# Semantic Pollution and Syntactic Purity

Stephen Read

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**Abstract**

Logical inferentialism claims that the meaning of the logical constants should be given, not model-theoretically, but by the rules of inference of a suitable calculus. It has been claimed that certain proof-theoretical systems, most particularly, labelled deductive systems for modal logic, are unsuitable, on the grounds that they are semantically polluted and suffer from an untoward intrusion of semantics into syntax. The charge is shown to be mistaken. It is argued on inferentialist grounds that labelled deductive systems are as syntactically pure as any formal system in which the rules define the meanings of the logical constants.

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## 1 Semantic Pollution

Poggiolesi and Restall (2012) discuss how to frame a proof theory for modal logic which accords with Gentzen's idea (1969) that the operational rules should be separable (that is, each rule should mention only one connective, so that there are separate rules for each logical constant), and should divide into those rules which define the meaning of the connective by introducing a formula exhibiting the connective as its main connective, and those which eliminate it. The standard rules, due to Curry (1950), Fitch (1952) and Prawitz (1965), do not do this, for two reasons: first, they do not easily generalize (if at all) from rules for the systems **S4** and **S5** to the full range of (even) normal modal logics; secondly, although the rules have the form of introduction and elimination rules, and indeed normalize,[1] it is clear that, e.g., the introduction-rule for '$\Diamond$' (possibility) does not capture its full meaning in the way inferentialism expects:

$$\frac{\alpha}{\Diamond\alpha}\ \Diamond\mathrm{I}^*$$

This says that $\Diamond\alpha$ is true whenever $\alpha$ is true, which is indeed so. However, $\Diamond\alpha$ can also be true when $\alpha$ is false. Inferentialist considerations turning on the idea that the introduction-rule(s) give the meaning of the connective in question suggest that if $\Diamond\mathrm{I}^*$ did indeed capture the (full) meaning of '$\Diamond$', the elimination-rule which it justifies would read:

$$\frac{\Diamond\alpha \quad \begin{matrix}(\alpha)\\ \vdots \\ \beta\end{matrix}}{\beta}\ \Diamond\mathrm{E}^*$$

[1] See, e.g., Prawitz (1965, ch. VI).

However, $\Diamond\mathrm{I}^*$ and $\Diamond\mathrm{E}^*$ suffice to show that $\alpha$ and $\Diamond\alpha$ are inter-derivable, so that '$\Diamond$' collapses into a truth-operator. To prevent this, Curry et al. restrict the elimination-rule by the form of the formulae which can replace '$\beta$' and which can be parametric assumptions in the minor premise. But nothing in the I-rule justifies these restrictions, which are added *ad hoc* solely to prevent the derivation of inferences which do not accord with the intended meaning. In brief, $\Diamond\mathrm{I}^*$ and $\Diamond\mathrm{E}^*$ (so restricted) are not harmonious.[2]

Poggiolesi and Restall (2012) consider three ways of giving rules for the modal connectives which accord better with their intended meaning. All three are in fact presented in sequent calculus versions, with right-introduction rules matching natural deduction I-rules and left-introduction rules matching the natural deduction E-rules. The first is a display calculus, which they dismiss as unnecessarily complex. (Poggiolesi and Restall, 2012, p. 46) The second method they canvass and reject is the use of a labelled deductive system of the sort presented in Read (2008), and originally put forward in Simpson (1994), Basin et al. (1997) and Viganò (2000). Their particular target is the labelled sequent calculus presented in Negri (2005). They marshal three objections against it, the first of which is that it unacceptably exploits semantic notions in the proof theory:

> "The labelled method ... is a semantic method [in that] it imports in its language the whole structure of Kripke semantics in an explicit and significant way." (Poggiolesi and Restall, 2012, p.49)

Rather, they protest, a purely syntactic method should "not make any use of semantic parameters beyond the language of formulas" (*loc.cit.*). Ironically, a similar complaint of impurity was rebutted in Negri (2007, see esp. p. 109).[3] A similar complaint against labelled deductive systems was endorsed in an early draft of (Humberstone, 2011, Remark 1.21.2) circulated online (though omitted from the published version) and dubbed "semantic pollution", an epithet attributed there to Rajeev Goré in conversation.[4] My aim is to contest this misconstruction. The rules of the labelled calculus give a clear and transparent account of the meaning of the modal connectives—their semantics. This is not pollution, but is in fact purer syntactically than Poggiolesi and Restall's third and preferred calculus, that of tree-hypersequents.

## 2 A Labelled Deductive System for Modal Logic

We have seen that the Curry-Fitch-Prawitz (CFP) rule $\Diamond\mathrm{I}^*$ does not capture the full meaning of '$\Diamond$'. $\Diamond\mathrm{I}^*$ suggests that $\Diamond\alpha$ means that $\alpha$ is true, not just possibly true. But, although the truth of $\alpha$ is sufficient for the truth of $\Diamond\alpha$, it is not necessary. If we take $\Diamond\mathrm{I}^*$ as the only case suitable for introduction (i.e., assertion) of $\Diamond\alpha$, we cannot help but suggest, when the rule is taken as meaning-conferring, that the truth of $\alpha$ is not only sufficient but necessary for that of $\Diamond\alpha$. We need to weaken $\Diamond\mathrm{I}^*$ so that this suggestion, or implication, is removed.

