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From the weak to the strong existence property

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Abstract

A hallmark of many an intuitionistic theory is the existence property, **EP**, i.e., if the theory proves an existential statement then there is a provably definable witness for it. However, there are well known exceptions, for example, the full intuitionistic Zermelo-Fraenkel set theory, **IZF**, does not have the existence property, where **IZF** is formulated with Collection. By contrast, the version of intuitionistic Zermelo-Fraenkel set theory formulated with Replacement, $\mathbf{IZF}_R$, has the existence property. Moreover, **IZF** does not even enjoy a weaker form of the existence property, **wEP**, defined by the slackened requirement of finding a provably definable set of witnesses for every existential theorem. In view of these results, one might be tempted to put the blame for the failure of the existence properties squarely on Collection. However, in this paper it is shown that several well known intuitionistic set theories with Collection have the weak existence property. Among these theories are $\mathbf{CZF}^-$, $\mathbf{CZF}_\mathcal{E}$, and $\mathbf{CZF}_\mathcal{P}$, i.e., respectively, constructive Zermelo-Fraenkel set theory (**CZF**) without subset collection, **CZF** formulated with Exponentiation and also **CZF** augmented by the Power Set axiom (basically **IZF** with only bounded separation). As a result, the culprit preventing the weak existence property from obtaining must consist of a combination of Collection and unbounded Separation.

To bring about these results we introduce a form of realizability based on general set recursive functions where a realizer for an existential statement provides a set of witnesses for the existential quantifier rather than a single witness. Moreover, this notion of realizability needs to be combined with truth to yield the desired results.

This form of realizability is also utilized, albeit shorn of its truth component, in showing partial conservativity results for $\mathbf{CZF}^-$, $\mathbf{CZF}_\mathcal{E}$, and $\mathbf{CZF}_\mathcal{P}$ over their intuitionistic counterparts **IKP**, $\mathbf{IKP}(\mathcal{E})$, and $\mathbf{IKP}(\mathcal{P})$, respectively.

As it turns out, the combination of the weak existence property and partial conservativity of $\mathbf{CZF}^-$ over **IKP** plus a further ingredient can be used to show that $\mathbf{CZF}^-$ actually has the existence property. The additional ingredient is an advanced techniques from proof theory (cut elimination and ordinal analysis of **IKP**). Roughly the same techniques can be deployed in showing that $\mathbf{CZF}_\mathcal{E}$ and $\mathbf{CZF}_\mathcal{P}$ have the stronger existence property, too. However, this requires a new form of ordinal analysis for theories with Power Set and Exponentiation (cf. [39]) and is beyond the scope of the current paper.

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1 Introduction

Intuitionistic theories are known to often possess very pleasing metamathematical properties such as the disjunction property and the numerical existence property. While it is fairly easy to establish these properties for arithmetical theories and theories with quantification over sets of natural numbers or Baire space (e.g. second order arithmetic and function arithmetic), set theories with their transfinite hierarchies of sets of sets and the extensionality axiom can pose considerable technical challenges.

Definition 1.1 Let T be a theory whose language, $L(T)$, encompasses the language of set theory. Moreover, for simplicity, we shall assume that $L(T)$ has a constant ω denoting the set of von Neumann natural numbers and for each n a constant $\bar{n}$ denoting the n-th element of ω.

1. T has the *disjunction property*, **DP**, if whenever $T \vdash B \vee D$ holds for sentences B and D of T, then $T \vdash B$ or $T \vdash D$.

2. T has the *numerical existence property*, **NEP**, if whenever $T \vdash (\exists x{\in}\omega)A(x)$ holds for a formula $A(x)$ with at most the free variable x, then $T \vdash A(\bar{n})$ for some n.

3. T has the *existence property*, **EP**, if whenever $T \vdash \exists x A(x)$ holds for a formula $A(x)$ having at most the free variable x, then there is a formula $C(x)$ with exactly x free, so that

$$T \vdash \exists! x\,[C(x) \wedge A(x)].$$

Realizability semantics are of paramount importance in the study of intuitionistic theories. They were first proposed by Kleene [17] in 1945. Friedman [12] showed metamathematical results for intuitionistic set theories by extending a notion of realizability developed by Kreisel and Troelstra [21]. A realizability-notion akin to Kleene's slash [18, 19] was extended to various intuitionistic set theories by Myhill [27, 28], whereby he also drew on work by J.R. Moschovakis [24]. We use **IZF** to denote intuitionistic Zermelo-Fraenkel set theory formulated with Collection. [27] showed that intuitionistic **ZF** with Replacement instead of Collection (dubbed $\mathbf{IZF}_R$ henceforth) has the **DP**, **NEP**, and **EP**. [28] proved that his constructive set theory **CST** enjoys the **DP** and the **NEP**, and that the theory without the axioms of countable and dependent choice, $\mathbf{CST}^-$, also has the **EP**. It was left open in [28] whether the full existence property holds in the presence of relativized dependent choice, **RDC**. Friedman and Ščedrov [15] then established that $\mathbf{IZF}_R$+**RDC** satisfies the **EP** also. Several systems of set theory for the constructive mathematical practice were propounded by Friedman in [14]. The metamathematical properties of these theories and several others as well were subsequently investigated by Beeson [5, 6]. In particular, Beeson showed that **IZF** has the **DP** and **NEP**. He used a combination of Kreisel-Troelstra realizability and Kleene's [17, 18, 19, 20] q-realizability. However, while Myhill and Friedman developed realizability directly for extensional set theories, Beeson engineered his realizability for non-extensional set theories and obtained results for the extensional set theories of [14] only via an interpretation in their non-extensional counterparts. This detour had the disadvantage that in many cases (where the theory does not have full Separation or Powerset) the **DP** and **NEP** for the corresponding extensional set theory T-*ext* could only be established for a restricted class of formulas. In [33, 36, 37] the author of the present paper developed a different machinery for showing the **DP** and the **NEP** (and several other properties) directly for extensional set theories. [36] introduced a self-validating semantics for constructive Zermelo-Fraenkel set theory, **CZF**, that combines realizability for extensional set theory and truth. In [37] this method was used to establish the **DP** and **NEP** for **CZF** and **IZF** augmented by familiar choice principles, i.e., any combination of the principles of Countable Choice, Relativized Dependent Choices and the Presentation Axiom (cf. [32]). Also Markov's principle may be added.

So far we haven't addressed the question whether the **EP** holds for **IZF** and **CZF**. Partial results were obtained in [35, Theorems 8.3,8.4] to the effect that **CZF** augmented via a strong form of the axiom of choice, the $\Pi\Sigma$ axiom of choice, has the **EP** for a very large collection of formulae. It was shown by Friedman and Ščedrov that the **EP** fails for **IZF**, intuitionistic Zermelo-Fraenkel set theory formulated with Collection. As $\mathbf{IZF}_R$ possesses the **EP**, Collection is clearly implicated in this failure. Beeson in [6, IX.1] posed the following question:

Does any reasonable set theory with collection have the existence property?

When investigating this problem, one is naturally led to a weaker form of the existence property, **wEP**, defined by the slackened requirement of finding a provably definable set of witnesses for every existential theorem.

Definition 1.2 Let T be a theory whose language, $L(T)$, encompasses the language of set theory.

1. T has the *weak existence property*, **wEP**, if whenever
$$T \vdash \exists x A(x)$$
holds for a formula $A(x)$ having at most the free variable x, then there is a formula $C(x)$ with exactly x free, so that
$$\begin{array}{lcl} T & \vdash & \exists! x\, C(x), \\ T & \vdash & \forall x\, [C(x) \to \exists u\, u \in x], \\ T & \vdash & \forall x\, [C(x) \to \forall u \in x\, A(u)]. \end{array}$$

2. We also consider a more general version of **wEP**. The *uniform weak existence property*, **uwEP**, is the following property: if
$$T \vdash \forall u\, \exists x A(u, x)$$
holds for a formula $A(u, x)$ having at most the free variables u, x, then there is a formula $C(u, x)$ with exactly u, x free, so that
$$\begin{array}{lcl} T & \vdash & \forall u\, \exists! x\, C(u, x), \\ T & \vdash & \forall u\, \forall x\, [C(u, x) \to \exists z\, z \in x], \\ T & \vdash & \forall u\, \forall x\, [C(u, x) \to \forall z \in x\, A(u, z)]. \end{array}$$

Obviously, if **uwEP** holds for T then T has the weak existence property.

As it turns out, **IZF** doesn't satisfy **wEP** either.

Proposition 1.3 **IZF** *does not have the weak existence property.*

Proof: We say that **IZF** has the existence property for a formula $\exists x A(x)$ if whenever $\mathbf{IZF} \vdash \exists x A(x)$ then there is a formula $C(x)$ such that $\mathbf{IZF} \vdash \exists! x\, [C(x) \wedge A(x)]$.

By [16, Theorem 1.1], **IZF** does not have the existence property for some sentence of the form
$$\exists x\, [\exists y\, D(y) \to \exists y \in x\, D(y)]\,.$$

But clearly, if **wEP** held for **IZF**, then the existence property would hold for this particular sentence, contradicting [16, Theorem 1.1]. □

The previous result shows that **wEP** is an interesting property. Again one might be tempted to put the blame for the failure of this property squarely on Collection. However, in this paper it is shown that several well known intuitionistic set theories with Collection have the weak existence property. Among these theories are constructive Zermelo-Fraenkel set theory, **CZF**, formulated with Exponentiation and also **CZF** augmented by the Power Set axiom (basically **IZF** with only bounded Separation).[1] As a result, the culprit preventing the weak existence property from obtaining in the case of **IZF** must consist of a combination of Collection and unbounded Separation.

To bring about these results we introduce in section 3 a form of realizability based on general set recursive functions (defined in section 2), where a realizer for an existential statement provides a set of witnesses for the existential quantifier rather than a single witness. Moreover, this notion of realizability needs to be combined with truth to yield the desired results.

This form of realizability is also utilized, albeit shorn of its truth component, in showing partial conservativity results in section 4 for $\mathbf{CZF}^{-}$, $\mathbf{CZF}_{\mathcal{E}}$, and $\mathbf{CZF}_{\mathcal{P}}$ over their intuitionistic counterparts **IKP**, $\mathbf{IKP}(\mathcal{E})$, and $\mathbf{IKP}(\mathcal{P})$, respectively.

As it turns out, the combination of the weak existence property and partial conservativity of $\mathbf{CZF}^{-}$ over **IKP** can be used to show that $\mathbf{CZF}^{-}$ actually has the existence property. A sketch of proof is provided in section 5. It uses methods from proof theory (ordinal analysis). The same techniques can be deployed in showing that $\mathbf{CZF}_{\mathcal{E}}$ and $\mathbf{CZF}_{\mathcal{P}}$ have the stronger existence property, too. However, this requires a new form of ordinal analysis for theories with Power Set and Exponentiation, respectively. This is beyond the scope of the current paper (cf. [39]).

The traditional system of constructive Zermelo-Fraenkel set theory, **CZF**, has an axiom scheme called Subset Collection (cf. [1, 2, 3]). Subset Collection implies Exponentiation and is a consequence of Power Set. It also follows from Exponentiation with the aid of the Presentation Axiom. On the basis of the other axioms of **CZF**, Subset Collection is equivalent to the *Fullness* Axiom which asserts that given any sets A and B there exists a set C (called *full*) of multi-valued functions from A to B such that for every multi-valued function R from A to B there exists $S \subseteq R$ with $S \in C$. The statement that for any two sets the class of multi-valued functions between them is a set is equivalent to Power Set. Proof-theoretically there is a huge gap between Exponentiation and Power Set. The Fullness Axiom simply postulates the existence of a full set. Since in general it does not seem possible to define a full set of multi-valued functions without assuming Powerset or choice (e.g. from ${}^{\mathbb{N}}\mathbb{N}$ to $\mathbb{N}$), we are led to surmise the following:

Conjecture: **CZF** *does not have the weak existence property.*[2]

1.1 The theory $\mathbf{CZF}_{\mathcal{E}}$

In this paper we look at constructive Zermelo-Fraenkel set theory formulated with Exponentiation, $\mathbf{CZF}_{\mathcal{E}}$. We briefly summarize the language and axioms of $\mathbf{CZF}_{\mathcal{E}}$. Its language is based on the same first order language as that of classical Zermelo-Fraenkel Set Theory, whose only non-logical symbol is $\in$. The logic of $\mathbf{CZF}_{\mathcal{E}}$ is intuitionistic first order logic with equality. Among its non-logical axioms are *Extensionality*, *Pairing* and *Union* in their usual forms. **CZF** has additionally axiom schemata which we will now proceed to summarize.

[1] Burr [7, Corollary 5.12] and Diller [10, Proposition 4.4] proved weak forms of term existence property for a higher type versions of **CZF** without Subset Collection and Exponentiation.

[2] Added in proof: This has recently be proved by A. Swan.

Infinity: $\exists x \forall u[u \in x \leftrightarrow (\emptyset = u \vee \exists v \in x\, u = v+1)]$ where $v+1 = v \cup \{v\}$.

Set Induction: $\forall x[\forall y \in x A(y) \to A(x)] \to \forall x A(x)$

Bounded Separation: $\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge A(x)]$

for all *bounded* formulae A. A set-theoretic formula is *bounded* or *restricted* if it is constructed from prime formulae using $\neg, \wedge, \vee, \to, \forall x \in y$ and $\exists x \in y$ only.

Strong Collection: For all formulae A,

$$\forall a[\forall x \in a \exists y A(x,y) \;\to\; \exists b\,[\forall x \in a\, \exists y \in b\, A(x,y) \wedge \forall y \in b\, \exists x \in a\, A(x,y)]].$$

Exponentiation: Let $\mathrm{Fun}(f,a,b)$ be the set-theoretic formula expressing that f is a function from the set a to the set b.

$$\forall a \forall b \exists c \forall f\, (\mathrm{Fun}(f,a,b) \to f \in c)\,.$$

We shall also study the theory augmented by the Power Set Axiom, **Pow**:

$$\forall x\, \exists y\, \forall z\, (z \subseteq y \to z \in y).$$

We denote the system with **Pow** added by $\mathbf{CZF}_{\mathcal{P}}$ rather than $\mathbf{CZF}_{\mathcal{E}} + \mathbf{Pow}$. The reason for this is that both Exponentiation and Subset Collection are consequences of **Pow** (see [2, Proposition 7.2]).

To save work when proving realizability of the axioms of $\mathbf{CZF}_{\mathcal{E}}$ it is useful to know that the axiom scheme of Bounded Separation can be deduced from a single instance (in the presence of Strong Collection).

