Anti-Realist Classical Logic and Realist Mathematics *

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Abstract. I sketch an application of a semantically anti-realist understanding of the classical sequent calculus to the topic of mathematics. The result is a semantically anti-realist defence of a kind of mathematical realism. In the paper, I begin the development of the view and compare it to orthodox positions in the philosophy of mathematics.

Keywords: logic, mathematics, proof theory, ontology

My aim in this paper is to apply a semantically anti-realist understanding of (classical) logical consequence, and to then use the change of perspective from the semantically realist concern of *truth-in-a-model* to the semantically anti-realist analysis in terms of *propriety-of-assertion* (or *denial*) as a position from which to view the philosophy of mathematics. The result is not so much a new position in the metaphysics or epistemology of mathematics, but instead a fresh perspective on traditional positions.

Let us start with logical consequence.

1. Logic

The relationship between logic and mathematics is remarkably close. The rise of classical logic, in the work of Frege, Russell, Gödel and Tarski, arose not so much from a desire to give a uniform account of judgement, to treat problems of quantification in natural languages, to treat vagueness — target was *mathematics*. From the development of the calculus and its rigorisation, to the paradoxes of set theory, the aim was to clarify, to make explicit the forms of deduction valid in

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mathematical reasoning.[1] How we think of mathematics and how we think of logic are intertwined. In this paper, a refigured view of logic will bring along with it a reconfigured position in the philosophy of mathematics.

1.1. Logic without Vocabulary

The logic of Frege, Russell, Gödel and Tarski is deductive logic, it is not only deductive logic, it is *classical* deductive logic. Our task here will not be to define a new relation of logical consequence, but to picture this relation in a distinctive way. We will reframe the central notion of logical consequence in terms of coherence. Coherence is a normative notion, making explicit a particular kind of *mistake* that one can make, or can avoid making, in a discourse.

The motivation, which will suffice to introduce the concept, is that coherence is a kind of virtue that a position in a discourse might have. A position — involving a collection of assertions and denials — is *coherent* when those assertions and denials hang together, when they are consistent. However, we will not think of coherence as defined in terms of possible truth or truth-in-an-interpretation. Instead, we will consider the upshot of taking coherence as a *starting* point for our analysis, as opposed to a notion defined in other terms.

Taking coherence as a starting point does not mean that there is nothing more we can say about it. On the contrary, it can be argued that coherence must satisfy certain criteria: there are *norms* of coherence [22].

NORMS: The norms are straightforward to state, given some individuation of the *content* of the assertions and denials of the discourse in question. We will call a collection of assertions and denials a POSITION. A position $[\Gamma : \Delta]$ is a pair of (finite) sets, Γ of things *asserted* and Δ of things *denied*. Positions are evaluated for *coherence*. Such an evaluation must satisfy the following three conditions:

IDENTITY: $[A : A]$ is not coherent.

WEAKENING: If $[\Gamma, A : \Delta]$ or $[\Gamma : A, \Delta]$ is coherent, then is $[\Gamma : \Delta]$ coherent too.

STRENGTHENING: If $[\Gamma : \Delta]$ is coherent, then either $[\Gamma, A : \Delta]$ or $[\Gamma : A, \Delta]$ is coherent too.

There are *many* different relations satisfying these norms. Perhaps the coherence relation on our target vocabulary is the smallest relation

[1] For an enlightening historical account of the development of the mathematical sciences and logic, along with with the rise of the "Semantic Tradition," see Coffa's *The Semantic Tradition from Kant to Carnap* [5].

satisfying these conditions (the propositional contents are totally independent of one another; think of atomic statements in some formal language), or the relation is richer than this (think of the content of judgements in some particular language; we may say that the position [this is red : this is coloured] is incoherent).