Leibniz's seminal idea was that the meaning of 'possibly $\alpha$' can be articulated in terms of the idea of possible worlds. If we allow ourselves to speculate

[2] See Read (2008). Nonetheless, as discussed there, the Curry-Fitch-Prawitz rules between them do capture the correct meaning of '$\Diamond$', as is shown by the semantics.

[3] Moreover, many of the desiderata on a suitable proof theory set out in Poggiolesi (2011, §1.7) can already be found in Negri (2007, p. 108).

[4] See also Goré (1999, p. 359).

that this world, though perhaps the "best of all possible worlds",[5] is only one among many alternatives, we can capture the meaning of '$\Diamond\alpha$' by the formula '$\alpha$ is true in some possible world', either this world or one of its alternatives. Kripke's particular insight,[6] some three hundred years later, was that the notion of alternative in play here is relative: that one world is alternative relative to another. The properties of this binary relation (sometimes called "accessibility") determine what model structures are permissible and so, by restricting the class of models, determine which consequences are valid and so which logic is modelled.

Accordingly, $\Diamond\alpha$ is true at world(-index) $w$ if $\alpha$ is true at some world(-index) possible relative to $w$. That condition spells out the meaning of '$\Diamond$' in terms of both necessary and sufficient conditions for the truth of $\Diamond\alpha$ relative to an index $w$ (written $\Diamond\alpha_w$):

$$\frac{\alpha_v \quad w < v}{\Diamond\alpha_w}\ \Diamond\mathrm{I}$$

'$w < v$' reads: $v$ is accessible from, that is, possible relative to, $w$.[7] We refer to wffs of the form $\Diamond\alpha_w$ as "labelled formulae", and wffs of the form $w < v$ as "relational formulae".[8] Leibniz relativized truth to worlds, or indexes, $w$, and Kripke relativized possibility to those indexes truth at which determines modal truth at other worlds. $\Diamond$I thus gives the necessary and sufficient conditions for the assertion that $\Diamond\alpha$ is true at $w$, that is, it gives the meaning of '$\Diamond$' inferentially. $\Diamond\alpha$ means that $\alpha$ is (relatively) possible.

$\Diamond$I does not specify the full inferential behaviour of '$\Diamond$', however. It tells us under what conditions $\Diamond\alpha$ may be asserted, but it does not say what may be inferred from an assertion of $\Diamond\alpha$. As Gentzen (1969, p. 80) remarked, "the introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions." This alludes to the importance of proof-theoretical harmony to an inferentialist account of meaning. The E-rules should permit no more and no less than is justified by the meaning encapsulated in and conferred by the I-rules. Let us consider the general case before returning to $\Diamond$I. Suppose some connective '$*$' forming wffs $*\vec{\alpha}$ (the vector indicating that '$*$' may take multiple arguments, $\geq 0$) has $m$ I-rules (as, e.g., '$\vee$' has 2) each with $n_i$ premises, $\pi_{ij}$ ($0 \leq j \leq n_i$), $n_i \geq 0$. Each premise may be a single wff, or a derivation of a wff from one or more assumptions that are discharged by the rule (think of the former case as a derivation in which nothing is discharged):

$$\frac{\pi_{i1} \quad \dots \quad \pi_{in_i}}{*\vec{\alpha}}\ *\mathrm{I}_i$$

The collection $\{*\mathrm{I}_i : i \leq m\}$ specifies the meaning of '$*$', and the E-rules in harmony with this collection should allow one to infer no more and no less than is justified by that meaning. Gentzen's insight was that such harmony would be guaranteed (he didn't have the term, which was introduced by Dummett (1973, p. 396), but he had the concept) if each minor premise of each case of $*$E should infer a common conclusion $\gamma$ from the assumption, discharged by the rule, of one of the grounds $\pi_{ij}$ ($a \leq j \leq n_i, 1 \leq i \leq m$) of each of the I-rules. For when

[5] See, e.g., Leibniz (1985, §168).

[6] Kripke (1963).

[7] This is an heuristic reading. With no constraints on '$<$', it no more means 'relatively possible' than '$\Diamond$' means 'possible' in the weakest normal modal logic, **K**. The rules constraining '$<$' given below affect the grounds on which $\Diamond\alpha$ may be asserted, and so affect the meaning of '$\Diamond$'.

[8] Note that the non-modal rules must now be rewritten to track the labels. See, e.g., Viganò (2000, chs. 2, 4).