Lemma 1.4 *Let* **Binary Intersection** *be the statement* $\forall x \forall y \exists z\, x \cap y = z$. *If* $\mathbf{CZF}_0$ *denotes* $\mathbf{CZF}_{\mathcal{E}}$ *without Bounded Separation and Exponentiation, then every instance of Bounded Separation is provable in* $\mathbf{CZF}_0 +$ Binary Intersection.

Proof: [2, Proposition 4.8] is a forerunner of this result. It is proved in the above form in [3, Corollary 9.5.7]. □

2 Intuitionistic Kripke-Platek set theories

A particularly interesting (classical) subtheory of **ZF** is Kripke-Platek set theory, **KP**. Its standard models are called *admissible sets*. One of the reasons that this is an important theory is that a great deal of set theory requires only the axioms of **KP**. An even more important reason is that admissible sets have been a major source of interaction between model theory, recursion theory and set theory (cf. [4]). **KP** arises from **ZF** by completely omitting the power set axiom and restricting separation and collection to bounded formulae. These alterations are suggested by the informal notion of 'predicative'. To be more precise, the axioms of **KP** consist of *Extensionality, Pair, Union, Infinity, Bounded Separation*

$$\exists x\, \forall u\, [u \in x \leftrightarrow (u \in a \;\wedge\; A(u))]$$

for all bounded formulae $A(u)$, *Bounded Collection*

$$\forall x \in a\, \exists y\, B(x,y) \;\rightarrow\; \exists z\, \forall x \in a\, \exists y \in z\, B(x,y)$$

for all bounded formulae $B(x,y)$, and *Set Induction*

$$\forall x\; [(\forall y \in x\, C(y)) \rightarrow C(x)] \;\rightarrow\; \forall x\, C(x)$$

for all formulae $C(x)$.
We denote by **IKP** the version of **KP** where the underlying logic is intuitionistic logic.

2.1 Power and Exponentiation Kripke-Platek set theory

We use subset bounded quantifiers $\exists x \subseteq y\ \ldots$ and $\forall x \subseteq y\ \ldots$ as abbreviations for $\exists x(x \subseteq y \wedge \ldots)$ and $\forall x(x \subseteq y \rightarrow \ldots)$, respectively.
We call a formula of $\mathcal{L}_{\in}$ $\Delta_0^{\mathcal{P}}$ if all its quantifiers are of the form $Q\, x \subseteq y$ or $Q\, x{\in}y$ where Q is $\forall$ or $\exists$ and x and y are distinct variables.
Let $\mathrm{Fun}(f,x,y)$ be a acronym for the bounded formula expressing that f is a function with domain x and co-domain y. We use exponentiation bounded quantifiers $\exists f \in {}^x y\ \ldots$ and $\forall f \in {}^x y\ \ldots$ as abbreviations for $\exists f(\mathrm{Fun}(f,x,y) \wedge \ldots)$ and $\forall x(\mathrm{Fun}(f,x,y) \rightarrow \ldots)$, respectively.

Definition 2.1 The $\Delta_0^{\mathcal{P}}$ formulae are the smallest class of formulae containing the atomic formulae closed under $\wedge, \vee, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall x \subseteq a,\ \exists x \subseteq a.$$

The $\Delta_0^{\mathcal{E}}$ formulae are the smallest class of formulae containing the atomic formulae closed under $\wedge, \vee, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall f \in {}^a b,\ \exists f \in {}^a b.$$

Definition 2.2 $\mathbf{IKP}(\mathcal{E})$ has the same language and logic as **IKP**. Its axioms are the following: Extensionality, Pairing, Union, Infinity, Exponentiation, $\Delta_0^{\mathcal{E}}$-Separation and $\Delta_0^{\mathcal{E}}$-Collection.
$\mathbf{IKP}(\mathcal{P})$ has the same language and logic as **IKP**. Its axioms are the following: Extensionality, Pairing, Union, Infinity, Powerset, $\Delta_0^{\mathcal{P}}$-Separation and $\Delta_0^{\mathcal{P}}$-Collection.

The transitive classical models of $\mathbf{IKP}(\mathcal{P})$ have been termed **power admissible** sets in [13].

Remark 2.3 Alternatively, $\mathbf{IKP}(\mathcal{P})$ can be obtained from **IKP** by adding a function symbol $\mathcal{P}$ for the powerset function as a primitive symbols to the language and the axiom

$$\forall y\, [y \in \mathcal{P}(x) \leftrightarrow y \subseteq x]$$

and extending the schemes of Δ_0 Separation and Collection to the Δ_0 formulae of this new language. Likewise, $\mathbf{IKP}(\mathcal{E})$ can be obtained from **IKP** by adding a primitive function symbol $\mathcal{E}$ for the exponentiation and the pertaining axioms.

Lemma 2.4 *(i)* **IKP** *is a subtheory of* $\mathbf{CZF}^-$.

(i) $\mathbf{IKP}(\mathcal{E})$ *is a subtheory of* $\mathbf{CZF}_{\mathcal{E}}$.

(i) $\mathbf{IKP}(\mathcal{P})$ *is a subtheory of* $\mathbf{CZF}_{\mathcal{P}}$.

Proof: (i) is obvious. For (ii) one has to show that $\mathbf{CZF}_{\mathcal{E}}$ proves $\Delta_0^{\mathcal{E}}$-Separation. This follows by induction on the buildup of the $\Delta_0^{\mathcal{E}}$-formula. Similarly, for (iii) one has to show that $\mathbf{CZF}_{\mathcal{P}}$ proves $\Delta_0^{\mathcal{P}}$-Separation. $\Box$

Definition 2.5 The Σ formulae are the smallest class of formulae containing the Δ_0-formulae closed under $\wedge, \vee$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \exists x.$$

The $\Sigma^{\mathcal{E}}$ formulae are the smallest class of formulae containing the $\Delta_0^{\mathcal{E}}$-formulae closed under $\wedge, \vee$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall f \in {}^a b,\ \exists f \in {}^a b,\ \exists x.$$

The $\Sigma^{\mathcal{P}}$ formulae are the smallest class of formulae containing the $\Delta_0^{\mathcal{P}}$-formulae closed under $\wedge, \vee$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall x \subseteq a,\ \exists x \subseteq a,\ \exists x.$$

To be able to formalize the notion of E-recursion in **IKP** as well as the corresponding extensions in $\mathbf{IKP}(\mathcal{E})$ and $\mathbf{IKP}(\mathcal{P})$, we need to know that certain (class) inductive definitions can be formalized in these theories.

Definition 2.6 An *inductive definition* Φ is a class of pairs. Intuitively an inductive definition is an abstract proof system, where $\langle x, A\rangle \in \Phi$ means that A is a set of premises and x is a Φ-consequence of these premises.
Φ is a Σ inductive definition if Φ is a Σ definable class. Φ is $\Sigma^{\mathcal{E}}$ ($\Sigma^{\mathcal{P}}$) if Φ is a $\Sigma^{\mathcal{E}}$ ($\Sigma^{\mathcal{P}}$) definable class.
A class X is said to be Φ-*closed* if $A \subseteq X$ implies $a \in X$ for every pair $\langle a, A\rangle \in \Phi$.

Theorem 2.7 (**IKP**) *For any Σ inductive definition Φ there is a smallest Φ-closed class $\mathbf{I}(\Phi)$; moreover, $\mathbf{I}(\Phi)$ is a Σ class as well.*

Proof: [2, Theorem 11.4] and [23]. $\Box$

Theorem 2.8 *(i)* ($\mathbf{IKP}(\mathcal{E})$) *For any $\Sigma^{\mathcal{E}}$ inductive definition Φ there is a smallest Φ-closed class $\mathbf{I}(\Phi)$; moreover, $\mathbf{I}(\Phi)$ is a $\Sigma^{\mathcal{E}}$ class as well.*

(ii) ($\mathbf{IKP}(\mathcal{P})$) *For any $\Sigma^{\mathcal{P}}$ inductive definition Φ there is a smallest Φ-closed class $\mathbf{I}(\Phi)$; moreover, $\mathbf{I}(\Phi)$ is a $\Sigma^{\mathcal{P}}$ class as well.*

Proof: Basically the same proof as for Theorem 2.7. $\Box$

2.2 E-recursive functions

We would like to have unlimited application of sets to sets, i.e. we would like to assign a meaning to the symbol $\{a\}(x)$ where a and x are sets. Here we use Kleene's curly bracket notation to convey that a is viewed as encoding the programme of a some kind of Turing machine which takes a set input x to produce a result $\{a\}(x)$. In generalized recursion theory this is known as *E-recursion* or *set recursion* (see, e.g., [29] or [40, Ch.X]). One point of deviation from the standard notion of E-computability though is that we will take the constant function with value ω as an initial function. There is a lot of leeway in setting up E-recursion. The particular schemes we use are especially germane to our situation. Very likely there is a lot of redundancy but any attempts at being economical wouldn't have any benefits for the purposes of this paper.

Our construction will provide a specific set-theoretic model for the elementary theory of operations and numbers **EON** (see, e.g., [6, VI.2], or the theory **APP** as described in [43, Ch.9, Sect.3]). We utilize encoding of finite sequences of sets by the usual pairing function $\langle\,,\rangle$ with $\langle x, y\rangle = \{\{x\},\{x,y\}\}$, letting $\langle x\rangle = x$ and $\langle x_1,\dots,x_n,x_{n+1}\rangle = \langle\langle x_1,\dots,x_n\rangle, x_{n+1}\rangle$. We use functions $()_0$ and $()_1$ to retrieve the left and right components, respectively, of an ordered pair $a = \langle x, y\rangle$, i.e., $(a)_0 = x$ and $(a)_1 = y$.

Below we use the notation $[\,x\,](y)$ rather than the more traditional $\{x\}(y)$ to avoid any ambiguity with the singleton set $\{x\}$.

It will also be convenient to assume that all systems of set theory are formulated in a language that has a constant $\bar{n}$ for each $n \in \mathbb{N}$ plus the pertaining axiom asserting that $\bar{n}$ is the n^{th} member of ω.

Definition 2.9 (**IKP**) First, we select distinct non-zero natural numbers $\mathbf{k}$, $\mathbf{s}$, $\mathbf{p}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{s_N}$, $\mathbf{p_N}$, $\mathbf{d_N}$, $\bar{\mathbf{0}}$, $\bar{\omega}$, $\boldsymbol{\gamma}$, $\boldsymbol{\rho}$, $\boldsymbol{\upsilon}$, $\boldsymbol{\pi}$, $\boldsymbol{i_1}$, $\boldsymbol{i_2}$ and $\boldsymbol{i_3}$, which will provide indices for special E-recursive partial (class) functions. Inductively we shall define a class $\mathbb{E}$ of triples $\langle e, x, y\rangle$. Rather than "$\langle e, x, y\rangle \in \mathbb{E}$", we shall write "$[\,e\,](x) \simeq y$", and moreover, if $n > 0$, we shall use $[\,e\,](x_1,\dots,x_n) \simeq y$ to convey that

$$[\,e\,](x_1) \simeq \langle e, x_1\rangle \ \wedge\ [\,\langle e, x_1\rangle\,](x_2) \simeq \langle e, x_1, x_2\rangle \ \wedge\ \dots\ \wedge\ [\,\langle e, x_1,\dots,x_{n-1}\rangle\,](x_n) \simeq y.$$

We shall say that $[\,e\,](x)$ is defined, written $[\,e\,](x)\downarrow$, if $[\,e\,](x) \simeq y$ for some y. Let $\mathbb{N} := \omega$. $\mathbb{E}$ is defined by the following clauses:

$$\begin{array}{rcl}
[\,\mathbf{k}\,](x,y) & \simeq & x \\
[\,\mathbf{s}\,](x,y,z) & \simeq & [\,[\,x\,](z)\,]([\,y\,](z)) \\
[\,\mathbf{p}\,](x,y) & \simeq & \langle x, y\rangle \\
[\,\mathbf{p_0}\,](x) & \simeq & (x)_0 \\
[\,\mathbf{p_1}\,](x) & \simeq & (x)_1 \\
[\,\mathbf{s_N}\,](n) & \simeq & n+1 \text{ if } n \in \mathbb{N} \\
[\,\mathbf{p_N}\,](0) & \simeq & 0 \\
[\,\mathbf{p_N}\,](n+1) & \simeq & n \text{ if } n \in \mathbb{N} \\
[\,\mathbf{d_N}\,](n,m,x,y) & \simeq & x \text{ if } n,m \in \mathbb{N} \text{ and } n = m \\
[\,\mathbf{d_N}\,](n,m,x,y) & \simeq & y \text{ if } n,m \in \mathbb{N} \text{ and } n \neq m \\
[\,\bar{\mathbf{0}}\,](x) & \simeq & 0 \\
[\,\bar{\omega}\,](x) & \simeq & \omega \\
[\,\boldsymbol{\pi}\,](x,y) & \simeq & \{x,y\} \\
[\,\boldsymbol{\upsilon}\,](x) & \simeq & \bigcup x \\
[\,\boldsymbol{\gamma}\,](x,y) & \simeq & x \cap (\bigcap y) \\
[\,\boldsymbol{\rho}\,](x,y) & \simeq & \{[\,x\,](u) \mid u \in y\} \text{ if } [\,x\,](u) \text{ is defined for all } u \in y \\
[\,\boldsymbol{i_1}\,](x,y,z) & \simeq & \{u \in x \mid y \in z\} \\
[\,\boldsymbol{i_2}\,](x,y,z) & \simeq & \{u \in x \mid u \in y \to u \in z\} \\
[\,\boldsymbol{i_3}\,](x,y,z) & \simeq & \{u \in x \mid u \in y \to z \in u\}.
\end{array}$$

Note that $[\,\mathbf{s}\,](x,y,z)$ is not defined unless $[\,x\,](z)$, $[\,y\,](z)$ and $[\,[\,x\,](z)\,]([\,y\,](z))$ are already defined. The clause for $\mathbf{s}$ is thus to be read as a conjunction of the following clauses: $[\,\mathbf{s}\,](x) \simeq \langle \mathbf{s}, x\rangle$, $[\,\langle \mathbf{s}, x\rangle\,](y) \simeq \langle \mathbf{s}, x, y\rangle$ and, if there exist a, b, c such that $[\,x\,](z) \simeq a$, $[\,y\,](z) \simeq b$, $[\,a\,](b) \simeq c$, then $[\,\langle \mathbf{s}, x, y\rangle\,](z) \simeq c$. Similar restrictions apply to $\boldsymbol{\rho}$.