THESE NORMS ARE SUFFICIENT FOR LOGICAL CONSEQUENCE If we take a discourse to be governed by a coherence relation satisfying these norms, then we thereby may evaluate it with respect to deductive validity. A logical consequence relation is *definable* in terms of coherence. For example, if we take a position $[\Gamma : A, \Delta]$ to be incoherent then, given that an agent has asserted (implicitly or explicitly) Γ and denied (implicitly or explicitly) Δ and is *coherent*, then the *only* coherent option available concerning A is to assert it. Once the question has come up, its answer is implicit. A is now undeniable, it *follows* from what has already been said. A kind of consequence is implicit in the notion of coherence.

THESE NORMS ARE NECESSARY FOR LOGICAL CONSEQUENCE We can make the connection in the other direction too. Consider the kind of grip a *deductive argument* from A to B ought have on a discourse. It is too much to think that an assertion of A need be followed by an assertion of B, or that anyone who accepts A must accept B. What we require is that the assertion of A is not to be combined with the denial of B — that a position in which A is asserted and B is denied is defective.

So, suppose that we take an argument from premises to a conclusion[2] to have the normative force of rendering 'defective' a position in which the premises are asserted and the conclusion denied in this sense.

What should *count* as denial, and how is it to be related to assertion? At the very least, the denial of a propositional content together with its assertion must count as defective in this salient sense, since the argument from A to A is valid (so we have the IDENTITY condition for this sense of 'defectiveness'). WEAKENING is straightforward too, since if a position is not defective, any position with fewer assertions or denials is also not defective in that sense. For STRENGTHENING, note that it is a condition showing us when adding a denial *is* coherent. If $[\Gamma : \Delta]$ is coherent and we cannot coherently assert A, then we must be able to coherently deny it. We might say then that A has been *implicitly* denied in a position in which Γ has been asserted and Δ has been denied.

Taking the necessity and sufficiency of the coherence and logical consequence, we have the following connection between coherence and con-

[2] Let us not beg the question in favour of multiple conclusion arguments at this point.

sequence: the claim that $[\Gamma : \Delta]$ is incoherent can be recast *positively* as the endorsement of the *sequent* $\Gamma \Rightarrow \Delta$. We will henceforth use this more familiar notation, but keep in mind that the validity of the sequent $\Gamma \Rightarrow \Delta$ is to be thought of as the verdict that asserting Γ and denying Δ is incoherent. With this formulation, the norms of coherence take a more familiar form as the structural rules of the sequent calculus.

Identity: $A \Rightarrow A$

Weakening: $$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

Stregthening: $$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

So, the strengthening rule is a formulation of the usual rule Cut from the sequent calculus. From bottom to top, we strengthen positions by adding a statement, either to the left or the right. From top to bottom, we cut out that statement.

We choose coherence as a starting point, because it enables us to *do* logic in such a way as to result in classical logic (as we will see soon), it coheres with mathematical practice, and it does not require truth conditional or model theoretic semantics while still managing to be recognisably *semantics*. With this perspective, we can 'do logic' as soon as we have a discourse that is recognisably bound by norms of coherence. Mathematical discourse is clearly such a discourse. So, let us now consider how to use the notion of coherence to clarify *semantics*. We start with the connectives of propositional logic.

1.2. Connectives

Now consider the sequent rules for the propositional connectives. Here are the the rules for negation and conjunction.

Negation: $$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

Conjunction: $$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

These clauses can be seen as ways to add vocabulary (here, propositional connectives) to a discourse, and to continue constraining that with respect to coherence. The negation rules tell you that if denying A is incoherent (in the context of asserting Γ and denying Δ) then so is asserting $\neg A$, and that if asserting A is incoherent (in the context of asserting Γ and denying Δ) then so is denying $\neg A$. Or to put it positively, if the assertion of $\neg A$ is coherent (along with asserting Γ and denying Δ) so is the denial of A, and if the denial of $\neg A$ is coherent (along with asserting Γ and denying Δ) so is the assertion of A. The rules for negation tell you how to treat the assertion of a negation and the denial of a negation, with respect to coherence.

Similarly the rules for conjunction dictate the behaviour of that connective with respect to coherence. If the assertion of $A \wedge B$ is coherent (in some context) then so is the separate assertion of A and of B. On the other hand if the denial of $A \wedge B$ is coherent (in some context) then either the denial of A is coherent (in that context) or the denial of B is coherent (in that context).