$*\vec{\alpha}$ (the major premise of the rule) was so inferred by $*\mathrm{I}_i$ and used in turn to infer $\gamma$, the introduction of $*\vec{\alpha}$ would be redundant, and $\gamma$ would be justified not by assertion of $*\vec{\alpha}$, but by the grounds for asserting $*\vec{\alpha}$ in the first place:[9]

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \pi_{i1}\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\\ \pi_{in_i}\end{matrix}}{*\vec{\alpha}}\,*\mathrm{I}_i \qquad \begin{matrix}[\pi_{1j_1}]\\ \vdots\\ \gamma\end{matrix} \quad \cdots \quad \begin{matrix}[\pi_{mj_m}]\\ \vdots\\ \gamma\end{matrix}}{\gamma}\,*\mathrm{E}_k$$

Provided the condition on $*\mathrm{E}_k$ is satified that, whichever E-rule is applied, at least one premise of $*\mathrm{I}_i$ matches a minor premise in which it is discharged, the detour via $*\vec{\alpha}$ may be avoided, and $\gamma$ inferred directly from $\pi_{ij}$:

$$\begin{matrix}\vdots\\ \pi_{ij_i}\\ \vdots\\ \gamma\end{matrix}$$

A straightforward calculation reveals that we need at most $\prod_{i=1}^{m} n_i$ E-rules to meet this condition.

In fact, fewer than $\prod_{i=1}^{m} n_i$ E-rules will be generated if $\pi_{ij}$ and $\pi_{ij'}$ are in some way connected. For example, in the case of $\Diamond$I: there is one I-rule with two premises, so we might expect there to be two E-rules. However, this would be to ignore the fact that '$v$' occurs in both premises, so that they are indeed connected. Moreover, '$v$' is in a certain sense arbitrary. To understand what is happening, consider the rule $\exists$I of existential introduction:

$$\dfrac{\alpha(t/x)}{(\exists x)\alpha}\,\exists\mathrm{I}$$

where $\alpha(t/x)$ results from $\alpha$ by replacing all free occurrences of '$x$' by '$t$'.[10] '$t$' is arbitrary in the sense that this rule is shorthand for the indefinite collection of rules $\{\exists\mathrm{I}_t : t \in T\}$, where $T$ is the set of terms in the language. The rule is valid for any term $t \in T$. So instead of one E-rule with just one minor premise, $\{\exists\mathrm{I}_t : t \in T\}$ generates one E-rule with infinitely many minor premises, one for each $t \in T$.[11] However, that infinitary rule can be simplified to the familiar rule $\exists$E by recognising the arbitrariness of '$t$' in $\exists$I, using a variable '$u$' in place of the various terms '$t$':

$$\dfrac{(\exists x)\alpha \qquad \begin{matrix}[\alpha(u/x)]\\ \vdots\\ \gamma\end{matrix}}{\gamma}\,\exists\mathrm{E}$$

where '$u$' does not occur (free) in $\gamma$ or in any parametric assumption, and the assumption $\alpha(u/x)$ is discharged by the rule. Provided this restriction is satisfied, the minor premise here can go proxy for any of the minor premises in the infinitary rule. '$u$' is arbitrary in the sense that the result of replacing '$u$'

[9] Note that the discharged assumptions $\pi_{ij_i}$ here are in general the assumptions of the existence of derivations. They cannot be simplified to the assumption of only the conclusions of those derivations (adding proofs of their discharged assumptions as further minor premises)—called in Read (2014) the "flattening" of the E-rule—as shown there and in Schroeder-Heister and Olkhovikov (2014).

[10] Provided '$t$' is free for '$x$' in $\alpha$, that is, '$x$' does not occur within the scope of a quantifier over '$t$'; else we need to replace '$x$' by '$t$' in a suitable variant of $\alpha$.

[11] Assuming $T$ is infinite. See Read (2000, §2.6).

throughout the subproof of the minor premise $\gamma$ by any term '$t$' is still a proof of $\gamma$.

Suppose we now want to adapt $\exists$I so that it characterizes the existential quantifier of free logic, in which the quantifier connotes existence, but not all terms $t \in T$ denote. We then need to restrict '$t$' as it occurs in $\exists$I to denoting terms:

$$\frac{\alpha(t/x) \quad \mathrm{E}!t}{(\exists x)\alpha}\ \exists\mathrm{I}^*$$

where 'E!$t$' states the non-emptiness of the term '$t$'.[12] 'E!$t$' restricts the set of terms on which it is legitimate to generalize. Once again, '$t$' can be any term whatever, so $\exists\mathrm{I}^*$ is shorthand for infinitely many rules $\{\exists\mathrm{I}^*_t : t \in T\}$. However, in composing the E-rule, $\exists\mathrm{E}^*$, we need to take the premises $\alpha(u/x)$ and E!$u$ in matching pairs so that, when we reformulate the E-rule in a finitary way, the variable '$u$' is uniformly restricted:

$$\frac{(\exists x)\alpha \qquad \begin{matrix}[\alpha(u/x), \mathrm{E}!u] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \exists\mathrm{E}^*$$

As before, '$u$' should not occur (free) in $\gamma$ or in any parametric assumption on which the minor premise depends, in order that it go proxy for the infinitely many minor premises in the infinitary rule. $\exists\mathrm{E}^*$ is in harmony with $\exists\mathrm{I}^*$, so that the scheme:

$$\frac{\dfrac{\begin{matrix}\mathcal{A} \\ \alpha(t/x)\end{matrix} \quad \begin{matrix}\mathcal{B} \\ \mathrm{E}!t\end{matrix}}{(\exists x)\alpha} \qquad \begin{matrix}[\alpha(u/x), \mathrm{E}!u] \\ \mathcal{C} \\ \gamma\end{matrix}}{\gamma} \qquad \text{reduces to} \qquad \begin{matrix}\underbrace{\begin{matrix}\mathcal{A} \\ \alpha(t/x)\end{matrix} \quad \begin{matrix}\mathcal{B} \\ \mathrm{E}!t\end{matrix}} \\ \mathcal{C}(t/u) \\ \gamma\end{matrix}$$

