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### *The World is the Totality of Facts, Not of Things*

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# THE WORLD IS THE TOTALITY OF FACTS, NOT OF THINGS

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Absolutism—the view that there is sense to be made of discourse concerning “absolutely everything”—plays an important role in contemporary philosophy. It is a natural backdrop for ontological claims (e.g. “there are no abstract objects”); it is used to ensure that rival metaphysical views pertain to a single subject-matter (Williamson 2013); it is the key assumption in standard arguments for the expressive richness of plural quantifiers (Boolos 1984, McKay 2006, Oliver & Smiley 2013); and it plays an important role in certain research programs in the philosophy of mathematics (McGee 1997) and the semantic paradoxes (Field 2008).

Although absolutism is not without its detractors, it seems to be an increasingly prominent view amongst contemporary metaphysicians.<sup>1</sup> The prevailing attitude is nicely summed up by Timothy Williamson: “The most serious concern about [unrestricted uses of quantifiers] is their close association with the paradoxes of set theory ... However, one cannot generate [the relevant] contradiction just by using unrestricted quantifiers. The contradiction always depends on auxiliary assumptions ... [and we] can reject those assumptions without rejecting unrestricted quantification.” (Williamson 2013, p 15)

I would like to suggest that this passage underplays the case against absolutism.<sup>2</sup> It seems to me that the most compelling anti-absolutist considerations come not from the set theoretic paradoxes but from a particular conception of relationship between our language and

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<sup>1</sup>For a recent overview, and a list of references, see Florio 2014.

<sup>2</sup>For relevant discussion, see Russell forthcoming.

the world it represents. The aim of this paper is to give a precise characterization of this conception, and to explain how it can be used to buttress a non-absolutist picture of ontology.

I begin with some preliminary remarks about realism and ontology (sections 1 and 2). I then develop a domain-free semantics for first-order quantifiers (section 3), and use it to develop a “facts first” conception of the world, according to which there is no such thing as an “absolutely general” domain of discourse (sections 4 and 5).

## I Realism

To a first approximation,<sup>3</sup> realism is the view that there is a definite fact of the matter about how that world is: a fact of the matter that is independent of our conception of the world, our values, and our representational resources.

I am a realist. I believe, for example, that there is a definite fact of the matter about whether Venus has moons. But my realism has its limits. Take Harry, who is a borderline case of baldness. I do *not* believe that there is a definite fact of the matter about whether he is bald. This is not because I think that reality is somehow incomplete, or that it is somehow mind- or value-dependent. It is because I think the sentence “Harry is bald” does not succeed in expressing a condition definite enough to make it possible for reality to settle the question of whether the condition has been met.

Here is another example. I believe there is no such thing as “metaphysical fashionability”: fashionability in a sense that transcends the tastes of some community or other. So I do not believe that there is a definite fact of the matter about whether ascots are metaphysically fashionable. As before, this is because I think the sentence “ascots are metaphysically fashionable” does not succeed in expressing a condition definite enough for reality to settle the question of whether the condition is met. In this case, however, the reason has nothing to

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<sup>3</sup>There are problems with this first approximation. For instance, the fact that we have such-and-such representational resources is, in a trivial sense, dependent on our representational resources. But we don't thereby want to count as anti-realists about what representational resources we have. Nothing in the present discussion will hinge on how one chooses to finesse such issues. For further discussion see Rosen 1994.

do with vagueness. It is instead to do with a false presupposition. “Ascots are metaphysically fashionable” presupposes a well-defined distinction between the metaphysically fashionable and the not. Since there no sense to be made of such a distinction, there is no way for reality to settle the question of which side of the distinction ascots belong on.<sup>4</sup> To put the point in set-theoretic terms: it is not that the set of metaphysically fashionable things is empty, and therefore fails to include ascots; it is rather that “metaphysically fashionable” does not succeed in characterizing a set at all.

Realism is a metaphysical view: it is a doctrine about the nature of reality. But, as our examples show, it is sometimes hard to separate the extent of one’s realism from one’s views about language, and, in particular, about which sentences succeed in specifying conditions definite enough to be settled by reality.

## 2 Ontology

What about *ontological* realism—realism about what there is?

It is clear that some ontological questions have definite answers. For example, an assertion of “there are seventeen eggs in the basket” (uttered in a situation in which nothing unusual is going on) can be expected to succeed in specifying a condition definite enough to be settled by the way the world is. And similarly for an assertion of “there are seventeen objects in the basket”, in the presence of suitable contextual cues.

I would like to suggest, however, that there are ontological questions that lack definite answers. To illustrate the point, consider the metaphysical picture I shall call *objectualism*. Objectualists believe that reality is “objectively structured”: it is articulated into constituents in a way that is significant independently of the manner in which *we* happen to represent it. They believe, moreover, that the objective constituents of reality can be classified in accordance with their “ontological character”: some of them enjoy “saturated” ontological

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<sup>4</sup>Compare addendum (a) to Kripke’s *Naming and Necessity*.

character, and are what we call objects; others have “unsaturated” ontological character, and are what we call properties. As a result, an objectualist’s conception of objecthood is informed by her metaphysics: she thinks that to be an object is to be an objective constituent of reality with saturated ontological character. (On a more sophisticated version of the proposal, she thinks that the relevant constituent of reality are “fundamental objects”, which “ground” further non-fundamental objects.<sup>5</sup>)

Suppose a group of objectualists is having a discussion about the objective constituents of reality, and that one of them asserts “the objects are inaccessiblely many in number”.<sup>6</sup> It is not clear to me that there is a fact of the matter about whether this assertion is true. The worry is not to do with whether reality is somehow incomplete, or whether it is somehow mind- or value-dependent. It is to do with whether the assertion succeeds in specifying a condition definite enough to make it possible for reality to settle the question of whether the condition is met. Let me explain.

I earlier claimed that there is no fact of the matter about whether ascots are metaphysically fashionable, on the grounds that an assertion of “ascots are metaphysically fashionable” suffers from a false presupposition: it assumes, incorrectly, that there is a well-defined distinction between the metaphysically fashionable and the not. Now suppose one is skeptical of the objectualist’s metaphysical picture: one thinks there is no sense to be made of a “saturated ontological character”, and therefore no sense to be made of the objectualist’s conception of object. Then one should think that the objectualist’s assertion of “the objects are inaccessiblely many in number” also suffers from a false presupposition: it assumes, incorrectly, that there is a well-defined distinction between the ontologically saturated and the not ontologically saturated. Such a presupposition failure would thwart one’s effort to specify a condition definite enough to make it possible for reality to settle the question of whether the condition is met.

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<sup>5</sup>See Fine 2001, Schaffer 2009, Sider 2012, Bennett forthcoming.

<sup>6</sup>The notion of inaccessibility I have in mind here is a plural version of the set-theoretic notion of a (strongly) inaccessible cardinal. For details, see Shapiro 1991.

In arguing that the objectualist's assertion fails to have a definite answer, I made use of the assumption that objectualists work with a metaphysically informed conception of object. Suppose we instead considered ontological questions that are posed using a "neutral" notion of object: a notion of object free from theoretical commitments. Would there then be a fact of the matter about whether the objects are inaccessibly many in number?

It is not clear to me that there is such a thing as a "neutral" notion of object. I certainly agree that words like "object", "item" and "thing", and associated quantificational resources, are part of natural language, and I agree that one can be a competent user of such expressions without associating them with anything like the objectualist's metaphysical picture. Notice, however, that ordinary use of quantification tends to presuppose highly restricted domains of discourse. (When I utter "there is something missing from my suitcase" in an ordinary context, it is understood that I am not concerned with the Milky Way, or Socrates's death, or my bank account, or the direction of the Earth's axis, or the concept of justice, or the number 17.) In contrast, when a metaphysician considers the question of whether the objects are inaccessibly many in number, she purports to be asking a much more general question: a question that concerns "absolutely everything". And it is not obvious that absolutely general claims of this kind occur in non-philosophical contexts. As a result, it is not obvious that mere linguistic competence gives us the resources to pose ontological questions of the sort that are our present concern.

One way to argue that mere linguistic competence affords us the resources to pose such questions is to assume a certain picture of quantification, a picture whereby ordinary (non-absolutely general) quantification is the result of adding a *restriction* to an absolutely general quantifier. Accordingly, all that is required to engage in absolutely general quantification is to find a way of keeping out the restriction. Since the restriction is often determined by context, this is partly a matter of positioning oneself in the right kind of context. And ontological investigation provides a context of just the right kind.

A key aim of this paper is to show that that there is an alternate picture of quantifica-

tion, one that does not see ordinary (non-absolutely-general) quantification as the result of restricting an absolutely general quantifier.

### 3 A domain-free account of quantification

The proposal I would like to consider is based on a very simple idea. It treats existential quantification as a generalized form of disjunction, and universal quantification as a generalized form of conjunction. (I will restrict my attention to first-order languages.)

Let me spell out the idea in a little further detail.<sup>7</sup> It is natural to think of the logical operators “ $\neg$ ” and “ $\vee$ ” as expressing *propositional operations*: operations that take propositions as input and yield a proposition as output. The operations in question can be characterized *algebraically*. One starts with a space of propositions. No assumptions are made about the internal structure of individual propositions (they are treated like “black boxes”), but one assumes that the space of propositions forms a complete Boolean Algebra.<sup>8</sup> This means, in particular, that every proposition has an algebraic complement, and that every set of propositions has a least upper bound.<sup>9</sup> (The easiest way to visualize this is to think of propositions as sets of “worlds”, and take the algebraic complementation and least upper bound operations to correspond to the set-theoretic complementation and union operations, respectively.) One then takes the logical operator “ $\neg$ ” to express algebraic complementation, and the logical operator “ $\vee$ ” to express the least upper bound operation of the algebra.

On the proposal I would like to consider, “ $\exists$ ” expresses the least upper bound operation of the algebra, just like “ $\vee$ ”. But whereas “ $\vee$ ” only takes two propositions as input, “ $\exists$ ” takes arbitrarily many. Because the least upper bound operation is a purely algebraic operation,

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<sup>7</sup>For different proposals which are similar in spirit, see Fraassen 1982, Dorr 2014 and Turner typescript.

<sup>8</sup>Importantly, we do *not* presuppose that the algebra is equipped with cylindrification operations, since this additional structure would mirror that of a domain. For more on cylindric algebras, see Henkin et al. (1971).