In this way, the rules for the connective tell you what to *do* with them. The traditional natural deduction rules for the connectives (infer $A \wedge B$ from A, B, and infer both A and B from $A \wedge B$) are instances of these rules: By identity, we have $A \Rightarrow A$ and $B \Rightarrow B$. It follows that $A, B \Rightarrow A \wedge B$. Similarly, $A, B \Rightarrow A$ (by weakening the identity $A \Rightarrow A$) so it follows that $A \wedge B \Rightarrow A$. Similarly, we have $A \wedge B \Rightarrow B$.

The result is classical propositional logic. (For example, we have $\neg\neg A \Rightarrow A$, via $\Rightarrow A, \neg A$.) As a matter of fact, I take it that there is a defensible natural deduction system in which *proofs* allow multiple premises and multiple conclusions [23], but discussing this would take us to far away from the present topic.

These rules have the attractive virtue that if we add them alone, then the relation of coherence defined on the new vocabulary satisfies the conditions of identity, weakening and strengthening. It follows that the addition of these rules is *conservative* over a base vocabulary without these logical connectives. (However, it is not necessarily conservative over a base vocabulary already containing logical connectives, such as a non-distribuitive pair of lattice connectives, or an intuitionistic conditional. There is significant debate over what this might mean [6].)

These rules tell us *something* about how to use the connectives. They do not tell you everything of what '$\neg$' or '$\wedge$' might "mean," but they do tell you how to use these connectives when it comes to evaluating assertions and denials featuring them, for coherence. In other words, rules such as these give you a starting point for the *practice* or *precisification* of a concept. For example, with these rules at hand we can see that a dialetheist or an intuitionist or a supervaluationist is not using negation this way when they diverge from these rules. A dialetheist may take $[p, \mathsf{not}\ p : \ ]$ to be coherent, and an intuitionist or a supervaluationist may take $[\ : p, \mathsf{not}\ p]$ to be coherent. It follows that they are not using their concept 'not' in a way that conforms to the rules above. However, it is another thing entirely to say that in the mouth of an intuitionist or a dialetheist or a supervaluationist, 'not' does not mean *not*. The sequent calculus rules are a very useful technique for constraining use, and for making precise a concept (in just the same way as we might present a truth table and say that we will take disjunction to behave like *this*). Sequent rules facilitate the introduction of connectives by a kind of *definition*. However, we well know that definition is not all that there is to say about meaning, as a definition might introduce a term into the vocabulary, and the vicissitudes of use might sweep it in another direction.[3]

This account of coherence and logical consequence does not appeal to *truth* or to *warrant*. It is semantically anti-realist in Dummett's sense [6], in that it does not take the preservation of warrant-transcendent truth—or indeed, any kind of truth—to be constitutitive of logical consequence. The approach is normatively inferentialist [3, 4] as the central notion (coherence) is evaluative, and it is understood in terms of the category of inference (at least, if we are prepared to understand the inference $A \Rightarrow B$ in terms of the incoherence of the assertion of A and the denial of B) rather than the category of representation.[4] This interpretation differs from the usual proof-first interpretation of intuitionist propositional logic: after all, the result is a defence of *classical* logic, and not *intuitionist* logic. The result is not the traditional BHK interpretation of intuitionist logic, in which the semantics of propositions is defined in terms of proof: a proof of a conjunction is a pair of proofs, one for each conjunct. A proof for a conditional is a function taking proofs of the antecedent to proofs of the consequent, and so on. This interpretation is well suited to the interpretation of logical consequence as preservation of *warrant*, and a conception of proofs according to which they have a number of premises and a single conclusion. If we take proofs to have a more rich structure (allowing for multiple conclusions, as we have seen), or if we allow for richer operations on proofs (such as *continuations* [28], for one example), then the clauses for the connectives in the BHK interpretation motivate classical logic, rather than intuitionist logic.