The restriction on '$u$' ensures that the result of replacing '$u$' by '$t$' throughout $C$ is still a proof of $\gamma$.[13]

The premises in $\Diamond$I are similarly connected, so that $w < v$ restricts the generality of the premise $\alpha_v$ in such a way that the premises must again be taken in pairs in composing $\Diamond$E; and again the minor premise of $\Diamond$E will go proxy for indefinitely many derivations of $\gamma_u$ from the pairs $\alpha_v$ and $w < v$:

$$\frac{\Diamond\alpha_w \qquad \begin{matrix}[\alpha_v, w < v] \\ \vdots \\ \gamma_u\end{matrix}}{\gamma_u}\ \Diamond\mathrm{E}$$

where '$v$' is not free in $\gamma$ nor in any parametric assumptions. That $\Diamond$I and $\Diamond$E are harmonious is shown by a suitable simplification (or local reduction):

$$\frac{\dfrac{\begin{matrix}\mathcal{A} \\ \alpha_{v'}\end{matrix} \quad \begin{matrix}\mathcal{B} \\ w < v'\end{matrix}}{\Diamond\alpha_w}\ \Diamond\mathrm{I} \qquad \begin{matrix}[\alpha_v, w < v] \\ \mathcal{C} \\ \gamma_u\end{matrix}}{\gamma_u}\ \Diamond\mathrm{E} \qquad \text{reduces to} \qquad \begin{matrix}\underbrace{\begin{matrix}\mathcal{A} \\ \alpha_{v'}\end{matrix} \quad \begin{matrix}\mathcal{B} \\ w < v'\end{matrix}} \\ \mathcal{C}(v'/v) \\ \gamma_u\end{matrix}$$

where once again the restriction on '$v$' ensures that $C(v'/v)$ is still a proof.

[12]See, e.g., Nolt (2014, §2.2).

[13]See, e.g., Prawitz (1965, pp. 37-38).

The corresponding harmonious rules for '$\Box$' are:[14]

$$\dfrac{\begin{matrix}[w<v]\\ \vdots \\ \alpha_v\end{matrix}}{\Box\alpha_w}\,\Box\mathrm{I} \qquad \text{and} \qquad \dfrac{\Box\alpha_w \quad w<v}{\alpha_v}\,\Box\mathrm{E}$$

where '$v$' in $\Box$I does not occur free in any parametric assumptions.[15] We can now prove $\Box(\alpha\to\beta)\vdash\Diamond\alpha\to\Diamond\beta$, valid in **K**, without resort to principles (such as $\Box\alpha\to\alpha$) valid only in stronger systems:[16]

$$\dfrac{\dfrac{\overline{\Diamond\alpha_w}^{\,3} \qquad \dfrac{\dfrac{\dfrac{\Box(\alpha\to\beta)_w \quad \overline{w<v}^{\,1}}{\alpha\to\beta_v}\,\Box\mathrm{E} \quad \overline{\alpha_v}^{\,2}}{\beta_v}\,{\to}\mathrm{E} \quad \overline{w<v}^{\,1}}{\Diamond\beta_w}\,\Diamond\mathrm{I}}{\Diamond\beta_w}\,\Diamond\mathrm{E}(1,2)}{\Diamond\alpha\to\Diamond\beta_w}\,{\to}\,\mathrm{I}(3)$$

The logic **K**, however, is too weak properly to capture the notion of possibility. The rules need strengthening at least to give the system **T**, if not **S4** or **S5**, for '$\Diamond$' to be interpretable as possibility. We do so by adding constraints on the relational symbol '$<$':[17]

$$\dfrac{\begin{matrix}[w<w]\\ \vdots \\ \alpha_u\end{matrix}}{\alpha_u}\,T \qquad\qquad \dfrac{w<v \quad v<u \quad \begin{matrix}[w<u]\\ \vdots \\ \alpha_t\end{matrix}}{\alpha_t}\,4$$

$$\dfrac{w<v \quad \begin{matrix}[v<w]\\ \vdots \\ \alpha_u\end{matrix}}{\alpha_u}\,B \qquad\qquad \dfrac{w<v \quad w<u \quad \begin{matrix}[v<u]\\ \vdots \\ \alpha_t\end{matrix}}{\alpha_t}\,5$$

Note that relational wffs such as $w<v$ can only occur as assumptions, being never the conclusion of a rule of inference. (To this extent, the local reductions displayed above are misleading, in suggesting they might.) So too for wffs such as E!$u$ in free logic. As we will see, $w<v$ has no meaning in itself, for there are no grounds for its assertion.

