Sheffer's Stroke: a study in proof-theoretic harmony

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Abstract

In order to explicate Gentzen's famous remark that the introduction-rules for logical constants give their meaning, the elimination-rules being simply consequences of the meaning so given, we develop natural deduction rules for Sheffer's stroke, alternative denial. The first system turns out to lack Double Negation. Strengthening the introduction-rules by allowing the introduction of Sheffer's stroke into a disjunctive context produces a complete system of classical logic, one which preserves the harmony between the rules which Gentzen wanted: all indirect proof reduces to direct proof.

1 Proof Theory for Alternative Denial

Gerhard Gentzen made the following insightful remark in 1934: "the introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions."[1] His insight was that the elimination-rules (for some logical constant, *) should allow one to infer from an occurrence of some formula with main connective * no more and no less than the introduction-rules for * warrant. When this situation obtains, one can say, following Dummett,[2] that the rules are in harmony. Gentzen was able to give substance to his insight by developing, in parallel with the system of natural deduction introduction- and elimination-rules, a sequent calculus in which all rules have the form of

[1] Gentzen *1969*, 80.

[2] Dummett *1993*, 162: "justification and commitment ought to be in harmony with one another." Dummett elaborates on the notion in Dummett *1991*. But note that I disagree with much of Dummett's elaboration of the notion, as will become clear in the paper.

introductions. There is a close connection between a natural deduction system and its corresponding sequent formulation.[3] When the natural deduction rules are in harmony, it is possible to show the eliminability of the Cut Rule from derivations in the corresponding sequent calculus: Gentzen's Hauptsatz (Main Theorem) was a constructive proof of the eliminability of Cut from his classical and intuitionistic systems LK and LJ.

Despite the elegance of Gentzen's systems, and the clarity of his exposition, his remark about the proof-theoretical meaning of the logical constants has been explicated and developed in a variety of ways. I want in this paper to illustrate what I believe to be the correct explication of his insight by applying it to the binary connective usually known as Sheffer's stroke, or alternative denial. It is often written $p|q$, for arbitrary wffs p and q, or in prefix notation (due to Łukasiewicz) as Dpq. Its truth table is:

p	q	$p\|q$
T	T	F
T	F	T
F	T	T
F	F	T

$p|q$ is true if p or q is false (or both), false only if both are true.

How can we capture the sense of $p|q$ proof-theoretically? That is, what do we need to establish in order to show that $p|q$ is true? What suffices to show that $p|q$ is true is to show either that p is false or that q is false (or both). First, recall Gentzen's introduction-rules for $p \vee q$ (Apq)—that p or q is true:

$$\frac{p}{p \vee q}\ \vee\text{I} \qquad\qquad \frac{q}{p \vee q}\ \vee\text{I}$$

or, setting the assumptions out explicitly:

$$\frac{X : p}{X : p \vee q}\ \vee\text{I} \qquad\qquad \frac{X : q}{X : p \vee q}\ \vee\text{I}$$

Expressions of the form $X : p$ are called sequents (in fact, single-conclusion sequents). A (single-conclusion) sequent is an ordered pair of a set of wffs (possibly empty) and a wff.

To deal with Sheffer's stroke, we have instead to show that p or q is false, that is, that $\neg p$ or $\neg q$ (Np or Nq) is true. It is useful to avail ourselves (for the present) of an absurdity constant, $\perp$, a wff of degree 1, containing no propositional variables, which is always false. Showing p to be false was done (in one form) by Gentzen by the rule:

$$\begin{array}{c}(p)\\ \dfrac{\perp}{\neg p}\ \neg\text{I}\end{array}$$

or, setting the assumptions out explicitly:

$$\frac{X, p : \perp}{X : \neg p}\ \neg\text{I}$$

[3] Prawitz *1965*; Zucker *1974*.

Combining these insights, the following grounds for asserting $p|q$, i.e. introduction-rules, are suggested:

$$\frac{\begin{array}{c}(p)\\ \bot\end{array}}{p|q}\ |\text{I-left} \qquad\qquad \frac{\begin{array}{c}(q)\\ \bot\end{array}}{p|q}\ |\text{I-left}$$

or, setting the assumptions out explicitly:

$$\frac{X, p : \bot}{X : p|q}\ |\text{I-left} \qquad\qquad \frac{X, q : \bot}{X : p|q}\ |\text{I-right}$$

In other words, one may assert $p|q$ if either p leads to absurdity (i.e. isn't true) or q does. $p|q$ is true just when p or q is false.

