS) via the Brouwer-Kripke Principle, and that deploys the extreme consequences of intuitionistic subjectivism, is classically valid! In other words, the extreme subjectivism deployed by the formalization of a creating subject, is not, constructively speaking, meaningful. Hence, the development of constructive logic and mathematics is beyond any suspect of subjectivity expressed by such an axiomatization of CS. So, though Popper's rejection of Brower's assumption of a CS – profusely discussed in the volume *Logic, Methodology and Philosophy of Science III*, of 1968 – can be now shown to be on the right track; to the dismay of Popperian arguments against epistemic approaches to the foundations of logic, it also shows that the foundations of constructive logic and mathematics are not loaded with subjectivism (on that account) after all: epistemic approaches to logic and mathematics in its constructivist version can and even should dispense of a formal system that includes CS among its axioms!

As the reader will have already noticed, we think that Popper's discussion of the notion objective knowledge still deserves to be reflected on, particularly so in the context that originated it: namely as a challenge to the purported subjectivist features of the intuitionist, constructivist or more generally of the antirealist epistemological frameworks. More generally the move of taking away the subjective features of epistemology is one we adhere to – let us mention here once more Sundholm's (2014) deep result concerning the dispensability of CS.
The main aim of our remarks concerns mainly point two – point one has been partially developed in Rahman/Redmond (2015) and Dango (2015), point three is covered by our responses to point two and while developing point 2 we do not assume the existence of CS – the remarks of point 4 show that it is wise to do so. Moreover, the general argument we will deploy in the present paper, can be adapted to the case of hypotheticals, though a thorough discussion requires a longer study that we will leave for another opportunity.

In fact independently of the general fallibilistic tenets of Popper, we claim that objective knowledge produced by argumentative interaction is not achieved by dispensing of knowing subjects engaged in such an interaction. On the contrary what we need is to face

the dialogical nature of objective knowledge, its argumentative feature. Indeed, within the dialogical framework we differentiate the play and the strategic levels of argumentation, where a distinction should be drawn between a particular concrete knowing subject, say Karl, who develops some play, and an arbitrary one who may, contrary to the singular individual Karl, build a winning strategy if (potentially) there is one. Moreover, the theory of meaning deployed at the level of plays, the so-called *local meaning*, provides already some kind of objective knowledge, since it is player-independent and it does not reduce to the purely logical one of the strategy level. Let us mention once more that, according to our view, the fallibilistic features of Popper's project seems to be closer to the dynamic feature of arguments based on *epistemic assumptions* (to be discussed in section 3), than to the static properties of truth in a model: Agentivity is the dynamic notion required by Popper's project but his conception of objective truth – at least as presented in *Epistemology Without Knowing Subject* – rejects it!

Let us now provide first a brief informal introduction to the dialogical conception of logic, that, so we claim, provides

1. the means to understand a non-subjectivist conception of epistemic agents engaged in an argumentative process
2. the means to tackle the challenge of logical omniscience
3. an argumentative framework where points 1 and 2 can be developed.

2 The dialogical conception of logic and the distinction between the play and the strategic levels.

2.1 Basic notions[5]

The dialogical approach to logic is not a specific logical system but rather a rule-based semantic framework in which different logics can be developed, combined and compared. An important point is that the rules that fix meaning can be of more than one

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[5] The following brief presentation of standard dialogical logic is based on Clerbout (2013). The main original papers on dialogical logic are collected in Lorenzen and Lorenz (1978). For a historical overview see Lorenz (2001). Other papers have been collected more recently in Lorenz (2008, 2010a, 2010b). A detailed account of recent developments can be found in Rahman and Keiff (2005) and Keiff (2009). For the underlying metalogic see Clerbout (2013, 2014). For a textbook presentation: Redmond and Fontaine (2011) and Rückert (2011). For the key role of dialogic in regaining the link between dialectics and logic, see Keiff (2009), Rahman and Keiff (2010. Fiutek *et al.* (2010) study the dialogical approach to belief revision. Clerbout *et al.* (2011) study Jain Logic in the dialogical framework. Popek (2012, p. 223-244) develops a dialogical reconstruction of medieval *obligationes*. See also Magnier (2013) on dynamic epistemic logic and legal reasoning in a dialogical framework.

kind. This feature of its underlying semantics quite often motivated the dialogical framework to be understood as a *pragmatist* semantics. More precisely, in a dialogue two parties argue about a thesis respecting certain fixed rules. The player that states the thesis is called Proponent (**P**), his rival, who contests the thesis, is called Opponent (**O**). In its original form, dialogues were designed in such a way that each of the plays end after a finite number of moves with one player winning, while the other loses. Actions or moves in a dialogue are often understood as speech-acts involving *declarative utterances or posits and interrogative utterances or requests*. The point is that the rules of the dialogue do not operate on expressions or sentences isolated from the act of uttering them. The rules are divided into particle rules or rules for logical constants (*Partikelregeln*) and structural rules (*Rahmenregeln*). The structural rules determine the general course of a dialogue game, whereas the particle rules regulate those moves (or utterances) that are requests and those moves that are answers (to the requests).