<sup>9</sup>Both of these operations can be defined on the basis of a reflexive, anti-symmetric and transitive entailment relation on propositions. The least upper bound of a set of propositions is the strongest (i.e. most entailing) proposition entailed by each proposition in the set. The complement of a proposition  $p$  is the strongest proposition  $\bar{p}$  such that the least upper bound of  $p$  and  $\bar{p}$  is entailed by every proposition.

it is characterized with no appeal to a domain of discourse. So everything we have done so far is consistent with our aim of specifying the meaning of the existential quantifier without appealing to domains of discourse.

Unfortunately, our work is not yet complete. We have given a precise specification of the operation expressed by the existential quantifier, but we have yet to give a precise specification of the range of propositions that the operation is to take as input when it comes to evaluating particular instances of existential quantification. Let me explain.

Suppose, first, that we wish to determine which proposition is expressed by the disjunctive sentence  $\lceil (A \vee B) \rceil$ . Since “ $\vee$ ” expresses the least upper bound operation, the proposition we’re looking for is the result of applying the least upper bound operation to certain input propositions. Which input propositions? Answer: the proposition expressed by  $A$  and the proposition expressed by  $B$ .

Now suppose we wish to determine which proposition is expressed by the existentially quantified sentence  $\lceil \exists x(P(x)) \rceil$ . Since the existential quantifier expresses the least upper bound operation, the proposition we’re looking for is the result of applying the least upper bound operation to certain input propositions. Which input propositions? Here the answer is less straightforward.

The standard approach is to bring in a domain of discourse, and characterize the input propositions as the propositions that say of some object in the domain that it has the property expressed by  $P$ . But this answer is unavailable in the present context, so we will have to do things differently.

### **A domain-free semantics**

Let us start with the question of how to interpret the *atomic* sentences of a first-order language. On a standard domain-based semantics, one starts by assigning an object from one’s domain to each individual constant, and an  $n$ -place property of such objects to each  $n$ -place predicate letter. One then maps the atomic sentence  $\lceil P(a_1, \dots, a_n) \rceil$  to a proposition that



ascribes the property that is assigned to  $P$  to the objects that are assigned to  $a_1, \dots, a_n$ .

On the picture I would like to develop here, the initial assignment of sub-propositional entities is just a middleman: it is a means for assigning propositions to sentences, which is what we are really after. The way to cut out the middleman is to think of the meanings of predicates and names as constraints on the propositions that one is allowed to express using atomic sentences, and to specify these constraints without invoking an assignment of sub-propositional entities to predicates and individual constants.

I spell out the details in the appendix, but the basic idea is straightforward. One starts with the notion of an *atomic assignment*: a function that assigns a proposition to each atomic sentence of the language in a way that respects the logical properties of “=”. One then characterizes one’s understanding of the predicates of the language by singling out a set  $I$  of atomic assignments. (Intuitively the atomic assignments in  $I$  are those one takes to respect one’s understanding of the predicates of the language.) Next, one characterizes one’s understanding of the individual constants of the language by selecting an assignment  $\iota \in I$ . (Intuitively,  $\iota$  is taken to respect both one’s understanding of the predicates of the language and one’s understanding of the individual constants.)

Each member  $i$  of  $I$  is an assignment of propositions to *atomic* sentences. But by taking “ $\neg$ ”, “ $\vee$ ” and “ $\exists$ ” to express suitable algebraic operations, one can easily expand  $i$  to a function  $\llbracket \dots \rrbracket_i$  which assigns a proposition to each sentence of the language:

- if  $A$  is an atomic sentence, then  $\llbracket A \rrbracket_i = i(A)$ ;
- $\llbracket \neg A \rrbracket_i$  is the algebraic complement of  $\llbracket A \rrbracket_i$ ;
- $\llbracket A \vee B \rrbracket_i$  is the least upper bound of  $\llbracket A \rrbracket_i$  and  $\llbracket B \rrbracket_i$ ;
- $\llbracket \exists x \phi(x) \rrbracket_i$  is the least upper bound of the set of propositions  $\llbracket c = c \rrbracket_j \cap \llbracket \phi(c) \rrbracket_j$ , where  $j \in I$  agrees with  $i$  on atomic sentences not involving  $c$ .

The *intended* interpretation of the language is the special case of  $\llbracket \dots \rrbracket_i$  in which  $i = \iota$ .

Sentential truth is characterized on the basis of propositional truth in the usual way: a sentence  $A$  is true if and only if  $\llbracket A \rrbracket_i$  is a true proposition.

### The Representation Theorems

Our domain-free semantics can be used to prove the following results. (See appendix for details.)

**Representation Theorem** Any assignment of propositions to sentences that can be generated using a domain-free semantics can also be generated using a domain-based semantics (given a suitable domain of objects).

**Converse Representation Theorem** Any assignment of propositions to sentences that can be generated using a domain-based semantics (given a suitable domain of objects) can also be generated using a domain-free semantics (without having to presuppose a domain).

I'll have more to say about these results below. But let me start by mentioning an analogy that I think helps bring out the significance of the Representation Theorem.

There is a familiar way of using objects and properties to characterize possible worlds, and using possible worlds to characterize propositions. Roughly speaking: one specifies a possible world by selecting a domain of objects (thereby specifying the objects that are to exist at the world) and selecting an assignment of properties to those objects (thereby specifying the properties are to be enjoyed by the objects in our domain at the world). Proposition can then be characterized as sets of possible worlds.

Could one go in the opposite direction? Could one start with an algebra of propositions, and use it to characterize a space of possible worlds, together with an assignment of objects and properties to each world? As Stalnaker (1976) points out, going from an algebra of propositions to a space of worlds is relatively straightforward: one can think of a world as a

maximally consistent proposition.<sup>10</sup>

The next step is to specify a domain of objects, and a suitable assignment of properties to objects, for each of the resulting worlds. This might turn out to be a relatively straightforward task if one starts with a space of *structured* propositions. Suppose, for example, that one works with Russellian propositions: one thinks of “basic” propositions as property/object pairs (or, more generally, as  $n$ -place relation/ $n$ -tuple pairs), and of non-basic propositions as Boolean combinations of basic propositions. One might then take the domain of a world to consist of any objects that figure in basic propositions that are entailed by that world, and use those basic propositions to characterize an assignment of properties to objects.

In the present context, however, we are not allowed to help ourselves to Russellian propositions. All we are allowed to assume about our space of propositions is that it enjoys a certain amount of algebraic structure: the propositions themselves are treated like black boxes. What the Representation Theorem shows is that one can get around this limitation by using a first-order language to “carve up” worlds into objects and properties in a principled way. A little more precisely: suppose one starts with an algebra of propositions, and uses it to interpret a first-order language by defining an interpretation function  $\llbracket \dots \rrbracket_{\mathcal{A}}$  which assigns a proposition in the algebra to each sentence, subject to the constraints described above; the theorem shows that there is a systematic way of using  $\llbracket \dots \rrbracket_{\mathcal{A}}$  to characterize a space of possible worlds and specify a suitable domain of objects and assignment of properties for each of the resulting worlds.<sup>11</sup>

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<sup>10</sup>In algebraic terms, this means that a possible world is an *atom* of one’s algebra: an element  $p$  of the algebra which is such that  $p$  does not entail  $\perp$  on its own but does entail it in conjunction with any element of the algebra that is not already entailed by  $p$ . What if one’s algebra is non-atomic? It is a consequence of Stone’s Representation Theorem that any Boolean Algebra can be embedded in an atomic Boolean Algebra in which every proposition is a set of atoms. So one can think of the original algebra as an (incomplete) collection of sets of worlds of the embedding algebra.

<sup>11</sup>The expressive resources of a first-order language are fairly limited. One manifestation of these expressive limitations is the Löwenheim–Skolem Theorem, which entails that a countable first-order theory with infinite models has models of every infinite cardinality. So although one can use a first-order language to state that there are infinitely many objects, one cannot go on to specify the size of that infinity. This has an important consequence in the present context. It means that if one uses a first-order language to “carve up” worlds into objects and properties in the way I’m suggesting, the language may be compatible with many different carvings, not all of which are isomorphic to one another. On the Facts First View, which I develop below,

## Taking stock

I earlier noted that one might be tempted to endorse a picture of quantification according to which ordinary English quantifiers presuppose an “absolutely general” domain: a domain consisting of “absolutely everything”. If this picture is correct, it is not clear how one could be a realist about whether, e.g. there are seventeen eggs in the basket without also being a realist about questions concerning the “absolutely general” domain that ordinary quantifiers presuppose. Our domain-free semantics offers an alternative to this picture by showing that one can specify the behavior of a first-order quantifier without presupposing a domain of discourse. The Representation Theorems add precision to that response by showing that the assignments of propositions to sentences that can be generated by a domain-based semantics (given a suitable domain) are exactly the assignments that can be generated by a domain-free semantics (without having to presuppose a domain).

## 4 Objects First vs. Facts First

The Representation Theorems also supply the formal groundwork for a “facts first” conception of the relationship between our language and the world it represents. The aim of this section is to explain what such a conception amounts to. I start with a pair of impressionistic slogans:

**Objects First** The world consists of the totality of objects, and their properties.

What about facts? A basic fact is the result of ascribing a property to one or more objects; a fact is a Boolean combination of basic facts.

**Facts First** The world consists of the totality of facts.

What about objects? All it takes for a singular term to refer is for it to occur in an atomic sentence that expresses a true proposition.

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there is nothing to privilege one of these carvings over the rest. Since ontology is language-relative, there is no sense to be made of ontological questions that transcend the expressive resources of our language.

How might these slogans be made more precise? One might be tempted to do so by bringing in a notion of fundamentality: on the Objects First View, objects are “more fundamental” than facts; on the Facts First View, facts are “more fundamental” than objects. But that is not how I will understand them here, because I’m not sure about the consequences that fundamentalist readings of the slogans would have for the portions of our theorizing that do not invoke talk of fundamentality, or related notions. I will instead spell out each of the slogans as a pair of claims about the relationship between our language and the world it represents. The first member of each pair is a claim about reference; the second member is a claim about what it takes for an assignment of propositions to sentences to count as *admissible*.

