UNAVOIDABLE SEQUENCES IN CONSTRUCTIVE ANALYSIS

JOAN RAND MOSCHOVAKIS

Introduction.

Kleene's formalization **FIM** of intuitionistic analysis ([3] and [2]) includes bar induction, countable and continuous choice, but is consistent with the statement that there are no non-recursive functions ([5]). Veldman ([12]) showed that in **FIM** the constructive analytical hierarchy collapses at Σ^1_2. These are serious obstructions to interpreting the constructive content of classical analysis, just as the collapse of the arithmetical hierarchy at Σ^0_3 in **HA** + MP_0 + ECT_0 (cf. [6]) limits the scope and effectiveness of recursive analysis. Bishop's constructive mathematics, now undergoing (partial) formalization, is consistent with intuitionistic analysis and also with recursive analysis so must have similar defects. It seems natural to ask whether e.g. intuitionistic analysis could incorporate more of classical mathematics without seriously compromising its constructive content.

Brouwer and Bishop agreed that constructive mathematics was an intellectual work in progress. Bishop and Markov agreed on the primary importance of computational content. All three recognized the constructive significance of continuity. Their insights can be interpreted as prescribing *admissible rules*, rather than restrictive axiom schemas, for constructive formal systems compatible with larger parts of classical mathematics.[1]

A theory based on intuitionistic logic may adhere to a constructive closure rule without proving the corresponding implication. For example, the recursive choice rule known as *Church's Rule for arithmetic* CR_0:

"If $\forall \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y})$ is provable where $\mathrm{A}(\mathrm{x}, \mathrm{y})$ is arithmetical and contains only x, y free, then $\exists \mathrm{e} \forall \mathrm{x} \exists \mathrm{y} \exists \mathrm{z}[\mathrm{T}(\mathrm{e}, \mathrm{x}, \mathrm{y})\ \&\ \mathrm{U}(\mathrm{y}) = \mathrm{z}\ \&\ \mathrm{A}(\mathrm{x}, \mathrm{z})]$ is also provable."

holds for intuitionistic arithmetic **HA**, while the arithmetical form CT_0 of Church's Thesis is unprovable. Similarly, **HA** satisfies *Markov's Rule for arithmetic* MR_0:

"If $\forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \vee \neg \mathrm{A}(\mathrm{x}))\ \&\ \neg\neg \exists \mathrm{x} \mathrm{A}(\mathrm{x})$ is provable then also $\exists \mathrm{x} \mathrm{A}(\mathrm{x})$ is provable."

but does not prove the corresponding implication MP_0.

One type up, a constructive theory of numbers and number-theoretic sequences ("constructive analysis") based on intuitionistic logic generally satisfies Brouwer's Rule of continuous choice, some form of Markov's Rule, and the Church-Kleene Rule asserting that only recursive sequences can be proved to exist; precise definitions are in the next section.

I am grateful to Michael Beeson and one anonymous referee for observing that a modern reader would prefer Troelstra's treatment of modified relative realizability to the original versions in [3] and [5], and to both anonymous referees for suggesting many improvements in the text. For many enthusiastic discussions about axioms for intuitionistic mathematics I thank Garyfallia Vafeiadou.

[1] Kohlenbach's "proof mining" implicitly uses this idea to extract constructive information from classical proofs. Kleene [1], [2] are important precursors.

Here we introduce a semi-constructive theory $\mathbf{T}_2$ extending **FIM** by axioms asserting that certain kinds of choice sequences are *unavoidable* (cannot fail to exist) and that no choice sequence can fail to be classically Σ^1_1 (hence also Δ^1_1). $\mathbf{T}_2$ is consistent simultaneously with first-order classical arithmetic **PA** and with Vesley's Schema, which refutes the analytical form of Markov's Principle. We conjecture that $\mathbf{T}_2$ satisfies Brouwer's Rule and the Church-Kleene Rule, so preserves the constructive sense of existence.

Consistency is established using a subtle kind of modified relative realizability related to [5]. A good modern exposition of modified relative realizability, for an axiomatization based on the logic of partial terms, is Troelstra [10] (with a few misprints); cf. also [9]. We feel more confident working with Kleene's original axiomatization of intuitionistic analysis ([3], [1]) and ask the reader's indulgence for a rather old-fashioned presentation. Following [5] our potential and actual realizers are implicitly rather than explicitly typed; most modern treatments also adopt this simplifying convention. We hope the expository material, definitions, and statements of theorems concerning five recursively axiomatizable extensions $\mathbf{T}_2$-$\mathbf{T}_6$ of **FIM** will suffice to give the casual reader an inkling of the possibilities.

1. Preliminaries

We work in a two-sorted language $\mathcal{L}$ with variables over numbers and one-place number-theoretic functions (*choice sequences*). Our base theory **M** is the minimal theory used by Kleene in [2] to formalize the theory of recursive partial functionals, function realizability and q-realizability. **M** extends Heyting arithmetic to the two-sorted language and includes defining axioms for finitely many primitive recursive function constants, a λ-reduction schema, and the function comprehension schema $\forall \mathrm{x} \exists ! \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}) \rightarrow \exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x}))$.[2]

An *$\mathcal{L}$-theory* is a consistent axiomatic extension of **M** in the language $\mathcal{L}$ (possibly enriched by additional primitive recursive function constants). Let us call an $\mathcal{L}$-theory *intuitionistic* if its logical axioms and rules are exactly those of two-sorted intuitionistic predicate logic; *classical* if its logical postulates are those of two-sorted classical predicate logic; and *intermediate* otherwise.

The $\mathcal{L}$-theories $\mathbf{T}$ which have been proposed so far to express parts of constructive mathematics typically have one or more of the following properties, none of which can hold for a classical $\mathcal{L}$-theory. An *explicit* $\mathcal{L}$-theory T provides explicit witnesses for existential theorems; in particular,

(a) If $\exists \mathrm{x} \mathrm{A}(\mathrm{x})$ is closed and $\vdash_{\mathbf{T}} \exists \mathrm{x} \mathrm{A}(\mathrm{x})$ then $\vdash_{\mathbf{T}} \mathrm{A}(\mathbf{n})$ for some numeral $\mathbf{n}$.
(b) If $\exists \alpha \mathrm{A}(\alpha)$ is closed and $\vdash_{\mathbf{T}} \exists \alpha \mathrm{A}(\alpha)$, then for some $\mathrm{B}(\alpha)$ with only α free:
$$\vdash_{\mathbf{T}} \forall \alpha[\mathrm{B}(\alpha) \rightarrow \mathrm{A}(\alpha)] \ \&\ \exists ! \alpha \mathrm{B}(\alpha).$$

A *Brouwerian* $\mathcal{L}$-theory $\mathbf{T}$ satisfies *Brouwer's Rule*:[3]
"If $\vdash_{\mathbf{T}} \forall \alpha \exists \beta \mathrm{A}(\alpha, \beta)$ then $\vdash_{\mathbf{T}} \exists \sigma \forall \alpha \exists \beta[\forall \mathrm{x}(\{\sigma\}[\alpha](\mathrm{x}) \simeq \beta(\mathrm{x}))\ \&\ \mathrm{A}(\alpha, \beta)]$."

A *recursively acceptable* $\mathcal{L}$-theory $\mathbf{T}$ satisfies *Markov's Rule*:
"If $\vdash_{\mathbf{T}} \neg\neg \exists \mathrm{x} \mathrm{A}(\mathrm{x})\ \&\ \forall \mathrm{x}[\mathrm{A}(\mathrm{x}) \vee \neg \mathrm{A}(\mathrm{x})]$ then $\vdash_{\mathbf{T}} \exists \mathrm{x} \mathrm{A}(\mathrm{x})$"
and *Church's Rule*:
"If $\vdash_{\mathbf{T}} \exists \alpha \mathrm{A}(\alpha)$ with $\exists \alpha \mathrm{A}(\alpha)$ closed, then

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[2]The ! denotes uniqueness. An essentially equivalent system is Troelstra's **EL** ([9] and [11]).