Lemma 2.10 (**IKP**) $\mathbb{E}$ *is an inductively defined class and* $\mathbb{E}$ *is functional in that for all* e, x, y, y',

$$\langle e, x, y\rangle \in \mathbb{E} \;\wedge\; \langle e, x, y'\rangle \in \mathbb{E} \quad\Rightarrow\quad y = y'.$$

Proof. The inductive definition of $\mathbb{E}$ falls under the heading of Theorem 2.8. If $[\,e\,](x) \simeq y$ the uniqueness of y follows by induction on the stages (cf. [2, Lemma 5.2]) of that inductive definition. □

Definition 2.11 *Application terms* are defined inductively as follows:

(i) The constants $\mathbf{k}$, $\mathbf{s}$, $\mathbf{p}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{s_N}$, $\mathbf{p_N}$, $\mathbf{d_N}$, $\bar{\mathbf{0}}$, $\bar{\omega}$, $\boldsymbol{\gamma}$, $\boldsymbol{\rho}$, $\boldsymbol{\upsilon}$, $\boldsymbol{\pi}$, $\boldsymbol{i_1}$, $\boldsymbol{i_2}$ and $\boldsymbol{i_3}$ singled out in Definition 2.9 are *application terms*;

(ii) variables are *application terms*;

(iii) if s and t are *application terms* then (st) is an *application term.*

Definition 2.12 Application terms are easily formalized in **IKP**. However, rather than translating application terms into the set–theoretic language, we define the translation of expressions of the form $t \simeq u$, where t is an application term and u is a variable. The translation proceeds along the way t was built up:

$$\begin{array}{rll} [c \simeq u]^\wedge & \text{is} & c = u \text{ if } c \text{ is a constant or a variable;} \\ [(st) \simeq u]^\wedge & \text{is} & \exists x \exists y([s \simeq x]^\wedge \wedge [t \simeq y]^\wedge \wedge [\,x\,](y) \simeq u). \end{array}$$

Abbreviations. For application terms s, t, $t_1, \ldots, t_n$ we will use:

$$\begin{array}{rll} s(t_1, \ldots, t_n) & \text{as a shortcut for} & ((\ldots(st_1)\ldots)t_n); \quad \textit{(parentheses associated to the left)}; \\ st_1 \ldots t_n & \text{as a shortcut for} & s(t_1, \ldots, t_n); \\ t{\downarrow} & \text{as a shortcut for} & \exists x(t \simeq x)^\wedge; \quad (t \textit{ is defined}) \\ (s \simeq t)^\wedge & \text{as a shortcut for} & (\,s{\downarrow} \vee t{\downarrow}) \to \exists x((s \simeq x)^\wedge \wedge (t \simeq x)^\wedge)\,. \end{array}$$

A *closed* application term is an application term that does not contain variables. If t is a closed application term and $a_1, \ldots, a_n, b$ are sets we use the abbreviation

$$\begin{array}{lll} t(a_1, \ldots, a_n) \simeq b & \text{for} & \exists x_1 \ldots x_n \exists y\, (x_1 = a_1 \;\wedge\; \ldots \;\wedge\; x_n = a_n \;\wedge\; y = b \\ & & \qquad\qquad \wedge\;\; [t(x_1, \ldots, x_n) \simeq y]^\wedge). \end{array}$$

Definition 2.13 Every closed application term gives rise to a partial class function. A partial n-place (class) function Υ is said to be an E_{exp}*-recursive partial function* if there exists a closed application term t_Υ such that

$$\mathrm{dom}(\Upsilon) \;\;=\;\; \{(a_1, \ldots, a_n) \mid t_\Upsilon(a_1, \ldots, a_n)\downarrow\}$$

and for all for all sets $(a_1, \ldots, a_n) \in \mathrm{dom}(\Upsilon)$,

$$t_\Upsilon(a_1, \ldots, a_n) \;\;\simeq\;\; \Upsilon(a_1, \ldots, a_n).$$

In the latter case, t_Υ is said to be an *index* for Υ.
If Υ_1, Υ_2 are E_{exp}-recursive partial functions, then $\Upsilon_1(\vec{a}) \simeq \Upsilon_2(\vec{a})$ iff neither $\Upsilon_1(\vec{a})$ nor $\Upsilon_2(\vec{a})$ are defined, or $\Upsilon_1(\vec{a})$ and $\Upsilon_2(\vec{a})$ are defined and equal.

The next two results can be proved in the theory **APP** and thus hold true in any applicative structure. Thence the particular applicative structure considered here satisfies the Abstraction Lemma and Recursion Theorem (see e.g. [11] or [6]).

Lemma 2.14 (Abstraction Lemma, cf. [6, VI.2.2])
For every application term $t[x]$ there exists an application term $\lambda x.t[x]$ with $\mathrm{FV}(\lambda x.t[x]) := \{x_1, \ldots, x_n\} \subseteq \mathrm{FV}(t[x]) \backslash \{x\}$ *such that the following holds:*

$$\forall x_1 \ldots \forall x_n (\lambda x.t[x]{\downarrow} \wedge \forall y\, (\lambda x.t[x])y \simeq t[y]).$$

Proof. (i) $\lambda x.x$ is $\mathbf{skk}$; (ii) $\lambda x.t$ is $\mathbf{k}t$ for t a constant or a variable other than x; (iii) $\lambda x.uv$ is $(\mathbf{s}(\lambda x.u))(\lambda x.v)$. □

Lemma 2.15 (Recursion Theorem, cf. [6, VI.2.7])
There exists a closed application term rec *such that for any f, x,*

$$\mathsf{rec} f \downarrow \;\wedge\; \mathsf{rec} f x \simeq f(\mathsf{rec} f)x.$$

Proof. Take rec to be $\lambda f.tt$, where t is $\lambda y \lambda x.f(yy)x$. □

2.3 Extended E-recursive functions

We shall introduce two extended notions of E-computability, christened E_{exp}-computability and $E_{\wp}$-computability, rendering the functions $\exp(a, b) = {}^a b$ and $\mathcal{P}(x) = \{u \mid u \subseteq x\}$ computable, respectively (where ${}^a b$ denotes the set of all functions from a to b). Indices for these functions will supply suitable for realizability interpretations of $\mathbf{CZF}_{\mathcal{E}}$ and $\mathbf{CZF}_{\mathcal{P}}$, respectively.
$E_{\wp}$-computability is closely related to power recursion, where the power set operation is regarded to be an initial function. The latter notion has been studied by Moschovakis [25] and Moss [26].

Definition 2.16 (i) ($\mathbf{IKP}(\mathcal{E})$) For E_{exp}-computability we add an additional constant $\boldsymbol{\varepsilon}$ and the clause

$$[\,\boldsymbol{\varepsilon}\,](x, y) \simeq {}^x y$$

to Definition 2.9. We thereby arrive at an inductively defined class $\mathbb{E}_{\mathrm{exp}}$.

(ii) ($\mathbf{IKP}(\mathcal{P})$) For $E_{\wp}$-computability we add an additional constant $\bar{\wp}$ and the clause

$$[\,\bar{\wp}\,](x) \simeq \mathcal{P}(x)$$

to Definition 2.9. We thereby arrive at an inductively defined class $\mathbb{E}_{\wp}$.

Lemma 2.17

(i) ($\mathbf{IKP}(\mathcal{E})$) $\mathbb{E}_{\mathrm{exp}}$ *is an inductively defined class and* $\mathbb{E}_{\mathrm{exp}}$ *is functional in that for all* e, x, y, y',

$$\langle e, x, y\rangle \in \mathbb{E}_{\mathrm{exp}} \;\wedge\; \langle e, x, y'\rangle \in \mathbb{E}_{\mathrm{exp}} \;\;\Rightarrow\;\; y = y'.$$

(ii) ($\mathbf{IKP}(\mathcal{P})$) $\mathbb{E}_{\wp}$ *is an inductively defined class and* $\mathbb{E}_{\wp}$ *is functional in that for all* e, x, y, y',

$$\langle e, x, y\rangle \in \mathbb{E}_{\wp} \;\wedge\; \langle e, x, y'\rangle \in \mathbb{E}_{\wp} \;\;\Rightarrow\;\; y = y'.$$

Proof: The same procedure as for Lemma 2.10. $\Box$

There is of course a notion of application term pertaining to $\mathbb{E}_{\text{exp}}$ and another one pertaining to $\mathbb{E}_{\wp}$. Constants $\boldsymbol{\varepsilon}$ and $\bar{\wp}$, respectively, have to be added in Definition 2.11. We call them $\boldsymbol{\varepsilon}$-application terms and $\wp$-applications terms, respectively. All the previous results then hold, grosso modo, for the two expanded notions of application term. In particular one has the following results.

Corollary 2.18 *For any E_{exp}-recursive partial function Υ there exists a closed $\boldsymbol{\varepsilon}$-application term τ_{fix} such that $\tau_{fix}\downarrow$ and for all $\vec{a}$,*

$$\Upsilon(\bar{e},\vec{a}) \quad \simeq \quad \tau_{fix}(\vec{a}),$$

where $\tau_{fix} \simeq \bar{e}$. Moreover, τ_{fix} can be effectively (e.g. primitive recursively) constructed from an index for Υ.

Corollary 2.19 *For any $E_{\wp}$-recursive partial function Υ there exists a closed $\bar{\wp}$-application term τ_{fix} such that $\tau_{fix}\downarrow$ and for all $\vec{a}$,*

$$\Upsilon(\bar{e},\vec{a}) \quad \simeq \quad \tau_{fix}(\vec{a}),$$

where $\tau_{fix} \simeq \bar{e}$. Moreover, τ_{fix} can be effectively (e.g. primitive recursively) constructed from an index for Υ.

Lemma 2.20 *For each Δ_0 formula $A(\vec{a},u)$ formula (with a all free variables among $\vec{a},u$) there is a closed application term t_A such that*

$$\mathbf{IKP} \vdash t_A(\vec{a},b)\downarrow \,\wedge \forall u\,[u \in t_A(\vec{a},b) \;\leftrightarrow\; (u \in b \wedge A(\vec{a},u))].$$

Proof: We use induction on the generation of $A(\vec{a},u)$. Owing to extensionality it suffices to show this for formulae that do not contain $=$.

1. Let $A(\vec{a},u)$ be atomic. First suppose that $A(\vec{a},u)$ is of the form $u \in a_i$. Then $\{u \in b \mid A(\vec{a},u)\} = a_i \cap b$. Let $t_A := \lambda\vec{x}y.\boldsymbol{\gamma}(x_i, \boldsymbol{\pi}(y,y))$. Then $t_A(\vec{a},b) = \boldsymbol{\gamma}(a_i, \boldsymbol{\pi}(b,b)) = a_i \cap (\bigcup\{b\}) = a_i \cap b$.
Now suppose $A(\vec{a},u)$ is of the form $a_i \in u$. Then $\{u \in b \mid A(\vec{a},u)\} = \{u \in b \mid u \in b \to a_i \in u\}$. Thus $t_A := \lambda\vec{x}y.\boldsymbol{i_3}(y,y,x_i)$ will do the job.
The other cases are where $A(\vec{a},u)$ is of either form $u \in u$ or $a_i \in a_j$. Here the terms $\lambda\vec{x}y.\mathbf{0}y$ and $\lambda\vec{x}y.\boldsymbol{i_1}(y,x_i,x_j)$ will work.

2. If $A(\vec{a},u)$ is of the form $A_1(\vec{a},u) \vee A_2(\vec{a},u)$ let $t_A := \lambda\vec{x}y.\boldsymbol{\upsilon}(\boldsymbol{\pi}(t_{A_1}(\vec{x},y), t_{A_2}(\vec{x},y)))$. This works since $t_A(\vec{a},b) = \bigcup\{t_{A_1}(\vec{a},b), t_{A_2}(\vec{a},b)\} = t_{A_1}(\vec{a},b) \cup t_{A_2}(\vec{a},b)$.

3. If $A(\vec{a},u)$ is of the form $A_1(\vec{a},u) \wedge A_2(\vec{a},u)$ let $t_A := \lambda\vec{x}y.\boldsymbol{\gamma}(t_{A_1}(\vec{x},y), \boldsymbol{\pi}(t_{A_2}(\vec{x},y), t_{A_2}(\vec{x},y)))$. This works as $t_A(\vec{a},b) = t_{A_1}(\vec{a},b) \cap \bigcap\{t_{A_2}(\vec{a},b)\} = t_{A_1}(\vec{a},b) \cap t_{A_2}(\vec{a},b)$.

4. If $A(\vec{a},u)$ is of the form $A_1(\vec{a},u) \to A_2(\vec{a},u)$ put $t_A := \lambda\vec{x}y.\boldsymbol{i_2}(y, t_{A_1}(\vec{x},y), t_{A_2}(\vec{x},y))$ since $t_A(\vec{a},b) = \{u \in b \mid u \in t_{A_1}(\vec{x},y) \to u \in t_{A_2}(\vec{x},y)\}$.

5. If $A(\vec{a},u)$ is of the form $\neg A_1(\vec{a},u)$ put $t_A := \lambda\vec{x}y.\boldsymbol{i_2}(y, t_{A_1}(\vec{x},y), \mathbf{0}(y))$ since $t_A(\vec{a},b) = \{u \in b \mid u \in t_{A_1}(\vec{x},y) \to u \in \emptyset\}$.

6. Suppose $A(\vec{a},u)$ is of the form $\forall w \in a_i\, B(\vec{a},w,u)$. Inductively we have a term t_B such that $t_B(\vec{a},d,b) = \{u \in b \mid B(\vec{a},d,u)\}$. Thus, letting $t_A := \lambda\vec{x}y.\boldsymbol{\gamma}(y, \boldsymbol{\rho}(\lambda z.t_B(\vec{x},z,y), x_i))$, we have

$$\begin{aligned} t_A(\vec{a},b) &= \boldsymbol{\gamma}(b, \boldsymbol{\rho}(\lambda z.t_B(\vec{a},z,b), a_i)) \;=\; b \cap \bigcap\{t_B(\vec{a},d,b) \mid d \in a_i\} \\ &= b \cap \bigcap\{\{u \in b \mid B(\vec{a},d,u)\} \mid d \in a_i\} \;=\; \{u \in b \mid \forall w \in a_i\, B(\vec{a},w,u)\}. \end{aligned}$$

If $A(\vec{a},u)$ is of the form $\forall w \in u\, B(\vec{a},w,u)$, we have

$$\{u \in b \mid \forall w \in u\, B(\vec{a},w,u)\} = \{u \in b \mid \forall w \in \bigcup b\, (w \in u \to B(\vec{a},w,u))\},$$

and hence this case can be reduced to the previous with the help of (4).