Gentzen's important insight was that, when read in this way, the introduction-rule(s) justify the elimination-rule, that is, they lead one to construct the elimination-rule to reflect the grounds for asserting the wff in question (here, $p|q$) exhibited in the introduction-rule(s). We proceed as follows: suppose we have a proof of $p|q$. Then we can infer from it whatever can be inferred from the grounds for its assertion. We know those grounds are either a proof that p is false or a proof that q is false. Let us work in the explicit notation. This is what we obtain at first:

$$\frac{X : p|q \quad Y, p \Rightarrow \bot : r \quad Z, q \Rightarrow \bot : r}{X, Y, Z : r}$$

That is, from a proof of $p|q$ (from assumptions X) and proofs of r from the assumption that p is false (that is, a derivation of $\bot$ from p, written $p \Rightarrow \bot$) and parametric wffs Y, and the same for q (and Z), we can infer r, discharging the assumptions about the falsity of p and q (i.e. from X, Y, Z alone). However, what exactly is meant by $p \Rightarrow \bot$ (and $q \Rightarrow \bot$), and how is it dealt with in actuality?

Gentzen showed how to deal with $p \Rightarrow \bot$ (in general, with $p \Rightarrow q$) in his sequent calculus. For what we have is an introduction of $p \Rightarrow \bot$ into the assumptions on which r is based. r is supposed to follow from $p \Rightarrow \bot$ in conjunction with other assumptions Y. But all $p \Rightarrow \bot$ connotes is that $\bot$ is derivable from p. Consequently, whatever follows from $\bot$ follows from whatever entails p:

$$\frac{Y : p \quad \bot : r}{Y, p \Rightarrow \bot : r} \Rightarrow :$$

Extending our earlier schema upwards, we obtain:

$$\frac{X : p|q \quad \dfrac{Y : p \quad \bot : r}{Y, p \Rightarrow \bot : r} \Rightarrow : \quad \dfrac{Z : q \quad \bot : r}{Z, q \Rightarrow \bot : r} \Rightarrow :}{X, Y, Z : r}$$

But anything follows from $\bot$, at least according to the canons of classical and intuitionistic logic. So we can suppress those premises and finally settle on the form of the elimination-rule for $p|q$:

$$\frac{p|q \quad p \quad q}{r}\ |\text{E}$$

or explicitly:

$$\frac{X : p|q \quad Y : p \quad Z : q}{X, Y, Z : r}\ |\text{E}$$

We also need to include a rule for $\bot$, *viz* $\bot$E:[4]

$$\frac{X : \bot}{X : p}\ \bot\text{E}$$

and a Thinning rule:

$$\frac{X : q}{p, X : q}\ \text{Thinning}$$

Let us call the resulting theory S_1. We define provability in S_1, $\vdash_{\mathsf{S}_1}$, as follows: $X \vdash_{\mathsf{S}_1} p$, that is, p is derivable from X in S_1, if there is a tree of sequents $Y : q$ whose every member is either of the form $q : q$, or is an immediate consequence by the rules Thinning, |I-left and -right, |E or $\bot$E of the sequents above it in the tree, and whose last member is $X' : p$, for some subset $X' \subseteq X$.

Definition 1 *Let* $\neg p =_{df} p|p$.

Lemma 1 $\neg I$ *and* $\neg E$ *are admissible rules of* S_1*, that is, if* $X, p \vdash_{\mathsf{S}_1} \bot$ *then* $X \vdash_{\mathsf{S}_1} \neg p$*, and if* $X \vdash_{\mathsf{S}_1} \neg p$ *and* $Y \vdash_{\mathsf{S}_1} p$ *then* $X, Y \vdash_{\mathsf{S}_1} \bot$.

Proof:

¬I

$$\frac{\begin{array}{c} p^1 \\ \bot \end{array}}{p|p}\ |\text{I-left}(1)$$

¬E

$$\frac{p|p \quad p \quad p}{\bot}\ |\text{E}$$

□

However, the theory of $p|q$ given by the above rules is too weak. We cannot prove, for example, the commutativity of |, that $p|q \vdash_{\mathsf{S}_1} q|p$.