[3]Here "$\{\sigma\}[\alpha](\mathrm{x}) \simeq \mathrm{z}$" expresses "$\sigma(\langle x\rangle * \overline{\alpha}(\mu y \sigma(\langle x\rangle * \overline{\alpha}(y)) > 0)) \simeq z+1$" where $\langle x\rangle * \overline{\alpha}(x)$ codes the sequence $x, \alpha(0), \ldots, \alpha(x-1)$, so every σ codes a continuous partial functional.

$\vdash_{\mathbf{T}} \exists e[\forall x \exists ! y T(e, x, y)\ \&\ \forall \alpha[\forall x \forall y[T(e, x, y) \to \alpha(x) = U(y)] \to A(\alpha)]]$."

Any explicit theory **T** for which Church's Rule is admissible evidently satisfies the *Church-Kleene Rule*:

"If $\vdash_{\mathbf{T}} \exists \alpha A(\alpha)$ where $\exists \alpha A(\alpha)$ is closed, then for a suitable number e:

$\vdash_{\mathbf{T}} \forall x \exists ! y T(\mathbf{e}, x, y)\ \&\ \forall \alpha[\forall x \forall y[T(\mathbf{e}, x, y) \to \alpha(x) = U(y)] \to A(\alpha)]$."

Kleene and Vesley's formal theory **FIM** for intuitionistic analysis has all these properties. So do the $\mathcal{L}$-theory $\mathbf{T}_1 = \mathbf{FIM} + \mathrm{MP}_1$ and its classically correct $\mathcal{L}$-subtheory $\mathbf{T}_0 = \mathbf{M} + \mathrm{BI}_1 + \mathrm{MP}_1$, which prove that the constructive arithmetical hierarchy is proper (cf. [7]). Here BI_1 is the axiom schema (26.3b in [3]) of bar induction, so that $\mathbf{M} + \mathrm{BI}_1$ comes from **FIM** by weakening countable choice to function comprehension and omitting Brouwer's principle of continuous choice; and MP_1 is the strong analytical form $\forall \alpha(\neg\neg \exists x \alpha(x) = 0 \to \exists x \alpha(x) = 0)$ of Markov's Principle.

This note concerns five recursively axiomatizable intermediate $\mathcal{L}$-theories $\mathbf{T}_2$-$\mathbf{T}_6$ which are Brouwerian (in the strong sense of extending **FIM**) but do not prove MP_1. $\mathbf{T}_3$ and $\mathbf{T}_5$, which include all of first-order Peano arithmetic, are not explicit and fail to satisfy Church's Rule. We conjecture that Markov's Rule with sequence parameters is not admissible for any of $\mathbf{T}_2$-$\mathbf{T}_6$, and that $\mathbf{T}_2$, $\mathbf{T}_4$ and $\mathbf{T}_6$ satisfy the Church-Kleene Rule and hence are explicit.

2. Unavoidable sequences

Definition. If **T** is an $\mathcal{L}$-theory and $A(x, y)$ a formula (perhaps with other free variables of both sorts) such that $\vdash_{\mathbf{T}} \forall x \neg\neg \exists ! y A(x, y)$ (equivalently, such that $\vdash_{\mathbf{T}} \forall x \neg\neg \exists y A(x, y)\ \&\ \forall x \forall y \forall z[A(x, y)\ \&\ A(x, z) \to y = z]$), then we say that $A(x, y)$ *classically defines an infinite sequence in* **T** (*from* the other free variables, if any).

Proposition. If **T** is an $\mathcal{L}$-theory and $\vdash_{\mathbf{T}} \neg\neg \exists ! \alpha \forall x A(x, \alpha(x))$, then $A(x, y)$ classically defines an infinite sequence in **T**.

Proof. From $\neg\neg \exists ! \alpha \forall x A(x, \alpha(x))$ follow $\neg\neg \forall x \forall y \forall z[A(x, y)\ \&\ A(x, z) \to y = z]$ and $\neg\neg \forall x \exists y A(x, y)$, so $\forall x \forall y \forall z[A(x, y)\ \&\ A(x, z) \to y = z]$ and $\forall x \neg\neg \exists y A(x, y)$ by intuitionistic logic with the stability of number-theoretic equality.

Remarks:

(1) The converse fails. The predicate

$$A(x, y) \ \equiv\ [y \leq 1\ \&\ [y = 0 \leftrightarrow \exists z(T(x, x, z)\ \&\ U(z) = 1)]]$$

classically defines an infinite sequence in **M** but $\neg\neg \exists \alpha \forall x A(x, \alpha(x))$ contradicts weak Church's Thesis $\forall \alpha \neg\neg \exists e \forall x \exists y[T(e, x, y)\ \&\ U(y) = \alpha(x)]$, which is consistent with **M** and even with **FIM** by [5].

(2) $A(x, y)$ classically defines an infinite sequence in **T** if and only if $\neg\neg A(x, y)$ classically defines an infinite sequence in **T**.

(3) If **T** is a Brouwerian theory and $\vdash_{\mathbf{T}} \neg\neg \exists ! \alpha A(\alpha)$ then $\nvdash_{\mathbf{T}} \forall \alpha[A(\alpha) \vee \neg A(\alpha)]$.
Proof. Assume $\vdash_{\mathbf{T}} \neg\neg \exists ! \alpha A(\alpha)$ and $\vdash_{\mathbf{T}} \forall \alpha[A(\alpha) \vee \neg A(\alpha)]$. By Brouwer's Rule, **T** proves that $A(\alpha)$ has a continuous characteristic function depending only on an initial segment of α, and hence $\vdash_{\mathbf{T}} \neg \exists ! \alpha A(\alpha)$, violating the consistency of **T**.

(4) $\vdash_{\mathbf{M}} \exists ! x A(x) \to \forall x(A(x) \vee \neg A(x))$.

(5) $\vdash_{\mathbf{FIM}} \neg\neg \exists ! \alpha A(\alpha) \to \neg \forall \alpha[A(\alpha) \vee \neg A(\alpha)]$.

If $\mathrm{A(x,y)}$ classically defines an infinite sequence in $\mathbf{T}$ and α is a choice sequence such that $\forall x A(x,\alpha(x))$ holds under an interpretation $\mathcal{I}$ of $\mathbf{T}$, we may say that α is classically defined by $\mathrm{A(x,y)}$ under the interpretation.

Definition. If $\mathbf{T}$ is an $\mathcal{L}$-theory, $\mathrm{A}(\alpha)$ is a formula with α as its only free variable, and $\vdash_{\mathbf{T}} \neg\neg\exists!\alpha\mathrm{A}(\alpha)$, then the sequence classically defined by $\forall\beta[\mathrm{A}(\beta) \rightarrow \beta(\mathrm{x}) = \mathrm{y}]$ under any interpretation of $\mathbf{T}$ will be called *unavoidable over* $\mathbf{T}$.

More generally, if $\vdash_{\mathbf{T}} \neg\neg\exists\alpha\mathrm{A}(\alpha)$ we may say "A sequence α satisfying $\mathrm{A}(\alpha)$ is unavoidable over $\mathbf{T}$." Only classically recursive sequences are unavoidable over **FIM** (Moschovakis [5]). In contrast, the characteristic functions of all arithmetical relations (with or without sequence parameters), and of all classically Δ^1_1 relations, are unavoidable over **FIM** + MP_1 and over **M** + BI_1 + MP_1 (Solovay and Moschovakis, in [7]).

We are interested in the general question of determining all the unavoidable sequences over an arbitrary constructive $\mathcal{L}$-theory including bar induction BI_1. As an example, consider the $\mathcal{L}$-theory $\mathbf{T}_2$ which is obtained by adjoining to **FIM** one axiom schema and two axioms:

I. $\neg\neg\forall\mathrm{x}[\mathrm{A(x)} \vee \neg\mathrm{A(x)}]$ for arithmetical $\mathrm{A(x)}$ with parameters allowed.