1.3. Names, variables and quantifiers

If we wish to give an account of the logical features of the first-order quantifiers (and surely we must if we are to do justice to the logic of mathematics), then we go beyond the combination of propositions with

3 Consider the changes in the meaning of the terms *force* and *mass* in physics [11].
4 For other work on the sequent calculus, assertion and denial: [12, 14, 19, 24].

other propositions and we analyse some of the internal structure of propositions. In the language of first-order logic, we compose propositions out of *predicates* and *terms*.[5] Some terms are *variables*, which play a role in quantified expressions. Some of the terms may play a special role in mathematical theories, such as the term "0" in arithmetic, or terms built up using function symbols, such as "$x' + (y \times z)$". We wish to understand the addition of quantificational vocabulary in terms of the rules for the coherence of packages involving quantifiers. For the rules of the quantifiers to work, we need one last piece of terminology. We need *arbitrary* names, which are able to stand in assertions and denials. One way to think of arbitrary names is as *unbound* variables.[6] Think of the fragment of discourse: "suppose x is a number, then x is either even or odd. If x is not odd, then ..." — "x" here can be understood as an arbitrary name. If we wish to treat the fragment 'if x is not odd, then...' as a statement, with its own inferential properties—rather than as a component of a larger expression in which a quantifier binds the *variable* x—then an understanding like this seems appropriate. If you hold to an understanding of variables according to which they are always bound, you must choose a stock of names that have no inferential capacities of their own. What is means is simple: if a is an arbitrary name then whenever $[\Gamma : \Delta]$ is coherent, it would remain coherent with the replacement of some term occuring in Γ and Δ by the name a. (So, in the traditional vocabulary of arithmetic, 0 is not an arbitrary name but variables such as x and y behave as arbitrary names. In classical first-order *logic*, or higher-order logics, all names are arbitrary.)

Clearly, in mathematical practice, we have arbitrary names. We use them all of the time in reasoning when we make suppositions and reason under hypotheses. With this concept in mind, we can now examine the rules governing the universal quantifier.

$$\textit{Universal Quantifier}: \frac{\Gamma, Bt \Rightarrow \Delta}{\Gamma, (\forall x)Bx \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow Ba, \Delta}{\Gamma \Rightarrow (\forall x)Bx, \Delta}.$$

In the second rule, we require a to be *arbitrary* and not present in $\Gamma \Rightarrow (\forall x)Bx, \Delta$. The motivation for these rules is clear. If the assertion of $(\forall x)Bx$ is coherent, then so is the asseriton of Bt, which is a specific case of the general claim. Conversely, if the denial of Ba is incoherent (where

[5] In what follows, we will use the following shorthand notation. For a formula B with some occurrences of a term t marked out, we will write "Bt", and we will write "Bs" for the result of replacing the selected instances of t in Bt by s.

[6] Or, as Schütte calls them, *free object variables* [25].

we have assumed *nothing* about the object a) then it is incoherent to deny that *everything* has property B.[7]

These rules give the usual properties of the universal quantifier. We have $(\forall x)Bx \Rightarrow Ba$ by the assertion rule and the identity $Ba \Rightarrow Ba$. By the denial rule, if we derive Fa for an arbitrary name a, then this derivation applies generally. Is there any *object* which we cannot therefore show to be B? Suppose there were: call such an object c. Run through the proof for Ba, except with 'c' in place of 'a.' We can do that since the name 'a' is arbitrary.

You may be concerned that the possiblity of non-denoting terms invalidate this account. Perhaps we should modify our rules by adding an "existence" predicate as seen in free logics. Then we would say that $[\Gamma, (\forall x)Fx : \Delta]$ is incohrent if $[\Gamma, Fa, E!a : \Delta]$ is incoherent, and $[\Gamma : (\forall x)Fx, \Delta]$ is incoherent if $[\Gamma, E!a : Fa, \Delta]$ is incoherent for an arbitrary name a not in $[\Gamma : \Delta]$. I will ignore this complication, since non-denoting terms do not seem to play a significant role in mathematical discourse.[8]

This kind of account of the quantifier sidesteps the usual debate between "objectual" and "substitutional" accounts of the quantifiers. We have not relied upon an "intended domain" of quantification, yet neither have we given a substitutional account identifying the *truth* of all $(\forall x)Bx$ with the *truth* instances of Bx. We have explained the inferential capacity of $(\forall x)Bx$ in terms of substitutions, not its *truth* [15].