Theorem 1 *Assuming* S_1 *is consistent,* $p|q : q|p$ *is not derivable in* S_1.[5]

[4] $\bot$ has no introduction-rule. Hence, in accordance with Gentzen's remark, p follows from $\bot$ provided that p follows from whatever entails $\bot$. Since nothing entails $\bot$, the 'provided that ...' clause here places no restriction. So p follows from $\bot$, for all p. See Prawitz *1973*, 243.

[5] I am indebted to Roy Dyckhoff for the observation that Theorem 1 holds, for the suggestion of moving to system S_2 below, and generally for helpful discussions in the composition of this paper.

Proof: suppose there were a derivation of $p|q : q|p$. Consider the final inference:

Thinning then $: q|p$ would be derivable, which is clearly impossible, if S_1 is consistent.

|I then w.l.g. $p|q, p : \bot$ would be derivable, and again S_1 would be inconsistent.

⊥E then $p|q : \bot$ would be derivable, which is again impossible.

|E

$$\frac{p|q : r|s \quad p|q : r \quad p|q : s}{p|q : q|p}\ |\mathrm{E}$$

Then we could equally derive $p|q : \bot$, as follows:

$$\frac{p|q : r|s \quad p|q : r \quad p|q : s}{p|q : \bot}\ |\mathrm{E}$$

and again S_1 would be inconsistent. □

That S_1 is consistent follows by a proof similar to that of Theorem 2 below.

Clearly, we need to strengthen S_1. How can we do so, yet preserve the harmony between |I and |E? Only by strengthening |I, and seeing what changes that warrants in |E. Considering how the derivation of $p|q : q|p$ fails in S_1, we see that what is needed is to discharge both p and q in |I. What warrants this is to think of $p|q$ not as $\neg p \vee \neg q$, i.e., as alternative denial, but as nand, $\neg(p\&q)$:

$$\frac{X, p, q : \bot}{X : p|q}\ |\mathrm{I}$$

The schema for the elimination-rule is now:

$$\frac{X : p|q \quad Y, (p\&q) \Rightarrow \bot : r}{X, Y : r}$$

Now apply Gentzen's sequent calculus observations to $(p\&q) \Rightarrow \bot$, and we have:

$$\frac{X : p|q \quad \dfrac{\dfrac{Y' : p \quad Y'' : q}{Y : p\&q}\ {:}\&}{Y, (p\&q) \Rightarrow \bot : r}\ \Rightarrow:}{X, Y : r}$$

where $Y = Y' \cup Y''$. We obtain the same elimination-rule as before:

$$\frac{X : p|q \quad Y : p \quad Z : q}{X, Y, Z : r}\ |\mathrm{E}$$

Let S_2 be the system consisting of |I, |E, ⊥E and Thinning.

Lemma 2 *$p|q : q|p$ is derivable in S_2.*

Proof:

$$\dfrac{\dfrac{p|q : p|q \qquad \dfrac{p : p}{p|q, q, p : p}\,\text{Thinning} \qquad \dfrac{q : q}{p|q, q, p : p}\,\text{Thinning}}{p|q, q, p : \bot}\,|\text{E}}{p|q : q|p}\,|\text{I}$$

□

However, S_2 is still not a system of classical logic. Suppose we try to introduce $p \vee q$ by definition in the standard way, as $(p|p)|(q|q)$. Then, although the introduction-rules for $\vee$ are admissible, that is, if $X \vdash_{\mathsf{S}_2} p$ then $X \vdash_{\mathsf{S}_2} p \vee q$ (so defined), and the same for q, the elimination-rule is not admissible. The same is true for $p \& q$ defined as $(p|q)|(p|q)$, and for $p \supset q$ defined as $p|(q|q)$. The reason is that Double Negation Elimination is not derivable in S_2.

Theorem 2 $\neg\neg p \nvdash_{\mathsf{S}_2} p$.