II. "There are no sequences which are not classically Σ^1_1":

$$\forall\alpha\neg\neg\exists\mathrm{e}\forall\mathrm{x}\forall\mathrm{y}[\alpha(\mathrm{x}) = \mathrm{y} \leftrightarrow \neg\neg\exists\beta\forall\mathrm{z}\neg\mathrm{T}(\mathrm{e},\mathrm{x},\mathrm{y},\overline{\beta}(\mathrm{z}))].$$

III. "Every sequence classically defined by a Π^1_1 formula is unavoidable":

$$\forall\mathrm{e}[\forall\mathrm{x}\neg\neg\exists!\mathrm{y}\forall\beta\exists\mathrm{z}\mathrm{T}(\mathrm{e},\mathrm{x},\mathrm{y},\overline{\beta}(\mathrm{z})) \rightarrow \neg\neg\exists!\alpha\forall\mathrm{x}\forall\mathrm{y}[\alpha(\mathrm{x}) = \mathrm{y} \leftrightarrow \forall\beta\exists\mathrm{z}\mathrm{T}(\mathrm{e},\mathrm{x},\mathrm{y},\overline{\beta}(\mathrm{z}))]].$$

Remarks. (I), which is equivalent over **M** to arithmetical double negation shift (with parameters) DNS_0, ensures that the characteristic function of every arithmetical predicate, with or without sequence parameters, is unavoidable over $\mathbf{T}_2$. If $\mathrm{A(x)}$ is such a predicate then

$$\vdash_{\mathbf{T}_2} \neg\neg\exists!\alpha\forall\mathrm{x}[\alpha(\mathrm{x}) \leq 1 \ \&\ (\alpha(\mathrm{x}) = 0 \leftrightarrow \mathrm{A(x)})].$$

(II) guarantees that only classically Δ^1_1 sequences are unavoidable over $\mathbf{T}_2$, since every classically Σ^1_1 sequence is classically Δ^1_1.[4] (III) entails classical function comprehension for Π^1_1 formulas.

3. Δ^1_1 REALIZABILITY

In a nutshell, a recursive realizability interpretation implements the Brouwer-Heyting-Kolmogorov interpretation of the logical connectives and quantifiers by ultimately attaching, to each closed theorem of a constructive theory, a recursive object verifying that the theorem is correct.[5] To prove the independence of MP_0, Kreisel modified Kleene and Nelson's original number-realizability for **HA** by introducing auxiliary potential realizers which agree in type with a formula but may give misleading information about it. Kleene extended both interpretations to **FIM**, using number-theoretic functions as potential and actual realizers, and suggested methods of relativization.

[4]The "classical quantifiers" $\forall\mathrm{x}\neg\neg$, $\neg\neg\exists\mathrm{x}$, $\forall\beta\neg\neg$ and $\neg\neg\exists\beta$ were developed and used to express classical theorems in an intuitionistic setting by Krauss [4], unpublished; cf. [7]. Note the difference between "classically Π^1_1" and "classically defined by a (constructively) Π^1_1 formula."

[5]Troelstra's [10] gives a general framework for realizability interpretations, with historical references and major results.

We prove consistency of $\mathbf{T}_2$ (and later of $\mathbf{T}_3$-$\mathbf{T}_6$) by providing a classical modified relative realizability interpretation satisfying all of first-order Peano arithmetic **PA** but not MP_1. This new ${}^{\Delta^1_1}$realizability is analogous to the ${}^{\mathrm{G}}$realizability of [5] with the same potential realizers, but with Δ^1_1 sequences in place of recursive sequences as the actual realizing objects (though in the end every closed theorem of $\mathbf{T}_2$ will be shown to have a recursive realizer). Along with Kleene's brackets, we use his informal Λ notation to indicate an arbitrary choice of a primitive recursive modulus of continuity for a recursive partial functional.[6] Recall that the collection of (classically) Δ^1_1 sequences is closed under "recursive in."[7]

Definition. We define when a sequence ε *agrees with* a formula E of $\mathcal{L}$, by formula induction as in [5], weakening "properly defined" to "completely defined" as in [2].

(1) ε *agrees with* a prime formula P, for each ε.
(2) ε *agrees with* A & B, if $(\varepsilon)_0$ *agrees with* A and $(\varepsilon)_1$ *agrees with* B.
(3) ε *agrees with* A $\vee$ B, if $(\varepsilon(0))_0 = 0$ implies that $(\varepsilon)_1$ *agrees with* A, while $(\varepsilon(0))_0 \neq 0$ implies that $(\varepsilon)_1$ *agrees with* B.
(4) ε *agrees with* A $\rightarrow$ B, if, whenever α *agrees with* A, $\{\varepsilon\}[\alpha]$ is defined and *agrees with* B.
(5) ε *agrees with* $\neg$A, if ε *agrees with* A $\rightarrow 1 = 0$ by the preceding clause.
(6) ε *agrees with* $\exists$xA(x), if $(\varepsilon)_1$ *agrees with* A(x).
(7) ε *agrees with* $\forall$xA(x), if, for each x, $\{\varepsilon\}[x]$ is completely defined and *agrees with* A(x).
(8) ε *agrees with* $\exists\alpha$A(α), if $\{(\varepsilon)_0\}$ is completely defined and $(\varepsilon)_1$ *agrees with* A(α).
(9) ε *agrees with* $\forall\alpha$A(α), if, for each sequence α, $\{\varepsilon\}[\alpha]$ is completely defined and *agrees with* A(α).

Definition. Let ε be a Δ^1_1 sequence and E a formula of $\mathcal{L}$ containing free at most the distinct number and sequence variables Ψ. Let $\mathit{\Psi}$ be natural numbers and Δ^1_1 sequences corresponding to Ψ. We define when ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ E, by induction:

(1) ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ a prime formula P, if P is *true-*$\mathit{\Psi}$.
(2) ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ A & B, if $(\varepsilon)_0$ ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ A and $(\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ B.
(3) ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ A $\vee$ B, if $(\varepsilon(0))_0 = 0$ implies that $(\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ A, while $(\varepsilon(0))_0 \neq 0$ implies that $(\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ B.
(4) ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ A $\rightarrow$ B, if ε agrees with A $\rightarrow$ B and, whenever α (is Δ^1_1 and) ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ A, $\{\varepsilon\}[\alpha]$ is defined and ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ B.
(5) ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ $\neg$A, if ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ A $\rightarrow 1 = 0$ by the preceding clause.
(6) ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ $\exists$xA(x), if $(\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}, (\varepsilon(0))_0$ A(x).
(7) ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ $\forall$xA(x), if, for each x, $\{\varepsilon\}[x]$ is defined (and therefore Δ^1_1) and ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}, x$ A(x).
(8) ε ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}$ $\exists\alpha$A(α), if $\{(\varepsilon)_0\}$ is defined (and therefore Δ^1_1) and $(\varepsilon)_1$ ${}^{\Delta^1_1}$*realizes-*$\mathit{\Psi}, \{(\varepsilon)_0\}$ A(α).

[6]See §8.2 of Kleene and Vesley [3]. Formalization of the argument could probably be carried out in an appropriate classical extension of **M**, based on the detailed formal treatment in [2] of recursive functionals within **M**.

[7]Any recursively closed class F of sequences could be used instead of Δ^1_1 to give a corresponding notion of ${}^{\mathrm{F}}$realizability satisfying **FIM** and more. We need Δ^1_1 here to verify (I) - (III).

(9) ε ${}^{\Delta^1_1}$*realizes-Ψ* $\forall\alpha \mathrm{A}(\alpha)$, if ε agrees with $\forall\alpha\mathrm{A}(\alpha)$ and, for each Δ^1_1 sequence α, $\{\varepsilon\}[\alpha]$ is defined (and therefore Δ^1_1) and ${}^{\Delta^1_1}$*realizes-Ψ,α* $\mathrm{A}(\alpha)$.

Definition. A closed formula E is ${}^{\Delta^1_1}$*realizable* if and only if some Δ^1_1 sequence ε ${}^{\Delta^1_1}$realizes E. An open formula is ${}^{\Delta^1_1}$*realizable* if and only if its universal closure is.

Lemma 1. If ε ${}^{\Delta^1_1}$realizes-Ψ E then ε agrees with E.

Lemma 2. If s is a term free for y in A(y), then ε agrees with A(y) if and only if ε agrees with A(s). Similarly for v a functor free for β in $\mathrm{A}(\beta)$.