Crucial for the dialogical approach are the following points:

1. The distinction between *local* (rules for logical constants) and *global* meaning (included in the structural rules that determine how to play)
2. The player independence of local meaning
3. The distinction between the play level (local winning or winning of a play) and the strategic level (existence of a winning strategy).
4. A notion of validity that amounts to the conception of winning strategy *independently of any model* instead of winning strategy for *every* model.
5. The distinction between non formal and formal plays – neither latter nor the first kind concerns plays where the actions of positing an elementary sentences require a meta-language level that provides their truth.

2.2   Local Meaning and Player Independence

Particle rules:
In dialogical logic, the particle rules are said to state the *local meaning*: what is at stake is only the request and the answer corresponding to the utterance of a given logical constant, rather than the whole context where the logical constant is embedded.

- The standard terminology makes use of the terms *challenge* or *attack* and *defence*. However let me point out that at the local level (the level of the particle rules) this terminology should be devoid of strategic underpinning.

- *Declarative utterances* involve the use of formulae, *interrogative utterances* do not involve the use of formulae

The following table displays the particle rules, where X and Y stand for any of the players **O** or **P**:

∨, ∧, →, ¬, ∀, ∃	Challenge	Defence
X: α∨β	**Y**: ?-∨	**X**: α *or* **X**: β (**X** chooses)
X: α∧β	**Y**: ?∧1 or **Y**: ?∧2 (**Y** chooses)	**X**: α *respectively* **X**: β
X: α→β	**Y**: α (**Y** challenges by uttering α and requesting β)	**X**: β
X: ¬α	**Y**: α	— (no defence available)
X: ∀xα	**Y**: ?-∀x/k (**Y** chooses)	**X**: α [x/k]
X: ∃xα	**Y** ?∃	**X**: α [x/k] (**X** chooses)

In the diagram, α[x/k] stands for the result of substituting the constant *k* for every occurrence of the variable *x* in the formula A.

Crucial for our discussion on Popper's notion of objective knowledge is the following feature of local meaning:

- **Player independence:** The particle rules are symmetric in the sense that they are player independent – that is why they are formulated with the help of variables for players.

This player independence is double, on one hand it is independent of the role of proponent and opponent but it is also independent of the particular player or agent performing the moves.
Notice that we have a kind of player-independence that is not reducible to the logical validity of the propositions involved. There is a notion of knowledge of propositions involved based on the local meaning; that is objective, in the sense of player independent.

2.3   Global Meaning

Structural rules:

(SR 0) (starting rule):
The initial formula is uttered by **P** (if possible). It provides the topic of the argumentation. Moves are alternately uttered by **P** and **O**. Each move that follows the initial formula is either a request or an answer.

> **Comment**: The proviso *if possible* relates to the utterance of elementary propositions. See formal rule *(SR 2)* below.

(SR 1) (no delaying tactics rule):

Both **P** and **O** may only make moves that change the situation.

Comments: This rule should assure that plays are finite (though there might an infinite number of them). There are several formulations of it with different advantages and disadvantages. The original formulation of Lorenz made use of ranks; other devices introduced explicit restrictions on repetitions (see Clerbout (2013,2014). Let us describe here the rule that implements the use of ranks.

- After the move that sets the thesis players **O** and **P** each choose a natural number n and m respectively (termed their repetition ranks). Thereafter the players move alternately, each move being a request or an answer.

- In the course of the dialogue, **O** (**P**) may attack or defend any single (token of an) utterance at most n (or m) times.

(SR 2) (formal rule):[6]
P may not utter an elementary proposition unless **O** uttered it first. Elementary propositione can not be challenged.

(SR 3) (winning rule):
X wins iff it is Y's turn but he cannot move (either challenge or defend).

Global meaning
These rules determine the meaning of a formula where a particle occurs as a main operator in every possible play.