Proof: We interpret S_2 in frames $F = \langle W, R \rangle$. Let W be a non-empty set and R a transitive relation over W. Let A be the set of atoms of S_2. An assignment on A is a map $f : A \times W \to 2$ such that if $f(p, w) = 1$ and Rwu then $f(p, u) = 1$ (i.e. f is required to be hereditary over R). We extend f to a valuation $v : \mathsf{S}_2 \times W \to 2$ such that:

1. $v(p, w) = f(p, w)$ for all $p \in A$
2. $v(\bot, w) = 0$ for all $w \in W$
3. $v(p|q, w) = \begin{cases} 1 & \text{if for all } u \in W\text{, if } Rwu \text{ then } v(p, u) = 0 \text{ or } v(q, u) = 0, \\ 0 & \text{otherwise.} \end{cases}$

$X \models p$ if for all frames $F = \langle W, R \rangle$ and all $w \in W$, $v(p, w) = 1$ whenever $v(q, w) = 1$ for all $q \in X$.

Lemma 3 *v is hereditary over R, i.e. if $v(p, w) = 1$ and Rwu, then $v(p, u) = 1$.*

Proof: by induction on the degree of p.

Base: $p \in A$. Immediate, from the definition of assignment.

Induction step: Note that $v(\bot, w) = 0$ for all w.

Let $p = q|r$, and suppose $v(q|r, w) = 1$ and Rwu.

Then for all $x \in W$ such that Rwx, either $v(q, x) = 0$ or $v(r, x) = 0$. We have to show that $v(q|r, u) = 1$.
Suppose Ruy. Then Rwy, since R is transitive. So either $v(q, y) = 0$ or $v(r, y) = 0$. So $v(q|r, u) = 1$, as required. □

Returning to the proof of Theorem 2: we can now show that |I, |E and $\bot$E are sound w.r.t. these frames.

|I : We have first to show that whenever $X \cup \{p, q\} \models \bot$, $X \models p|q$. So suppose $X \cup \{p, q\} \models \bot$. Then there is no frame F such that $\forall w \in W, v(p, w) = 1$, $v(q, w) = 1$ and $v(r, w) = 1$ for all $r \in X$. Take a frame F and $w \in W$ such that $v(r, w) = 1$ for all $r \in X$. Let Rwu. Then by Lemma 3, $v(r, u) = 1$ for all $r \in X$. So either $v(p, u) = 0$ or $v(q, w) = 0$. Hence, by clause 3, $v(p|q, w) = 1$, since u was arbitrary.

$|$**E** : We have to show that whenever $X \models p|q$, $Y \models p$ and $Z \models q$, $X, Y, Z \models r$. So suppose there is a frame F and $w \in W$ such that $v(s, w) = 1$ for all $s \in X \cup Y \cup Z$. Then $v(p|q, w) = 1$, $v(p, w) = 1$ and $v(q, w) = 1$. So for all u such that Rwu, $v(p, u) = 0$ or $v(q, u) = 0$. But by Lemma 3, $v(p, u) = 1$ and $v(q, u) = 1$. Contradiction. Hence there is no frame such that $v(s, w) = 1$ for all $w \in W$ and all $s \in X \cup Y \cup Z$. Thus $X, Y, Z \models r$.

$\perp$**E** : We have to show that whenever $X \models \perp$, $X \models p$. So suppose $X \models \perp$. Then there is no frame such that for all $w \in W$, $v(q, w) = 1$ for all $q \in X$. Hence $v(p, w) = 1$ in any frame in which $v(q, w) = 1$ for all $w \in W$ and all $q \in X$. So $X \models p$.

We now show that $\neg\neg p \not\models p$. Let $W = \{0, 1\}$ such that $R11$ and $R01$. $\langle W, R\rangle$ is a frame. Let $v(p, 0) = 0$ and $v(p, 1) = 1$. Note that $v(p|p, 1) = 0$ and so $v(\neg\neg p, 0) = 1$. But $v(p, 0) = 0$. So $\neg\neg p \not\models p$. Hence $\neg\neg p \nvdash_{\mathsf{S}_2} p$. □

Clearly, S_2 is at most an intuitionistic account of alternative denial. It is therefore no surprise that $|$ is insufficient to introduce &, $\vee$ and $\supset$, for no two-place connective of intuitionistic logic is functionally complete.[6]

2 The Classical Theory of Alternative Denial

The results of §1 appear to bear out the remark of Prawitz' that "there is no known procedure that justifies ... the classical rule of indirect proof (i.e. the rule of inferring A given a derivation of a contradiction from $\neg A$."[7] For if $\neg p \vdash_{\mathsf{S}_2} \perp$, we can infer $\vdash_{\mathsf{S}} \neg\neg p$ by $|$I, but we need the rule of Double Negation to infer p. It appears that the claims of Dummett's and Prawitz', that classical logic is unharmonious,[8] and so proof-theoretically suspect, are borne out by our development of S_2, a calculus for nand based directly on Gentzen's remarks about proof-theoretic meaning and the harmony between the introduction- and elimination-rules.