Lemma 3. ε agrees with E if and only if ε agrees with the result of replacing each part of E of the form $\neg$A by ($\mathrm{A} \rightarrow 1 = 0$). Similarly for "$\varepsilon$ ${}^{\Delta^1_1}$realizes-Ψ E."

Lemma 4. For each formula E there is a primitive recursive sequence ε^{E} which agrees with E.

Proof. By induction on the logical complexity of E, for example: If E is prime then ε^{E} is $\lambda t.0$. Given ε^{A} and ε^{B} agreeing with A and B respectively, $\varepsilon^{\mathrm{A}\vee\mathrm{B}}$ is $\langle\lambda t.0,\varepsilon^{\mathrm{A}}\rangle$ and $\varepsilon^{\mathrm{A}\rightarrow\mathrm{B}}$ is $\Lambda\alpha\varepsilon^{\mathrm{B}}$. Given $\varepsilon^{\mathrm{A}(\alpha)}$ agreeing with $\mathrm{A}(\alpha)$, then $\varepsilon^{\exists\alpha\mathrm{A}(\alpha)}$ is $\langle\Lambda.\lambda t.0,\varepsilon^{\mathrm{A}(\alpha)}\rangle$ and $\varepsilon^{\forall\alpha\mathrm{A}(\alpha)}$ is $\Lambda\alpha\{\varepsilon^{\mathrm{A}(\alpha)}\}[\alpha]$.

Lemma 5. Let Ψ be a list of distinct variables including all those occurring free in E, let Ψ' be those which actually occur free in E, let ε be a Δ^1_1 sequence and $\mathit{\Psi}$ be numbers and Δ^1_1 sequences corresponding to Ψ. Then ε ${}^{\Delta^1_1}$realizes-$\mathit{\Psi}$ E if and only if ε ${}^{\Delta^1_1}$realizes-$\mathit{\Psi}'$ E.

Lemma 6. For no formula E are there Δ^1_1 sequences $\varepsilon_1,\varepsilon_2$ and numbers and Δ^1_1 sequences $\mathit{\Psi}$ corresponding to the variables Ψ free in E, such that ε_1 ${}^{\Delta^1_1}$realizes-$\mathit{\Psi}$ E and ε_2 ${}^{\Delta^1_1}$realizes-$\mathit{\Psi}$ $\neg$E.

Lemma 7. Let E contain free only Ψ. Then E is ${}^{\Delta^1_1}$realizable if and only if there is a recursive partial functional $\varphi[\mathit{\Psi},\gamma] \simeq \lambda t.\varphi(\mathit{\Psi},\gamma,t)$ such that, for some Δ^1_1 sequence δ: $\varphi[\mathit{\Psi},\delta]$ is completely defined and agrees with E for every choice of $\mathit{\Psi}$, and if every sequence in the list $\mathit{\Psi}$ is Δ^1_1 then $\varphi[\mathit{\Psi},\delta]$ ${}^{\Delta^1_1}$realizes-$\mathit{\Psi}$ E.

Proof. Suppose for concreteness that E is $\mathrm{A}(\alpha,\mathrm{x})$ so the universal closure of E is $\forall\alpha\forall\mathrm{x}\mathrm{A}(\alpha,\mathrm{x})$. Let $\varphi[\alpha,x,\gamma] \simeq \{\{\gamma\}[\alpha]\}[x]$. If ε ${}^{\Delta^1_1}$realizes $\forall\alpha\forall\mathrm{x}\mathrm{A}(\alpha,\mathrm{x})$, then $\varepsilon \in \Delta^1_1$ and $\{\{\varepsilon\}[\alpha]\}[x]$ is completely defined and agrees with $\mathrm{A}(\alpha,\mathrm{x})$ for every α, x; moreover, for each $\alpha \in \Delta^1_1$ and $x \in \omega$, $\varphi[\alpha,x,\varepsilon]$ (is Δ^1_1 and) ${}^{\Delta^1_1}$realizes-α,x $\mathrm{A}(\alpha,\mathrm{x})$.

Conversely, if $\varphi[\alpha,x,\gamma]$ is a recursive partial functional and ε a Δ^1_1 sequence such that $\varphi[\alpha,x,\varepsilon]$ is completely defined and agrees with $\mathrm{A}(\alpha,\mathrm{x})$ for every α and x, and ${}^{\Delta^1_1}$realizes-α,x $\mathrm{A}(\alpha,\mathrm{x})$ for every $\alpha \in \Delta^1_1$ and every x, then $\Lambda\alpha\Lambda x\varphi[\alpha,x,\varepsilon]$ (is recursive in ε, hence is Δ^1_1 and) ${}^{\Delta^1_1}$realizes $\forall\alpha\forall\mathrm{x}\mathrm{A}(\alpha,\mathrm{x})$.

Lemma 8. (a) Let A(y) be a formula containing free at most the distinct variables Ψ, y, let s be a term containing free at most Ψ, y and free for y in A(y), let $\mathit{\Psi}, y$ be Δ^1_1 sequences and natural numbers, and let $s(\mathit{\Psi},y)$ be the number expressed by s when Ψ, y are interpreted by $\mathit{\Psi},y$. Then a sequence ε ${}^{\Delta^1_1}$realizes-$\mathit{\Psi},y$ A(s) if and only if ε ${}^{\Delta^1_1}$realizes-$s(\mathit{\Psi},y)$ A(y). (b) Similarly if $\mathrm{A}(\beta)$ contains free at most Ψ,β, and v is a functor containing free at most Ψ,β and free for β in $\mathrm{A}(\beta)$ and expressing $\varphi[\mathit{\Psi},\beta]$, then for Δ^1_1 sequences and numbers $\mathit{\Psi},\beta$: the sequence ε ${}^{\Delta^1_1}$realizes-$\mathit{\Psi},\beta$ A(v) if and only if ε ${}^{\Delta^1_1}$realizes-$\mathit{\Psi},\varphi[\mathit{\Psi},\beta]$ $\mathrm{A}(\beta)$.

Lemma 9. (a) For each arithmetical formula $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$ with no free variables other than $\beta, x_1, \ldots, x_k$, and for each Δ^1_1 sequence β, there is a Δ^1_1 function ϑ_β of $t, x_1, \ldots, x_k$ such that if $\vartheta[x_1, \ldots, x_k] = \lambda t.\vartheta_\beta(t, x_1, \ldots, x_k)$ then for all $x_1, \ldots, x_k$:

(i) $\vartheta[x_1, \ldots, x_k]$ agrees with $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$.

(ii) $\vartheta[x_1, \ldots, x_k]$ ${}^{\Delta^1_1}$realizes-$\beta, x_1, \ldots, x_k$ $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$ if and only if, under the intended classical interpretation, $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$ is true-$\beta, x_1, \ldots, x_k$.

Similarly with $\beta_1, \ldots, \beta_m$ in place of β.

(b) With the same conditions on $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$ and β, there is a Δ^1_1 sequence ψ which ${}^{\Delta^1_1}$realizes-β $\forall \mathrm{x}_1 \ldots \forall \mathrm{x}_k[\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k) \vee \neg \mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)]$. In particular, if $\mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_k)$ is purely arithmetical, then $\mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_k) \vee \neg \mathrm{A}(\mathrm{x}_1, \ldots, \mathrm{x}_k)$ is ${}^{\Delta^1_1}$realizable.

Proof of (a), by induction on the logical form of A.

(1) If A is prime then $\vartheta[x_1, \ldots, x_k]$ is $\lambda t.0$.

(2) If A is B & C where ϑ_1, ϑ_2 satisfy (i) and (ii) for B, C respectively, then $\vartheta[x_1, \ldots, x_k]$ is $\langle \vartheta_1, \vartheta_2 \rangle$.