7. Suppose $A(\vec{a},u)$ is of the form $\exists w \in a_i\, B(\vec{a},w,u)$. Inductively we have a term t_B such that $t_B(\vec{a},d,b) = \{u \in b \mid B(\vec{a},d,u)\}$. Let $t_A := \lambda \vec{x}y.\boldsymbol{\upsilon}(\boldsymbol{\rho}(\lambda z.t_B(\vec{x},z,y),x_i))$, we have

$$\begin{aligned} t_A(\vec{a},b) &= \boldsymbol{\upsilon}(\boldsymbol{\rho}(\lambda z.t_B(\vec{a},z,b),a_i)) = \bigcup\{t_B(\vec{a},d,b) \mid d \in a_i\} \\ &= \bigcup\{\{u \in b \mid B(\vec{a},d,u)\} \mid d \in a_i\} = \{u \in b \mid \exists w \in a_i\, B(\vec{a},w,u)\}. \end{aligned}$$

If $A(\vec{a},u)$ is of the form $\exists w \in u\, B(\vec{a},w,u)$, we have

$$\{u \in b \mid \exists w \in u\, B(\vec{a},w,u)\} = \{u \in b \mid \exists w \in \bigcup b\, (w \in u \wedge B(\vec{a},w,u))\},$$

so that this case can be reduced to the previous with the help of (3). $\square$

Lemma 2.21 *(i) For each $\Delta_0^{\mathcal{E}}$ formula $A(\vec{a},u)$ formula (with a all free variables among $\vec{a},u$) there is a closed $\boldsymbol{\varepsilon}$-application term t_A such that*

$$\mathbf{IKP}(\mathcal{E}) \vdash t_A(\vec{a},b)\downarrow \wedge \forall u\,[u \in t_A(\vec{a},b) \leftrightarrow (u \in b \wedge A(\vec{a},u))].$$

(ii) For each $\Delta_0^{\mathcal{P}}$ formula $A(\vec{a},u)$ formula (with a all free variables among $\vec{a},u$) there is a closed $\bar{\wp}$-application term t_A such that

$$\mathbf{IKP}(\mathcal{P}) \vdash t_A(\vec{a},b)\downarrow \wedge \forall u\,[u \in t_A(\vec{a},b) \leftrightarrow (u \in b \wedge A(\vec{a},u))].$$

Proof: The proof expands that of Lemma 2.20. For instance, in (ii) $A(\vec{a},u)$ could be of the form $\forall w \subseteq a_i\, B(\vec{a},w,u)$. Then, inductively we have a term t_B such that $t_B(\vec{a},d,b) = \{u \in b \mid B(\vec{a},d,u)\}$. Thus, letting $t_A := \lambda \vec{x}y.\boldsymbol{\gamma}(y,\boldsymbol{\rho}(\lambda z.t_B(\vec{x},z,y),\bar{\wp}(x_i)))$, we have

$$\begin{aligned} t_A(\vec{a},b) &= \boldsymbol{\gamma}(b,\boldsymbol{\rho}(\lambda z.t_B(\vec{a},z,b),\bar{\wp}(a_i))) = b \cap \bigcap\{t_B(\vec{a},d,b) \mid d \in \mathcal{P}(a_i)\} \\ &= b \cap \bigcap\{\{u \in b \mid B(\vec{a},d,u)\} \mid d \in \mathcal{P}(a_i)\} = \{u \in b \mid \forall w \subseteq a_i\, B(\vec{a},w,u)\}. \end{aligned}$$

If $A(\vec{a},u)$ is of the form $\forall w \subseteq u\, B(\vec{a},w,u)$, we have

$$\{u \in b \mid \forall w \subseteq u\, B(\vec{a},w,u)\} = \{u \in b \mid \forall w \subseteq \bigcup b\, (w \subseteq u \to B(\vec{a},w,u))\},$$

and hence this case can be reduced to the previous one.
The other cases are similar. $\square$

3 Defining realizability with sets of witnesses for set theory

Realizability semantics are a crucial tool in the study of intuitionistic theories. We introduce a form of realizability based on general set recursive functions where a realizer for an existential statement provides a set of witnesses for the existential quantifier rather than a single witness. Realizability based on indices of general set recursive functions was introduced in [34] and employed to prove, inter alia, metamathematical properties for **CZF** augmented by strong forms of the axiom of choice in [35, Theorems 8.3,8.4]. There are points of contact with a notion of realizability used by Tharp [42] who employed (indices of) Σ_1 definable partial (class) functions as realizers, though there are important differences, too, as Tharp works in a classical context and assumes a definable search operation on the universe which basically amounts to working under the hypothesis $V = L$. Moreover, there are connections with Lifschitz' realizability [22] where a realizer for an existential arithmetical statement provides a finite co-recursive set of witnesses (see [30, 8] for extensions to analysis and set theory). Another important type of semantics or interpretation for intuitionistic systems is functional interpretation. The Diller-Nahm interpretation [9] also employs sets of witnesses to interpret existential quantifiers. It has been extended to set theories by Burr, Diller and Schulte [7, 10, 41]. Burr [7, Corollary 5.12] and Diller [10, Proposition 4.4] prove weak forms of term existence property for a higher type versions of $\mathbf{CZF}^-$. Interestingly, Diller conjectures ([10] Conjectures 4.8,4.9) that the term existence property fails for higher type versions of $\mathbf{CZF}^-$. By contrast, in this paper it shown that $\mathbf{CZF}^-$ does have the existence property (Corollary 6.1).

A further important aspect of our realizability is that it is combined with truth so that realizability entails truth and thereby yields the desired results.

We adopt the conventions and notations from the previous section. However, we prefer to write $\jmath_0 e$ and $\jmath_1 e$ rather than $(e)_0$ and $(e)_1$, respectively, and instead of $[\,a\,](b) \simeq c$ we shall write $a \bullet b \simeq c$.

Definition 3.1 Bounded quantifiers will be treated as quantifiers in their own right, i.e., bounded and unbounded quantifiers are treated as syntactically different kinds of quantifiers.

We use the expression $a \neq \emptyset$ to convey the positive fact that the set a is inhabited, that is $\exists x\, x \in a$. We define a relation $a \Vdash_{\mathfrak{wt}} B$ between sets and set-theoretic formulae. $a \bullet f \Vdash_{\mathfrak{wt}} B$ will be an abbreviation for $\exists x[a \bullet f \simeq x \wedge x \Vdash_{\mathfrak{wt}} B]$.

$$
\begin{array}{rll}
a \Vdash_{\mathfrak{wt}} A & \text{iff} & A \text{ holds true, whenever } A \text{ is an atomic formula} \\
a \Vdash_{\mathfrak{wt}} A \wedge B & \text{iff} & \jmath_0 a \Vdash_{\mathfrak{wt}} A \ \wedge\ \jmath_1 a \Vdash_{\mathfrak{wt}} B \\
a \Vdash_{\mathfrak{wt}} A \vee B & \text{iff} & a \neq \emptyset \ \wedge\ (\forall d \in a)([\jmath_0 d = 0 \ \wedge\ \jmath_1 d \Vdash_{\mathfrak{wt}} A] \ \vee \\
 & & \qquad\qquad\qquad [\jmath_0 d = 1 \ \wedge\ \jmath_1 d \Vdash_{\mathfrak{wt}} B]) \\
a \Vdash_{\mathfrak{wt}} \neg A & \text{iff} & \neg A \ \wedge\ \forall c\, \neg c \Vdash_{\mathfrak{wt}} A \\
a \Vdash_{\mathfrak{wt}} A \rightarrow B & \text{iff} & (A \rightarrow B) \ \wedge\ \forall c\,[c \Vdash_{\mathfrak{wt}} A \ \rightarrow\ a \bullet c \Vdash_{\mathfrak{wt}} B] \\
a \Vdash_{\mathfrak{wt}} (\forall x \in b)\ A & \text{iff} & (\forall c \in b)\ a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} (\exists x \in b)\ A & \text{iff} & a \neq \emptyset \ \wedge\ (\forall d \in a)[\jmath_0 d \in b \ \wedge\ \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d] \\
a \Vdash_{\mathfrak{wt}} \forall x A & \text{iff} & \forall c\ \ a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} \exists x A & \text{iff} & a \neq \emptyset \ \wedge\ (\forall d \in a)\ \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d] \\
 & & \\
\Vdash_{\mathfrak{wt}} B & \text{iff} & \exists a\, a \Vdash_{\mathfrak{wt}} B.
\end{array}
$$

The preceding realizability notion is based on E-computability, i.e., $e \bullet x \simeq y$ stands for $\langle e, x, y\rangle \in \mathbb{E}$. If instead we use the corresponding realizability notion based on E_{exp}–computability, where $e \bullet x \simeq y$ stands for $\langle e, x, y\rangle \in \mathbb{E}_{\mathrm{exp}}$, we notate this by writing $e \Vdash^{\varepsilon}_{\mathfrak{wt}} B$. In the same vein, realizability based on $E_{\wp}$–computability with $e \bullet x \simeq y$ standing for $\langle e, x, y\rangle \in \mathbb{E}_{\wp}$, will be indicated by $e \Vdash^{\wp}_{\mathfrak{wt}} \psi$.

Remark 3.2 The above notion of realizability stripped of its truth component was employed in [38] to obtain proof-theoretic results relating intuitionistic and and classical set theories.

Corollary 3.3 *(i)* $\mathbf{CZF}^{-} \vdash (\Vdash_{\mathfrak{wt}} B) \to B$.

(ii) $\mathbf{CZF}_{\mathcal{E}} \vdash (\Vdash^{\varepsilon}_{\mathfrak{wt}} B) \to B$.

(iii) $\mathbf{CZF}_{\mathcal{P}} \vdash (\Vdash^{\wp}_{\mathfrak{wt}} B) \to B$.

Proof: This is immediate by induction on the complexity of B. $\square$

Lemma 3.4 *Let $\vec{x} = x_1, \ldots, x_r$ and $\vec{a} = a_1, \ldots, a_r$. To each formula $A(\vec{x})$ of set theory (with all free variables among $\vec{x}$) we can effectively assign (a code of) an E-recursive partial function χ_A such that the following hold:*

(i) $\mathbf{CZF}^{-} \vdash \forall\vec{a}\,\forall c \neq \emptyset\,[\,(\forall d \in c\; d \Vdash_{\mathfrak{wt}} A(\vec{a})) \to \chi_A(\vec{a}, c) \Vdash_{\mathfrak{wt}} A(\vec{a})]$.

(ii) $\mathbf{CZF}_{\mathcal{E}} \vdash \forall\vec{a}\,\forall c \neq \emptyset\,[\,(\forall d \in c\; d \Vdash^{\varepsilon}_{\mathfrak{wt}} A(\vec{a})) \to \chi_A(\vec{a}, c) \Vdash^{\varepsilon}_{\mathfrak{wt}} A(\vec{a})]$.

(iii) $\mathbf{CZF}_{\mathcal{P}} \vdash \forall\vec{a}\,\forall c \neq \emptyset\,[\,(\forall d \in c\; d \Vdash^{\wp}_{\mathfrak{wt}} A(\vec{a})) \to \chi_A(\vec{a}, c) \Vdash^{\wp}_{\mathfrak{wt}} A(\vec{a})]$.

Proof: We prove (i). (ii) and (iii) are almost identical. We use induction on the buildup of A.
If A is atomic, let $\chi_A(\vec{a}, c) := 0$.
Let $A(\vec{x})$ be $B(\vec{x}) \wedge C(\vec{x})$ and χ_B and χ_C be already defined. Then

$$\chi_A(\vec{a}, c) := \jmath(\chi_B(\vec{a}, \{\jmath_0 x \mid x \in c\}), \chi_C(\vec{a}, \{\jmath_1 x \mid x \in c\}))$$

will do the job.
Let $A(\vec{x})$ be $B(\vec{x}) \to C(\vec{x})$ and suppose χ_B and χ_C have already been defined. Assume that $c \neq \emptyset$ and $(\forall d \in c)\, d \Vdash_{\mathfrak{wt}} [B(\vec{a}) \to C(\vec{a})]$. Suppose $e \Vdash_{\mathfrak{wt}} B(\vec{a})$. Define the E-recursive partial function ϑ by

$$\vartheta(c, e) \simeq \{d \bullet e \mid d \in c\}.$$

Then $\vartheta(c, e) \neq \emptyset$ and hence, by the inductive assumption, $\chi_C(\vec{a}, \vartheta(c, e)) \Vdash_{\mathfrak{wt}} C(\vec{a}\,)$, so that

$$\lambda e.\chi_C(\vec{a}, \vartheta(c, e)) \Vdash_{\mathfrak{wt}} A(\vec{a}\,).$$

Now let $A(\vec{x}\,)$ be of the form $\forall y\, B(\vec{x}, y)$. Suppose that $c \neq \emptyset$ and $(\forall d \in c)\, d \Vdash_{\mathfrak{wt}} A(\vec{a}\,)$. Fixing b, we then have $(\forall d \in c)\, d \bullet b \Vdash_{\mathfrak{wt}} B(\vec{a}, b)$, thus, $\forall d' \in \vartheta(c, b)\; d' \Vdash_{\mathfrak{wt}} B(\vec{a}, b)$, and therefore, by the inductive assumption, $\chi_B(\vec{a}, \vartheta(c, b)) \Vdash_{\mathfrak{wt}} B(\vec{a}, b)$. As a result

$$\lambda b.\chi_B(\vec{a}, \vartheta(c, b)) \Vdash_{\mathfrak{wt}} A(\vec{a}\,).$$

The case of $A(\vec{x}\,)$ starting with a bounded universal quantifier is similar to the previous case.
In all the remaining cases, $\chi_A(\vec{a}, c) := \bigcup c$ will work owing to the definition of realizability in these cases. $\square$

Lemma 3.5 (IKP) *Realizers for equality laws:*

(i) $0 \Vdash_{\mathfrak{wt}} x = x$.

(ii) $\lambda u.u \Vdash_{\mathfrak{wt}} x = y \to y = x$.

(iii) $\lambda u.u \Vdash_{\mathfrak{wt}} (x = y \wedge y = z) \;\to\; x = z$.

(iv) $\lambda u.u \Vdash_{\mathfrak{wt}} (x = y \wedge y \in z) \to x \in z$.

(v) $\lambda u.u \Vdash_{\mathfrak{wt}} (x = y \wedge z \in x) \to z \in y$.

(vi) $\lambda u.\jmath_1 u \Vdash_{\mathfrak{wt}} (x = y \wedge A(x)) \to A(y)$
for any formula A.