Such an inference would be mistaken, however. Clearly, the rules of S_2 are inadequate to yield the full classical theory of alternative denial. But, if Gentzen was right, $|$E did no more than spell out the consequences of the meaning given to Sheffer's stroke by the introduction-rules, $|$I. We all know the dangers of going beyond that licence. They were shown by Prior in his famous paper on 'tonk'.[9] The introduction-rule for 'tonk' had the form:

$$\frac{p}{p \text{ tonk } q}\ \text{tonk-I}$$

[6]See Kuznetsov *1965*. There are countably many functionally complete three-place connectives in intuitionistic logic: see Cubric *1988*.

[7]Prawitz *1977*, 34.

[8]Dummett *1991*, 291, 299.

[9]Prior *1960*.

By Gentzen's lights, this would justify the elimination-rule:

$$\frac{p \text{ tonk } q \quad \begin{array}{c}(p)\\ r\end{array}}{r}\ \text{tonk-E}$$

These rules yield a perfectly consistent, if dull, calculus—p tonk q is true if p is, otherwise false. Clearly, $p \vdash p$ tonk q. Conversely, p tonk $q \vdash p$ as follows:

$$\frac{p \text{ tonk } q \quad p^1}{p}\ \text{tonk-E(1)}$$

Prior, however, proposed a stronger tonk-E rule:

$$\frac{p \text{ tonk } q}{q}$$

This rule is not justified proof-theoretically in Gentzen's manner. Moreover, it leads to triviality, as Prior showed. By his rules, any two wffs, p and q, are equivalent.

To return to "stroke": we need to strengthen the rules for '|' while at the same time preserving their harmony. Clearly, the only way to do so, is to strengthen the introduction-rule yet further, which will in itself justify reconsideration of the elimination-rule. But how can it be strengthened and in what way?

We can see what to do by considering the sequent calculus analogue, LS, of our natural deduction system, S_2.

Definition 2 *Sequents, written $X : p$, now consist of a (possibly empty) set of wffs, X and a singleton or empty set of wffs, p. In $p, X : q$ it is assumed, unless stated to the contrary, that $p \notin X$; the succedent, q may be a single wff, or empty.*

Operational Rules

$$\frac{X, p, q :}{X : p|q}\ |\text{right}$$

$$\frac{X : p \quad X : q}{X, p|q :}\ |\text{left}$$

Structural Rules

$$\frac{X : q}{p, X : q}\ \text{Thinning(-left)} \qquad \frac{X :}{X : p}\ \text{Thinning(-right)}$$

$$\frac{X : p \quad p, Y : q}{X, Y : q}\ \text{Cut}$$

We say that $X \vdash_{\mathsf{LS}} p$ if there is a sequence of sequents whose last member is $X' : p$ where $X' \subseteq X$, and whose every member is either of the form $Y : q$ where $q \in Y$ or is an immediate consequence by |left, |right, Thinning or Cut of earlier members of the sequence.

Theorem 3 $X \vdash_{\mathsf{S}_2} p$ *iff* $X \vdash_{\mathsf{LS}} p$.

Proof: by inspection. □

It should be no surprise that LS gives an intuitionistic account of consequence and of "stroke", since it is a single-conclusion calculus. That was how Gentzen obtained his sequent calculus LJ for intuitionistic logic from LK, by restricting sequents to single-conclusion (one or no s-wffs). In fact, this formulation somewhat obscures what the real restriction is. Consider the analogue of |right in a multiple-succedent calculus:

$$\frac{X, p, q : Y}{X : p|q, Y}\ |\text{right}_m$$

Comma in the succedent in sequent calculus has a disjunctive interpretation, while that in the antecedent is conjunctive, so the inference here has the form:

$$\frac{\neg(p\&q) \vee r}{(p|q) \vee r}\ (*)$$

introducing '|' into a disjunctive context.