What is admissibility? Start by considering the distinction between language use and abuse. Consider, for example, the English sentence “The *Times* has decided to change its format” (Nunberg 1978). This sentence can be used to communicate that a publisher decided to change the format of its publication. One might take the view that any attempt to use the sentence to *express* such a content, without the intervention of some sort of pragmatic repair strategy, would constitute an instance of language abuse, since it would require using “the *Times*” to refer to two different entities. Another example: consider the question of whether the first-order sentence “ $2 + 2 = 4$ ” can be used to express truth-conditions that impose no substantive ontological demands on the world. One might take the view that it cannot, on the grounds that an atomic first-order sentence can only be true if its singular terms refer, and that the reference of singular terms always imposes a non-trivial ontological demand on the world. This is not my own view. But if it is correct, then any attempt to use “ $2 + 2 = 4$ ” to express ontologically innocent truth-conditions without pragmatic intervention would constitute an instance of language abuse. An *admissible* assignment of propositions to sentences is one that allows for language use, rather than abuse. It is an assignment that respects the relationship that ought to obtain between a properly functioning language and the world it represents.

In explicating Objects First and Facts First I will make an important simplifying assumption. I will restrict my attention to first-order languages, and presuppose a negative free logic.<sup>12</sup> I will also take the notion of objecthood under discussion to be captured by the first-order existential quantifier.

### **The Objects First View**

The Objects First View presupposes that there is a domain of objects whose significance is independent of the way in which we happen to represent the world. I shall take it to be the “absolutely general” domain. This domain is then used to set forth the following two theses:

#### **Reference Thesis (Objects First Version)**

For a singular term to refer is for it to be paired with an object from the absolutely general domain.

#### **Admissibility Thesis (Objects First Version)**

An assignment of propositions to sentences is only admissible if it is generable by: (a) choosing a subdomain of the absolutely general domain as the range of one’s quantifiers, (b) assigning an object from this subdomain as a referent for each individual constant, (c) assigning and an  $n$ -place property to each  $n$ -place predicate, and (d) assigning propositions to sentences compositionally, in the usual way.

### **The Facts First View**

The Facts First View constitutes a rejection of the Objects First View. It is based on the somewhat inchoate idea that our conception of objecthood is essentially tied to the language we use to represent the world, and has been discussed in different forms by a many different philosophers, including Gottlob Frege, Michael Dummett, Crispin Wright, Robert

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<sup>12</sup>A free logic is a classical logic that allows for empty individual constants. A negative free logic is a free logic in which an atomic sentence can only be true if its individual constants are non-empty. This means, in particular, that  $\ulcorner a = a \urcorner$  is logically equivalent to  $\ulcorner \exists x(x = a) \urcorner$ , for  $a$  a constant.

Stalnaker, Gideon Rosen and John Burgess.<sup>13</sup> In the next few paragraphs I will try to add some precision to the picture, by drawing on the domain-free semantics of section 3.

On the version of the Facts First View I will develop here, there is an analogy between objecthood and fashionability: just like there is no sense to be made of the question of what counts as fashionable in a sense of fashionability that transcends the tastes of some community or other, so there is no sense to be made of the question of what counts as an object in a sense of objecthood that transcends a particular language.

It is no part of the view, however, that “ $\exists$ ” has a hidden contextual parameter. The existential quantifier is modeled using a domain-free semantics according to which it expresses the least upper bound operation of one’s algebra of propositions—a parameter-free operation which is fully determined by the structure of the algebra. Where then does the language-relativity of our conception of objecthood come from? The key observation is that even if different first-order languages agree about treating the existential quantifier as a least upper bound operator, they might disagree about the range of propositions that the least upper bound operation should take as input when it comes to assessing a particular existential sentence.

The upshot is that even though “ $\exists$ ” is interpreted uniformly across different first order-languages, “ $\exists x(x = x)$ ” doesn’t always express the same proposition. What does the difference turn on? On a domain-free semantics, the proposition expressed by “ $\exists x(x = x)$ ” is determined by one’s understanding of the *predicates* of the language. More specifically, it is defined as the least upper bound of the propositions  $\llbracket \ulcorner c = c \urcorner \rrbracket_j$ , where  $j$  is an interpretation of the language which respects one’s understanding of the predicates of the language, but may or may not respect one’s understanding of  $c$ . Recall, moreover, that on a domain-free semantics one’s understanding of the predicates of the language is specified with no recourse to a domain of discourse. One does so by setting forth a constraint on assignments of propo-

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<sup>13</sup>Frege, *Die Grundlagen der Arithmetik*; Dummett, *Frege: Philosophy of Language*; Wright, *Frege’s Conception of Numbers as Objects*; Stalnaker, “What Possible Worlds Could Not Be”; Rosen, “The Refutation of Nominalism (?)”; Glanzberg, “Quantification and Realism”; and Burgess “Being Explained Away”.

sitions to atomic sentences. And the constraint is identified by selecting a set  $I$  of functions from atomic sentences to propositions.

Are there any restrictions on one's choice of  $I$ ? A proponent of the Objects First View, would recommend an absolutist restriction. Recall that her Admissibility Thesis entails that an assignment of propositions to sentences is only admissible if it can be generated by selecting a subdomain of the absolutely general domain, and using it to assign referents to one's singular terms. So she thinks that the only admissible choices of  $I$  are those that would lead to an interpretation of the language that can be generated in that way.

A proponent of the Facts First View disagrees. She rejects the Objects First version of the Admissibility Thesis in favor of the following:

**Admissibility Thesis (Facts First Version)**

An assignment of propositions to first-order sentences is admissible just in case it is generable by a domain-free semantics.

Accordingly, she thinks that the admissibility of an assignment of propositions to sentences does not depend on whether the assignment could have been generated by starting with a subdomain of the “absolutely general” domain and using it to fix a referent to each individual constant of the language.

The Representation Theorems help clarify the connection between the two versions of the Admissibility Thesis. They entail that an assignment of propositions to first-order sentences is generable by a function  $[[ \dots ]]$  if and only if it could have also been generated by a domain-based semantics, given a suitable domain. This means that the Facts First View and the Objects First View agree about a key issue: an assignment of propositions to sentences is only admissible if it is generable by a domain-based semantics, given a suitable domain. Where they come apart is on the question of whether admissibility also requires that the relevant domain exist as part of the world's “absolutely general” domain.

A second component of the Facts First View is needed to complete the picture: a con-



ception of reference. As I noted above, a friend of the Facts First View thinks that there is no sense to be made of the notion of objecthood independently of a particular language. Similarly, she thinks that there is no sense to be made of the notion of reference independently of a particular language. Within a particular language, however, one can elucidate the notion of reference by asserting a version of the following:

**Reference Thesis (Facts First Version)**

For a singular term of the language I am now speaking to refer is for it to figure in an atomic sentence of my language, and for that sentence to express a true proposition.

In the case of a first-order language, the Reference Thesis might seem trivial. For suppose some singular term  $a$  is taken to refer by speakers of the language. Then they will take  $\ulcorner a = a \urcorner$  to express a true proposition, and so think that  $a$  figures an atomic sentence that expresses a true proposition. Conversely, suppose that  $a$  figures in an atomic sentence  $\ulcorner P(a) \urcorner$ , which expresses a true proposition. When one assumes a negative free logic, as we are doing here,  $\ulcorner P(a) \urcorner$  entails  $\ulcorner \exists x(x = a) \urcorner$ . So speakers of the language must take  $a$  to refer.

Taken in conjunction with the Admissibility Thesis, however, the Reference Thesis becomes much more interesting. The best way to see this is to consider an example. Let  $L^{\mathbb{N}}$  be the language of first-order arithmetic,<sup>14</sup> and let  $\iota$  be the following atomic assignment for  $L^{\mathbb{N}}$ :

- $\iota(A) = \top$ , if  $A$  figures in one of these lists:
  - “ $0 = 0$ ”, “ $1 = 1$ ”, “ $2 = 2$ ”, ...
  - “ $+(0, 0, 0)$ ”, “ $+(0, 1, 1)$ ”, “ $+(1, 0, 1)$ ”, “ $+(1, 1, 2)$ ”, “ $+(0, 2, 2)$ ”, ...
  - “ $\times(0, 0, 0)$ ”, “ $\times(0, 1, 0)$ ”, “ $\times(1, 0, 0)$ ”, “ $\times(1, 1, 1)$ ”, “ $\times(0, 2, 0)$ ”, ...
- $\iota(A) = \perp$ , otherwise.

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<sup>14</sup>In addition to the identity symbol “=”,  $L$  consists of the three-place predicates “+” and “ $\times$ ”, corresponding to addition and multiplication, and the individual constants “0”, “1”, “2”, etc.

where  $\top$  and  $\perp$  are the top and bottom elements of our algebra of propositions, respectively. On natural assumptions,<sup>15</sup> this delivers the result that  $\llbracket \phi \rrbracket_\iota = \top$  if  $\phi$  is an arithmetical truth, and  $\llbracket \phi \rrbracket_\iota = \perp$  otherwise.

It follows immediately from the Facts First version of the Admissibility Thesis that  $\llbracket \cdot \cdot \cdot \rrbracket_\iota$  is an admissible interpretation of  $L^{\mathbb{N}}$ . So there is an admissible interpretation of  $L^{\mathbb{N}}$  on which every standard arithmetical truth counts as necessarily true, and every standard arithmetical falsehood counts as necessarily false. This means, in particular, that each of the atomic sentences “ $0 = 0$ ”, “ $1 = 1$ ”, ... expresses a necessarily true proposition. Now suppose we are speaking a language that extends  $L^{\mathbb{N}}$ . A version of the Reference Thesis for our language entails that the numerals “0”, “1”, ... are all genuinely referential, and therefore the conclusion that natural numbers exist.<sup>16</sup> And, crucially, we have arrived at this conclusion without ever considering the question of whether the “absolutely general” domain includes numbers amongst its constituents.

The point generalizes. According to the Facts First View, when propositions have been assigned to the sentences of a first-order language in a suitably systematic way, the fact that an atomic sentence has been assigned a proposition that turns out to be true is enough to guarantee that the sentence’s singular terms are genuinely referential, and therefore that the feature of the world that verifies the truth of the proposition is accurately described as containing objects.

The Facts First View vindicates a version of the Fregean idea that a given content might be “carved up” into constituents in more than one way. Suppose, for example, that we wish to talk about whether Susan and Bob are happy and wise, and set up a first-order language to do so. We want our language to be distinctive in a certain respect, however. We want

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<sup>15</sup>Specifically: we let  $I$  be the set of atomic assignments that agree with  $\iota$  except perhaps for a relabelling of the individual constants.