Proof: (i) - (v) are obvious. (vi) follows by a trivial induction on the buildup of A. $\square$

Lemma 3.6 (IKP) *Realizers for logical axioms: Below we use the E-recursive function $\mathfrak{sg}(a) := \{a\}$.*

(IPL1) $\mathbf{k} \Vdash_{\mathfrak{wt}} A \to (B \to A)$.

(IPL2) $\mathbf{s} \Vdash_{\mathfrak{wt}} [A \to (B \to C)] \to [(A \to B) \to (A \to C)]$.

(IPL3) $\lambda e.\lambda d.\jmath(e,d)\rangle \Vdash_{\mathfrak{wt}} A \to (B \to A \wedge B)$.

(IPL4) $\lambda e.\jmath_0 e \Vdash_{\mathfrak{wt}} A \wedge B \to A$.

(IPL5) $\lambda e.\jmath_1 e \Vdash_{\mathfrak{wt}} A \wedge B \to B$.

(IPL6) $\lambda e.\mathfrak{sg}(\jmath(0,e)) \Vdash_{\mathfrak{wt}} A \to A \vee B$.

(IPL7) $\lambda e.\mathfrak{sg}(\jmath(1,e))\rangle \Vdash_{\mathfrak{wt}} B \to A \vee B$.

(IPL8) $\mathfrak{k}(\vec{a}\,) \Vdash_{\mathfrak{wt}} (A \vee B) \to [(A \to C) \to ((B \to C) \to C)]$, *for some $E_{\exp}$-recursive partial function $\mathfrak{k}$, where $\vec{a}$ comprises all parameters appearing in the formula.*

(IPL9) $\lambda e.\lambda d.0 \Vdash_{\mathfrak{wt}} (A \to B) \to ((A \to \neg B) \to \neg A)$.

(IPL10) $\lambda e.0 \Vdash_{\mathfrak{wt}} A \to (\neg A \to B)$.

(IPL11) $\lambda e.e \bullet b \Vdash_{\mathfrak{wt}} \forall x A(x) \to A(b)$.

(IPL12) $\lambda e.\mathfrak{sg}(e) \Vdash_{\mathfrak{wt}} A(a) \to \exists x A(x)$.

Proof: As for IPL1 and IPL2, this justifies the combinators **s** and **k**. Combinatory completeness of these two combinators is equivalent to the fact that these two laws together with modus ponens generate the full set of theorems of propositional implicational intuitionistic logic.
Except for IPL8, one easily checks that the proposed realizers indeed realize the pertaining formulae. So let's check IPL8. $A \vee B \to ((A \to C) \to ((B \to C) \to C))$. Suppose $e \Vdash_{\mathfrak{wt}} A \vee B$. Then $e \neq \emptyset$. Let $d \in e$. Then $\jmath_0 d = 0 \wedge \jmath_1 d \Vdash_{\mathfrak{wt}} A$ or $\jmath_0 d = 1 \wedge \jmath_1 d \Vdash_{\mathfrak{wt}} B$. Suppose $f \Vdash_{\mathfrak{wt}} A \to C$ and $g \Vdash_{\mathfrak{wt}} B \to C$. Define an $E_{\exp}$-recursive partial function $\mathfrak{f}$ by

$$\mathfrak{f}(d', f', g') = [\mathbf{d_N}](\jmath_0 d', 0, f' \bullet (\jmath_1 d'), g' \bullet (\jmath_1 d')).$$

Then

$$\mathfrak{f}(d', f', g') = \begin{cases} f' \bullet (\jmath_1 d') & \text{if } \jmath_0 d' = 0 \\ g' \bullet (\jmath_1 d') & \text{if } \jmath_0 d' = 1\,. \end{cases}$$

As a result, $\mathfrak{f}(d,f,g) \Vdash_{\mathfrak{wt}} C$ and hence $\lambda f.\lambda g.\mathfrak{f}(d,f,g) \Vdash_{\mathfrak{wt}} (A \to C) \to ((B \to C) \to C)$. Thus, $[\boldsymbol{\rho}](\lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g), e) \neq \emptyset$ and for all $p \in [\boldsymbol{\rho}](\lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g), e)$ we have

$$p \Vdash_{\mathfrak{wt}} (A \to C) \to ((B \to C) \to C).$$

Let $E(\vec{a}\,) := (A \to C) \to ((B \to C) \to C)$, where $\vec{a}$ comprises all parameters appearing in the formula on the right hand side. The upshot is that by Lemma 3.4 we can conclude

$$\chi_E(\vec{a}, [\boldsymbol{\rho}](\lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g), e)) \Vdash_{\mathfrak{wt}} E(\vec{a}\,).$$

And consequently we have

$$\lambda e.\chi_E(\vec{a}, [\boldsymbol{\rho}](\lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g), e)) \Vdash_{\mathfrak{wt}} A \vee B \to E(\vec{a}\,).$$

□

Theorem 3.7 *Let $D(u_1, \ldots, u_r)$ be a formula of $\mathcal{L}_\in$ all of whose free variables are among $u_1, \ldots, u_r$. If*

$$\mathbf{CZF}^- \vdash D(u_1, \ldots, u_r),$$

then one can effectively construct an index of an E-recursive function g such that

$$\mathbf{CZF}^- \vdash \forall a_1, \ldots, a_r\, g(a_1, \ldots, a_r) \Vdash_{\mathfrak{wt}} D(a_1, \ldots, a_r)\,.$$

Proof: We use a standard Hilbert-type systems for intuitionistic predicate logic. The proof proceeds by induction on the derivation. For the logical axioms and the equality axioms we have already produced appropriate $E_{\exp}$-recursive functions in Lemmata 3.5 and 3.6. It remains to deal with logical inferences and set-theoretic axioms. We start with the rules.
The only rule from propositional logic is modus ponens. Suppose that we have E-recursive functions g_0 and g_1 such that for all $\vec{a}$, $g_0(\vec{a}\,) \Vdash_{\mathfrak{wt}} A(\vec{a}\,) \to B(\vec{a}\,)$ and $g_1(\vec{a}\,) \Vdash_{\mathfrak{wt}} A(\vec{a}\,)$. Then $g(\vec{a}\,) \Vdash_{\mathfrak{wt}} B(\vec{a}\,)$ holds with the E-recursive function $g(\vec{a}\,) := g_0(\vec{a}\,) \bullet g_1(\vec{a}\,)$.
For the $\forall$ quantifier we have the rule: from $B(\vec{u}) \to A(x,\vec{u})$ infer $B(\vec{u}) \to \forall x A(x,\vec{u})$ if x is not free in $B(\vec{u})$. Inductively we have an E-recursive function $\mathfrak{h}$ such that for all $b, \vec{a}$,

$$\mathfrak{h}(b,\vec{a}\,) \Vdash_{\mathfrak{wt}} B(\vec{a}) \to A(b,\vec{a}).$$

Suppose $d \Vdash_{\mathfrak{wt}} B(\vec{a})$. Then $\mathfrak{h}(b,\vec{a}\,) \bullet d \Vdash_{\mathfrak{wt}} A(b,\vec{a})$ holds for all b, whence $\lambda x.(\mathfrak{h}(x,\vec{a}\,) \bullet d) \Vdash_{\mathfrak{wt}} \forall x A(x,\vec{a})$. As a result,

$$\lambda d.\lambda x.(\mathfrak{h}(x,\vec{a}\,) \bullet d) \Vdash_{\mathfrak{wt}} B(\vec{a}) \to \forall x A(x,\vec{a}).$$

For the $\exists$ quantifier we have the rule: from $A(x,\vec{u}) \to B(\vec{u})$ infer $\exists x\, A(x,\vec{u}) \to B(\vec{u})$ if x is not free in $B(\vec{u})$. Inductively we then have an E-recursive function $\mathfrak{g}$ such that for all $b, \vec{a}$,

$$\mathfrak{g}(b,\vec{a}\,) \Vdash_{\mathfrak{wt}} A(b,\vec{a}) \to B(\vec{a}).$$

Suppose $e \Vdash_{\mathfrak{wt}} \exists x A(x,\vec{a})$. Then $e \neq \emptyset$ and for all $d \in e$, $\jmath_1 d \Vdash_{\mathfrak{wt}} A(\jmath_0 d, \vec{a})$. Consequently, $(\forall d \in e)\, \mathfrak{g}(\jmath_0 d, \vec{a}\,) \bullet \jmath_1 d \Vdash_{\mathfrak{wt}} B(\vec{a})$. We then have $\Phi(e, \lambda d.\mathfrak{g}(\jmath_0 d,\vec{a}\,) \bullet \jmath_1 d) \neq \emptyset$ and

$$(\forall y \in \Phi(e, \lambda d.\mathfrak{g}(\jmath_0 d,\vec{a}\,) \bullet \jmath_1 d)\; y \Vdash_{\mathfrak{wt}} B(\vec{a})\,.$$

Using Lemma 3.4 we arrive at $\chi_B(\vec{a}, \Phi(e, \lambda d.\mathfrak{g}(\jmath_0 d,\vec{a}\,) \bullet \jmath_1 d)) \Vdash_{\mathfrak{wt}} B(\vec{a})$; whence

$$\lambda e.\chi_B(\vec{a}, \Phi(e, \lambda d.\mathfrak{g}(\jmath_0 d,\vec{a}\,) \bullet \jmath_1 d))\; \Vdash_{\mathfrak{wt}} \exists x A(x,\vec{a}) \to B(\vec{a})\,.$$

Next we show that every axiom of $\mathbf{CZF}^-$ is realized by an E-recursive function. We treat the axioms one after the other.

(**Extensionality**): Since $e \Vdash_{\mathfrak{wt}} \forall x(x \in a \ \leftrightarrow \ x \in b)$ implies $a = b$, and hence $0 \Vdash_{\mathfrak{wt}} a = b$, it follows that

$$\lambda u.0 \Vdash_{\mathfrak{wt}} [\forall x(x \in a \ \leftrightarrow \ x \in b) \to a = b].$$

(**Pair**): There is an E-recursive function ℓ such that

$$\ell(a,b,c) \ := \ \{\jmath(0,a) \mid c = a\} \ \cup \ \{\jmath(1,b) \mid c = b\}.$$

We have $\forall u \in \{a,b\}\ \ell(a,b,u) \Vdash_{\mathfrak{wt}} (u = a \vee u = b)$ and hence, letting $c := \{a,b\}$,

$$\lambda u.\ell(a,b,u) \Vdash_{\mathfrak{wt}} \forall x \in c(x = u \vee x = b)\,.$$

We also have $\jmath(0,0) \Vdash_{\mathfrak{wt}} (a \in c \ \wedge \ b \in c)$, so that

$$\jmath(\lambda u.\ell(a,b,u), \jmath(0,0)) \Vdash_{\mathfrak{wt}} \forall x \in c(x = a \vee x = b) \ \wedge \ (a \in c \ \wedge \ b \in c).$$

Thus we arrive at

$$\mathfrak{sg}(\jmath(\mathfrak{p}(a,b), \jmath(\lambda u.\ell(a,b,u), \jmath(0,0))) \Vdash_{\mathfrak{wt}} \exists y\,[\forall x \in y(x = a \vee x = b) \ \wedge \ (a \in y \ \wedge \ b \in y)].$$

(**Union**): Let ℓ_U be the E-recursive function defined by

$$\ell_U(a,u) \quad = \quad \{\jmath(x, \jmath(0,0)) \mid x \in a \ \wedge \ u \in x\}.$$

For $u \in \bigcup a$ we then have $\ell_U(a,u) \Vdash_{\mathfrak{wt}} \exists x \in a\ u \in x$, and therefore

$$\lambda u.\ell_U(u,a) \Vdash_{\mathfrak{wt}} (\forall u \in \bigcup a)(\exists x \in a)\ u \in x.$$

Obviously $\lambda u.\lambda v.0 \Vdash_{\mathfrak{wt}} (\forall x \in a)(\forall y \in x)\ y \in \bigcup a$. Therefore we have

$$\mathfrak{sg}(\jmath(\bigcup a, \jmath(\lambda u.\ell_U(u,a), \lambda u.\lambda v.0))) \Vdash_{\mathfrak{wt}} \exists w\,[(\forall u \in w)(\exists x \in a)\ u \in x \ \wedge \ (\forall x \in a)(\forall y \in x)\ y \in w].$$

(**Empty Set**): Obviously $\mathfrak{sg}(\jmath(\emptyset, \lambda v.0)) \Vdash_{\mathfrak{wt}} \exists x\,(\forall u \in x) u \neq u$.