Generalizing |left to its multiple-conclusion form:

$$\frac{X : p, Y \quad X : q, Y}{X, p|q : Y}\ |\text{left}_m$$

it is straightforward to prove Double Negation Elimination:

$$\dfrac{\dfrac{\dfrac{p : p}{p, p : p}\ \text{Thinning}}{: p|p, p}\ |\text{right}_m \qquad \dfrac{\dfrac{p : p}{p, p : p}\ \text{Thinning}}{: p|p, p}\ |\text{right}_m}{(p|p)|(p|p) : p}\ |\text{left}_m$$

However, it is clear that the admission of multiple succedents is crucial here. If we call the new system based on $|\text{left}_m$ and $|\text{right}_m$, LSC, we can easily check that $X \vdash_{\mathsf{LSC}} p$ iff p follows from X classically.

How can we extend our natural deduction system, S_2, to allow the introduction of '|' into a disjunctive context, as (*) permits, and so incorporate the added power of derivability that a multiple-conclusion system permits? One way is to replace the single wffs at the nodes of the natural deduction tree with sets of wffs.[10] The sequents then constituting a proof are multiple-conclusion sequents $X : Y$, where Y is non-empty (the empty succedent of sequent calculus is matched by letting $Y = \bot$). But the effect of multiple-succedent can be achieved in natural deduction without such a radical departure from the normal single-conclusion format, where what is proved at each juncture is a (single) wff on certain assumptions. The solution is, quite literally, to allow the introduction of '|' into a disjunctive context. A first attempt would give:

$$\frac{\begin{array}{c}(p, q)\\ \bot \vee r\end{array}}{(p|q) \vee r}$$

[10]See von Kutschera *1962*, Boricic *1985* and Cellucci *1992*.

and similarly for q. But think: $\bot \vee r$ is derivable from p if and only if r itself is derivable from p (clearly, r entails $\bot \vee r$, and conversely, $\bot \vee r$ entails r by $\vee$E and $\bot$E). So we can simplify the rule to read:

$$\frac{\begin{array}{c}(p,q)\\ r\end{array}}{(p|q) \vee r}\ |\mathrm{I}_c$$

that is,

$$\frac{X, p, q : r}{X : (p|q) \vee r}\ |\mathrm{I}_c$$

What elimination-rule does $|\mathrm{I}_c$ justify, by Gentzen's proposal? A first attempt reads:

$$\dfrac{X : p|q \qquad \dfrac{\dfrac{Y' : p \quad Y'' : q}{Y : p\&q}\ :\& \qquad r : s}{Y, (p\&q) \Rightarrow r : s}\ \Rightarrow :}{X, Y, Z : s}$$

Let $r = s$. We obtain our original $|$E rule as a special case, which we will find to suffice for (classical) completeness:

$$\frac{X : p|q \quad Y : p \quad Z : q}{X, Y, Z : r}\ |\mathrm{E}$$

Call the new system, based on $\vee$I, $\vee$E, $\bot$E, $|\mathrm{I}_c$ and $|$E, SC. Clearly, SC is at least as strong as S_2. But we can now derive Double Negation:

Lemma 4 (DN) *If $X \vdash_{\mathsf{SC}} \neg\neg p$ then $X \vdash_{\mathsf{SC}} p$.*

Proof:

$$\dfrac{\dfrac{p^1}{\neg p \vee p}\ |\mathrm{I}_c(1,1) \qquad \dfrac{\neg\neg p \quad \neg p^2 \quad \neg p^2}{p}\ |\mathrm{E} \qquad p^3}{p}\ \vee\mathrm{E}(2,3)$$ □

In fact, the system SC is complete for classical logic. We show this in Theorem 4; first, we show the admissibility of certain rules, which will make the derivations in Theorem 4 easier to display:

Lemma 5 **|I-left** *If $X, p \vdash \bot$ then $X \vdash p|q$*

|I-right *if $X, q \vdash \bot$ then $X \vdash p|q$.*

¬E *If $X \vdash \neg p$ and $Y \vdash p$ then $X, Y \vdash \bot$.*

SZ *If $X \vdash p|q$, $Y, r \vdash p$ and $Z \vdash q$ then $X, Y, Z \vdash r|s$ and*
if $X \vdash p|q$, $Y \vdash p$ and $Z, s \vdash q$ then $X, Y, Z \vdash r|s$.

|I *If $X, p, q \vdash \bot$ then $X \vdash p|q$.*

MO *If $X, r \vdash p$ and $Y, r \vdash q$ then $X, Y \vdash r|(p|q)$.*

Proof:

|I-left

$$\dfrac{\dfrac{\dfrac{X, p : \bot}{X, p, q : \bot}\ \text{Thinning}}{X : (p|q) \vee \bot}\ |\mathrm{I}_c \qquad p|q : p|q \qquad \dfrac{\bot : \bot}{\bot : p|q}\ \bot\mathrm{E}}{X : p|q}\ \vee\mathrm{E}$$

|I-right is similar.