# 3 Syntactic Purity

My aim is to show that the system of modal logic in §2 is syntactically pure, that is, as syntactically pure as any other system of logic, and not polluted by the (apparent) semantic allusions in contains. Before we proceed, however, we need to clarify what exactly the charge of "semantic pollution" amounts to, especially in an inferentialist setting. By 'semantics', Poggiolesi and Restall mean "model-theoretic semantics" or "model theory". Anyone who accepts the

[14] The simplified form of $\Box$E here results from the general-elimination rule in a similar manner to the simplification of the GE-rule for '$\to$', as spelled out in Read (2014, §3).

[15] Thus '$v$' in $\Box$I is arbitrary, the premise going proxy for the indefinite collection of specific I-rules, one for each index '$v$'; and $\Box$E is an indefinite collection of rules, one for each '$v$'.

[16] For the difficulties in incorporating '$\Diamond$' in the CFP systems, see Read (2008, p. 296).

[17] See Simpson (1994, p. 73). Cf. Viganò (2000, p. 24).

application of model theory to the expressions of a logical system has to concede that model theory interprets and gives meaning to those expressions. An exception might be an intuitionist such as Dummett or Brouwer, who rejects such semantic methods as inherently realist and hence to be eschewed.[18] Some inferentialists, such as Brandom or Prawitz,[19] may endorse this attitude. They believe that model theory incorporates realist assumptions which should be resisted. Viewed from this perspective, Kripke's semantics for intuitionistic logic, for example, may help a classical logician, with his Platonist commitment to the assumptions of model theory, to gain some grasp of the constructivist point of view. But from that constructivist viewpoint, it is a false model, arguably (*pace* Brouwer[20]) validating the right logical principles, but giving them a determinate and objectual foundation which belies the open-ended nature of the operations which are represented.

From a less austere inferentialist perspective, however, model theory has a role to play. For example, Poggiolesi (2011, p. 6) endorses a model-theoretic (*aka* "semantic") proof of cut-elimination, even if it is a less illuminating one than its constructive demonstration. Accordingly, she rejects Arnon Avron's demand for such a strong syntactic purity that the calculus "should be independent of any particular semantics,"[21] since that would seem to entail that even the sequent calculus for classical logic is semantically polluted. For, as Beth (1969, p. 18) points out, classical sequent calculus is little more than a notational variant of his semantic tableaux or Smullyan's signed trees. Consider the following demonstration of one of the De Morgan laws, by sequent calculus, compared with those by semantic tableaux in the style of Beth and the corresponding signed tree (*aka* analytic tableau) which Smullyan (1968, II §1) adapted from Beth's method:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha, \neg\beta}{\Rightarrow \alpha, \neg\alpha, \neg\beta} \quad \dfrac{\beta \Rightarrow \beta, \neg\alpha}{\Rightarrow \beta, \neg\alpha, \neg\beta}}{\Rightarrow \alpha \wedge \beta, \neg\alpha, \neg\beta}}{\Rightarrow \alpha \wedge \beta, \neg\alpha \vee \neg\beta}}{\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta}$$

| Valid | | Invalid | |
|---|---|---|---|
| (1) $\neg(\alpha \wedge \beta)$ | | (2) $\neg\alpha \vee \neg\beta$<br>(3) $\alpha \wedge \beta$<br>(4) $\neg\alpha$<br>(5) $\neg\beta$ | |
| (i) | (ii) | (i) | (ii) |
| (8) $\alpha$ | (6) $\alpha$ | (9) $\beta$ | (7) $\beta$ |

$$\begin{array}{c|c}
\multicolumn{2}{c}{(1)\ \mathrm{T}\neg(\alpha \wedge \beta)} \\
\multicolumn{2}{c}{(2)\ \mathrm{F}\neg\alpha \vee \neg\beta} \\
\multicolumn{2}{c}{(3)\ \mathrm{F}\alpha \wedge \beta} \\
\multicolumn{2}{c}{(4)\ \mathrm{F}\neg\alpha} \\
\multicolumn{2}{c}{(5)\ \mathrm{F}\neg\beta} \\
(6)\ \mathrm{F}\alpha & (7)\ \mathrm{F}\beta \\
(8)\ \mathrm{T}\alpha & (9)\ \mathrm{T}\beta \\
\times & \times
\end{array}$$

As Beth (1969, p. 21) remarks, the rules for the tableaux have a semantic basis:

> "The sentence: '$X$ and $Y$' is true (and the formula: '$X \& Y$' is valid), if and only if both $X$ and $Y$ are true (or valid). However, the rules assume a completely formal character, if instead of speaking about an individual $p$ we speak about a symbol '$p$'."

One can read the semantics directly off the proof theory, contrary to Avron's proposal for syntactic purity.

[18]See, e.g., Dummett (1991, pp. 239 ff.).

[19]See, e.g., Brandom (2000, ch. 1), Prawitz (1979).

[20]See, e.g., Fraenkel et al. (1973, pp. 239-40).

[21]Avron (1996, p. 2). He continues: "One should not be able to guess, just from the form of the [syntactic] structures which are used, the intended semantics [i.e., models] of a given proof system."