(**Binary Intersection**): Let $c := a \cap b$. As

$$\lambda v.\jmath(0,0) \Vdash_{\mathfrak{wt}} \forall x \in c\,(x \in a \ \wedge \ x \in b)$$

and $\lambda u.0 \Vdash_{\mathfrak{wt}} \forall x\,(x \in a \ \wedge \ x \in b \to x \in c)$ hold, we conclude that

$$\mathfrak{sg}(\jmath(a \cap b, \jmath(\lambda v.\jmath(0,0), \lambda u.0))) \Vdash_{\mathfrak{wt}} \exists y\,[\forall x \in y\,(x \in a \ \wedge \ x \in b) \ \wedge \ \forall x\,(x \in a \ \wedge \ x \in b \to x \in y)].$$

(**Set Induction**): Suppose $e \Vdash_{\mathfrak{wt}} \forall x[\forall y(y \in x \to A(y)) \to A(x)]$. Then, for all a,

$$e \bullet a \Vdash_{\mathfrak{wt}} [\forall y(y \in a \to A(y)) \to A(a)]\,.$$

Suppose we have an index e^* such that for all $b \in a$, $e^* \bullet b \Vdash_{\mathfrak{wt}} A(b)$. As $v \Vdash_{\mathfrak{wt}} b \in a$ entails $b \in a$, we get

$$\lambda u.\lambda v.e^* \bullet u \Vdash_{\mathfrak{wt}} \forall y(y \in a \to A(y)),$$

and hence

$$(e \bullet a) \bullet (\lambda u.\lambda v.e^* \bullet u) \quad \Vdash_{\mathfrak{wt}} \quad A(a). \tag{1}$$

By the recursion theorem we can effectively cook up an index $\mathfrak{q}$ such that

$$(\mathfrak{q} \bullet e) \bullet a \simeq (e \bullet a) \bullet (\lambda u.\lambda v.(\mathfrak{q} \bullet e) \bullet u).$$

In view of the above it follows by set induction that for all a, $(\mathfrak{q} \bullet e) \bullet a \downarrow$ and $(\mathfrak{q} \bullet e) \bullet a \Vdash_{\mathfrak{wt}} A(a)$. As a result we have $\lambda w.(\mathfrak{q} \bullet e) \bullet w \Vdash_{\mathfrak{wt}} \forall x\, A(x)$, yielding

$$\lambda e \lambda w.(\mathfrak{q} \bullet e) \bullet w \Vdash_{\mathfrak{wt}} \forall x[\forall y(y \in x \to A(y)) \to A(x)] \to \forall x\, A(x).$$

(**Strong Collection**): Suppose

$$e \Vdash_{\mathfrak{wt}} \forall u(u \in a \to \exists y\, B(u, y)). \tag{2}$$

Then we have, for all $b \in a$, $(e \bullet b) \bullet 0 \Vdash_{\mathfrak{wt}} \exists y\, B(b, y)$, and so $(e \bullet b) \bullet 0 \neq \emptyset$ and

$$(\forall d \in (e \bullet b) \bullet 0) \qquad \jmath_1 d \Vdash_{\mathfrak{wt}} B(b, \jmath_0 d). \tag{3}$$

Let

$$C^* := \{\jmath_0 d \mid (\exists x \in a)[d \in (e \bullet x) \bullet 0\,]\}.$$

C^* is a set in our background theory, using Replacement or Strong Collection.
Now assume $e' \Vdash_{\mathfrak{wt}} b \in a$. Then $b \in a$ and hence, by the above, $(e \bullet b) \bullet 0 \neq \emptyset$ and

$$(\forall d \in (e \bullet b) \bullet 0) \;\; \jmath(0, \jmath_1 d) \Vdash_{\mathfrak{wt}} [\jmath_0 d \in C^* \, \wedge \, B(b, \jmath_0 d)]. \tag{4}$$

There is an E-recursive function ℓ_2 defined by

$$\ell_2(e, b) \simeq \{\jmath(\jmath_0 d, \jmath(0, \jmath_1 d)) \mid d \in (e \bullet b) \bullet 0\}.$$

From (4) we can infer that $\ell_2(e, b) \Vdash_{\mathfrak{wt}} \exists y\, [y \in C^* \, \wedge \, B(b, y)]$ and hence, with the help of Corollary 3.3,

$$\lambda u.\lambda v.\ell_2(e, u) \quad \Vdash_{\mathfrak{wt}} \quad \forall x(x \in a \to \exists y\, [y \in C^* \, \wedge \, B(x, y)]). \tag{5}$$

Now assume $c \in C^*$. Then there exist $b \in a$ and $d \in (e \bullet b) \bullet 0$ such that $c = \jmath_0 d$. Moreover, by (3), whenever $b \in a$, $d \in (e \bullet b) \bullet 0$ and $\jmath_0 d = c$, then $\jmath_1 d \Vdash_{\mathfrak{wt}} B(b, c)$. Letting ℓ_3 be the E-recursive function defined by

$$\ell_3(a, c, e) \simeq \{\jmath(b, \jmath(0, \jmath_1 d)) \mid b \in a \, \wedge \, \exists d \in (e \bullet b) \bullet 0 \; \jmath_0 d = c\},$$

we then have

$$\ell_3(a, c, e) \Vdash_{\mathfrak{wt}} \exists x(x \in a \, \wedge \, B(x, c)), \tag{6}$$

thus, again with the help of Corollary 3.3,

$$\lambda u.\lambda v.\ell_3(a, u, e) \Vdash_{\mathfrak{wt}} \forall y[y \in C^* \to \exists x(x \in a \, \wedge \, B(x, y))]. \tag{7}$$

Finally observe that there is an E-recursive function $\mathfrak{l}$ such that

$$\mathfrak{l}(a,e) := \{\jmath_0 d \mid d \in \bigcup_{x\in a}((e\bullet x)\bullet 0)\} = \{\jmath_0 d \mid (\exists x \in a)[d \in (e\bullet x)\bullet 0\,]\} = C^*.$$

Thus in view of (5) and (7) we arrive at

$$\begin{aligned}\mathfrak{sg}(\jmath(\mathfrak{l}(a,e),\jmath(\lambda u.\lambda v.\ell_2(e,u),\lambda u.\lambda v.\ell_3(a,u,e)))) \quad \Vdash_{\mathfrak{wt}} \quad & \exists z\,[\forall x(x\in a \to \exists y\,[y\in z \,\wedge\, B(x,y)]) \\ & \wedge \forall y[y\in z \to \exists x(x\in a \,\wedge\, B(x,y))]].\end{aligned}$$

As a result, $\lambda w.\lambda q.\mathfrak{sg}(\jmath(\mathfrak{l}(w,q),\jmath(\lambda u.\lambda v.\ell_2(q,u),\lambda u.\lambda v.\ell_3(w,u,q))))$ is a realizer for each instance of Strong Collection.

(**Infinity**): By [3, Lemma 9.2.2] it suffices to find a realizer for the formula

$$\exists z\,\forall x(x\in z \;\leftrightarrow\; [x=\emptyset \,\vee\, \exists y\in z\, x = y\cup\{y\}]).$$

Here $x=\emptyset$ is an abbreviation for $\forall y(y\in x \to y\neq y)$ and $(\exists y\in z)\,x = y\cup\{y\}$ is an abbreviation for

$$\exists y(y\in z \,\wedge\, [\forall w(w\in x \to [w\in y \,\vee\, w=y]) \,\wedge\, [\forall w(w\in y \to w\in x) \,\wedge\, y\in x\,]]).$$

We have

$$\lambda u'.\lambda v'.0 \quad \Vdash_{\mathfrak{wt}} \quad \forall y(y\in\emptyset \to y\neq y). \tag{8}$$

For $n+1\in\omega$ we have

$$\ell_4(n+1) \quad \Vdash_{\mathfrak{wt}} \quad \forall w(w\in n+1 \to (w\in n \,\vee\, w=n)) \tag{9}$$

for the E-recursive function

$$\ell_4(u) := \lambda w.\lambda v'.\{\jmath(0,0)\mid w\in[\mathbf{p_N}](u)\} \,\cup\, \{\jmath(1,0)\mid w=[\mathbf{p_N}](u)\}.$$

We also have $\jmath(\lambda w'.\lambda v'.0,0)\Vdash_{\mathfrak{wt}} \forall w(w\in n\to w\in n+1) \,\wedge\, n\in n+1$. Thus

$$\begin{aligned}\ell_5(n+1) \quad \Vdash_{\mathfrak{wt}} \quad & n\in\omega \,\wedge\, [\forall w(w\in n+1\to(w\in n \,\vee\, w=n)) \\ & \wedge\, [\forall w(w\in n\to w\in n+1) \,\wedge\, n\in n+1]].\end{aligned} \tag{10}$$

with $\ell_5(n+1) := \jmath(0,\jmath(\ell_4(n+1),\jmath(\lambda w'.\lambda v'.0,0)))$. From (10) we conclude that

$$\ell_6(n+1) \quad \Vdash_{\mathfrak{wt}} \quad (\exists y\in\omega)(n+1 = y\cup\{y\}), \tag{11}$$

where $\ell_6(m) := \mathfrak{sg}(\jmath([\mathbf{p_N}](m),\ell_5(m)))$. Now from (8) and (11) we conclude that for every $m\in\omega$:

$$\mathfrak{sg}([\mathbf{d_N}](0,m,\jmath(0,\lambda u'.\lambda v'.0),\jmath(1,\ell_6(m)))) \quad \Vdash_{\mathfrak{wt}} \quad m=\emptyset \,\vee\, \exists y\in\omega\, m = y\cup\{y\}.$$

If $e\Vdash_{\mathfrak{wt}} a\in\omega$ then $a\in\omega$, and hence with $\ell_7(\omega) := \lambda u.\mathfrak{sg}([\mathbf{d_N}](0,u,\jmath(0,\lambda u'.\lambda v'.0),\jmath(1,\ell_6(u))))$,

$$\ell_7(\omega) \quad \Vdash_{\mathfrak{wt}} \quad (\forall x\in\omega)[x=\emptyset \,\vee\, \exists y\in\omega\, x = y\cup\{y\}]. \tag{12}$$

Conversely, if $e\Vdash_{\mathfrak{wt}} \forall y(y\in a\to y\neq y)$, then really $\forall y\in a\, y\neq y$, and hence $a=\emptyset$, so that $a\in\omega$. Also, if $e'\Vdash_{\mathfrak{wt}} \exists y\in\omega\; a = y\cup\{y\}$ then by unraveling this definition it turns out that $a\in\omega$ holds.

As a result, if $d \Vdash_{\mathfrak{wt}} [a = \emptyset \vee \exists y \in \omega\, a = y \cup \{y\}]$ then there exists $f \in d$ such that $\jmath_0 f = 0$ and $\jmath_1 f \Vdash_{\mathfrak{wt}} a = \emptyset$ or $\jmath_0 f = 1$ and $\jmath_1 f \Vdash_{\mathfrak{wt}} \exists y \in \omega\, a = y \cup \{y\}$. In either case we have $a \in \omega$, and so

$$\lambda x.\lambda e.0 \quad \Vdash_{\mathfrak{wt}} \quad \forall x([x = \emptyset \vee \exists y \in \omega\, x = y \cup \{y\}] \to x \in \omega). \tag{13}$$

Combining (12) and (13), we have with $\mathfrak{h} := \mathfrak{sg}(\jmath(\omega, \lambda v.\jmath(\lambda d.(\ell_7(\omega) \bullet v), \lambda e.0)))$ that

$$\mathfrak{h} \quad \Vdash_{\mathfrak{wt}} \quad \exists z\, \forall x(x \in z \; \leftrightarrow \; [x = \emptyset \vee \exists y \in z\, x = y \cup \{y\}]). \tag{14}$$

$\square$

Theorem 3.8 *Let $D(u_1, \ldots, u_r)$ be a formula of $\mathcal{L}_\in$ all of whose free variables are among $u_1, \ldots, u_r$. If*

$$\mathbf{CZF}_{\mathcal{E}} \vdash D(u_1, \ldots, u_r),$$

then one can effectively construct an index of an E_{exp}-recursive function g such that

$$\mathbf{CZF}_{\mathcal{E}} \vdash \forall a_1, \ldots, a_r\, g(a_1, \ldots, a_r) \Vdash^{\varepsilon}_{\mathfrak{wt}} D(a_1, \ldots, a_r)\,.$$

Proof: We just have to extend the proof of Theorem 3.7 by finding a realizer for Exponentiation: It suffices to find a realizer for the formula

$$\exists y\, \forall f(\mathrm{Fun}(f, a, b) \to f \in y)$$

since realizability of the exponentiation axiom follows then with the help of Δ_0 Separation. Let $c = exp(b, a)$. If $e \Vdash^{\varepsilon}_{\mathfrak{wt}} \mathrm{Fun}(f, a, b)$ then, by Lemma 3.3, $\mathrm{Fun}(f, a, b)$ holds, and hence $f \in c$. Thus

$$\lambda v.0 \Vdash^{\varepsilon}_{\mathfrak{wt}} \forall f(\mathrm{Fun}(f, a, b) \to f \in c)\,,$$

so that

$$\mathfrak{sg}(\jmath(exp(b, a), \lambda v.0)) \Vdash^{\varepsilon}_{\mathfrak{wt}} \exists y(\mathrm{Fun}(f, a, b) \to f \in y)\,.$$

$\square$

Theorem 3.9 *Let $D(u_1, \ldots, u_r)$ be a formula of $\mathcal{L}_\in$ all of whose free variables are among $u_1, \ldots, u_r$. If*

$$\mathbf{CZF}_{\mathcal{P}} \vdash D(u_1, \ldots, u_r),$$

then one can effectively construct an index of an $E_{\wp}$-recursive function g such that

$$\mathbf{CZF}_{\mathcal{P}} \vdash \forall a_1, \ldots, a_r\, g(a_1, \ldots, a_r) \Vdash^{\wp}_{\mathfrak{wt}} D(a_1, \ldots, a_r)\,.$$

Proof: This is the same proof as for Theorem 3.7, except that we also have to take care of Powerset. It suffices to find a realizer for the formula

$$\exists y\, \forall x(x \subseteq a \to x \in y)$$

since realizability of the power set axiom follows then with the help of Δ_0 Separation. $e \Vdash_{\mathfrak{wt}} \forall u(u \in b \to u \in a)$ implies $b \subseteq a$ and consequently $b \in \mathcal{P}(a)$. Therefore we have

$$\lambda u.\lambda v.0 \Vdash_{\mathfrak{wt}} \forall x[x \subseteq a \to x \in \mathcal{P}(a)],$$

thus $\mathfrak{sg}(\jmath(\mathcal{P}(a), \lambda u.\lambda v.0)) \Vdash_{\mathfrak{wt}} \exists y\, \forall x[x \subseteq a \to x \in y]$. $\square$

Theorem 3.10 **CZF**$^-$, **CZF**$_\mathcal{E}$, *and* **CZF**$_\mathcal{P}$ *have the weak existence property. Indeed, they satisfy the stronger property* **uwEP**.

Proof: Suppose

$$\mathbf{CZF}^- \vdash \forall u\, \exists x D(u,x)$$

holds for a formula $D(u,x)$ having at most the free variables u,x. According to Theorem 3.7, one can effectively construct an index of an E-recursive function g such that

$$\mathbf{CZF}^- \vdash \forall a\, g(a) \Vdash_{\mathfrak{wt}} \exists x\, D(a,x)$$

and hence

$$\mathbf{CZF}^- \vdash \forall a\, [\exists d\, d \in g(a) \,\wedge\, \forall d \in g(a)\, \jmath_0 d \Vdash_{\mathfrak{wt}} D(a, \jmath_1 d)]\,.$$

In view of Corollary 3.3 we conclude that

$$\mathbf{CZF}^- \vdash \forall a\, [\exists d\, d \in g(a) \,\wedge\, \forall d \in g(a)\, D(a, \jmath_1 d)]\,.$$

Letting $C(a,y)$ be the formula $y = \{\jmath_1 d \mid d \in g(a)\}$ we then have

$$\begin{array}{lcl} \mathbf{CZF}^- & \vdash & \forall u\, \exists! y\, C(u,y), \\ \mathbf{CZF}^- & \vdash & \forall u \forall y\, [C(u,y) \rightarrow \exists z\, z \in y], \\ \mathbf{CZF}^- & \vdash & \forall u \forall y\, [C(u,y) \rightarrow \forall x \in y\, D(u,x)]\,, \end{array}$$

as desired. The proofs for **CZF**$_\mathcal{E}$ and **CZF**$_\mathcal{P}$ are the same but use Theorem 3.8 and Theorem 3.9, respectively. $\square$

4 Conservativity over intuitionistic Kripke-Platek set theories

In this section we shall show that **CZF**$^-$, **CZF**$_\mathcal{E}$ and **CZF**$_\mathcal{P}$ are conservative for restricted classes of formulae over their intuitionistic Kripke-Platek counterparts.