(3) If A is $\mathrm{B} \vee \mathrm{C}$ where ϑ_1, ϑ_2 satisfy (i) and (ii) for B, C respectively, then $\vartheta[x_1, \ldots, x_k]$ is $\langle \lambda t.\chi_\beta(x_1, \ldots, x_k), \psi[x_1, \ldots, x_k] \rangle$ where

$$\chi_\beta(x_1, \ldots, x_k) = \begin{cases} 0 & \text{if } \mathrm{B}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k) \text{ is true-}\beta, x_1, \ldots, x_k, \\ 1 & \text{otherwise.} \end{cases}$$

$$\psi[x_1, \ldots, x_k] = \begin{cases} \vartheta_1[x_1, \ldots, x_k] & \text{if } \chi_\beta(x_1, \ldots, x_k) = 0, \\ \vartheta_2[x_1, \ldots, x_k] & \text{otherwise.} \end{cases}$$

(4) If A is $\mathrm{B} \rightarrow \mathrm{C}$ where ϑ_1, ϑ_2 satisfy (i) and (ii) for B, C respectively, then $\vartheta[x_1, \ldots, x_k]$ is $\Lambda\pi\, \vartheta_2[x_1, \ldots, x_k]$.

(5) If A is $\exists \mathrm{y} \mathrm{B}(\mathrm{y}, \beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$ where $\vartheta_1[y, x_1, \ldots, x_k]$ satisfies (i) and (ii) for B, then $\vartheta[x_1, \ldots, x_k]$ is $\langle \lambda t.\nu(x_1, \ldots, x_k), \vartheta_1[\nu(x_1, \ldots, x_k), x_1, \ldots, x_k] \rangle$ where $\nu(x_1, \ldots, x_k)$ is the least y such that $\mathrm{B}(\mathrm{y}, \beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$ is true-$y, \beta, x_1, \ldots, x_k$ if such a y exists (classically), otherwise 0.

(6) If A is $\forall \mathrm{y} \mathrm{B}(\mathrm{y}, \beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$ where $\vartheta_1[y, x_1, \ldots, x_k]$ satisfies (i) and (ii) for B, then $\vartheta[x_1, \ldots, x_k]$ is $\Lambda y\, \vartheta_1[y, x_1, \ldots, x_k]$.

Proof of (b): Given a Δ^1_1 sequence β, let ϑ satisfy (i) and (ii) for $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$, let $\chi_\beta(x_1, \ldots, x_k)$ be the characteristic function of the standard classical interpretation of $\mathrm{A}(\beta, \mathrm{x}_1, \ldots, \mathrm{x}_k)$ with respect to β, and let $\pi[x_1, \ldots, x_k]$ be $\langle \lambda t.\chi_\beta(x_1, \ldots, x_k), (1 \dot{-} \chi_\beta(x_1, \ldots, x_k))\vartheta[x_1, \ldots, x_k] + \chi_\beta(x_1, \ldots, x_k)\Lambda\tau\lambda t.0 \rangle$. Then the sequence $\psi \simeq \Lambda x_1 \ldots \Lambda x_k \pi[x_1, \ldots, x_k]$ satisfies the conclusion of (b).

Theorem 1. If $\Gamma \vdash_{\mathbf{T}_2} \mathrm{E}$ and the formulas Γ are ${}^{\Delta^1_1}$realizable, then E is ${}^{\Delta^1_1}$realizable.

Proof. First, for each axiom E (or instance E of an axiom schema) containing free only Ψ we give a recursive partial function $\varphi[\Psi, \gamma]$ and a particular Δ^1_1 sequence δ satisfying the condition of Lemma 7; we call such a $\varphi[\Psi, \delta]$ a ${}^{\Delta^1_1}$*realizer for* E. Then, assuming that a ${}^{\Delta^1_1}$realizer exists for each premise of a rule of inference, we give a ${}^{\Delta^1_1}$realizer for the conclusion.

For each of the axiom schemas 1a, 1b, 3-7 of intuitionistic propositional logic, a ${}^{\Delta^1_1}$realizer is $\varphi[\Psi, \lambda t.0]$ where $\varphi[\Psi, \gamma] \simeq \vartheta[\Psi]$ is the primitive recursive realizing functional given by Kleene in the proof of Theorem 9.3(a) of [3]. For axiom schema

8^{I}, let $\varphi[\varPsi,\gamma] \simeq \varepsilon^{\neg\mathrm{A}\to(\mathrm{A}\to\mathrm{B})} \simeq \varLambda\sigma\varepsilon^{\mathrm{A}\to\mathrm{B}}$ and observe that if σ ${}^{\varDelta^1_1}$realizes-$\varPsi$ $\neg$A then no ϱ can ${}^{\varDelta^1_1}$realize-$\varPsi$ A; so $\varphi[\varPsi,\lambda t.0]$ is a ${}^{\varDelta^1_1}$realizer for the axiom.

The predicate logic schemas 10N, 10F, 11N and 11F require Lemma 8. As an example, consider 11F: $\mathrm{A(v)} \to \exists\alpha\mathrm{A}(\alpha)$ where v is free for α in $\mathrm{A}(\alpha)$. If Ψ,α are all the distinct variables occurring free in an instance of the axiom, and if $\nu[\varPsi,\alpha]$ is the primitive recursive functional expressed by v, let $\varphi[\varPsi,\alpha,\gamma] \simeq \varLambda\sigma\langle\varLambda.\nu[\varPsi,\alpha],\sigma\rangle$; then $\varphi[\varPsi,\alpha,\lambda t.0]$ is a ${}^{\varDelta^1_1}$realizer for the axiom. Agreement is a consequence of Lemma 2 with the fact that $\nu[\varPsi,\alpha]$ is totally defined. Suppose $\varPsi,\alpha$ are $\varDelta^1_1$ sequences and numbers interpreting Ψ,α, and suppose σ ${}^{\varDelta^1_1}$realizes-$\varPsi,\alpha$ A(v); then $\nu[\varPsi,\alpha]$ is $\varDelta^1_1$ and σ ${}^{\varDelta^1_1}$realizes-$\varPsi,\alpha,\nu[\varPsi,\alpha]$ $\mathrm{A}(\alpha)$ by Lemma 8(b), so $\varphi[\varPsi,\alpha,\lambda t.0]$ is a ${}^{\varDelta^1_1}$realizer for the axiom.

If E is an instance of the induction schema 13: $\mathrm{A}(0)\ \&\ \forall\mathrm{x}(\mathrm{A(x)} \to \mathrm{A(x')}) \to \mathrm{A(x)}$ with only Ψ, x free, let $\varphi[\varPsi,x,\gamma] \simeq \varLambda\sigma\vartheta[\varPsi,x,\sigma]$ where $\vartheta[\varPsi,x,\sigma]$ is defined by the functional recursion

$$\vartheta[\varPsi,0,\sigma] \simeq (\sigma)_0 \text{ and } \vartheta[\varPsi,x',\sigma] \simeq \{\{(\sigma)_1\}[x]\}[\vartheta[\varPsi,x,\sigma]].$$

Then $\varphi[\varPsi,x,\lambda t.0]$ is a ${}^{\varDelta^1_1}$realizer for the axiom. If E is a number-theoretic axiom by any of the schemas 14-21, 0.1, 1.1, or an axiom from Group D, then one of $\lambda t.0$, $\varLambda\sigma\lambda t.0$, $\varLambda\sigma\varLambda\rho\lambda t.0$ is a ${}^{\varDelta^1_1}$realizer for E.

If E is an instance of axiom schema 2.1: $\forall\mathrm{x}\exists\alpha\mathrm{A}(\mathrm{x},\alpha) \to \exists\beta\forall\mathrm{x}\mathrm{A}(\mathrm{x},\lambda\mathrm{y}.\beta((\mathrm{x},\mathrm{y})))$ with only Ψ free, define $\varphi[\varPsi,\gamma] \simeq \varLambda\sigma\langle\varLambda.\lambda t\{(\{\sigma\}[(t)_0])_0\}[(t)_1],\varLambda x(\{\sigma\}[x])_1\rangle$; then $\varphi[\varPsi,\lambda t.0]$ is a ${}^{\varDelta^1_1}$realizer for E.