¬E

$$\dfrac{p|p \quad p \quad p}{\bot}\ |\mathrm{E}$$

SZ

$$\dfrac{\dfrac{p|q \quad \overset{r^1}{p} \quad q}{\bot}\ |\mathrm{E}}{r|s}\ |\text{I-left}(1)$$

The other case is similar.

|I

$$\dfrac{\dfrac{\dfrac{p^1 \quad q^2}{\bot}}{(p|q) \vee \bot}\ |\mathrm{I}_c(1,2) \qquad p|q^3 \qquad \dfrac{\bot^4}{p|q}\ \bot\mathrm{E}}{p|q}\ \vee\mathrm{E}(3{,}4)$$

MO

$$\dfrac{\dfrac{p|q^2 \quad \overset{r^1}{p} \quad \overset{r^1}{q}}{\bot}\ |\mathrm{E}}{r|(p|q)}\ |\mathrm{I}(1{,}2)$$

□

We can now show that SC is complete by deriving the sole axiom and showing admissible the sole rule of inference of Wajsberg's formulation of classical propositional logic.[11]

Theorem 4 *1. If* $X \vdash p|(q|r)$ *and* $Y \vdash p$ *then* $X, Y \vdash r$.

2. $\vdash (p|(q|r))|(((s|r)|\neg(p|s))|(p|(p|q)))$.

Proof:

1.

$$\dfrac{\dfrac{p|(q|r) \quad p \qquad \dfrac{\neg r^2 \quad r^1 \quad r^1}{q|r}\ \mathrm{SZ}(1)}{\neg\neg r}\ \mathrm{SZ}(2)}{r}\ \mathrm{DN}$$

[11] See Church *1956*, 138.

2.

$$\dfrac{\dfrac{\dfrac{\neg(p|s)^1 \qquad \dfrac{\dfrac{p^2}{\neg p \vee p}\,|\mathrm{I}(2) \qquad \dfrac{p|(q|r)^3 \quad p^4 \quad \dfrac{s|r^5 \quad s^6 \quad r^7}{q|r}\,\mathrm{SZ}(7)}{p|s}\,\mathrm{SZ}(6) \qquad \dfrac{\dfrac{\neg p^8 \quad p^9}{\bot}\,\neg\mathrm{E}}{p|s}\,|\text{I-left}(9)}{p|s}\,\vee\mathrm{E}(4,8)}{\bot}\,\neg\mathrm{E}}{(s|r)|\neg(p|s)}\,|\mathrm{I}(1,5) \qquad \dfrac{\dfrac{p|(q|r)^3 \quad p^{10} \quad \dfrac{p|q^{11} \quad p^{10} \quad q^{12}}{q|r}\,\mathrm{SZ}(12)}{\bot}\,|\mathrm{E}}{p|(p|q)}\,|\mathrm{I}(10,11)}{(p|(q|r))|(((s|r)|\neg(p|s))|(p|(p|q)))}\,\mathrm{MO}(3)$$

□

Moreover, SC is harmonious—we've designed it so to be. We do not have a Normalization Theorem for SC—proofs like that of **DN** in Lemma 4 contain a maximal wff, in that case $\neg p \vee p$, introduced by $|\mathrm{I}_c$ and major premise of $\vee$E. That is because $\vee$ is performing the role of structural connective as well as its standard role as disjunction. Harmony consists rather in the justificatory relation between the -I and -E rules.