<sup>16</sup>Here I assume that our metalanguage contains singular terms corresponding to the numerals of  $L^{\mathbb{N}}$ . The assumption is certainly satisfied in the present context, since English has its own numerals. But it is worth noting that it would be satisfied even if English didn’t contain its own numerals because we could also use, e.g. “the referent of ‘0’”.

the proposition that is actually expressed in English using “Susan is happy” to be expressed by the first-order “Susanishly(happiness)” (read “Happiness is susanishly instantiated”), and similarly for other cases. We proceed by letting  $L^P$  be a first-order language containing the individual constants “happiness” and “wisdom”, and the predicates “Susanishly” and “Bobishly”. We then let  $\iota$  be the following atomic assignment for  $L^P$ :

- $\iota(\text{“Susanishly(happiness)”}) = \llbracket \text{“Susan is happy”} \rrbracket^{En}$
- $\iota(\text{“Bobishly(happiness)”}) = \llbracket \text{“Bob is happy”} \rrbracket^{En}$
- $\iota(\text{“Susanishly(wisdom)”}) = \llbracket \text{“Susan is wise”} \rrbracket^{En}$
- $\iota(\text{“Bobishly(wisdom)”}) = \llbracket \text{“Bob is wise”} \rrbracket^{En}$

where  $\llbracket A \rrbracket^{En}$  is the element of our algebra of propositions that is expressed by the English sentence  $A$ .

As before, it follows from the Facts First version of the Admissibility Thesis that  $\llbracket . . . \rrbracket_{\iota}$  is an admissible interpretation of  $L^P$ .<sup>17</sup> So we know that there is an admissible way of using the atomic sentence “susanishly(happiness)” to express the proposition that Susan is happy. Now suppose we are speaking a language that extends  $L^P$ . On the assumption that Susan is indeed happy, a version of the Reference Thesis for our language entails that “happiness” is genuinely referential. This gives us a precise sense in which one might say that the proposition that Susan is happy can be “carved up” into constituents in different ways. When we describe it using “Susan is happy”, we take it to have Susan as a constituent and we take it to say, of her, that she is happy; when we describe it using “susanishly(happiness)” we take it to have the property of happiness as a constituent, and we take it to say, of it, that it is susanishly instantiated.

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<sup>17</sup>To the assignments above, we add  $\iota(\text{“happiness = happiness”}) = \iota(\text{“wisdom = wisdom”}) = \top$ . For technical reasons, we must assume that  $L^P$  contains infinitely many individual constants. So we add the constants “ $a_1$ ”, “ $a_2$ ”,  $\dots$ , and treat them as empty names.  $I$  is the set of atomic assignments that agree with  $\iota$  except perhaps for a relabelling of the individual constants.

### But are they *really* objects?

A critic might complain that just because a singular term has been shown to refer in the Facts First sense of reference, there is no reason to think that it *really* refers. Such a complaint would beg the question in the present context. According to the Facts First View, what it is for a singular term of one's language to *really* refer is for it to figure in an atomic sentence that expresses a true proposition.

Return to an earlier analogy. I take the notion of metaphysical fashionability to be irreparably misguided, so I think that only the community-relative notion of fashionability deserves a place in our cognitive lives: to be fashionable *just is* to be fashionable in the community-relative sense. Similarly the Facts First View takes an absolutist conception of objecthood to be irreparably misguided. So only the language-relative conception of objecthood deserves a place in our cognitive lives: to be an object *just is* to be an object in the language-relative sense.

There is a second line of attack against the Facts First View that should also be resisted, but for a different reason. It starts by construing the Facts First conception of objecthood as the claim that to be an object is to be the referent of a singular term of some *possible* language. It then uses this claim to argue that proponents of the Facts First View don't really manage to avoid commitment to an "absolutely general" domain of discourse. For the absolutely general domain is simply the result of taking the space of all possible languages and pooling together the referents of all their singular terms.

I'm not sure this argument succeeds on its own terms, because I'm not sure that there is a definite fact of the matter about what counts as a "possible language".<sup>18</sup> Either way, it seems to me that the argument misses its target by mischaracterizing the Facts First View. A proponent of the Facts First View does *not* claim that to be an object (in a language-transcendent sense) is to be the referent of a singular term of some possible language. She claims instead that there is no sense to be made of a language-transcendent notion of objecthood. She might

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<sup>18</sup>See Eklund 2014 for a related point, and Rayo 2014 for an effort to respond to it.

add that (within a given language) there isn't much to be said about what being an object consists in. What there is to be said is what one says when one uses the language in question to assert "*everything* is an object", or some analogue sentence.

Actually, one can say a little bit more by engaging in semantic ascent; namely: that all there is to using one's language to correctly describe the world as containing objects is to correctly describe the facts that make up the world using sentences of one's language that include singular terms (or variables taking singular term positions).

## 5 Further Discussion

### Why accept a facts-first conception of the world?

I would like to suggest that the Facts First View does a better job of explaining our linguistic practice than its Objects First counterpart.

Consider, in particular, the ease with which we feel free to expand our domain discourse when it suits our communicative purposes. We do not hesitate to go from an event-free statement like "they started arguing" to an event-loaded counterpart, like "an argument broke out". We routinely expand our domain of discourse to keep track of our social practices: we talk about commitments, and bank accounts, and universities, and votes, and cabinet positions. We routinely expand our domain of discourse to streamline our descriptions of the natural world—as when we talk about orbits or tides—or to abstract away from irrelevant differences between objects—as when we talk about particle-types or letter-types. And we do not hesitate to introduce talk of mathematical objects when it is expedient to do so.

On the Objects First View, this leaves us with an awkward choice. We must choose between the claim that much of ordinary discourse is to be understood non-literally, and the claim that much of our ordinary discourse might turn out to be false for reasons that speakers would fail to recognize as germane to their purposes, since we might end up speaking falsely when we talk about Socrates's death, or my bank account, or the moon's orbit, or Newton's

*Principia*, or Hilbert spaces.

On the Facts First View, in contrast, there is nothing objectionable about expanding one's domain of discourse to suit one's communicative aims. The expansion is risk-free because our singular terms are guaranteed to refer, as long the facts that make up the world are accurately described by our sentences.<sup>19</sup>

## Complementarity

As we have seen, the Representation Theorems entail that the assignments of propositions to sentences that can be generated by a domain-based semantics (given a suitable domain) are exactly the assignments that can be generated by a domain-free semantics (with no need to presuppose a domain). But domain-based and domain-free semantics are more than just convergent. There is an important sense in which they are complementary to one another: they each suffer from a limitation that can be overcome by appeal to its counterpart.

Domain-free semantics are not well-suited to figure in an account of linguistic competence because they are only compositional in a limited sense: they are compositional in their treatment of complex sentences, but not in their treatment of atomic sentences. This limited compositionality makes a domain-free semantics implausible as a piece of human psychology: whatever our linguistic competence consists in, it does not consist in a grasp of  $I$  and  $\iota$  and an understanding of how they might be used to derive  $\llbracket \dots \rrbracket_{\iota}$ . So a domain free-semantics cannot stand on its own. It needs a domain-based counterpart to account for linguistic competence. Happily, the Representation Theorem guarantees that a suitable counterpart will always be available.

It is worth noting, however, that domain-based semantics have limitations of their own. Let me illustrate the point with an example. Suppose that the arithmetical language  $L^{\mathbb{N}}$  of Section 4 is used by a community of speakers as part of a linguistic practice. Suppose, more-

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<sup>19</sup>Related views have been recently defended by Sidelle (2002), Chalmers (2009), Eklund (2009), Hirsch (2010) and Thomasson (2015).

over, that we—the theorists—favor a metasemantics that assigns propositions to sentences before assigning referents to subsentential expressions (Lewis 1973). Suppose, finally, that when we apply our metasemantics to the relevant linguistic practice we get the following assignment of propositions to sentences:

- every mathematical truth in  $L^{\mathbb{N}}$  is assigned the necessary proposition;
- every mathematical falsehood in  $L^{\mathbb{N}}$  is assigned the impossible proposition.

Since human speakers have finite cognitive capabilities, we should expect this assignment to be generable compositionally, from an assignment of semantic values to basic lexical items. And, of course, it is not hard to find a domain-based semantics that does the job: one takes one’s domain of discourse to consist of the natural numbers, and assigns interpretations to the non-logical vocabulary of  $L^{\mathbb{N}}$  in the usual way.

Notice, however, that our domain-based semantics for  $L^{\mathbb{N}}$  relies on an ontological assumption: it presupposes that the natural numbers exist.<sup>20</sup> A friend of the Objects First View will think that this ontological assumption opens the door to a certain kind of challenge: unless we are justified in thinking that the “absolutely general” domain includes natural numbers, we won’t be justified in using a semantics based on a domain of natural numbers to show that our assignment of propositions to sentences counts as an adequate interpretation of  $L^{\mathbb{N}}$ .

On the Facts First View, the worry can be defused. By the Converse Representation Theorem, the possibility of giving a domain-based semantics for  $L^{\mathbb{N}}$  (given a domain of natural numbers) entails that it is also possible to give a domain-free semantics for  $L^{\mathbb{N}}$  (with no need for such a domain). And by the Facts First version of the Adequacy Thesis, this is enough to guarantee that our assignment of propositions to sentences of  $L^{\mathbb{N}}$  is in good

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<sup>20</sup>In fact, one needs the stronger assumption that the natural numbers exist necessarily. In order to vindicate this assumption on the basis of the bootstrapping procedure described below, we need to use a slight strengthening of the Facts First version of the Reference Thesis. When an assignment of propositions to sentences maps an atomic sentence  $\ulcorner P(c) \urcorner$  to a true proposition, we need not just the claim that  $c$  refers, but also the claim that  $c$  refers *at every world at which the proposition is true*.

standing. Notice, moreover, that this puts a friend of the Facts First View in a position to run a *bootstrapping* argument. She can use the Facts First version of the Reference Thesis to go from the observation that the sentences “ $0 = 0$ ”, “ $1 = 1$ ”, ... of  $L^{\mathbb{N}}$  all express true propositions to the conclusion that the numerals “0”, “1”, ... are all referential, and therefore to the conclusion that the natural numbers exist. So she is in a position to justify the very ontological assumption that is needed to get our domain-based semantics to work.

And, of course, the point generalizes. The bootstrapping argument will always be available to proponents of the Facts First View as a way of using a domain-free semantics to justify the ontological assumptions of its domain-based counterpart.<sup>21</sup> This is in keeping with the spirit of the Facts First View, on which expanding our domain of discourse is risk free: our singular terms are guaranteed to refer, as long the facts that make up the world are accurately described by our sentences.