Beth's talk of the rules' having "a completely formal character" is an example of what Dutilh Novaes (2011, §2.2.3) (cf. 2012, §1.1) calls the "formal as desemantification": treating symbols as having no meaning at all, in other words, ironically, as non-symbolic. She draws an example (and the term 'desemantification') from Krämer (2003): the introduction of the symbol '0' as a (formal) aid to computation before it was thought of as referring to a number, zero.[22] But if inferentialism is right, this is a mistake: meaning (semantics) does not consist solely or properly in a reference to a realm of objects, a model theory, but meaning is already given by the rules for the use of the expression. Dutilh Novaes (2012, p. 57) suggests that only a "very broad conception of semantics" would admit non-representational semantic import to count as meaningful, but in fact, it is the very use of expressions that makes them so. Whether they serve to denote something is a separate matter, arguably applicable only to a subset of expressions and itself a matter of metaphysics, not of semantics—the can of worms that follows from the "reification of entities we may not be prepared to reify" (Dutilh Novaes, 2012, p. 92). She quotes Leibniz with approval (2012, p. 200): "There is no need to let mathematical analysis depend on metaphysical controversies."[23] But the infinitesimal calculus (to which Leibniz was referring) is still meaningful whether or not infinitesimal quantities exist. Semantics should not be identified with model theory.

Poggiolesi is rightly opposed to charging classical sequent calculus with semantic pollution. Instead, she argues (2011, p. 29) for a weaker purity claim, that "a sequent calculus should not make any use of explicit semantic elements." She sees this requirement as a descendant of Aristotle's demand for "purity of method" (2011, p. 13), whereby "the theory of any one science [cannot] be demonstrated by means of another science" (*An. Post.* I 7, 75b14-15). Again, the general thesis is too strong, as was recognised both by Aristotle himself (*An. Post.* I 9 and 13: e.g., the use of arithmetic in music, or of geometry in medicine) and by Bolzano (e.g., that the real basis of the intermediate value theorem in geometry lies in analysis).[24] Poggiolesi offers two arguments for her particular claim: first, she appeals (2011, p. 31) to Ockham's razor: why complicate things for no good reason? But as we saw in §2, there is good reason, for the use of labels in the deductive system can achieve more than did the CFP systems, and more clearly, permitting harmony between the I- and E-rules and so making the meaning conferred by the I-rule transparent. Secondly, Poggiolesi (2011, p. 19) appeals to two conditions on a good calculus, one being that the rules be formal, the other an expression of inferentialism:

> (III) *The left and the right introduction rules of the sequent calculus together can be considered as a definition of the symbol they introduce since they both give the grounds for asserting a sentence containing the connective they define.*

(As we have seen, this is not a good way to express inferentialism: the CFP rules $\Diamond I^*$ and $\Diamond E^*$ together characterise '$\Diamond$' correctly, but neither gives the meaning of '$\Diamond$'. It is the I-rules that define the meaning by specifying the grounds for assertion, as Gentzen observed, and the E-rules should be justified by them.) Poggiolesi claims that her conditions (II) and (III) contain the tacit assumption that the rules of a good calculus operate on wffs, not on semantic elements. But this is circular. It assumes that the harmonious rules mentioned in her

[22] See Dutilh Novaes (2012, p. 200).

[23] Gerhardt (1859, IV, p. 91): letter from Leibniz to Varignon (2 February 1701): "On n'a point besoin de faire dependre l'analyse Mathematique des controverses metaphysiques."

[24] For a discussion of Bolzano's observations, see Paoli (1991).

conditions exclude labelled and relational formulas. We have seen, however, that the labelled modal rules in §2 are harmonious.

A further consideration to which Poggiolesi and Restall (2012, p. 49) appeal is that the admission of labels into formulae permits the construction of expressions which have no use in proof, and introduces pairs of expressions different in form but identical in meaning. This leads to an increase in expressive power (2012, p. 48), e.g., the introduction of expressions valid only on irreflexive frames—there famously being no pure formula (or even set of formulas) which does this. This is true; but it is a familiar situation, and it is unclear why such an increase in expressive power should in itself be seen as a drawback. A similar objection was made by, e.g., Rumfitt (2008, §7) and Tennant (1997, p. 320) to the multiple conclusions of classical sequent calculus and natural deduction systems on the ground that they create expressions which do not naturally occur. Even if this is true, however, they facilitate a clear and effective proof system for classical logic and other systems, e.g., linear logic.[25]

Poggiolesi (2011, p. 15) also claims that eschewing labels provides a criterion of logicality of an expression, namely, that "it can be analysed in purely structural terms". But consider Poggiolesi and Restall's preferred calculus for modal logic, employing so-called tree-hypersequents. The language permits the construction of hypersequents, that is, multisets of sequents. Tree-hypersequents add a further complexity:

- if $\Gamma$ is a sequent $M \Rightarrow N$, then $\Gamma$ is a tree-hypersequent
- if $\Gamma$ is a sequent and $G_1, \ldots, G_n$ are tree-hypersequents, then $\Gamma/G_1; \ldots; G_n$ is a tree-hypersequent.