(SR 4i) (intuitionist rule):
In any move, each player may challenge a (complex) formula uttered by his partner or he may defend himself against the last challenge that has not yet been defended.
or
(SR 4c) (classical rule):
In any move, each player may challenge a (complex) formula uttered by his partner or he may defend himself against any challenge (including those challenges that have already been defended once).

Remark: Notice that the dialogical framework offers a fine-grained answer to the question: Are intuitionist and classical negation the same negations? Namely: The particle rules are the same but it is the global meaning that changes.

In the dialogical approach validity is defined via the notion of *winning strategy*, where winning strategy for X means that **for any choice** of moves by Y, X has at least one possible move at his disposal such that he (X) wins:

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[6] For a discussion on this rule see section three of the present paper.

Validity (definition):
A formula is valid in a certain dialogical system iff **P** has a formal winning strategy for this formula.

Thus,

α is classically valid if there is a winning strategy for **P** in the formal dialogue Dc(α).

α is intuitionistically valid if there is a winning strategy for **P** in the formal dialogue Dint (α).

Objective Knowledge based on Logical Validity:
Notice that the notion of winning strategy is based on the conception of winning by *any choice* by an *arbitrary opponent.* This is what Popper's aiming at: the argumentative function operative in such kind of objective knowledge does not dispense of a knowing subject but assumes that the knowing subject is an arbitrary one.

In the following section we will discuss this point much more thoroughly.

3   **The dialogical conception of objective knowledge and the challenge of logical omniscience**

        The idea that animates the present section is that the notion of *epistemic assumption*, when deployed in an argumentative context, is closely related to the so-called *formal rule*, that, though formal is not void of content. Furthermore, in such contexts, that, as explained further on, we call contexts of *immanent reasoning*, epistemic assumptions can be extended to an argumentative conception of the equality of pieces of evidence brought forward in support of a given proposition. This allows delving more deeply into the distinction between the play and the strategic level. Furthermore, the reflection on the distinction between both levels also leads to elucidate the different levels of objective knowledge mentioned in the preceding section.

The notion of *epistemic assumption* (to be introduced below), that must be distinguished from the notion of hypothetical presents a new challenge to the nature of the entities inhabiting Popper's third world. In fact, one of the main points underlying our development is that epistemic assumptions provide the basis of the so-called "objective content" deployed by argumentative (critical) processes so dear to Popper's take on the dynamic of knowledge.

Let us thus start by taking up some results of work in progress where we relate the formal-rule with real definitions and the notion of epistemic assumption.

3.1   *Immanent Reasoning* and the formal rule revisited

When introducing equations in the way we are used to in mathematics there are two main different notions at stake. On the one hand we use equality when introducing both *nominal definitions* (that establish a relation between linguistic expressions – such a relation yields abbreviations) and *real definitions* (that establish a relation between objects within a type – this relation yields equivalences in the type). But definitions are neither true nor false, though real definitions can make propositions true. For example, the following equalities are not propositions but certainly constitute an assertion:

$a + 0$ and a are equal objects in the set of numbers

Which we can write – using the notation of chapter 2 – as:

$a + 0 = a$: *number*

Since it is an assertion we can formulate the following inference rule:

a : *number*

$a + 0 = a$: *number*

Once more, a real definitional equality is a relation between objects it does not express a proposition. In other words, it is not the dyadic-predicate as found in the usual presentation of first-order logic. However in mathematics, we do have, and even need, an equality predicate. For example when we assert that $a + b = b + a$. In fact, we can prove it: we prove it by induction. It is proving the proposition that expresses the commutativity of equality. Thus equality expresses here a dyadic predicate.[7]

- Since we do not have much to add to the subject of nominal definitions, in the following, when we speak of *definitional equality* we mean those equalities that express a *real definition*.

It is the Constructive Type Theory of Per Martin-Löf that enabled us to express these different forms of equality in the object language. In recent work by Rahman/ Clerbout/Redmond (2015), it has been claimed that these distinctions can be seen as the result of the different forms that a specific kind of dynamic process can take when (what we call) *immanent reasoning* is deployed.

Immanent reasoning is the reasoning where the speaker endorses his responsibility of grounding the conclusion by rooting it in the assertion of the relevant premises made by the antagonist. In fact the point of such a kind of reasoning is that the speaker accepts the assertions of the premises brought forward by the antagonist and he has now the duty to develop his reasoning towards the conclusion based on this acceptance. We call this kind of reasoning *immanent* since there is no other authority that links premises and conclusion

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[7] For a thorough discussion on this issue see Granström (2011, pp. 30-36, and pp. 63-69).

beyond the intertwining of acceptance and responsibility during the interaction. Göran Sundholm (1997) called such premises *epistemic assumptions*, since with them we assume that the proposition involved is known, though no demonstration backing the assumption has been (yet) produced.