## 6 Conclusion

I have tried to articulate an anti-absolutist picture of the world. It is based on the Facts First View, which is a conception of the relationship between our language and the world it represents. According to the Facts First View, all there is to correctly describing the world as containing objects is to correctly describe the facts that make up the world using sentences that include singular terms (or variables taking singular term positions). Although claims of this kind have been defended by many different philosophers over the years, I hope to have added precision to the underlying picture by articulating it on the basis of the formal framework that emerges from a domain-free account of quantification.<sup>22</sup>

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<sup>21</sup>If this is right, there is no need to worry about Chomsky’s notorious *London* argument (Chomsky 2000, 37), as interpreted by Stoljar (2014).

<sup>22</sup>For their many helpful comments, I am grateful to Andrew Bacon, Cian Dorr, Adam Elga, Salvatore Florio, Peter Fritz, Jeremy Goodman, Bob Hale, Benj Hellie, Kathrin Koslicki, David Liebesman, Rose Lenehan, Øystein Linnebo, Matthew Mandelkern, Gideon Rosen, Jeff Russell, Graham Priest, Brad Skow, Raul Saucedo, Jonathan Schaffer, Ted Sider, Jack Spencer, Judy Thomson, Robert Truman, Jason Turner, and to audiences at MIT, Oxford University, SOFIA, the University of Arizona, the University of California, Irvine, the University of Pavia, and the University of Oslo. Special thanks are due to Matthias Jenny, who I was lucky



## Appendix

The purpose of this appendix is to spell out the details of the domain-free semantics I describe in Section 3.

### Definitions

We work with a first-order language  $L$  whose individual constants are  $a_0, a_1, a_2, \dots$ . (To keep things simple, I shall assume that  $L$  contains no function-letters.) We will work with the following definitions:

- An *atomic assignment* is a function that assigns a proposition to each atomic sentence of  $L$ .
- An *interpretation* of the predicates and individual constants of  $L$  is a triple  $\langle I, \iota, \circ \rangle$ , where  $I$  is a non-empty set of atomic assignments,  $\iota$  is a distinguished member of  $I$ , and  $\circ$  is a function that takes each pair of atomic assignments in  $I$  to an atomic assignment in  $I$ .

( $I$  is to be thought of as the set of atomic assignments that are compatible with the intended interpretation of the *predicates* of  $L$ .  $\iota$  is to be thought of as the unique atomic assignment in  $I$  which is not only compatible with the intended interpretation of the predicates of  $L$ , but is also compatible with the intended interpretation of the *individual constants* of  $L$ .)

- I assume that every atomic assignment in  $I$  respects the logical properties of the identity predicate “=”.

This is to be spelled out as follows. Let  $\top$  and  $\perp$  be the top and bottom elements of one’s algebra, respectively, let “ $\wedge$ ”, “ $\vee$ ” and “ $\neg$ ” express the algebra’s meet, join and

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enough to have as a research assistant on this project. Matthias performed a careful review of the manuscript and proofs, and added much needed bibliographical references.

not operations, respectively, and let  $\ulcorner p \leq q \urcorner$  express the thought that proposition  $p$  algebraically entails proposition  $q$ . Let  $a, b$  and  $c$  be arbitrary individual constants of  $L$ ,  $S_a$  be an atomic sentence of  $L$  in which  $a$  occurs and  $S_b$  be the result of replacing some occurrences of  $a$  in  $S_a$  by  $b$ . Then we require each  $i \in I$  to satisfy the following constraints:

1. *Negative free logic*

$$i(S_a) \leq i(\ulcorner a = a \urcorner);$$

2. *Necessity of identity and distinctness*

$$\text{either } i(\ulcorner a = b \urcorner) = i(\ulcorner a = a \urcorner) \text{ or } i(\ulcorner a = b \urcorner) = \perp;$$

3. *Transitivity*

$$(i(\ulcorner a = b \urcorner) \wedge i(\ulcorner b = c \urcorner)) \leq i(\ulcorner a = c \urcorner);$$

4. *Intersubstitutivity*

$$i(\ulcorner a = b \urcorner) \leq i(S_a) \leftrightarrow i(S_b)$$

(where  $\ulcorner p \leftrightarrow q \urcorner$  is a syntactic abbreviation for  $\ulcorner (p \wedge q) \vee (\neg p \wedge \neg q) \urcorner$ ).

Note that condition 4 entails symmetry:  $i(\ulcorner a = b \urcorner) = i(\ulcorner b = a \urcorner)$ . Note also that if one wanted to work with classical logic, rather than negative free logic, it suffices to strengthen condition 1 to “ $i(\ulcorner a = a \urcorner) = \top$ ”.

- $\circ$  is a “fusion” function defined on members of  $I$ .

The intuitive idea is that  $i \circ j$  is an atomic assignment that “combines”  $i$  and  $j$ ’s interpretations of the individual constants, by interpreting the constant  $a_{2n}$  the way  $i$  interprets constant  $a_n$  and interpreting  $a_{2n+1}$  the way  $j$  interprets constant  $a_n$ .

We implement this idea formally by introducing the individual constant relabelling operations “ $\leftarrow$ ” and “ $\rightarrow$ ”:

$$\overleftarrow{a}_k = a_{2k} \quad \overrightarrow{a}_k = a_{2k+1}$$

and requiring  $\circ$  to satisfy the following condition:

$$i \circ j(\ulcorner P(\overleftarrow{a}_{k_0}, \dots, \overleftarrow{a}_{k_n}) \urcorner) = i(\ulcorner P(a_{k_0}, \dots, a_{k_n}) \urcorner)$$

$$i \circ j(\ulcorner P(\overrightarrow{a}_{k_0}, \dots, \overrightarrow{a}_{k_n}) \urcorner) = j(\ulcorner P(a_{k_0}, \dots, a_{k_n}) \urcorner)$$

We also need to make sure that the fusion of an atomic assignment with itself does no more than relabel individual constants. We implement this idea formally by requiring a generalized version of the following:<sup>23</sup>

$$i \circ i(\ulcorner \overleftarrow{a}_k = \overrightarrow{a}_k \urcorner) = i(\ulcorner a_k = a_k \urcorner)$$

Finally, we need to make sure that the order in which atomic assignments are brought together by the fusion operator doesn't matter. We do this by requiring  $\circ$  to satisfy a generalized version of the following symmetry condition:<sup>24</sup>

$$i \circ j(\ulcorner P(\overleftarrow{a}_k, \overrightarrow{a}_l) \urcorner) = j \circ i(\ulcorner P(\overrightarrow{a}_k, \overleftarrow{a}_l) \urcorner)$$

An *interpretation* of  $L$  is a function that assigns a proposition to every sentence of  $L$ . I will now explain how to extend an interpretation  $\langle I, \iota, \circ \rangle$  of the predicates and individual constants of  $L$  to an interpretation of the entire language.

For  $i \in I$ , we let  $\llbracket \dots \rrbracket_i$  be defined as follows:

- if  $A$  is an atomic sentence, then  $\llbracket A \rrbracket_i = i(A)$ ;
- $\llbracket \ulcorner \neg A \urcorner \rrbracket_i$  is the algebraic complement of  $\llbracket A \rrbracket_i$ ;

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<sup>23</sup>The generalization is as follows:

$$i \circ j_1 \circ \dots \circ j_n \circ i(\ulcorner \overleftarrow{a}_k = \overrightarrow{a}_k \urcorner) = i(\ulcorner a_k = a_k \urcorner)$$

Here and throughout,  $i \circ j \circ h = (i \circ j) \circ h$ , and  $\overleftarrow{\phantom{x}}$  and  $\overrightarrow{\phantom{x}}$  are the  $n$ th iterations of  $\leftarrow$  and  $\rightarrow$ , respectively.

<sup>24</sup>The generalization is as follows, for  $r < 0$ :

$$j_1 \circ j_2 \circ \dots \circ j_k \circ i \circ j_{k+1} \dots \circ j_n(\ulcorner P(\overleftarrow{a}_{s_1}, \dots, \overleftarrow{a}_{s_k}, \overrightarrow{a}_m, \overrightarrow{a}_{s_{k+1}}, \dots, \overrightarrow{a}_{s_n}) \urcorner) =$$

$$j_1 \circ j_2 \circ \dots \circ j_{k+r} \circ i \circ j_{k+r+1} \dots \circ j_n(\ulcorner P(\overleftarrow{a}_{s_1}, \dots, \overleftarrow{a}_{s_k}, \overrightarrow{a}_m, \overrightarrow{a}_{s_{k+1}}, \dots, \overrightarrow{a}, \overrightarrow{a}, \dots, \overrightarrow{a}_{s_n}) \urcorner)$$

(Here finitely many constants can take the place of  $a_m$  and each of the  $a_{s_i}$ .)

- $\llbracket \ulcorner A \vee B \urcorner \rrbracket_i$  is the least upper bound of  $\llbracket A \rrbracket_i$  and  $\llbracket B \rrbracket_i$ ;
- $\llbracket \ulcorner \exists x \phi(a_{k_0}, \dots, a_{k_n}, x) \urcorner \rrbracket_i$  is the least upper bound of the set of propositions  $\llbracket \ulcorner \overrightarrow{a_m} = \overrightarrow{a_m} \urcorner \rrbracket_{i \circ j} \cap \llbracket \ulcorner \phi(\overleftarrow{a_{k_0}}, \dots, \overleftarrow{a_{k_n}}, \overrightarrow{a_m}) \urcorner \rrbracket_{i \circ j}$  for  $m \in \mathbb{N}$  and  $j \in I$ .

The intended interpretation of  $L$  is  $\llbracket \dots \rrbracket_\iota$ , which will sometimes be referred to as “ $\llbracket \dots \rrbracket$ ” (with no subscript).

## Proofs

Let  $\langle I, \iota, \circ \rangle$  be an interpretation of the predicates and individual constants of  $L$ . For  $i, j \in I$  we shall say that  $j$  is *reachable* from  $i$  if, for some  $n \in \mathbb{N}$ , there exist  $j_1, \dots, j_n \in I$  such that  $j = i \circ j_1 \circ \dots \circ j_n$ .