Now suppose E is an instance of the bar induction schema $\mathrm{BI}_1!$:[8]

$$[\forall\alpha\exists!\mathrm{x}\rho(\overline{\alpha}(\mathrm{x})) = 0\ \&\ \forall\mathrm{a}(\mathrm{Seq(a)}\ \&\ \rho(\mathrm{a}) = 0 \to \mathrm{A(a)})$$
$$\&\ \forall\mathrm{a}(\mathrm{Seq(a)}\ \&\ \forall\mathrm{sA}(\mathrm{a} * \langle\mathrm{s}\rangle) \to \mathrm{A(a)})] \to \mathrm{A}(\langle\ \rangle)$$

containing free only the variables Ψ,ρ where ρ is not free in A(a). Define the recursive partial functionals

$$\xi[\sigma,w] \simeq \{((\sigma)_0)_0\}[\lambda t.(w)_t \dot{-} 1],$$

and $\zeta[\sigma,w]$, which will be defined only for sequence numbers w using the recursion theorem:

$$\zeta[\sigma,w] \simeq \begin{cases} \varepsilon^{\mathrm{A}(\langle\rangle)} & \text{if } lh(w) > (\xi[\sigma,w](0))_0, \\ \{\{((\sigma)_0)_1\}[w]\}[\langle\lambda t.0,((\xi[\sigma,w])_1)_0\rangle] & \text{if } lh(w) = (\xi[\sigma,w](0))_0, \\ \{\{(\sigma)_1\}[w]\}[\langle\lambda t.0,\varLambda s\,\zeta[\sigma,w*\langle s\rangle]\rangle] & \text{if } lh(w) < (\xi[\sigma,w](0))_0. \end{cases}$$

We claim that $\varphi[\varPsi,\rho,\lambda t.0] \simeq \varLambda\sigma\,\zeta[\sigma,\langle\,\rangle]$ is a ${}^{\varDelta^1_1}$realizer for the axiom schema.

Assume σ agrees with the hypothesis. Then for every α, $\{((\sigma)_0)_0\}[\alpha]$ is completely defined and $((\sigma)_0)_1$ agrees with the second premise and $(\sigma)_1$ with the third; so $\xi[\sigma,w]$ and $\zeta[\sigma,w]$ are totally defined for every sequence code w. For each α let

$$\vartheta(\alpha) \simeq max((\{((\sigma)_0)_0\}[\alpha](0))_0, \mu x(((\sigma)_0)_0(\langle 0\rangle * \overline{\alpha}(x)) > 0)),$$

so $\xi[\sigma,\overline{\alpha}(\vartheta(\alpha))](0) = \{((\sigma)_0)_0\}[\alpha](0)$. For each sequence code w let

$$\tau(w) = lh(w) + 1 \dot{-} \vartheta(\lambda t.(w)_t \dot{-} 1).$$

We use the informal analogue of Kleene's bar induction schema 26.3b with τ in place of ρ, and the inductive predicate "$\zeta[\sigma,w]$ agrees with A(w)," to show that

[8]This variant is equivalent in **M** to Kleene's schema 26.3b.

$\zeta[\sigma, \langle\,\rangle]$ agrees with $\mathrm{A}(\langle\,\rangle)$. Evidently $\tau(\overline{\alpha}(\vartheta(\alpha))) > 0$ for every α. If w is a sequence code and $\tau(w) > 0$ then $lh(w) \geq (\xi[\sigma, w](0))_0$. By definition, if $lh(w) > (\xi[\sigma, w](0))_0$ then $\zeta[\sigma, w] = \varepsilon^{\mathrm{A}(\langle\,\rangle)}$ which agrees with $\mathrm{A}(\langle\,\rangle)$ by Lemma 4, and hence with A(w) by Lemma 2; and if $lh(w) = (\xi[\sigma, w](0))_0$ then $\zeta[\sigma, w] \simeq ((\xi[\sigma, w])_1)_0$ which agrees with A(w) by the hypothesis on σ. Finally, if w is a sequence code such that $\zeta[\sigma, w * \langle s\rangle]$ agrees with $\mathrm{A}(\mathrm{w} * \langle \mathrm{s}\rangle)$ for every s, then either $lh(w) \geq (\xi[\sigma, w](0))_0$ so $\zeta[\sigma, w]$ agrees with A(w) by the preceding arguments, or $lh(w) < (\xi[\sigma, w](0))_0$ and $\Lambda s\, \zeta[\sigma, w * \langle s\rangle]$ agrees with $\forall \mathrm{sA}(\mathrm{w} * \langle \mathrm{s}\rangle)$ so $\zeta[\sigma, w]$ agrees with A(w) by the hypothesis on σ with the definition of ζ. Thus $\zeta[\sigma, \langle\,\rangle]$ agrees with $\mathrm{A}(\langle\,\rangle)$ as claimed.

Now assume that σ ${}^{\Delta^1_1}$realizes-Ψ, ρ the hypothesis, so **(i)** for every Δ^1_1 sequence α: $\rho(\overline{\alpha}(x)) > 0$ if and only if $x = (\{((\sigma)_0)_0\}[\alpha](0))_0)$; **(ii)** if w is any sequence code with $\rho(w) > 0$ then for every κ: $\{\{((\sigma)_0)_1\}[w]\}[\langle \lambda t.0, \kappa\rangle]$ ${}^{\Delta^1_1}$realizes-Ψ, ρ, w A(w); and **(iii)** if w is a sequence code and ν ${}^{\Delta^1_1}$realizes-Ψ, ρ, w $\forall \mathrm{sA}(\mathrm{w} * \langle \mathrm{s}\rangle)$, then A(w) is ${}^{\Delta^1_1}$realized-Ψ, ρ, w by $\{\{(\sigma)_1\}[w]\}[\langle \lambda t.0, \nu\rangle]$. We must show that $\zeta[\sigma, \langle\,\rangle]$ ${}^{\Delta^1_1}$realizes-Ψ, ρ $\mathrm{A}(\langle\,\rangle)$.

First observe that if w is a sequence code then $w = \overline{(\lambda t.(w)_t \dot{-} 1)}(lh(w))$, so by (i): $\rho(w) > 0$ if and only if $lh(w) = (\{((\sigma)_0)_0\}[\lambda t.(w)_t \dot{-} 1](0))_0) = (\xi[\sigma, w](0))_0$. All three hypotheses for an informal bar induction corresponding to 26.3b!, with ρ determining the (thin) bar and with the inductive predicate "$lh(w) \leq (\xi[\sigma, w](0))_0$ and $\zeta[\sigma, w]$ ${}^{\Delta^1_1}$realizes-Ψ, ρ, w A(w)," follow by (i), (ii) and (iii) with Lemma 1 and the definition of ζ. Thus $\zeta[\sigma, \langle\,\rangle]$ ${}^{\Delta^1_1}$realizes-Ψ, ρ $\mathrm{A}(\langle\,\rangle)$, and the argument that $\varphi[\Psi, \rho, \lambda t.0]$ is a ${}^{\Delta^1_1}$realizer for the axiom schema of (thin) bar induction is complete.

Essentially as for ${}^{\mathrm{G}}$realizability, Brouwer's continuous choice principle 27.1:

$$\begin{aligned}&\forall \alpha \exists \beta \mathrm{A}(\alpha, \beta) \rightarrow \\ &\qquad \exists \tau \forall \alpha [\forall \mathrm{t} \exists ! \mathrm{y} \tau(\langle \mathrm{t}\rangle * \overline{\alpha}(\mathrm{y})) > 0 \ \&\ \forall \beta [\forall \mathrm{t} \exists \mathrm{y} \tau(\langle \mathrm{t}\rangle * \overline{\alpha}(\mathrm{y})) = \beta(\mathrm{t}) + 1 \rightarrow \mathrm{A}(\alpha, \beta)]]\end{aligned}$$

is ${}^{\Delta^1_1}$realized-Ψ by $\Lambda\sigma\langle \Lambda\tau, \Lambda\alpha\langle \rho_0, \rho_1\rangle\rangle$ where $\tau \simeq \Lambda\alpha\{(\{\sigma\}[\alpha])_0\}$, $\rho_0 \simeq \Lambda t\langle \lambda s.\mu y\, \tau(\langle t\rangle * \overline{\alpha}(y)) > 0,\ \langle \lambda s.0, \Lambda z \Lambda\pi\lambda s.0\rangle\rangle$ and $\rho_1 \simeq \Lambda\beta\Lambda\pi(\{\sigma\}[\alpha])_1$.