It is important to recall that in the context of CTT a distinction must be drawn between open assumptions, that involve *hypothetical judgements*, judgements that are true, that involve *categorical judgements*, and *epistemic assumptions*. What distinguishes open assumptions from true judgements is that open judgments contain variables: we do not know the proof-object that corresponds to the hypothesis. Open assumptions are different from epistemic assumptions, since with the former we express that we do not know the hypothesis to be true, while the latter we express that we *take* it to be true. Moreover, epistemic assumptions are not *part* of a judgement. It is a whole judgement that is *taken to be true*. A hypothetical judgement; can be object of an epistemic assumption. This naturally leads to think of epistemic assumptions as related to the force of a given judgement.

Notice that distinguishing hypotheticals from epistemic assumptions adds a new challenge to the nature of the entities that inhabit Popper's third world. Are the epistemically assumed hypotheticals or hypotheticals?
Be that as it may, let us come back to the notion of epistemic assumption that in a recent talk in Paris by Martin-Löf (2015) received the following dialogical interpretation:

> […] *the speaker is under an obligation, he is undertaking a certain duty when he makes the assertion, whereas the hearer has the right to trust that he can fulfil his obligation. So the speaker has a duty, whereas the hearer has a right, and right I take to be the same as permission. So, you see that from this dialogical perspective these deontic notions of obligation and permission come in, and they are of course central notions of normative ethics* […].
> […] *I have thought about them in this way, because I have been plagued, since six years ago in connection with a meeting organized by Maria van der Schaar, called Days of Judgement, in Leiden, that was September 2009, six years ago when preparing that lecture I became acutely aware of a circularity problem which I had not seen before* […].
> […] *When you are giving an account of the notion of immediate inference, the notion of demonstration is not yet at your disposal. So, to say, Assume that $J_1, ..., J_n$ have already been demonstrated makes you accusable of trying to explain things in a circle. The solution to this circularity problem, it seems to me now, comes naturally out of this dialogical analysis – once you have seen it, you can go to the normal logical situation and explain things properly there also, but at least I have seen it via the dialogical analysis. The solution is that the premisses here should not be assumed to be known in the qualified sense, that is, to be demonstrated, but we should simply assume that they have been asserted, which is to say that others have taken responsibility for them, and then the question for me is whether I can take responsibility for the conclusion. So, the assumption is merely that they have been asserted, not that they have been demonstrated. That seems to me to be the appropriate definition of epistemic assumption in Sundholm's sense.*

These paragraphs deploy in a deontic language[8] one of the main features of the dialogical framework: the proponent is entitled to use the opponent's moves in order to develop the

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[8] Let us point out that one of the main philosophical assumptions of the constructivist school of Erlangen was precisely the tight interconnection between logic and ethics, see among others: Lorenzen/Schwemmer (1975) and Lorenzen (1984).

defence of his own thesis. [9] According to this perspective the proponent takes the assertions of the opponent as *epistemic assumptions* (to put it into Sundholm's happy terminology), and this means that the proponent trusts them only because of its force, just because she claims that she has some grounds for them.[10] More generally, the conceptual links between equality and the *formal rule*, nowadays called by Marion/Rückert (2015) more aptly the *Socratic rule*, is one of the many lessons Plato and Aristotle left us concerning the meaning of expressions taking place during an argumentative process.

The main aim of the recent study of Rahman/Clerbout/Redmond (2015) is to show that in logical contexts the Π- and Σ-rules of definitional equality as deployed in the context of CTT can be seen as highlighting the dialogical interaction between entitlements and duties mentioned above.[11] Under this perspective the standard monological presentation of these rules for definitional equality encodes implicitly an underlying process – by the means of which the proponent "copies" some of the opponent's choices – that provides its dialogical and normative roots. Moreover, this can be extended to the dialogical interpretation of the equality-predicate. We are tracing back, in other words, the systematic origins of the dialogical interpretation recently stressed by Göran Sundholm and Per Martin-Löf.