4.1 Defining realizability with sets of witnesses (omitting truth)

We shall strip the definition of realizability given in Definition 3.1 of its truth component. This will enable us to to establish realizability interpretations in the pertaining Kripke-Platek versions.

Definition 4.1

$$\begin{array}{rcl} a \Vdash_{\mathfrak{w}} A & \text{iff} & A \text{ holds true, whenever } A \text{ is an atomic formula} \\ a \Vdash_{\mathfrak{w}} A \wedge B & \text{iff} & \jmath_0 a \Vdash_{\mathfrak{w}} A \,\wedge\, \jmath_1 a \Vdash_{\mathfrak{w}} B \\ a \Vdash_{\mathfrak{w}} A \vee B & \text{iff} & a \neq \emptyset \,\wedge\, (\forall d \in a)([\jmath_0 d = 0 \,\wedge\, \jmath_1 d \Vdash_{\mathfrak{w}} A] \,\vee \\ & & \qquad\qquad\qquad [\jmath_0 d = 1 \,\wedge\, \jmath_1 d \Vdash_{\mathfrak{w}} B]) \\ a \Vdash_{\mathfrak{w}} \neg A & \text{iff} & \forall c\, \neg c \Vdash_{\mathfrak{w}} A \\ a \Vdash_{\mathfrak{w}} A \rightarrow B & \text{iff} & \forall c\, [c \Vdash_{\mathfrak{w}} A \;\rightarrow\; a \bullet c \Vdash_{\mathfrak{w}} B] \\ a \Vdash_{\mathfrak{w}} (\forall x \in b)\, A & \text{iff} & (\forall c \in b)\, a \bullet c \Vdash_{\mathfrak{w}} A[x/c] \\ a \Vdash_{\mathfrak{w}} (\exists x \in b)\, A & \text{iff} & a \neq \emptyset \,\wedge\, (\forall d \in a)[\jmath_0 d \in b \,\wedge\, \jmath_1 d \Vdash_{\mathfrak{w}} A[x/\jmath_0 d] \\ a \Vdash_{\mathfrak{w}} \forall x A & \text{iff} & \forall c\;\, a \bullet c \Vdash_{\mathfrak{w}} A[x/c] \\ a \Vdash_{\mathfrak{w}} \exists x A & \text{iff} & a \neq \emptyset \,\wedge\, (\forall d \in a)\, \jmath_1 d \Vdash_{\mathfrak{w}} A[x/\jmath_0 d] \\ & & \\ \Vdash_{\mathfrak{w}} B & \text{iff} & \exists a\, a \Vdash_{\mathfrak{w}} B. \end{array}$$

If we use indices of $E_{\exp}$-recursive and $E_{\wp}$-recursive functions rather than E-recursive functions, we shall notate the corresponding notion of realizability by $a \Vdash^{\varepsilon}_{\mathfrak{w}} B$ and $a \Vdash^{\wp}_{\mathfrak{w}} B$, respectively.

Theorem 4.2 *Let $D(u_1, \dots, u_r)$ be a formula of $\mathcal{L}_\in$ all of whose free variables are among $u_1, \dots, u_r$.*

(i) If $\mathbf{CZF}^- \vdash D(u_1, \dots, u_r)$, then one can effectively construct an index of an E-recursive function f such that $\mathbf{IKP} \vdash \forall a_1, \dots, a_r\, f(a_1, \dots, a_r) \Vdash_{\mathfrak{w}} D(a_1, \dots, a_r)$.

(ii) If $\mathbf{CZF}_{\mathcal{E}} \vdash D(u_1, \dots, u_r)$, then one can effectively construct an index of an $E_{\exp}$-recursive function g such that $\mathbf{IKP}(\mathcal{E}) \vdash \forall a_1, \dots, a_r\, g(a_1, \dots, a_r) \Vdash^{\varepsilon}_{\mathfrak{w}} D(a_1, \dots, a_r)$.

(iii) If $\mathbf{CZF}_{\mathcal{P}} \vdash D(u_1, \dots, u_r)$, then one can effectively construct an index of an $E_{\wp}$-recursive function h such that $\mathbf{IKP}(\mathcal{P}) \vdash \forall a_1, \dots, a_r\, h(a_1, \dots, a_r) \Vdash^{\wp}_{\mathfrak{w}} D(a_1, \dots, a_r)$.

Proof: This follows by the obvious simplifications of the proof of Theorem 3.7 $\square$

Definition 4.3 To each Δ_0 formula $D(x_1, \dots, x_r)$ of $\mathcal{L}_\in$ all of whose free variables are among $\vec{x} = x_1, \dots, x_r$, we assign a total E-recursive function $\mathfrak{k}_D$ of arity r as follows:

1. $\mathfrak{k}_D(\vec{x}) = \{0\}$ if $D(\vec{x})$ is atomic.
2. $\mathfrak{k}_D(\vec{x}) = \{\{\langle 0, z\rangle\} \mid z \in \mathfrak{k}_A(\vec{x}) \wedge A(\vec{x})\} \cup \{\{\langle 1, z\rangle\} \mid z \in \mathfrak{k}_B(\vec{x}) \wedge B(\vec{x})\}$ if $D(\vec{x})$ is of the form $A(\vec{x}) \vee B(\vec{x})$.
3. $\mathfrak{k}_D(\vec{x}) = \{\langle z, w\rangle \mid z \in \mathfrak{k}_A(\vec{x}) \wedge w \in \mathfrak{k}_B(\vec{x})\}$ if $D(\vec{x})$ is of the form $A(\vec{x}) \wedge B(\vec{x})$.
4. $\mathfrak{k}_D(\vec{x}) = \{\lambda v.\chi_B(\vec{x}, \mathfrak{k}_B(\vec{x}))\}$ if $D(\vec{x})$ is of the form $A(\vec{x}) \rightarrow B(\vec{x})$.
5. $\mathfrak{k}_D(\vec{x}) = \{\{\langle z, v\rangle\} \mid z \in x_i \wedge v \in \mathfrak{k}_A(\vec{x}, z) \wedge A(\vec{x}, z)\}$ if $D(\vec{x})$ is of the form $\exists z \in x_i\, A(\vec{x}, z)$.
6. $\mathfrak{k}_D(\vec{x}) = \{\lambda z.\chi_A(\vec{x}, z, \mathfrak{k}_A(\vec{x}, z))\}$ if $D(\vec{z})$ is of the form $\forall z \in x_i\, A(\vec{x}, z)$.

To each $\Delta_0^{\mathcal{P}}$ formula $D(x_1, \dots, x_r)$ we assign a total $E_{\wp}$-recursive function $\mathfrak{k}^{\wp}_D$ of arity r by adding the following clauses to the above:

7. $\mathfrak{k}^{\wp}_D(\vec{x}) = \{\{\langle z, \langle \lambda y.0, v\rangle\rangle\} \mid z \in \mathcal{P}(x_i) \wedge v \in \mathfrak{k}_A(\vec{x}, z) \wedge A(\vec{x}, z)\}$ if $D(\vec{x})$ is of the form $\exists z \subseteq x_i\, A(\vec{x}, z)$.
8. $\mathfrak{k}^{\wp}_D(\vec{x}) = \{\lambda y.\lambda z.\chi_A(\vec{x}, z, \mathfrak{k}_A(\vec{x}, z))\}$ if $D(\vec{z})$ is of the form $\forall z \subseteq x_i\, A(\vec{x}, z)$.

Likewise, to each $\Delta_0^{\mathcal{E}}$ formula $D(x_1, \dots, x_r)$ we assign a total $E_{\exp}$-recursive function $\mathfrak{k}^{\varepsilon}_D$ of arity r by adding the following clauses to 1.-6. above:

7'. $\mathfrak{k}^{\varepsilon}_D(\vec{x}) = \{\{\langle f, \langle \mathfrak{k}_F(f, \vec{x}), v\rangle\rangle\} \mid f \in {}^{x_i}x_j \wedge v \in \mathfrak{k}_A(\vec{x}, f) \wedge A(\vec{x}, z)\}$ if $D(\vec{x})$ is of the form $\exists f \in {}^{x_i}x_j\, A(\vec{x}, f)$ and $F(f, \vec{x})$ is the formula $f \in {}^{x_i}x_j$.

8'. $\mathfrak{k}^{\varepsilon}_D(\vec{x}) = \{\lambda y.\lambda f.\chi_A(\vec{x}, f, \mathfrak{k}_A(\vec{x}, f))\}$ if $D(\vec{z})$ is of the form $\forall f \in {}^{x_i}x_j\, A(\vec{x}, f)$.

For Δ_0-formulae realizability and truth coincide as the following Proposition shows.

Proposition 4.4 *Let $D(\vec{x})$ be a Δ_0 formula whose free variables are among $\vec{x} = x_1, \dots, x_r$. Then the following are provable in* **IKP***:*

(i) $D(\vec{x}) \rightarrow \mathfrak{k}_D(\vec{x}) \neq \emptyset \wedge \forall u \in \mathfrak{k}_D(\vec{x})\, u \Vdash_{\mathfrak{w}} D(\vec{x})$.

(ii) $(\exists e\, e \Vdash_{\mathfrak{w}} D(\vec{x})) \to D(\vec{x})$.

Proof: We show (i) and (ii) simultaneously by induction on the complexity of D.

1. For atomic D this is obvious.

2. Let $D(\vec{x})$ be of the form $A(\vec{x}) \vee B(\vec{x})$. First suppose that $D(\vec{x})$ holds. Then the induction hypothesis entails that $A(\vec{x})$ and $\mathfrak{k}_A(\vec{x}) \neq \emptyset$ or $B(\vec{x})$ and $\mathfrak{k}_B(\vec{x}) \neq \emptyset$. In every case we have $\mathfrak{k}_D(\vec{x}) \neq \emptyset$.
If $u \in \mathfrak{k}_D(\vec{x})$ then either $u = \{\langle 0, z\rangle\}$ and $A(\vec{x})$ for some $z \in \mathfrak{k}_A(\vec{x})$ or $u = \{\langle 1, z\rangle\}$ and $B(\vec{x})$ for some $z \in \mathfrak{k}_B(\vec{x})$. In the first case the inductive assumption yields $z \Vdash_{\mathfrak{w}} A(\vec{x})$ and hence $u \Vdash_{\mathfrak{w}} D(\vec{x})$. In the second case the inductive assumption yields $z \Vdash_{\mathfrak{w}} B(\vec{x})$ and hence also $u \Vdash_{\mathfrak{w}} D(\vec{x})$. This shows (i).
As to (ii), suppose that $e \Vdash_{\mathfrak{w}} D(\vec{x})$. Then there exists $u \in e$ such that $u = \langle 0, d\rangle \wedge d \Vdash_{\mathfrak{w}} A(\vec{x})$ or $u = \langle 1, d\rangle \wedge d \Vdash_{\mathfrak{w}} B(\vec{x})$ for some d. The induction hypothesis yields $A(\vec{x})$ or $B(\vec{x})$, thus $D(\vec{x})$.

3. Let $D(\vec{x})$ be of the form $A(\vec{x}) \wedge B(\vec{x})$. Then (i) and (ii) are immediate by the induction hypothesis.

4. Let $D(\vec{x})$ be of the form $A(\vec{x}) \to B(\vec{x})$. By definition, $\mathfrak{k}_D(\vec{x}) = \{\lambda v.\chi_B(\vec{x}, \mathfrak{k}_B(\vec{x}))\} \neq \emptyset$. As to (i), assume that $D(\vec{x})$ holds and $e \Vdash_{\mathfrak{w}} A(\vec{x})$. The induction hypothesis (ii) applied to $A(\vec{x})$ yields that $A(\vec{x})$ holds, which implies that $B(\vec{x})$ holds. The induction hypothesis (i) for the latter formula yields that $\mathfrak{k}_B(\vec{x}) \neq \emptyset$ and $\forall u \in \mathfrak{k}_B(\vec{x})\; u \Vdash_{\mathfrak{w}} B(\vec{x})$. An application of Lemma 3.4 thus yields $\chi_B(\vec{x}, \mathfrak{k}_B(\vec{x})) \Vdash_{\mathfrak{w}} B(\vec{x})$. As a result, $\lambda v.\chi_B(\vec{x}, \mathfrak{k}_B(\vec{x})) \Vdash_{\mathfrak{w}} D(\vec{x})$ confirming (i).
For (ii), suppose $e \Vdash_{\mathfrak{w}} (A(\vec{x}) \to B(\vec{x}))$ and $A(\vec{x})$ holds. By the induction hypothesis (i) for the latter formula, $\mathfrak{k}_A(\vec{x}) \neq \emptyset$ and $\forall u \in \mathfrak{k}_A(\vec{x})\; u \Vdash_{\mathfrak{w}} A(\vec{x})$. Thus, picking $u_0 \in \mathfrak{k}_A(\vec{x})$ we have $e \bullet u_0 \Vdash_{\mathfrak{w}} B(\vec{x})$, and hence the induction hypothesis (ii) for the latter formula yields that $B(\vec{x})$ holds.

5. Let $D(\vec{x})$ be of the form $\exists z \in x_i\; A(\vec{x}, z)$. To verify (i), suppose $\exists z \in x_i\; A(\vec{x}, z)$ holds. Then there is $z \in x_i$ such that $A(\vec{x}, z)$. The induction hypothesis (i) for the latter formula yields that $\mathfrak{k}_A(\vec{x}, z) \neq \emptyset$, and hence $\mathfrak{k}_D(\vec{x}) \neq \emptyset$. Now suppose $u \in \mathfrak{k}_D(\vec{x})$. Then $u = \{\langle z, v\rangle\}$ for some $z \in x_i$ and $v \in \mathfrak{k}_A(\vec{x}, z)$. As $A(\vec{x}, z)$ holds, the induction hypothesis (i) yields that $v \Vdash_{\mathfrak{w}} A(\vec{x}, z)$, whence $u \Vdash_{\mathfrak{w}} \exists z \in x_i\; A(\vec{x}, z)$.
For (ii), assume $e \Vdash_{\mathfrak{w}} \exists z \in x_i\; A(\vec{x}, z)$. Then $e \neq \emptyset$. Picking $d \in e$ we have $\jmath_0 d \in x_i$ and $\jmath_1 d \Vdash_{\mathfrak{w}} A(\vec{x}, \jmath_0 d)$, thus $A(\vec{x}, \jmath_0 d)$ by the induction hypothesis (ii), thence $\exists z \in x_i\; A(\vec{x}, z)$ holds.