Let  $I^i$  be the set of  $j \in I$  such that  $j$  is reachable from  $i$ , and let  $L^+$  be the result of enriching  $L$  with a new constant  $c_n^j$  for each  $j \in I^i$  and  $n \in \mathbb{N}$ . We will extend the interpretation  $\llbracket \dots \rrbracket_i$  of  $L$  to an interpretation  $\llbracket \dots \rrbracket_i^+$  of  $L^+$ , as follows:

- Let  $A$  be an atomic sentence with individual constants  $a_{l_1}, \dots, a_{l_m}, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}$ .<sup>25</sup> Then:

$$\llbracket A \rrbracket_i^+ = \llbracket A^* \rrbracket_{i \circ j_1 \circ \dots \circ j_n}$$

where  $A^*$  is the result of substituting  $\overleftarrow{a_{l_1}}, \dots, \overleftarrow{a_{l_m}}, \overleftarrow{a_{k_1}}, \overrightarrow{a_{k_2}}, \dots, \overleftarrow{a_{k_{n-1}}}, \overrightarrow{a_{k_n}}$  for  $a_{l_1}, \dots, a_{l_m}, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}$  in  $A$ , respectively.

- $\llbracket \ulcorner \neg A \urcorner \rrbracket_i^+$  is the algebraic complement of  $\llbracket A \rrbracket_i^+$ ;
- $\llbracket \ulcorner A \vee B \urcorner \rrbracket_i^+$  is the least upper bound of  $\llbracket A \rrbracket_i^+$  and  $\llbracket B \rrbracket_i^+$ ;
- $\llbracket \ulcorner \exists x \phi(x) \urcorner \rrbracket_i^+$  is the least upper bound of the set of propositions  $\llbracket \ulcorner c_t^r = c_t^r \urcorner \rrbracket_i^+ \cap \llbracket \ulcorner \phi(c_t^r) \urcorner \rrbracket_i^+$ , for  $t \in \mathbb{N}$  and  $r \in I^i$ .

<sup>25</sup>I assume first occurrence of  $c_{k_q}^{j_q}$  in  $A$  occurs before the first occurrence of  $c_{k_r}^{j_r}$  in  $A$ , for  $q < r$ .

**Proposition 1** For any sentence  $\ulcorner \phi(a_{l_1}, \dots, a_{l_m}, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}) \urcorner$  of  $L^+$  and any  $i \in I$ ,

$$\llbracket \ulcorner \phi(a_{l_1}, \dots, a_{l_m}, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}) \urcorner \rrbracket_i^+ = \llbracket \ulcorner \phi(\overset{\leftarrow n}{a_{l_1}}, \dots, \overset{\leftarrow n}{a_{l_m}}, \overset{\leftarrow n-1}{a_{k_1}}, \overset{\leftarrow n-2}{a_{k_2}}, \dots, \overset{\leftarrow 1}{a_{k_{n-1}}}, \overset{\rightarrow}{a_{k_n}}) \urcorner \rrbracket_{i \circ j_1 \circ \dots \circ j_n}$$

**Proof** We proceed by induction on the complexity of sentences. The only non-trivial case is

$$\ulcorner \phi(a_{l_1}, \dots, a_{l_m}, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}) \urcorner = \ulcorner \exists x \psi(a_{l_1}, \dots, a_{l_m}, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}, x) \urcorner$$

So we need to verify that the least upper bound of

$$\llbracket \ulcorner c_t^r = c_t^r \urcorner \rrbracket_i^+ \cap \llbracket \ulcorner \psi(a_{l_1}, \dots, a_{l_m}, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}, c_t^r) \urcorner \rrbracket_i^+$$

(for  $t \in \mathbb{N}$  and  $r \in I^i$ ) equals the least upper bound of

$$\llbracket \ulcorner \overrightarrow{a_m} = \overrightarrow{a_m} \urcorner \rrbracket_{i \circ j_1 \circ \dots \circ j_n \circ j} \cap \llbracket \ulcorner \psi(\overset{\leftarrow n+1}{a_{l_1}}, \dots, \overset{\leftarrow n+1}{a_{l_m}}, \overset{\leftarrow n}{a_{k_1}}, \overset{\leftarrow n-1}{a_{k_2}}, \dots, \overset{\leftarrow 2}{a_{k_{n-1}}}, \overset{\leftarrow 1}{a_{k_n}}, \overrightarrow{a_m}) \urcorner \rrbracket_{i \circ j_1 \circ \dots \circ j_n \circ j}$$

(for  $m \in \mathbb{N}$  and  $j \in I$ ). The result follows from the following two observations:

- Fix  $r \in I^i$  and  $t \in \mathbb{N}$ . Then it follows from the definition of  $\llbracket \dots \rrbracket_i^+$  that  $\llbracket \ulcorner c_t^r = c_t^r \urcorner \rrbracket_i^+$  equals  $\llbracket \ulcorner \overrightarrow{a_t} = \overrightarrow{a_t} \urcorner \rrbracket_{i \circ j_1 \circ \dots \circ j_n \circ j}$ , and it follows from our inductive hypothesis that  $\llbracket \ulcorner \psi(a_{l_1}, \dots, a_{l_m}, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}, c_t^r) \urcorner \rrbracket_i^+$  equals

$$\llbracket \ulcorner \psi(\overset{\leftarrow n+1}{a_{l_1}}, \dots, \overset{\leftarrow n+1}{a_{l_m}}, \overset{\leftarrow n}{a_{k_1}}, \overset{\leftarrow n-1}{a_{k_2}}, \dots, \overset{\leftarrow 2}{a_{k_{n-1}}}, \overset{\leftarrow 1}{a_{k_n}}, \overrightarrow{a_t}) \urcorner \rrbracket_{i \circ j_1 \circ \dots \circ j_n \circ r}$$

- Fix  $m \in \mathbb{N}$  and  $j \in I$ . Then it follows from the definition of  $\llbracket \dots \rrbracket_i^+$  that  $\llbracket \ulcorner \overrightarrow{a_m} = \overrightarrow{a_m} \urcorner \rrbracket_{i \circ j_1 \circ \dots \circ j_n \circ j}$

equals,  $\llbracket \ulcorner c_m^j = c_m^j \urcorner \rrbracket_i^+$ , and it follows from our inductive hypothesis that

$$\llbracket \ulcorner \psi(\overset{\leftarrow n+1}{a_{l_1}}, \dots, \overset{\leftarrow n+1}{a_{l_m}}, \overset{\leftarrow n}{\underset{\rightarrow}{a_{k_1}}}, \overset{\leftarrow n-1}{\underset{\rightarrow}{a_{k_2}}}, \dots, \overset{\leftarrow 2}{\underset{\rightarrow}{a_{k_{n-1}}}}, \overset{\leftarrow 1}{\underset{\rightarrow}{a_{k_n}}}, a_m) \urcorner \rrbracket_{i \circ j_1 \circ \dots \circ j_n \circ j}$$

equals  $\llbracket \ulcorner \psi(a_{l_1}, \dots, a_{l_m}, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}, c_m^j) \urcorner \rrbracket_i^+$ . ■

**Corollary 1** For any sentence  $A$  of  $L$ ,  $\llbracket A \rrbracket = \llbracket A \rrbracket^+$ .

**Proof** Our corollary is the special case of Proposition 1 in which  $i = \iota$  and  $n = 0$ . ■

It is a consequence of the Stone Representation Theorem that no generality is lost if we assume that the propositions in our algebra are sets of “worlds”  $w \in W$ , and that the meet, join and not operations of our algebra are set-theoretic intersection, union and complementation, respectively. When our algebra of propositions is thought of in this way, we can define the notion of truth-at-a-world: a sentence  $\phi$  of  $L^+$  is true at world  $w$  just in case  $w \in \llbracket \phi \rrbracket^+$ .

**Proposition 2** For each world  $w \in W$ , the sentences true at  $w$  form a maximally consistent set with witnesses (i.e. a maximally consistent set in which every existential sentence true at  $w$  has an instance which is true at  $w$ ).

**Proof** All we need to do is check that  $\ulcorner \exists x \phi(x) \urcorner$  is true at  $w$  if and only if  $\ulcorner d = d \urcorner$  and  $\ulcorner \phi(d) \urcorner$  are both true at  $w$ , for some individual constant  $d$  of  $L^+$ .

The left to right direction follows immediately from the definition of  $\llbracket \dots \rrbracket_\iota^+$ . For the right to left direction, there are two cases:  $d = a_k$  ( $k \in \mathbb{N}$ ) and  $d = c_t^r$  ( $t \in \mathbb{N}$ ,  $r \in I^\iota$ ). If  $d = c_t^r$ , the result follows immediately from the definition of  $\llbracket \dots \rrbracket_\iota^+$ . So it is enough for present purposes to assume that  $d = a_k$  and show that  $\llbracket \ulcorner \phi(a_k) \urcorner \rrbracket_\iota^+ = \llbracket \ulcorner \phi(c_k^t) \urcorner \rrbracket_\iota^+$ . The proof is by induction on the complexity of sentences, but the only interesting case is when  $\ulcorner \phi(a_k) \urcorner$  is an atomic sentence. With no real loss of generality, we let  $\ulcorner \phi(a_k) \urcorner = \ulcorner P(a_{l_1}, \dots, a_{l_m}, a_k, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}) \urcorner$ .

We start by introducing some notational abbreviations:

$$\begin{aligned}
P^0(x) &= P(a_{l_1}, \dots, a_{l_m}, x, c_{k_1}^{j_1}, \dots, c_{k_n}^{j_n}) \\
P^n(x) &= P(\overset{\leftarrow}{a}_{l_1}^n, \dots, \overset{\leftarrow}{a}_{l_m}^n, x, \overset{\leftarrow}{a}_{k_1}^{n-1}, \overset{\leftarrow}{a}_{k_2}^{n-2}, \dots, \overset{\leftarrow}{a}_{k_{n-1}}^1, \overset{\leftarrow}{a}_{k_n}^0) \\
P^{n+1}(x) &= P(\overset{\leftarrow}{a}_{l_1}^{n+1}, \dots, \overset{\leftarrow}{a}_{l_m}^{n+1}, x, \overset{\leftarrow}{a}_{k_1}^n, \overset{\leftarrow}{a}_{k_2}^{n-1}, \dots, \overset{\leftarrow}{a}_{k_n}^2, \overset{\leftarrow}{a}_{k_n}^1)
\end{aligned}$$

Accordingly, we wish to show that  $\llbracket \ulcorner P^0(a_k) \urcorner \rrbracket_l^+ = \llbracket \ulcorner P^0(c_k^l) \urcorner \rrbracket_l^+$ . We proceed in two steps.