In a nutshell

- The Socratic rule underlying immanent reason deploys equality in action

Furthermore, we might also allow the Proponent to challenge the elementary expression and its purported piece of evidence. This leads to *material dialogues*, where, as we will mention below, the proponent will *adhere* to the elementary proposition at stake, only after

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[9] In fact, Martin-Löf's discussion is a further development of Sundholm's (2013, p. 17) remark on the links between some pragmatist tenets and inferentialism, that emerge from the following insight of J. L Austin (1946, p. 171):

> *If I say "S is P" when I don't even believe it, I am lying: if I say it when I believe it but am not sure of it, I may be misleading but I am not exactly lying. When I say "I know" ,I give others my word: I give others my authority for saying that "S is P".*

Sundholm's extension of Austin's remark on acts of assertion to inferences yields the following forceful formulation:

> *When I say therefore, I give others my authority for asserting the conclusion, given theirs for asserting the premises.*

[10] The recent terminology in dialogical logic, following a suggestion of Sundholm, uses the denomination "posit" rather than epistemic assumption, but this presumably is only a terminological variant.

[11] This work extends the results of Clerbout/Rahman(2015) that provide the first thorough dialogical approach to CTT. The reader eager to go into the technical details should consult that work.

it has been grounded following the accepted scientific standards that provide the background to the argumentative process involved.

3.2   Player independence and the distinction of the play and the strategic levels

This journey to the origins also engages us to study the whole process at the level of plays, that is, the stuff which winning-strategies (the dialogical notion of demonstration) are made of.

- While a winning strategy for the proponent can be seen as linked to a CTT-proof with epistemic assumptions, the play-level constitute a level is reducible neither to formal truth (that amounts to the existence of a winning strategy) nor to material truth (that comes close to model-theoretic truth).

- What characterizes the play-level (of non-material dialogues) are speech-acts of acceptance that lead to games where the proponent, when he wins a play, he might do so because he *accepts* some specific moves brought forward by the opponent during that play (without asking for the evidence that supports the propositions brought forward) – this leads to some kind of pragmatic-truth, if we wish to speak of truth.

- The strategy level is a level where the proponent wins accepting whatever the player might posit in every play that constitutes that strategy. Zoe McConaughey suggests that one other way to put the difference between the play and the strategic level; is that at the play-level we might have real concrete players, and that the strategic level only considers an arbitrary idealized one.[12] – this also has been suggested in a different formulation (less explicitly) by Keiff/ Rahman (2010).[13] Perhaps one should formulate it in the following way: at the strategy level, the concrete player might be replaced by an arbitrary one.

Let us focus on what we called immanent reasoning, where, as explained, the claims of the opponent are taken to be grounded, without requiring a defence for them.

However, we could also develop a similar kind of analysis for the case of the so-called

- *material dialogues and the critical role of Popper's argumentative process*: here the claims of the opponent engage her to put under dialogical scrutiny that what supports her claim. As pointed out by McConaughey, once the claim has been grounded (according to some agreed standards), the proponent *adheres* rather than

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[12] McConaughey interpretation stems from her dialogical reading of Aristotle; where, according to this reading, while e dialectics deploy the play level the syllogistic, deploy the strategy level.

[13] Keiff/Rahman (2010, vol. 2, n° 66, pp. 149 à 178).

accepts the claim under consideration.[14] Grounded claims of material dialogues provide the most basic form of real definitional equality, and not only of the definitional equality as displayed in logical contexts. In fact, the notion of winning in a material dialogue can be seen as the dialogical analogue of model-theoretical truth that Popper requires for the deployment of the critical argumentative process that links his third and second worlds.

However, material dialogues, lead to a rather static conception of knowledge: what is true in a model is true and cannot change under the conditions set by that model. Certainly the dialogical feature of material dialogues – different to properties of model-theoretic truth – opens the possibility of a more dynamic interpretation by the means of which models can change. But how should we understand and describe such kind of changes? A straightforward approach, on our view, faithful to Popper's own fallibilism, is the one that understands grounded, as acceptance of being grounded. On such a background it is sensible to conceive that this form of acceptances might change, when the grounding conditions do. But then we are back again in the realm of immanent reasoning where epistemic assumptions are deployed and out of the scope of material dialogues.

Now here the main points

What is the play-level?

- The play level of immanent reasoning is a level where the notion of winning is based on acceptances, not adherences.
- Acceptances are brought forward, in a given context, by some concrete player, with the role of the proponent against another concrete player, with the role of the opponent – recall once more that the acceptances are based on posits that the opponent proposes for the start of a play and/or brings forward during that play.
- The winning of a play does not lead to logical truth (validity), nor to local- or model-theoretical truth. It is a level only possible in a framework with an underlying pragmatist theory of meaning.
- Different initial posits proposed by the Opponent lead to different plays and winning strategies for the thesis under consideration. This seems to provide the means to develop Popper's dynamic (fallibilistic) conception of the critical role undertaken by argumentative processes.