You can do the same thing for higher-order quantifiers, using rules of the same general shape as the quantifier rules we have seen. We can also add very natural rules for identity [20], using arbitrary *predicates*, instead of arbitrary names.[9]

The quantifier rules have the usual harmony properties. For IDENTITY, we have $(\forall x)Bx \Rightarrow Ba$ (if we have at least one name). Since a is arbitrary, we have $(\forall x)Bx \Rightarrow (\forall x)Bx$. For STRENGTHENING, suppose $[\Gamma : (\forall x)Bx, \Delta]$ and $[\Gamma, (\forall x)Bx : \Delta]$ are both incoherent. This means that $[\Gamma, Bt : \Delta]$ is incoherent for some t, and $[\Gamma : Ba, \Delta]$ is incoherent

[7] We require the restriction to *arbitrary* names a, since proving that 0 is even (on the basis of no assumptions) should not be enough to prove that every number is even.

[8] However, consider $\frac{1}{0}$ or $\lim_{x\to 0}\frac{1}{x}$. If these are not eliminated from the vocabulary, the something like Beeson and Feferman's logic of 'definedness' seems appropriate [7].

[9] In sequent form, using X as an arbitrary predicate, the rules are $\dfrac{\Gamma, Xs \Rightarrow Xt, \Delta}{\Gamma \Rightarrow s = t, \Delta}$ and $\dfrac{\Gamma \Rightarrow Bs, \Delta}{\Gamma, s = t \Rightarrow Bt, \Delta}$.

for an arbitrary a. Then, $[\Gamma : Bt, \Delta]$ is incoherent (a has no distinctive coherence properties of its own) and hence, by strengthening for Bt, $[\Gamma : \Delta]$ is coherent.

2. Mathematical Practice and Mathematical Theories

Now, let us turn to mathematical reasoning. Clearly, mathematicians *assert* and *deny*. Mathematical discourse is well suited (perhaps it is *uniquely* suited) to evaluation for coherence by the canon of classical first-order logic. Consider someone who engages in the practice of reasoning in the language of arithmetic. Let us be completely *agnostic* on the ontology of numbers. However, let us also take it that the discourse has the kind of form that it appears to take, on face value. A statement of the form '$2 + 3 = 5$' contains three terms, one function symbol and the relation symbol of identity. It is not a disguised statement about numerical quantifiers, and in the mouth of a mathematical reasoner it is asserted, and not merely 'play'-asserted. Accounts of the structure of mathematical statements that do not take them at something *like* their face value must explain what kind of structure they have, and how this structure suffices for the correctness of classical logic when reasoning with these statements.[10] So, we will take the analysis of the structure of mathematical claims at face value.

2.1. Introducing mathematical vocabulary

Here is how you can "do arithmetic." You can join in to the discourse of arithmetic by adding the term '0', the function symbol '$'$' and the predicate N to your vocabulary, and by following the following norms.

$$\frac{}{\Rightarrow N0} \qquad \frac{}{\Rightarrow 0 = 0} \qquad \frac{}{x' = 0 \Rightarrow}$$

$$\frac{\Gamma \Rightarrow Nx, \Delta}{\Gamma \Rightarrow Nx', \Delta} \qquad \frac{\Gamma \Rightarrow x = y, \Delta}{\Gamma \Rightarrow x' = y', \Delta} \qquad \frac{\Gamma, x = y \Rightarrow \Delta}{\Gamma, x' = y' \Rightarrow \Delta}$$

$$\frac{}{Nx \Rightarrow x + 0 = x} \qquad \frac{}{Nx, Ny \Rightarrow x + y' = (x + y)'}$$