The role of $\vee$ and $\bot$ is entirely auxiliary—it is to enable us to construct a single-conclusion natural deduction system. The corresponding sequent system LSC can dispense with them, and concentrate entirely on the matching rules, $|\mathrm{left}_m$ and $|\mathrm{right}_m$. Cut is eliminable from LSC by the usual method of reducing the degree of the Cut formula. The schema:

$$\dfrac{\dfrac{X, p : Y}{X : p|q, Y}\,|\mathrm{right}_m \qquad \dfrac{Z : p, W \quad Z : q, W}{Z, p|q : W}\,|\mathrm{left}_m}{X, Z : Y, W}\,\mathrm{Cut}(p|q)$$

reduces to:

$$\dfrac{Z : p, W \quad X, p : Y}{X, Z : Y, W}\,\mathrm{Cut}(p)$$

The rule $|\mathrm{I}_c$ is not, in Dummett's terminology,[12] pure or simple. It is not pure, since both '|' and '$\vee$' figure in its conclusion; it is not simple, since '|' is not necessarily the principal operator of its conclusion. To demand that (introduction-)rules be simple (Dummett does not) is to misunderstand what Gentzen means by an introduction-rule. The role of such a rule is to introduce an occurrence of a logical constant, that is, to infer a conclusion containing an occurrence of that constant with no match in the premise. One might object, therefore, to $|\mathrm{I}_c$; for not only does the displayed occurrence of '|' have no match in the premises; neither does that of '$\vee$'. But recall our development of the rule. '$\vee$' is present as an auxiliary symbol to assist in the manipulation of single conclusions as if they were multiple. The premise has the logical effect of $\bot \vee r$. So '$\vee$' is implicitly present in the premise. What $|\mathrm{I}_c$ introduces is the occurrence of '|'. '$\vee$' is introduced by $\vee$I.

Purity is a different matter. But the crucial issue was identified by Dummett (*loc.cit.*). What must be avoided is circularity. Specifying the meaning of '|' by $|\mathrm{I}_c$ depends on a prior understanding of '$\vee$'; so we must be sure that there is no

[12] Dummett *1991*, 257.

corresponding dependency of the meaning of '$\vee$' on that of '$|$'. There is not. '$|$' does not feature in the rules for '$\vee$'.

To conclude: SC, constructed with the rules $\vee$I, $\vee$E, $\perp$E, $|\mathrm{I}_c$ and $|$E is a harmonious, single-conclusion natural deduction system of classical logic, from which all the theses of classical logic in '$|$' (and $\vee$ and $\perp$) can be derived. Harmony in classical logic is an achievable goal.

References

[1] Boricic, B. *1985*: "On sequence-conclusion natural deduction systems", *Journal of Philosophical Logic*, 14, 359-77.

[2] Cellucci, C. *1992*: "Existential quantification and normalization in sequent natural deduction", *Annals of Pure and Applied Logic*, 58, 111-48.

[3] Church, A. *1956*: *Introduction to Mathematical Logic*, vol. I. Princeton U.P.

[4] Cubric, D. *1988*: 'There are denumerably many ternary intuitionistic Sheffer functions', *Notre Dame Journal of Formal Logic*, 29, 579-81.

[5] Dummett, M. *1991*: *The Logical Basis of Metaphysics*. London: Duckworth. Cambridge, Mass.: Harvard University Press.

[6] Dummett, M. *1993*: *The Seas of Language*. Oxford: Clarendon.

[7] Gentzen, G. *1969*: "Untersuchungen über das logische Schliessen", in *The Collected Papers of Gerhard Gentzen*, tr. M. Szabo. Amsterdam: North-Holland.

[8] Kuznetsov, A.V. *1965*: "Analogs of the 'Sheffer stroke' in constructive logic", *Soviet Mathematics*, 6, 70-4.

[9] Prawitz, D. *1965*: *Natural Deduction*. Stockholm: Almqvist and Wiksell.

[10] Prawitz, D. *1973*: "Towards a foundation of general proof theory", in *Logic, Methodology and Philosophy of Science*, IV, ed. P. Suppes et al. Amsterdam: North-Holland.

[11] Prawitz, D. *1977*: "Meaning and proofs: on the conflict between intuitionistic and classical logic", *Theoria*, 43, 2-40.

[12] Prior, A. *1960-61*: "The runabout inference ticket", *Analysis*, 21, 38-9.

[13] von Kutschera, F. *1962*: "Zum Deduktionsbegriff der klassischen Prädikatenlogik erster Stufe", in *Logik und Logikkalkül*, ed. H. von Max Käsbauer and F. von Kutschera. Freiburg: Alber, 211-236.

[14] Zucker, J. *1974*: "The Correspondence between Cut-elimination and Normalization" I, *Annals of Mathematical Logic*, 7, 1-112.