The schema (I) asserts the classical decidability of arithmetical predicates with sequence parameters (i.e. with free sequence variables). A ${}^{\Delta^1_1}$realization function for an instance of (I) with only Ψ free is $\varphi[\Psi] \simeq \Lambda\sigma\lambda t.0$. For example, if $\mathrm{A}(\beta, \mathrm{x})$ has no sequence quantifiers and contains free only β, x, then $\Lambda\beta\Lambda\sigma\lambda t.0$ ${}^{\Delta^1_1}$realizes $\forall \beta \neg\neg \forall \mathrm{x}[\mathrm{A}(\beta, \mathrm{x}) \vee \neg \mathrm{A}(\beta, \mathrm{x})]$. Agreement is obvious, and for each Δ^1_1 sequence β Lemma 9(b) gives a Δ^1_1 sequence ψ which ${}^{\Delta^1_1}$realizes-β $\forall \mathrm{x}[\mathrm{A}(\beta, \mathrm{x}) \vee \neg \mathrm{A}(\beta, \mathrm{x})]$, so no sequence ${}^{\Delta^1_1}$realizes-β $\neg \forall \mathrm{x}[\mathrm{A}(\beta, \mathrm{x}) \vee \neg \mathrm{A}(\beta, \mathrm{x})]$.

The function $\varphi \simeq \varphi[\lambda t.0] \simeq \Lambda\alpha\Lambda\pi\lambda t.0$ is a ${}^{\Delta^1_1}$realizer for the axiom (II) asserting that every sequence is classically Σ^1_1. Agreement is obvious. Consider an arbitrary sequence α which is classically Δ^1_1, hence in particular Π^1_1. Then there exists an f and, by the Spector-Gandy Theorem, also an e such that for all x, y:

$$\begin{aligned}\alpha(x) = y \quad &\Leftrightarrow \quad (\gamma)(Ez)T(f, x, y, \overline{\gamma}(z)) \\ &\Leftrightarrow \quad (E\beta \in \Delta^1_1)(z)\overline{T}(e, x, y, \overline{\beta}(z))\end{aligned}$$

It follows that the Δ^1_1 sequence $\rho \simeq \langle \lambda t.e, \Lambda x \Lambda y\langle \Lambda\zeta\Lambda\pi\lambda t.0, \Lambda\pi\lambda t.0\rangle\rangle$ ${}^{\Delta^1_1}$realizes-α $\exists \mathrm{e} \forall \mathrm{x} \forall \mathrm{y}[\alpha(\mathrm{x}) = \mathrm{y} \leftrightarrow \neg\neg\exists \beta \forall \mathrm{z} \neg \mathrm{T}(\mathrm{e}, \mathrm{x}, \mathrm{y}, \overline{\beta}(\mathrm{z}))]$, so φ ${}^{\Delta^1_1}$realizes the axiom.

Finally, $\varphi \simeq \Lambda\rho\Lambda\sigma\Lambda\pi\lambda t.0$ ${}^{\Delta^1_1}$realizes axiom (III).

The rules of inference 2, 9N, 9F, 12N, 12F pose no difficulty. Taking Rule 9F as an example, if $\delta \in \Delta^1_1$ and $\varphi_1[\Psi, \alpha, \delta]$ is a ${}^{\Delta^1_1}$realizer for the hypothesis $\mathrm{C} \rightarrow \mathrm{A}(\alpha)$ where

α is not free in C, then $\Lambda\sigma\Lambda\alpha(\{\varphi_1[\Psi,\alpha,\delta]\}[\sigma])$ is a Δ^1_1realizer for $\mathrm{C} \rightarrow \forall\alpha\mathrm{A}(\alpha)$. For Rule 12F, if $\delta \in \Delta^1_1$ and $\varphi_2[\Psi,\alpha,\delta]$ is a Δ^1_1realizer for $\mathrm{A}(\alpha) \rightarrow \mathrm{C}$ with α not free in C, then $\Lambda\sigma(\{\varphi_2[\Psi,\alpha,\delta]\}[(\sigma)_1])$ is a Δ^1_1realizer for $\exists\alpha\mathrm{A}(\alpha) \rightarrow \mathrm{C}$.

Corollary 1. Every theorem of $\mathbf{T}_2$ has a recursive Δ^1_1realizer.

Proof. Just observe that in the proof of Theorem 1, the parameter δ used in defining a Δ^1_1realizer for an axiom of $\mathbf{T}_2$ can always be taken to be recursive, and this property is preserved by the rules of inference.

Now let $\mathbf{T}_3$ be obtained from $\mathbf{T}_2$ by adjoining the law of excluded middle for purely arithmetical predicates (no sequence variables), so $\mathbf{T}_3$ contains all of Peano arithmetic (including purely arithmetical Markov's Principle MP_0). Both $\mathbf{T}_2$ and $\mathbf{T}_3$ are Brouwerian $\mathcal{L}$-theories which do not prove MP_1, by the next corollary.

Corollary 2. Every theorem of $\mathbf{T}_3$ is Δ^1_1realizable, but MP_1 is not.

Proof. The first statement follows from Theorem 1 by Lemma 9(b). To see directly that MP_1 is not Δ^1_1realizable, suppose it has a Δ^1_1realizer π, so for every α: $\{\pi\}[\alpha]$ is completely defined and agrees with $[\neg\neg\exists x\alpha(x) = 0 \rightarrow \exists x\alpha(x) = 0]$; and if $\alpha \in \Delta^1_1$ then $\alpha((\{\{\pi\}[\alpha]\}[\Lambda\sigma\Lambda\rho\lambda t.0](0))_0) = 0$ if not every value of α is different from 0. Then $y \simeq (\{\{\pi\}[\lambda t.1]\}[\Lambda\sigma\Lambda\rho\lambda t.0](0))_0$ is completely determined by some finite initial segment $\overline{(\lambda t.1)}(m)$; so if $n = max(y, m) + 1$ and α agrees with $\lambda t.1$ at all arguments smaller than n, but $\alpha(n) = 0$, we have a contradiction.

Corollary 3. $\mathbf{T}_3$ is not recursively acceptable and does not satisfy the Church-Kleene Rule.

Proof. $\mathbf{T}_3$ proves $\exists!\alpha\forall\mathrm{x}[\alpha(\mathrm{x}) \leq 1\ \&\ [\alpha(\mathrm{x}) = 0 \leftrightarrow \exists\mathrm{y}(\mathrm{T}(\mathrm{x},\mathrm{x},\mathrm{y})\ \&\ \mathrm{U}(\mathrm{y}) = 1)]]$ and therefore $\mathbf{T}_3$ proves $\exists\alpha\neg\exists\mathrm{e}\forall\mathrm{x}\exists\mathrm{y}(\mathrm{T}(\mathrm{e},\mathrm{x},\mathrm{y})\,\&\,\mathrm{U}(\mathrm{y}) = \alpha(\mathrm{x}))$. Since $\mathbf{T}_3$ is consistent by Corollary 2, $\mathbf{T}_3$ fails to satisfy Church's Rule, and the conclusion follows.

4. Vesley's Schema and "independence of premise"

In [13] Richard Vesley proposed adding to **FIM** a new axiom schema VS:

$$\forall\mathrm{w}(\mathrm{Seq}(\mathrm{w}) \rightarrow \exists\alpha(\overline{\alpha}(\mathrm{lh}(\mathrm{w})) = \mathrm{w}\ \&\ \neg\mathrm{A}(\alpha)) \rightarrow [\forall\alpha(\neg\mathrm{A}(\alpha) \rightarrow \exists\beta\mathrm{B}(\alpha,\beta)) \rightarrow \forall\alpha\exists\beta(\neg\mathrm{A}(\alpha) \rightarrow \mathrm{B}(\alpha,\beta))]$$

(with β not free in $\mathrm{A}(\alpha)$) and proved the consistency of the resulting system using an intuitionistic model in which the choice sequence variables ranged over all not not recursive sequences. In **FIM** + VS he could derive $\neg\mathrm{MP}_1$ and other results for which Brouwer used "creating subject" arguments. He argued that VS was preferable for this purpose to Kripke's Schema KS^-, which asserted the existence of nonrecursive functions and was inconsistent with the strong form of Brouwer's continuous choice principle assumed in **FIM**.[9]

Vesley observed that VS is derivable from either KS^- or IP using the countable axiom of choice (axiom schema 2.1 of **FIM**), where IP is the "independence of premise" schema

$$(\neg\mathrm{A} \rightarrow \exists\beta\mathrm{B}(\beta)) \rightarrow \exists\beta(\neg\mathrm{A} \rightarrow \mathrm{B}(\beta))$$

with β not free in A. Let $\mathbf{T}_4 = \mathbf{T}_2$ + VS, $\mathbf{T}_5 = \mathbf{T}_3$ + VS and $\mathbf{T}_6 = \mathbf{T}_2$ + IP.