[10] For example, consider an approach that uncovers the "meaning" of a mathematical statement in terms of a conditional (if there is an ω sequence, then ...). You must show, for example, that the statement (if there is an ω sequence then $\neg A$) should either be equivalent to the negation $\neg$(if there is an ω sequence then A) or there should be an explanation of the divergence, for the mathematical statement $\neg A$ *appears* to be the negation of A, but on the conditional analysis of mathematical statements, appearances are deceiving.

$$\frac{}{Nx \Rightarrow x \times 0 = 0} \quad \frac{}{Nx, Ny \Rightarrow x \times y' = x \times y + x}$$

$$\frac{\Gamma \Rightarrow B(0), \Delta \quad \Gamma, Nx, B(x) \Rightarrow B(x'), \Delta}{\Gamma, Nx \Rightarrow B(x), \Delta}$$

This is a way to *add* arithmetic vocabulary to your already existing inferential repertoire. (This is why we included the predicate N for 'is a number'.) What we have is a sequent system for Peano Arithmetic, with one exception. We have placed no restriction on the judgement $B(x)$ to appear in the induction rule. $B(x)$ can be *any* statement, as long as the x is arbitrary.

What is the upshot of treating the new vocabulary as constrainted by these rules of coherence? The first feature making this differ from any *logic* you have already seen is that by following these rules, you are committed to $\Rightarrow (\exists x)(x = 0)$. We have *proved* that there is a number. What else can you prove? If your vocabulary merely contains arithmetical terms, then you will be committed to Peano Arithmetic (PA). If the predicates used in induction rule range more widely, you may commit yourself to something stronger.

Can we govern coherence in this way? Is it legitimate to "define" objects (like the number zero) into existence? It makes sense if we recall the treatment of other 'definitions'. Recall the connectives. If we find someone who is prepared to assert A, B but to deny $A \wedge B$, then we may be confident that the person is using '$\wedge$' in a non-standard way. The same goes with the predicate N and the term 0. If someone rejects $N0$ (as opposed to *eschewing* the vocabulary in which the claim is couched), then we may hold that she either has made a mistake, or she has not understood the predicate "N" or the term "0".

Now consider the benefits of understanding mathematical theorising and practice in this manner. It is telling you something about the *significance* of mathematical discourse, without explaining this using some particular model or class of models. The idea is not ontological economy[11] it is the direction of explanation. I shall discuss this further, below.

So, let us suppose that I teach you how to use arithmetical vocabulary according to these rules. You now judge arithmetical discourse using these canons of reasoning. (You take the denial of $2 + 2 = 4$ to be incoherent. You are committed to the validity of induction over all

[11] Think about it: the formal language itself provides us with an *omega* sequence of formulas. We already have enough ontology when we have a language to speak. Numbers add no more.

predicates in the vocabulary, etc.) Is there anything *else* needed for you to become a competent user of arithmetical vocabulary?

We can put the point colourfully: suppose we have two users of arithmetical vocabulary, equally competent with the rules we have considered, one of whom is in touch with '*the* ω sequence' and the other who is not. How would this difference manifest itself? How could we tell that we are reasoning like the one or the other?

Once you have added numerical vocabulary, there is no reason to stop there. You could add new prinicples to treat analysis, second order quantification, theroies of sets. I will not go into detail on this, but the aim ought to be to find a natural smooth axiomatisation of the extended mathematical theory, that goes as far as possible towards fixing the behaviour of the new vocabulary.