**Step 1:** verify  $\llbracket \ulcorner P^0(a_k) \urcorner \rrbracket_l^+ = \llbracket \ulcorner P^{n+1}(\overset{\rightarrow}{a}_k) \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l}$ . **Step 2:** verify  $\llbracket \ulcorner P^{n+1}(\overset{\rightarrow}{a}_k) \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l} = \llbracket \ulcorner P^0(c_k^l) \urcorner \rrbracket_l^+$ .

*Step 1.* The Intersubstitutivity condition on atomic assignments guarantees that:

$$\llbracket \ulcorner \overset{\leftarrow}{a}_k = \overset{\rightarrow}{a}_k \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l} \subseteq \llbracket \ulcorner P^{n+1}(\overset{\leftarrow}{a}_k) \leftrightarrow P^{n+1}(\overset{\rightarrow}{a}_k) \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l}$$

By the generalized version of the just-relabelling constraint on “ $\circ$ ”, we have  $\llbracket a_k = a_k \rrbracket_l = \llbracket \ulcorner \overset{\leftarrow}{a}_k = \overset{\rightarrow}{a}_k \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l}$ . So we can simplify the above to:

$$\llbracket a_k = a_k \rrbracket_l \subseteq \llbracket \ulcorner P^{n+1}(\overset{\leftarrow}{a}_k) \leftrightarrow P^{n+1}(\overset{\rightarrow}{a}_k) \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l} \quad (1)$$

The Negative Free Logic condition on atomic assignments gives us

$\llbracket \ulcorner P^{n+1}(\overset{\leftarrow}{a}_k) \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l} \subseteq \llbracket \ulcorner \overset{\leftarrow}{a}_k = \overset{\leftarrow}{a}_k \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l}$ . Since it follows from our characterization of  $\circ$  that  $\llbracket \ulcorner \overset{\leftarrow}{a}_k = \overset{\leftarrow}{a}_k \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l} = \llbracket a_k = a_k \rrbracket_l$ , this means that we can use equation (1) to show:

$$\llbracket \ulcorner P^{n+1}(\overset{\leftarrow}{a}_k) \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l} \subseteq \llbracket \ulcorner P^{n+1}(\overset{\rightarrow}{a}_k) \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l} \quad (2)$$

The Negative Free Logic condition also gives us  $\llbracket \ulcorner P^{n+1}(\overset{\rightarrow}{a}_k) \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l} \subseteq \llbracket \ulcorner \overset{\rightarrow}{a}_k = \overset{\rightarrow}{a}_k \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l}$ . Since it follows from our characterization of  $\circ$  that  $\llbracket \ulcorner \overset{\rightarrow}{a}_k = \overset{\rightarrow}{a}_k \urcorner \rrbracket_{l \circ j_1 \circ \dots \circ j_n \circ l} = \llbracket a_k = a_k \rrbracket_l$ , this

means that we can use equation (1) to show:

$$\llbracket \ulcorner P^{n+1}(\overrightarrow{a_k}) \urcorner \rrbracket_{\iota \circ j_1 \circ \dots \circ j_n \circ \iota} \subseteq \llbracket \ulcorner P^{n+1}(\overleftarrow{a_k}^{n+1}) \urcorner \rrbracket_{\iota \circ j_1 \circ \dots \circ j_n \circ \iota} \quad (3)$$

Combining (2) and (3) gives us

$$\llbracket \ulcorner P^{n+1}(\overleftarrow{a_k}^{n+1}) \urcorner \rrbracket_{\iota \circ j_1 \circ \dots \circ j_n \circ \iota} = \llbracket \ulcorner P^{n+1}(\overrightarrow{a_k}) \urcorner \rrbracket_{\iota \circ j_1 \circ \dots \circ j_n \circ \iota} \quad (4)$$

This is enough to give us what we want, since the definition of  $\llbracket \dots \rrbracket^+$  guarantees that

$\llbracket \ulcorner P^0(a_k) \urcorner \rrbracket^+ = \llbracket \ulcorner P^n(\overleftarrow{a_k}^n) \urcorner \rrbracket_{\iota \circ j_1 \circ \dots \circ j_n}$  and the constraints on  $\circ$  guarantee that

$$\llbracket \ulcorner P^n(\overleftarrow{a_k}^n) \urcorner \rrbracket_{\iota \circ j_1 \circ \dots \circ j_n} = \llbracket \ulcorner P^{n+1}(\overleftarrow{a_k}^{n+1}) \urcorner \rrbracket_{\iota \circ j_1 \circ \dots \circ j_n \circ \iota}.$$

*Step 2.* The definition of  $\llbracket \dots \rrbracket_i^+$  entails:

$$\llbracket \ulcorner P^0(c_k^i) \urcorner \rrbracket^+ = \llbracket \ulcorner P(\overleftarrow{a_{l_1}}^{n+1}, \dots, \overleftarrow{a_{l_m}}^{n+1}, \overleftrightarrow{a_k}^n, \overleftrightarrow{a_{k_1}}^{n-1}, \dots, \overleftrightarrow{a_{k_{n-1}}}^{n-1}, \overrightarrow{a_{k_n}}) \urcorner \rrbracket_{\iota \circ \iota \circ j_1 \circ \dots \circ j_n}$$

But the generalized symmetry condition on  $\circ$  entails that the right-hand term of this equation denotes the same proposition as

$$\llbracket \ulcorner P(\overleftarrow{a_{l_1}}^{n+1}, \dots, \overleftarrow{a_{l_m}}^{n+1}, \overrightarrow{a_k}^n, \overleftrightarrow{a_{k_1}}^{n-1}, \overleftrightarrow{a_{k_2}}^{n-1}, \dots, \overleftrightarrow{a_{k_n}}^{n-1}) \urcorner \rrbracket_{\iota \circ j_1 \circ \dots \circ j_n \circ \iota}$$

which is just  $\llbracket \ulcorner P^{n+1}(\overrightarrow{a_k}) \urcorner \rrbracket_{\iota \circ j_1 \circ \dots \circ j_n \circ \iota}$ . ■

**Proposition 3** For  $w \in W$ , consider the set  $C_w$  of constants  $d_k$  of  $L^+$  such that  $\ulcorner d_k = d_k \urcorner$  is true at  $w$ . Then “=” is an equivalence relation over  $C_w$ , in the following sense. For any  $d_k, d_l, d_m \in C_w$ :

- (a)  $\ulcorner d_k = d_k \urcorner$  is true at  $w$ ;
- (b) if  $\ulcorner d_k = d_l \urcorner$  is true at  $w$ , then  $\ulcorner d_l = d_k \urcorner$  is true at  $w$ ,



(c) if  $\ulcorner d_k = d_l \urcorner$  and  $\ulcorner d_l = d_m \urcorner$  are both true at  $w$ ; then  $\ulcorner d_l = d_m \urcorner$  is true at  $w$ .

**Proof** Condition (a) is trivial.

To verify condition (b), we assume that  $d_k = c_k^i$  and  $d_l = c_l^j$ , and show  $\llbracket \ulcorner c_k^i = c_l^j \urcorner \rrbracket^+ = \llbracket \ulcorner c_l^j = c_k^i \urcorner \rrbracket^+$ . (The case where one or both of the constants  $d_i$  and  $d_k$  is of the form  $a_m$  is analogous.) By the definition of  $\llbracket \dots \rrbracket_i^+$ ,  $\llbracket \ulcorner c_k^i = c_l^j \urcorner \rrbracket^+ = \llbracket \ulcorner \overrightarrow{a_k} = \overrightarrow{a_l} \urcorner \rrbracket_{\iota \circ i \circ j}$  and  $\llbracket \ulcorner c_l^j = c_k^i \urcorner \rrbracket^+ = \llbracket \ulcorner \overrightarrow{a_l} = \overrightarrow{a_k} \urcorner \rrbracket_{\iota \circ j \circ i}$ . But by the symmetry condition on  $\circ$  and the fact that atomic assignments satisfy the logical properties of “=”,  $\llbracket \ulcorner \overrightarrow{a_k} = \overrightarrow{a_l} \urcorner \rrbracket_{\iota \circ i \circ j} = \llbracket \ulcorner \overrightarrow{a_l} = \overrightarrow{a_k} \urcorner \rrbracket_{\iota \circ j \circ i}$ .

To verify condition (c), we shall assume that  $d_k = c_k^i$ ,  $d_l = c_l^j$  and  $d_m = c_m^h$ . (The case where some or all of the constants  $d_i$ ,  $d_k$  and  $d_m$  is of the form  $a_m$  is analogous.) We proceed by supposing that  $w \in \llbracket \ulcorner c_k^i = c_l^j \urcorner \rrbracket^+$  and  $w \in \llbracket \ulcorner c_l^j = c_m^h \urcorner \rrbracket^+$ , and proving that  $w \in \llbracket \ulcorner c_k^i = c_m^h \urcorner \rrbracket^+$ .