6. Let $D(\vec{x})$ be of the form $\forall z \in x_i\; A(\vec{x}, z)$. To verify (i), suppose $\forall z \in x_i\; A(\vec{x}, z)$ is true. By definition, $\mathfrak{k}_D(\vec{x}) = \{\lambda z.\chi_A(\vec{x}, z, \mathfrak{k}_A(\vec{x}, z))\} \neq \emptyset$. If $z_0 \in x_i$ we have $A(\vec{x}, z_0)$, so that inductively $\mathfrak{k}_A(\vec{x}, z_0) \neq \emptyset$ and $\forall d \in \mathfrak{k}_A(\vec{x}, z_0)\; d \Vdash_{\mathfrak{w}} A(\vec{x}, z_0)$. Whence, by Lemma 3.4, $\chi_A(\vec{x}, z_0, \mathfrak{k}_A(\vec{x}, z_0)) \Vdash_{\mathfrak{w}} A(\vec{z}, z_0)$. As a result, $\lambda z.\chi_A(\vec{x}, z, \mathfrak{k}_A(\vec{x}, z)) \Vdash_{\mathfrak{w}} D(\vec{x})$.
As for (ii), suppose $e \Vdash_{\mathfrak{w}} \forall z \in x_i\; A(\vec{x}, z)$. Thus $e \bullet z \Vdash_{\mathfrak{w}} A(\vec{x}, z)$ for all $z \in x_i$, so that inductively $\forall z \in x_i\; A(\vec{x}, z)$ holds. □

Proposition 4.5 *Let $D(\vec{x})$ be a $\Delta_0^{\mathcal{P}}$ formula whose free variables are among $\vec{x} = x_1, \ldots, x_r$. Then the following are provable in* $\mathbf{IKP}(\mathcal{P})$:

(i) $D(\vec{x}) \to \mathfrak{k}^{\wp}_D(\vec{x}) \neq \emptyset \wedge \forall u \in \mathfrak{k}^{\wp}_D(\vec{x})\; u \Vdash^{\wp}_{\mathfrak{w}} D(\vec{x})$.

(ii) $(\exists e\, e \Vdash^{\wp}_{\mathfrak{w}} D(\vec{x})) \to D(\vec{x})$.

Proof: In addition to the previous proof we have to consider two more cases.

7. Let $D(\vec{x})$ be of the form $\exists z \subseteq x_i\; A(\vec{x}, z)$. To verify (i), suppose $\exists z \subseteq x_i\; A(\vec{x}, z)$ holds. Then

there is $z \in \mathcal{P}(x_i)$ such that $A(\vec{x}, z)$. The induction hypothesis (i) for the latter formula yields that $\mathfrak{k}^{\wp}_A(\vec{x}, z) \neq \emptyset$, and hence $\mathfrak{k}^{\wp}_D(\vec{x}) \neq \emptyset$. Now suppose $u \in \mathfrak{k}^{\wp}_D(\vec{x})$. Then $u = \{\langle z, \langle \lambda y.0, v\rangle\rangle\}$ for some $z \subseteq x_i$ and $v \in \mathfrak{k}^{\wp}_A(\vec{x}, z)$. As $A(\vec{x}, z)$ holds the induction hypothesis (i) yields that $v \Vdash^{\wp}_{\mathfrak{w}} A(\vec{x}, z)$. Also $\lambda y.0 \Vdash^{\wp}_{\mathfrak{w}} z \subseteq x_i$. Whence $u \Vdash^{\wp}_{\mathfrak{w}} \exists z\,(z \subseteq x_i \;\wedge\; A(\vec{x}, z))$.
For (ii), assume $e \Vdash^{\wp}_{\mathfrak{w}} \exists z\,[z \subseteq x_i \;\wedge\; A(\vec{x}, z)]$. Then $e \neq \emptyset$. Picking $d \in e$ we have $\jmath_1 d \Vdash^{\wp}_{\mathfrak{w}} [\jmath_0 d \subseteq x_i \;\wedge\; A(\vec{x}, \jmath_0 d)]$. This entails $\jmath_0 d \subseteq x_i$ and $\jmath_1(\jmath_1 d) \Vdash^{\wp}_{\mathfrak{w}} A(\vec{x}, \jmath_0 d)$. Thus $A(\vec{x}, \jmath_0 d)$ by the induction hypothesis (ii), thence $\exists z \subseteq x_i\, A(\vec{x}, z)$ holds.

8. Let $D(\vec{x})$ be of the form $\forall z \in x_i\, A(\vec{x}, z)$. To verify (i), suppose $\forall z \in x_i\, A(\vec{x}, z)$ is true. By definition, $\mathfrak{k}^{\wp}_D(\vec{x}) = \{\lambda y.\lambda z.\chi_A(\vec{x}, z, \mathfrak{k}^{\wp}_A(\vec{x}, z))\} \neq \emptyset$. If $y \Vdash^{\wp}_{\mathfrak{w}} z_0 \subseteq x_i$, then $z_0 \subseteq x_i$ holds and we have $A(\vec{x}, z_0)$, so that inductively $\mathfrak{k}^{\wp}_A(\vec{x}, z_0) \neq \emptyset$ and $\forall d \in \mathfrak{k}_A(\vec{x}, z_0)\; d \Vdash^{\wp}_{\mathfrak{w}} A(\vec{x}, z_0)$. Whence, by Lemma 3.4, $\chi_A(\vec{x}, z_0, \mathfrak{k}^{\wp}_A(\vec{x}, z_0)) \Vdash^{\wp}_{\mathfrak{w}} A(\vec{z}, z_0)$. As a result, $\lambda y.\lambda z.\chi_A(\vec{x}, z, \mathfrak{k}^{\wp}_A(\vec{x}, z)) \Vdash^{\wp}_{\mathfrak{w}} D(\vec{x})$.
As for (ii), suppose $e \Vdash^{\wp}_{\mathfrak{w}} \forall z \subseteq x_i\, A(\vec{x}, z)$. Thus $e \bullet z \Vdash^{\wp}_{\mathfrak{w}} [z \subseteq x_i \to A(\vec{x}, z)]$ for all z. If $z \subseteq x_i$, then $\lambda y.0 \Vdash^{\wp}_{\mathfrak{w}} z \subseteq x_i$, so that $(e \bullet z) \bullet (\lambda y.0) \Vdash^{\wp}_{\mathfrak{w}} A(\vec{x}, z)$, and therefore, by the inductive assumption, $A(\vec{x}, z)$ holds. As a result, $\forall z \in x_i\, A(\vec{x}, z)$ holds. □

Proposition 4.6 *Let $D(\vec{x})$ be a $\Delta^{\mathcal{E}}_0$ formula whose free variables are among $\vec{x} = x_1, \ldots, x_r$. Then the following are provable in* $\mathbf{IKP}(\mathcal{E})$*:*

(i) $D(\vec{x}) \;\to\; \mathfrak{k}^{\varepsilon}_D(\vec{x}) \neq \emptyset \wedge \forall u \in \mathfrak{k}^{\varepsilon}_D(\vec{x})\; u \Vdash^{\varepsilon}_{\mathfrak{w}} D(\vec{x})$.

(ii) $(\exists e\, e \Vdash^{\varepsilon}_{\mathfrak{w}} D(\vec{x})) \;\to\; D(\vec{x})$.

Proof: This can be proved in the same vein as Proposition 4.5. □

Definition 4.7 We say that a formula D is Π_2, $\Pi^{\mathcal{E}}_2$, or $\Pi^{\mathcal{P}}_2$ if it is of the form $\forall x\, \exists y\, A(\vec{x}, \vec{y})$ with $A(\vec{x}, \vec{y})$ being, respectively, Δ_0, $\Delta^{\mathcal{E}}_0$, and $\Delta^{\mathcal{P}}_0$.

Theorem 4.8 *(i)* $\mathbf{CZF}^-$ *is conservative over* $\mathbf{IKP}$ *for* Π_2 *sentences.*

(ii) $\mathbf{CZF}_{\mathcal{E}}$ *is conservative over* $\mathbf{IKP}(\mathcal{E})$ *for* $\Pi^{\mathcal{E}}_2$ *sentences.*

(iii) $\mathbf{CZF}_{\mathcal{P}}$ *is conservative over* $\mathbf{IKP}(\mathcal{P})$ *for* $\Pi^{\mathcal{P}}_2$ *sentences.*

Proof: (i) Suppose

$$\mathbf{CZF}^- \vdash \forall x \exists y\, A(x, y)$$

with $A(x, y)$ Δ_0. By Theorem 4.2 there is an E-recursive function f such that $\mathbf{IKP} \vdash \forall x\, f(x) \Vdash_{\mathfrak{w}} \exists y\, A(x, y)$. Then

$$\mathbf{IKP} \vdash \forall x\, [f(x) \neq \emptyset \wedge \forall e \in f(x)\, (\jmath_1 e \Vdash_{\mathfrak{w}} A(x, \jmath_0 e)].$$

By Proposition 4.4 we get

$$\mathbf{IKP} \vdash \forall x\, [f(x) \neq \emptyset \wedge \forall e \in f(x)\, A(x, \jmath_0 e)]$$

which entails $\mathbf{IKP} \vdash \forall x\, \exists y\, A(x, y)$.
The proofs for (ii) and (iii) are similar. □

5 Targeting the existence property

The previous sections provide much more information than has been made explicit. Let T be one of the theories $\mathbf{CZF}^-$, $\mathbf{CZF}_{\mathcal{E}}$, or $\mathbf{CZF}_{\mathcal{P}}$. The question of whether T has the existence property can be reduced to the more manageable question of whether the Kripke-Platek version of T has the **EP** for Σ, $\Sigma^{\mathcal{P}}$, and $\Sigma^{\mathcal{E}}$ existential theorems, respectively.

Definition 5.1 Let Ξ be a collection of formulae. A theory T has the **EP** for Ξ if whenever $T \vdash \exists x A(x)$ for a sentence $\exists x A(x)$ with $A(x)$ in Ξ, then there exists a formula $C(x)$ in Ξ (with at most x free) such that

$$T \vdash \exists! x\, [C(x) \wedge A(x)].$$

Theorem 5.2 *If* **IKP** *has the* **EP** *for* Σ *formulae then* $\mathbf{CZF}^-$ *has the* **EP**.

Proof: Assume that **IKP** has the **EP** for Σ formulae. Suppose that

$$\mathbf{CZF}^- \vdash \exists y B(y) \tag{15}$$

with $\exists y B(y)$ a closed formula. It then follows that we can explicitly construct the index of an E-recursive function f such that

$$\mathbf{CZF}^- \vdash f(0)\downarrow \wedge f(0) \Vdash_{\mathfrak{rt}} \exists y B(y).$$

Hence, letting $C(x)$ stand for $\forall z \in x \exists u \in f(0)\; z = \jmath_0 u \wedge \forall u \in f(0)\, \jmath_0 u \in x$ we arrive at

$$\mathbf{CZF}^- \vdash \exists! x\, C(x) \wedge \forall x\, [C(x) \to \exists u\, u \in x] \wedge \forall x [C(x) \to \forall y \in x\, B(y)]. \tag{16}$$

In particular we have $\mathbf{CZF}^- \vdash \exists y\, \exists x\, [C(x) \wedge y \in x]$. Let D be the closed formula $\exists y\, \exists x\, [C(x) \wedge y \in x]$. D is a Σ formula. Using Σ-reflection (see [2, Theorem 11.4]) we have

$$\mathbf{IKP} \vdash D \leftrightarrow \exists a\, D^a \tag{17}$$

where D^a arises from D by restricting all unbounded (existential) quantifiers in D by a. Thus $\mathbf{CZF}^- \vdash \exists a\, D^a$ and therefore, owing to Σ_1 conservativity, $\mathbf{IKP} \vdash \exists a\, D^a$, whence

$$\mathbf{IKP} \vdash \exists y\, \exists x\, [C(x) \wedge y \in x]. \tag{18}$$

Since we assumed that **IKP** has the **EP** for Σ formulae, there exists a Σ formula $F(y)$ such that

$$\mathbf{IKP} \vdash \exists! y\, (F(y) \wedge \exists x\, [C(x) \wedge y \in x]),$$

so that

$$\mathbf{CZF}^- \vdash \exists! y\, (F(y) \wedge \exists x\, [C(x) \wedge y \in x]). \tag{19}$$

Combining (16) and (19) we have

$$\mathbf{CZF}^- \vdash \exists! y\, (F(y) \wedge \exists x\, [C(x) \wedge y \in x] \wedge B(y)).$$

$\square$

Theorem 5.3 *(i) If* $\mathbf{IKP}(\mathcal{E})$ *has the* **EP** *for* $\Sigma^{\mathcal{E}}$ *formulae then* $\mathbf{CZF}_{\mathcal{E}}$ *has the* **EP**.

(i) If $\mathbf{IKP}(\mathcal{P})$ *has the* **EP** *for* $\Sigma^{\mathcal{P}}$ *formulae then* $\mathbf{CZF}_{\mathcal{P}}$ *has the* **EP**.

Proof: This is similar to the previous one. $\square$

6 A proof sketch that IKP has the existence property for Σ formulae

To show this we use a much more elaborate technology than realizability. It is possible to carry out an ordinal analysis of **IKP** just as for **KP** as in [31]. It involves a term structure built from the backbone of an ordinal representation system that mimics the constructible hierarchy. For every theorem Σ theorem of **IKP** of the form $\exists x A(x)$ one can effectively determine an ordinal α from the representation system (which is smaller than the Bachmann-Howard ordinal) and an infinitary cut-free derivation $\mathcal{D}\vert\frac{\alpha}{0}\, \exists\, xA(x)$. Since this is a derivation in infinitary intuitionistic logic one obtains from the proof an explicit term t in the term structure and a proof $\mathcal{D}'\vert\frac{\alpha}{0}\, A\,(t)$. The canonical interpretation of t in the constructible hierarchy as defined in [31, 3.5 Soundness Theorem] then provides the explicit witness for $\exists x A(x)$. As the entire reasoning can be carried out in **IKP** this entails that **IKP** has the **EP** for Σ formulae.

Corollary 6.1 $\mathbf{CZF}^-$ *has the* **EP**.

Proof: This follows from Theorem 5.2 and the foregoing considerations. $\square$

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