2.2. Consistency and conservativity

This addition of arithmetical vocabulary is not necessarily conservative over your pre-arithmetical vocabulary. As an extreme case, consider a finitist, committed to $(\forall x)(x = a_1 \vee \cdots \vee x = a_n)$ for some n. Given that you can prove the existence of more than n numbers, the theory becomes inconsistent with the addition of the rules. However, it is possible to regain conservativity, if you are a little more careful in the way in which you add your language, even this is possible. Divide the terms into two *types*. We have two forms of quantification, one for each type. Add the numerical vocabulary to the language as inhabitants of a new type, so you do not substitute an arithmetical name in the positions taken by names in the original discourse, and you don't substitute your original names in your arithmetical positions. You have two kinds of quantifiers, the quantifiers of arithmetic, and those of the original discourse. If arithmetic is consistent, then this addition is conservative: adjoin to any model of the old language, a disjoint domain of numbers, and let one set of quantifier range over the old domain, and the other over the new domain of numbers. This is a model of the expanded language, and the interpretation of the formulas from the original language is unchanged. So, the addition of numerical vocabulary and arithmetical theory in this way is conservative, if consistent.

Of course, it is possible to add a truly expansive *existential* existential quantifier, that binds an *ambiguous* variable which may be substituted into a spot appropriate for a variable of either type. The question then arises: which of the existential quantifiers is the appropriate one? Why choose a bifurcated language over one with a single categoy of object variables and a single category of objectual quantifiers?

Here is the view concerning the behaviour of mathematical theories and their semantics. We adopt a mathematical theory by (1) introducing new vocabulary (2) constraining our patterns of assertion and denial in that vocabulary in such a way that it (3) remains conservative over the pre-mathematical vocabulary.

3. Consequences of the View

Now I'll chart out consequences of the view, by comparing it to a number of extant positions.

3.1. Realism and 'Platonism'

On this view, to *use* arithmetical vocabulary is to commit yourself to mathematical ontology. Given that you use the vocabulary in this way, there is no sense in which you need further information as to whether or not there are any numbers. They exist, mind independently and necessarily. It is a kind of "thin realism" to use Maddy's vocabulary [18], combined with a semantic anti-realism. Given that existence is what is expressed by the existential quantifier, this is the natural and default position if we take the grammar of mathematical claims seriously.

(We have nothing against the project of people who tell us that there is a *stronger* notion of existence, which is not shared by mathematical objects. It remains for the proponent of these positions to articulate the kind of *stronger* notion of existence that they have in mind.)

The semantic anti-realism of this position means that we may use the machinery of the classical sequent calculus without starting with a notion of a *model* in which arithmetic must be interpreted, or by presuming that there is a structure out of which arithmetical claims inherit their truth. As a matter of fact, there *is* a model of our arithmetical theory (at least, you will see that there is, once you adopt the vocabulary of sets), but to say that arithmetical claims are true *because* of their relationship to some particular structure is to get the order of explanation backwards.

We agree with Platonism about the existence—and even, perhaps the *necessary* existence, and mind-independence, etc.—of mathematical objects. We part from Platonism concerning the direction of explanation of mathematical truth. One need not explain the significance of mathematical vocabulary by way of reference or representation of mathematical objects. To explain the truth of a mathematical claim in terms of the properties of the mathematical objects referred to in that claim is to not offer the only kind of explanation … (and in fact, to leave the core concept unexplained at all). Rather, another explanation is possible, in terms of coherence of the assertion and denial of mathematical claims: in other words, we take the *proof* to be the explanation.

3.2. Formalism

It might seem to follow that since we have identified mathematical explanation with *proof* that I am committed to a kind of formalism, since a proof here is a formal proof in a deductive system. And formalism, at least as far as the commitment to formally articulated proof as the means for establishing mathematical truth is concerned, is plausible.

However, the view articulated here does not mean that arithmetic is merely a particular first-order theory such as Peano Arithmetic (PA). One can prove facts about arithmetic using non-arithmetic vocabulary, as predicates in non-arithmetic vocabulary may be substituted into the induction rule as necessary.

So, for example, if arithmetic is embedded into a reasonably strong set theory (ZF will do), then, substituting sentences involving set membership and other predicates not definable in arithmetical vocabulary into the induction scheme will enable us to prove more claims—in arithmetical vocabulary—than we can otherwise. As one example, we can prove CON(PA) (which can be expressed in a first-order sentence in the language of PA) by using stronger induction schemes allowing for induction over sets as well as first-order sentences in arithmetic vocabulary.