Play level, knowledge based on local meaning and the failure of *Logical Omniscience*:

Assume the thesis $A \wedge B \rightarrow B$

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[14] Zoe McConaughey suggested in a discussion at my office the 13/11/2015 that while *acceptance* seems to correspond to the speech act linked to epistemic assumptions, *adherence* corresponds to the speech act linked to material dialogues.

Assume also that the proponent chooses rank 1

> **P**: $A \wedge B \rightarrow B$
> **O**: $A \wedge B$
> **P**: L?
>
> *P* lost the play.

P lost this play, since he made two mistakes, the crucial one is that the has chosen left instead of right, and moreover since he has chosen the rank 2 he cannot repair his first bad choice, by launching a new challenge

One can imagine contexts where this might happen (time restrictions for example) How to explain such kind of outcomes? There is no truth value that explains this, nor is it explained by logical truth. Does this mean that **P** does not master the meaning of the conjunction? From the purely proof-theoretical point of view of meaning explanations, he has no such a knowledge. But the loss of the play is not incompatible with the fact that though he h**as local knowledge** on the logical constants involved; because of some contextual constraints he cannot fully develop this knowledge in his favour or does not come to realize how to do so. It is not the case that the there is a winning strategy that is not accessible by an epistemic subject, but rather that a concrete player in some concrete situation does not find it. However, the mere possibility to enter into dialogical interaction, assumes some level of objective knowledge that allows the deployment of the play.

In fact, the standard way to deal with such situation is to think an impossible world where the negation of logical truths can be true.

But this is the case since the pragmatist play-level of meaning is not present in most of the standard frameworks. Notice that **P**'s losing cannot be understood as establishing that there is a situation where its negation is the case!
Deductive closure is reasonable; when we are thinking strategically: when we think in the way of Kant on the formal: what was possible and what was, normatively speaking the best of the possible choices. Or *whoever* knows *A*, ought to know whatever can be inferred from *A*. This is what a winning strategy is about. However, this does not mean that some concrete player in a given situation will find the winning strategy.

4   Conclusions

The failure of logical omniscience is perfectly compatible with deductive closure if we link the latter with the strategy level and the failure of logical omniscience with the play level. Since Popper does not distinguish both levels, he identifies deductive closure with what he calls the autonomy of the denizens of the third world. But, according to our analysis, such an autonomy does not require Popper's dispensability of any form of epistemic approaches based on agentivity and/or assertibility conditions.

Indeed, on one hand, those formal systems that includes in its formalization Brower's *creating subject* as axiomatized via the Kreisel-Kripke Principle are, from the constructivist point of view, not epistemic– since this principle has been shown to be classically valid (Sundholm (2014). On the other, epistemic approaches to objective knowledge achieved by argumentative interaction do not require the dispensability of knowing subjects engaged in that kind of interaction but it requires to extend the notion of such subjects to arbitrary actors of dialogues on which winning strategies might be built. Moreover, the acts of acceptance that characterize immanent reasoning produce two different forms of objective knowledge: one based on the local meaning deployed at the play-level and the second one on the strategic level. Both of them deploy the critical role that argumentative processes Popper is aiming at.

Let us point out that the objective knowledge produced by the argumentative process that takes place at the play-level – on our view, the most fundamental level of a critical argumentative process – is based on the simple fact that any dialogical interaction, that aims at some kind of objective knowledge requires a basis shared by any player prepared to enter in an argumentative interaction of asking and giving reasons.

Where does this all leave us in relation to Popper's proposal of an *Epistemology without a Knowing Subject*?

Well, either Popperians give up the dynamic features of the fallibilistic project or argumentative interaction has to be conceived within the framework of some form of game-theoretical approach to knowledge and meaning. Marcel Nguimbi (2014) takes up the latter corn of the dilemma and develops a dialogical approach to Popperian fallibilism. However within the conceptual setting of this form of Popperianism knowing subjects, the actors of the argumentative process, cannot be dispensed of.

In fact, the main conceptual background underlying the approach developed above is that interaction is at the basis of every knowledge formation. Moreover, on our view this interaction is one that takes place in the environment where philosophy started, namely a dialogue constituted by a dynamic of giving and asking for reasons, as a consequence of taking responsibility for our own claims and actions.

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