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[9] KS^- is $\exists\beta[(\forall\mathrm{x}\beta(\mathrm{x}) = 0 \leftrightarrow \neg\mathrm{A})\ \&\ (\exists\mathrm{x}\beta(\mathrm{x}) \neq 0 \rightarrow \mathrm{A})]$, where β does not occur not free in A.

Corollary 4. Every closed theorem of $\mathbf{T}_4$ or $\mathbf{T}_6$ is ${}^{\Delta^1_1}$realized by a recursive function, and each theorem of $\mathbf{T}_5$ is ${}^{\Delta^1_1}$realizable. Thus $\mathbf{T}_4$, $\mathbf{T}_5$ and $\mathbf{T}_6$ are Brouwerian $\mathcal{L}$-theories refuting MP_1.

Proof. By Theorem 1, Corollaries 1 and 2, and the fact that IP entails VS over $\mathbf{T}_2$, since a recursive ${}^{\Delta^1_1}$realizer for IP is $\Lambda\sigma\langle(\{\sigma\}[\Lambda\rho\lambda t\,0])_0, \Lambda\tau(\{\sigma\}[\Lambda\rho\lambda t\,0])_1\rangle$.

5. Strong inadmissibility

Kleene's famous example of an infinite subtree of the binary tree with no infinite recursive branches evidently proves that Markov's Rule with sequence parameters is not admissible for the theory **FIM** + "there are no nonrecursive functions" of [5]. Kleene also gave an example of a subtree of the universal spread having infinite branches but no infinite hyperarithmetical branches. This example should yield a corresponding result for $\mathbf{T}_2$.

Let us call a rule *strongly inadmissible* for an $\mathcal{L}$-theory $\mathbf{T}$ if for some instance of the rule, $\mathbf{T}$ proves (the universal closures of) all the hypotheses and the *negation* of the universal closure of the conclusion. By the proof of Corollary 3 to Theorem 1, Church's Rule is strongly inadmissible for $\mathbf{T}_3$ and hence inadmissible for $\mathbf{T}_5$, and no consistent extension of $\mathbf{T}_3$ is recursively acceptable. We conjecture that Markov's Rule with sequence parameters is strongly inadmissible for $\mathbf{T}_2$ (hence also for$\mathbf{T}_3$-$\mathbf{T}_6$), so no consistent extension of $\mathbf{T}_2$ is recursively acceptable.

Conjecture. For a suitable formula $\mathrm{A}(\mathrm{w},\beta)$ with no free variables but w,β:

(a) $\vdash_{\mathbf{T_2}} \forall\alpha\forall\beta\neg\neg\exists \mathrm{n}\mathrm{A}(\overline{\alpha}(\mathrm{n}),\beta)$,
(b) $\vdash_{\mathbf{T_2}} \forall\alpha\forall\beta\forall \mathrm{n}[\mathrm{A}(\overline{\alpha}(\mathrm{n}),\beta)\vee\neg\mathrm{A}(\overline{\alpha}(\mathrm{n}),\beta)]$,
(c) $\vdash_{\mathbf{T_2}} \neg\forall\alpha\forall\beta\exists \mathrm{n}\mathrm{A}(\overline{\alpha}(\mathrm{n}),\beta)$.

We conjecture that the Church-Kleene Rule is admissible for $\mathbf{T}_2$, $\mathbf{T}_4$ and $\mathbf{T}_6$, and hence that these theories are explicit and satisfy Church's Rule.

6. Concluding remarks

Each inhabited class F of one-place number-theoretic functions closed under "recursive in" determines a corresponding notion of ${}^{\mathrm{F}}$realizability. The definition (cf. [8]) is like that for ${}^{\Delta^1_1}$realizability but with F in place of Δ^1_1 everywhere. For $\mathrm{F} = {}^\omega\omega$ the notion is like Kleene's ${}_{\mathrm{S}}$realizability ([3] pp. 119ff) except that Kleene called a closed formula ${}_{\mathrm{S}}$realizable only if it had a *recursive* ${}_{\mathrm{S}}$realizer. Each axiom of **FIM** has a recursive ${}^{\mathrm{F}}$realizer, and if the hypotheses of a rule of inference are ${}^{\mathrm{F}}$realized by functions recursive in $\Phi \subseteq \mathrm{F}$, the conclusion also has an ${}^{\mathrm{F}}$realizer recursive in Φ. MP_1 is not ${}^{\mathrm{F}}$realizable for any recursively closed F. The proofs are essentially like those in the previous section.

Troelstra's axiomatization of Kleene's function-realizability uses an extension of continuous choice which he calls "generalized continuity" GC_1:

$$\forall\alpha[\mathrm{A}(\alpha)\rightarrow\exists\beta\mathrm{B}(\alpha,\beta)]\rightarrow\exists\sigma\forall\alpha[\mathrm{A}(\alpha)\rightarrow\exists\gamma[\forall \mathrm{x}(\{\sigma\}[\alpha](\mathrm{x})\simeq\gamma(\mathrm{x}))\ \&\ \mathrm{B}(\alpha,\gamma)]],$$

where $\mathrm{A}(\alpha)$ must be *almost negative* (containing no $\vee$, and no $\exists$ except immediately before a prime formula). Since MP_1 is realizable but not ${}^{\mathrm{F}}$realizable for any recursively closed F, it is natural to ask if GC_1 has the same properties. In fact, its weaker consequence $\mathrm{GC}_0!$:

$$\forall\alpha[\mathrm{A}(\alpha)\rightarrow\exists!\mathrm{x}\mathrm{B}(\alpha,\mathrm{x})]\rightarrow\exists\sigma\forall\alpha[\mathrm{A}(\alpha)\rightarrow\exists \mathrm{z}[\{\sigma\}(\alpha)\simeq \mathrm{z}\ \&\ \mathrm{B}(\alpha,\mathrm{z})]]$$

(where $\{\sigma\}(\alpha) \simeq \mathrm{z}$ expresses $\sigma(\overline{\alpha}(\mu y(\sigma(\overline{\alpha}(y)) > 0))) \simeq z + 1$ and $\mathrm{A}(\alpha)$ is almost negative) already fails to be $^{\mathrm{F}}$realizable, by a proof analogous to 3.4.14 of [9].[10]

Since GC_1 is realizable, it follows that neither GC_1 nor its negation is refutable in **FIM**. It is tempting to ask if $\mathbf{FIM} + \mathrm{GC}_1 \vdash \mathrm{MP}_1$, but Troelstra's axiomatization of Kleene's realizability (together with the formalized version of Lemma 8.4 of [3], cf. [1]) already shows that this is not the case.

Vesley's schema and Markov's Principle, both classically correct, have very different effects on the intuitionistic continuum. The theory $\mathbf{M} + \mathrm{BI}_1 + \mathrm{MP}_1 + \mathrm{GC}_1$ asserts that every partial functional defined at least on an almost negative species has a continuous partial extension, while **FIM** + VS asserts that every partial functional defined at least on a negative dense species has a continuous total extension.[11] A detailed comparison of these two superintuitionistic $\mathcal{L}$-theories from a reverse mathematics perspective should be an interesting project.

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[10] I thank a referee for this reference.

[11] If $\mathrm{A}(\alpha)$ contains no $\forall$ then $\neg\mathrm{A}(\alpha)$ is equivalent over $\mathbf{M}$ to an almost negative formula.