The idea, as Poggiolesi (2011, p. 119) puts it, is to reflect "at the proof-theoretical level, the structure of the tree-frames of Kripke semantics ... without the support of explicit semantic parameters" as employed in labelled deductive systems. For example, the proof of the **K**-valid sequent $\Box(\alpha \to \beta) \vdash \Diamond\alpha \to \Diamond\beta$ in our labelled system in §2 takes the following tree-hypersequent form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \beta \Rightarrow \beta}{\alpha \to \beta, \alpha \Rightarrow \beta}\to\mathrm{L}}{\Box(\alpha \to \beta) \Rightarrow /\alpha \Rightarrow \beta}\Box\mathrm{L}}{\Box(\alpha \to \beta) \Rightarrow \Diamond\beta/\alpha \Rightarrow}\Diamond\mathrm{R}}{\Box(\alpha \to \beta), \Diamond\alpha \Rightarrow \Diamond\beta}\Diamond\mathrm{L}}{\Box(\alpha \to \beta) \Rightarrow \Diamond\alpha \to \Diamond\beta}\to\mathrm{R}$$

This might appear purer syntactically than the labelled proof. It certainly accords with the letter of Poggiolesi's condition that there be no *explicit* semantic elements. But as the structural similarity of the proof to that in §2 reveals, the semantic content is still there, indicated by the apparatus of '/' (and of ';' in more complex proofs). On the one hand, Poggiolesi and Restall (2012, pp. 52, 59) write: "the sequents are purely logical ... the tree-hypersequent proof theory gives us a framework in which the logical behaviour of these operators can be exposed and precisely treated;" and Poggiolesi (2011, p. 187) claims, "the sequent calculus has all the advantages of [the labelled] calculus ... moreover, it is syntactic [*sic*] pure." On the other hand, Poggiolesi (2011, p. 120) admits, "we can intuitively interpret the object $\Gamma_1/\Gamma_2; \Gamma_3; \Gamma_4$ as the world-sequent $\Gamma_1$ being linked to three other world-sequents." First, we look at "a classical sequent as

[25]Moreover, Negri (2005, §5) shows that the presence of labels allows a contraction-free, cut-free and harmonious calculus for provability logic.

a world of a Kripke tree-frame" (Poggiolesi and Restall, 2012, p. 49): that was what connected sequent calculus with Beth tableaux above. Then "the dash [i.e., slash] will represent the accessibility relation in a tree-frame" (*loc.cit.*), and the semi-colon will represent accessibility to a multiplicity of world-sequents. The meaning of '$\Diamond$' is given by the $\Diamond$R-rule, corresponding to the natural deduction rule $\Diamond$I:

$$\frac{G[M \Rightarrow N/S \Rightarrow T, \alpha]}{G[M \Rightarrow N, \Diamond\alpha/S \Rightarrow T]}\ \Diamond\mathrm{R}$$

that is (thinking of tableaux), if $\Diamond\alpha$ is false (in a world-sequent) then $\alpha$ is false at any accessible world-sequent. The notation is less explicit and more opaque than it is in the rule $\Diamond$I given in §2, but the content is the same. The rule $\Diamond$L, corresponding to $\Diamond$E, then allows us to make inferences from statements of the form $\Diamond\alpha$ in accordance with that meaning:

$$\frac{G[M \Rightarrow N/\alpha \Rightarrow; \underline{X}]}{G[\Diamond\alpha, M \Rightarrow N/\underline{X}]}\ \Diamond\mathrm{L}$$

that is, if $\Diamond\alpha$ is true (here), then $\alpha$ is true at some accessible world-sequent. Harmony follows from the proof of Cut, the base case of which is shown by the fact that:

$$\dfrac{\dfrac{M \Rightarrow N/S \Rightarrow T, \alpha}{M \Rightarrow N, \Diamond\alpha/S \Rightarrow T}\ \Diamond\mathrm{R} \qquad \dfrac{M' \Rightarrow N'/\alpha \Rightarrow; \underline{X}}{\Diamond\alpha, M' \Rightarrow N'/\underline{X}}\ \Diamond\mathrm{L}}{M, M' \Rightarrow N, N'/S \Rightarrow T; \underline{X}}\ \mathrm{Cut}(\Diamond\alpha)$$

reduces to

$$\frac{M \Rightarrow N/S \Rightarrow T, \alpha \quad M' \Rightarrow N'/\alpha \Rightarrow; \underline{X}}{M, M' \Rightarrow N, N'/S \Rightarrow T; \underline{X}}\ \mathrm{Cut}(\alpha)$$

Thus the notation of tree-hypersequents disguises what the notation of labelled sequents makes clear and explicit, namely, the semantic analysis of modal formulae in terms of Kripke semantics.

Poggiolesi's final argument turns on the claim that the calculus of labelled sequents does not satisfy the spirit of the Subformula Property, even if it sticks to the letter. A calculus satisfies the Subformula Property if whenever there is a proof of $M \Rightarrow N$ (or natural deduction derivation of $N$ from $M$), there is a proof in which only subformulae of $M$ and $N$ occur. This is true of the labelled formulae of the labelled calculus, but it is not true in general, that is, when the relational wffs are taken into account. For example, consider the derivation in §2 of $\Diamond\alpha \to \Diamond\beta$ from $\Box(\alpha \to \beta)$. The labelled wffs $\alpha, \beta, \alpha \to \beta, \Diamond\alpha, \Diamond\beta$ are all subformulae of the premises and the conclusion; but the relational wffs $w < v$ are not.