From the fact that atomic assignments satisfy the logical properties of “=”, we know:

$$\llbracket \ulcorner \overrightarrow{a_k} = \overrightarrow{a_l} \wedge \overrightarrow{a_m} = \overrightarrow{a_m} \urcorner \rrbracket_{\iota \circ i \circ j \circ h} \cap \llbracket \ulcorner \overrightarrow{a_l} = \overrightarrow{a_m} \wedge \overrightarrow{a_k} = \overrightarrow{a_k} \urcorner \rrbracket_{\iota \circ i \circ j \circ h} \subseteq \llbracket \ulcorner \overrightarrow{a_k} = \overrightarrow{a_m} \wedge \overrightarrow{a_l} = \overrightarrow{a_l} \urcorner \rrbracket_{\iota \circ i \circ j \circ h}$$

But by Proposition 1 and the symmetry condition on  $\circ$  we have:

- $\llbracket \ulcorner c_k^i = c_l^j \wedge c_m^h = c_m^h \urcorner \rrbracket^+ = \llbracket \ulcorner \overrightarrow{a_k} = \overrightarrow{a_l} \wedge \overrightarrow{a_m} = \overrightarrow{a_m} \urcorner \rrbracket_{\iota \circ i \circ j \circ h}$
- $\llbracket \ulcorner c_l^j = c_m^h \wedge c_k^i = c_k^i \urcorner \rrbracket^+ = \llbracket \ulcorner \overrightarrow{a_l} = \overrightarrow{a_m} \wedge \overrightarrow{a_k} = \overrightarrow{a_k} \urcorner \rrbracket_{\iota \circ j \circ h \circ i} = \llbracket \ulcorner \overrightarrow{a_l} = \overrightarrow{a_m} \wedge \overrightarrow{a_k} = \overrightarrow{a_k} \urcorner \rrbracket_{\iota \circ i \circ j \circ h}$
- $\llbracket \ulcorner c_k^i = c_m^h \wedge c_l^j = c_l^j \urcorner \rrbracket^+ = \llbracket \ulcorner \overrightarrow{a_k} = \overrightarrow{a_m} \wedge \overrightarrow{a_l} = \overrightarrow{a_l} \urcorner \rrbracket_{\iota \circ i \circ h \circ j} = \llbracket \ulcorner \overrightarrow{a_k} = \overrightarrow{a_m} \wedge \overrightarrow{a_l} = \overrightarrow{a_l} \urcorner \rrbracket_{\iota \circ i \circ j \circ h}$

So we know:

$$\llbracket \ulcorner c_k^i = c_l^j \wedge c_m^h = c_m^h \urcorner \rrbracket^+ \cap \llbracket \ulcorner c_l^j = c_m^h \wedge c_k^i = c_k^i \urcorner \rrbracket^+ \subseteq \llbracket \ulcorner c_k^i = c_m^h \wedge c_l^j = c_l^j \urcorner \rrbracket^+$$

Since we are supposing  $w \in \llbracket \ulcorner c_k^i = c_l^j \urcorner \rrbracket^+$  and  $w \in \llbracket \ulcorner c_l^j = c_m^h \urcorner \rrbracket^+$ , the definition of  $\llbracket \dots \rrbracket_i^+$ , together with the fact that atomic assignments satisfy the logical properties of “=”, gives us:

- $w \in \llbracket \ulcorner c_k^i = c_l^j \urcorner \rrbracket^+ \leftrightarrow w \in \llbracket \ulcorner c_k^i = c_l^j \wedge c_m^h = c_m^h \urcorner \rrbracket^+$
- $w \in \llbracket \ulcorner c_l^j = c_m^h \urcorner \rrbracket^+ \leftrightarrow w \in \llbracket \ulcorner c_l^j = c_m^h \wedge c_k^i = c_k^i \urcorner \rrbracket^+$
- $w \in \llbracket \ulcorner c_k^i = c_m^h \urcorner \rrbracket^+ \leftrightarrow w \in \llbracket \ulcorner c_k^i = c_m^h \wedge c_l^j = c_l^j \urcorner \rrbracket^+$

So we have

$$w \in \llbracket \ulcorner c_k^i = c_l^j \urcorner \rrbracket^+ \cap \llbracket \ulcorner c_l^j = c_m^h \urcorner \rrbracket^+ \rightarrow w \in \llbracket \ulcorner c_k^i = c_m^h \urcorner \rrbracket^+$$

Our supposition therefore guarantees that  $w \in \llbracket \ulcorner c_k^i = c_m^h \urcorner \rrbracket^+$ . ■

A *Kripke Structure* for a first-order language is a tuple  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}, \mathfrak{p} \rangle$ , where  $W$  is a set of worlds,  $a \in W$  is the “actualized” world,  $D$  is a domain of objects,  $\mathfrak{d}$  is a function that assigns a subset  $D_w$  of  $D$  to each  $w \in W$ ,  $\mathfrak{c}$  is a (possibly partial) function that assigns objects in  $D$  to individual constants,  $\mathfrak{p}$  is a function that assigns to each pair  $\langle w, P \rangle$  ( $w \in W$ ,  $P$  an  $n$ -place predicate of the language) a set of  $n$ -tuples of  $D_w$ , where  $\mathfrak{p}(\langle w, “=” \rangle) = \{ \langle x, x \rangle : x \in D_w \}$ . Truth at a world is defined compositionally, in the usual way. For a Kripke structure to *verify* an assignment of propositions to sentences is for the proposition assigned to each sentence to consist of the set of worlds at which that sentence is true.

**Proposition 4** *There is a Kripke structure that verifies the intended interpretation  $\llbracket \dots \rrbracket^+$  of  $L^+$ .*

**Proof** Let  $w \in W$ . We know from Proposition 3 that “=” is an equivalence relation over  $C_w$ . For each  $w \in W$ , we let the domain  $D_w$  of  $w$  be the set of equivalence classes of constants in  $C_w$ , and we let  $D$  be the union of the  $D_w$ . Since, by Proposition 2, the sentences of  $L^+$  that are true at  $w$  form a maximally consistent set with witnesses, we can use the construction familiar from standard presentations of the completeness theorem for first-order languages to construct a model with domain  $D_w$  which verifies every sentence of  $L^+$  which is true in  $w$ . ■

**Corollary 2 (Representation Theorem)** *There is a Kripke structure that verifies the intended interpretation  $\llbracket \dots \rrbracket$  of  $L$ .*

**Proof** The result is an immediate consequence of Proposition 4 and Corollary 1. ■

The converse of Corollary 2 is also true:

**Proposition 5 (Converse Representation Theorem)** *For any Kripke Structure  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}, \mathfrak{p} \rangle$  for a first-order language  $L$ , there is a set of atomic assignments  $I$  for  $L$ , an assignment  $\iota \in I$  and a fusion operator  $\circ$  on the elements of  $I$  such that  $\llbracket \dots \rrbracket_{\iota}$  is verified by  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}, \mathfrak{p} \rangle$ .*

**Proof** If  $\mathfrak{c}'$  is a (possibly partial) function which assigns objects in  $D$  to individual constants of  $L$ , we shall say that atomic assignment  $i$  is *generated* from  $\mathfrak{c}'$  just in case, for each atomic sentence  $A$ ,  $i(A)$  is the set of worlds  $w$  such that  $A$  is true at  $w$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}', \mathfrak{p} \rangle$ . We let  $I$  consist of the atomic assignments  $i$  which are generated by some  $\mathfrak{c}'$ , and let  $\iota$  be the special case in which  $\mathfrak{c}' = \mathfrak{c}$ .

For each  $i \in I$ , choose a function  $\mathfrak{c}^i$  such that  $i$  is generated from  $\mathfrak{c}^i$ . The fusion operator “ $\circ$ ” is then defined in the natural way:  $i \circ j(A)$  is the set of worlds  $w$  such that  $A$  is true at  $w$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^*, \mathfrak{p} \rangle$ , where  $\mathfrak{c}^*(\overleftarrow{a}_k) = \mathfrak{c}^i(a_k)$  and  $\mathfrak{c}^*(\overrightarrow{a}_k) = \mathfrak{c}^j(a_k)$ . It follows immediately that  $i \circ j \in I$  whenever  $i, j \in I$ .

We now show that, for each  $i \in I$ , the function  $\llbracket \dots \rrbracket_i$  is verified by  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^i, \mathfrak{p} \rangle$ . The proof is by induction on the complexity of sentences. The only interesting case is when  $\phi = \ulcorner \exists x \psi(a_{k_1}, \dots, a_{k_n}, x) \urcorner$ . So we need to verify that the least upper bound of propositions  $\llbracket \ulcorner \psi(\overleftarrow{a}_{k_1}, \dots, \overleftarrow{a}_{k_n}, \overrightarrow{c}) \urcorner \rrbracket_{i \circ j}$  ( $j \in I$ ) is the set of worlds  $w$  such that  $\ulcorner \exists x \psi(a_{j_1}, \dots, a_{j_n}, x) \urcorner$  is true at  $w$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^i, \mathfrak{p} \rangle$ .

Suppose first that the least upper bound of propositions  $\llbracket \ulcorner \psi(\overleftarrow{a}_{k_1}, \dots, \overleftarrow{a}_{k_n}, \overrightarrow{c}) \urcorner \rrbracket_{i \circ j}$  includes  $w$ . Then, by inductive hypothesis,  $\ulcorner \psi(\overleftarrow{a}_{k_1}, \dots, \overleftarrow{a}_{k_n}, \overrightarrow{c}) \urcorner$  is true at  $w$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^{i \circ j}, \mathfrak{p} \rangle$ . By the definition of  $\circ$ , this means that  $\ulcorner \psi(a_{k_1}, \dots, a_{k_n}, c) \urcorner$  must be satisfied at  $w$  by some object in  $D_w$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^i, \mathfrak{p} \rangle$ . So  $\ulcorner \exists x \psi(a_{k_1}, \dots, a_{k_n}, x) \urcorner$  is true at  $w$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^i, \mathfrak{p} \rangle$ .

Now suppose that  $\lceil \exists x \psi(a_{j_1}, \dots, a_{j_n}, x) \rceil$  is true at  $w$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^i, \mathfrak{p} \rangle$ . This means that  $\lceil \psi(a_{k_1}, \dots, a_{k_n}, c) \rceil$  is satisfied at  $w$  by some  $o \in D_w$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^i, \mathfrak{p} \rangle$ . Let  $\mathfrak{c}^j$  be such that  $\mathfrak{c}^j(c) = o$ , and let  $\mathfrak{c}^*$  be such that  $\mathfrak{c}^*(\overleftarrow{a}_k) = \mathfrak{c}^i(a_k)$  and  $\mathfrak{c}^*(\overrightarrow{a}_k) = \mathfrak{c}^j(a_k)$ . Since  $\lceil \psi(a_{k_1}, \dots, a_{k_n}, c) \rceil$  is satisfied at  $w$  by  $o$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^i, \mathfrak{p} \rangle$ ,  $\lceil \psi(\overleftarrow{a}_{k_1}, \dots, \overleftarrow{a}_{k_n}, \overrightarrow{c}) \rceil$  must be true at  $w$  according to  $\langle W, a, D, \mathfrak{d}, \mathfrak{c}^*, \mathfrak{p} \rangle$ . But by the definition of  $I$ , there must be a  $j \in I$  such that  $j$  is generated from  $\mathfrak{c}^j$ . And by the definition of  $\circ$ ,  $i \circ j$  is generated from  $\mathfrak{c}^*$ . So the least upper bound of of propositions  $\llbracket \lceil \psi(\overleftarrow{a}_{k_1}, \dots, \overleftarrow{a}_{k_n}, \overrightarrow{c}) \rceil \rrbracket_{i \circ j}$  ( $j \in I$ ) must include  $w$ . ■

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