Why is the Subformula Property considered so important? One reason is its technical utility in establishing results such as consistency, and in proof search algorithms. But there are other ways of estblishing consistency, and other ways of speeding up proof search. Poggiolesi also links it to Aristotle's "purity of method": possession of the subformula property means that proofs use no concepts other than those employed in what they prove. We have already seen that such modularity is probably an illusion in most sciences, including logic. But we may also question whether this is really true of the presence of relational wffs in proofs of purely modal formulae. Consider the use of '$w < v$' in the labelled proof of $\Box(\alpha \to \beta) \vdash \Diamond\alpha \to \Diamond\beta$. The concept of possibility that Leibniz identified and arguably replaced the idea that "no genuine possibility can remain forever unrealized"[26] at the start of the fourteenth century has at its heart the idea of alternative states of affairs, so the alternativeness relation '$<$' and the alternative possible world $v$ are implicitly, or analytically, contained in the meaning of '$\Diamond$'. That is why $\Diamond$I gives the meaning of '$\Diamond$' according to logical inferentialism. Thus the lack of the subformula property does not here betoken a failure of purity of method.[27]

It is important to realise, however, that while the appeal to expressions of the form $w < v$ in the modal rules characterises the meaning of the wffs $\Box\alpha$ and $\Diamond\alpha$, these expressions need not be thought of as having any meaning themselves. They are an auxiliary apparatus, heuristically motivated by Leibniz's metaphor of possible worlds (and Kripke's idea of relative possibility). For relational wffs $w < v$ are never asserted, but serve only to restrict the range of the labels, themselves a purely syntactic device, which are deployed in a proof. The semantics lies in the shape of the rules, not in any entities we may mistakenly reify when beguiled by Leibniz's metaphor.

# 4 Conclusion

In Read (2008) I rejected the model-theoretic account of modality on grounds of circularity, and defended the inferentialist, or proof-theoretic, account on the grounds that the non-actual possible worlds apparently referred to in the labelled deductive system I advocated do not really exist (cf. Read, 2005). Poggiolesi and Restall (2012, p. 55) also reject model-theoretic semantics for modal concepts, in their case on the grounds that it fails to pin down the notion of relative possibility categorically, whereas the proof theory does not suffer the same drawback. We share the belief, in fact, that the meaning of the logical constants, including '$\Diamond$' and '$\Box$', should be given inferentially by the rules of a suitable proof theory. They argue, however, that importing the concepts of Kripke semantics into proofs, as the labelled deductive systems do, robs these proofs of a desirable syntactic purity. Indeed, Poggiolesi (2011, p. 31) claims that the impurity that results is incompatible with the core aim of inferentialism embodied in her conditions on a good calculus, in brief, that the formal rules of a sequent calculus satisfying the subformula property define the logical constants. In place of such labelled systems, they advocate a calculus of tree-hypersequents, and claim (Poggiolesi and Restall, 2012, p. 49) that a calculus of tree-hypersequents "is ... a syntactic method [that] does not make any use of semantic parameters beyond the language of formulas", and consequently is "purely logical" (p. 53).

I have challenged that claim: the Kripke semantics is not merely implicit in the very notation of tree-hypersequents, rather, it is explicit but opaquely disguised in the notation of '/' and ';'. But, opaqueness aside, this is as it should be. The challenge of modal logic to inferentialism is to spell out the grounds for assertion of $\Box\alpha$ and $\Diamond\alpha$ in terms of conditions on $\alpha$ alone. That cannot be done without invoking some further apparatus. Leibniz's insight, extended by Kripke, was to analyse modality in terms of relative possibility: whether or not worlds other than this one really exist, we can understand the truth of $\Diamond\alpha$ at one "world" as consisting in the truth of $\alpha$ at some "world" possible relative to the first. The tree-hypersequent introduction rule, $\Diamond$R, encodes that opaquely; the labelled deduction rule, $\Diamond$I, spells it out transparently and explicitly. Moreover,

[26] See, e.g., Knuuttila (1982).

[27] Negri (2005, §6) accordingly extends the notion of subformula for modal wffs to include relational wffs and wffs with other labels.

in contrast to the traditional CFP rule, $\Diamond I^*$, $\Diamond I$ captures the whole meaning of '$\Diamond$', lying in harmony with $\Diamond E$, not obscurely spreading that meaning between the grounds for assertion and the consequences of that assertion, as do $\Diamond I^*$ and $\Diamond E^*$. Thus the rules of the labelled system for modal logic, far from being semantically polluted, wear their meaning on their sleeves. Proof-theoretic semantics is the claim that meaning is given by, or encapsulated in, the rules of inference and the proofs that they permit. The labelled deduction rules do just that: they spell out in proof-theoretic terms the grounds for assertion of modal formulae and the harmoniously justified consequences of those assertions.

Stephen Read
University of St Andrews
Arché Research Centre
17-19 College St.
St Andrews KY16 9AA
Scotland, U.K.

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