Similarly, people have objected to logicism on the grounds that mathematical theorems are typically not proved inside an easily isolated formal system. Wiles' proof of Fermat's last theorem, uses a great deal of abstract mathematics beyond the first order theory of arithmetic. It seems that we cannot specify the formal theory in which Wiles proof obtains [27]. The response is straightforward: the rules of arithmetic *explicitly* give you a place to import other vocabulary. Arithmetic as I have defined it is open ended, depending on what *else* is in our vocabulary. If we extend our language to contain a new predicate B, and we did *not* concede that induction with respect to B worked, then this would be akin to not adding instances of *modus ponens* with respect to our new vocabulary when we expand our language.

The fact that the induction schema contains a space for *arbitrary* induction predicates means that our arithmetic theory is not static and fixed: it grows as our language grows. It cannot be identified with a particular first-order theory such as Peano arithmetic.

This does not mean that PA does not play a special role. Dan Isaacson holds that PA delineates the class of genuinely *arithmetic* truths [13].

For us, the reasoner who is committed to the vocabulary of arithmetic (and nothing else, or at least, nothing else that she can import into the induction rule) can prove only what is provable in PA. By her lights, denying CON(PA) is coherent, and so is asserting PROV$(0 = 1)$, at least when these are construed as sentences in the language of arithmetic.

3.3. Hilbert

This view is recognisably in the tradition of David Hilbert, because there is an important sense in which consistency (together with formality, which is required for conservative extension) is all that is required for mathematical existence. We may do for other mathematical theories what we did for arithmetic. If a mathematical theory (of sets, of categories, of whatever else) is consistent, we may add the new vocabulary to our own, giving rise to a richer vocabulary, conservative (if we are careful) over the old theory. This much of Hilbert's program is worth keeping.

Of course, Hilbert's program of finding certainty through finitist consistency proofs is dead. There are no finitist consistency proofs for any interestingly strong mathematical theory. This does not mean nothing to the general Hilbertian insight that there is no more to mathematical existence than consistency.

3.4. Carnap

This view is *Carnapian*, since the perspective of mathematical theories allows us to distinguish the internal and the external questions concerning mathematical existence. The question concerning whether there *really are* any numbers is answered *internally* by the user of the vocabulary (who is genuinely *asserting* and *predicating* and the like) in the affirmative. She can prove that $(\exists x)(x+5 = 12)$, so 7 *exists*. (Unless, of course, you have a stronger reading of existence claims, according to which we can existentially quantify over non-existent objects.) The *external* question is a different matter, and like Carnap, we answer the external question of whether or not we ought adopt a mathematical theory on pragmatic grounds. The nature of your answer will depend on the precise version of the question asked.

It's only a *modest* Carnapianism, and it is not refuted by Gödelian worries which spelled the end of Carnap's own program [10, Chapters 7 and 9]. For Carnap, there was taken to be a theory-independent and neutral analytic–synthetic distinction, a neutral perspective from which you could judge what was analytic in a theory. As we can see, the question of what is provable in a theory like PA requires *more* than PA to articulate, not less. Metamathematics is more mathematics, not a retreat from mathematical commitment.

Is this view beholden to a pernicious or implausible version of the analytic–synthetic distinction? It does not seem so. We do not need to identify the meaning of an expression with the rules governing coherence of assertions and denials involving the expression. We merely need to say that we can *introduce* (or *explicate*) vocabulary by treating it as constrained by some collection of rules for coherence.

3.5. Plenitudinous Platonism, and Fictionalism

Similarly, the more recent view of "plenitudinous platonism" holds that there the mathematical universe is as full as it can be. It is quite difficult to characterise 'plenitudinous platonism' [2, 21]. The motivating picture is that any kind of mathematical structure that *can* exist *does* exist. Our position provides a plausible reconstruction of the idea: any consistent mathematical theory may (if we like) be *adopted*, enriching our own mathematical vocabulary. The universe places no limit to the extent of mathematical theorising.

As Balaguer notes [2], fictionalism about mathematics is structurally quite similar to plenitudious platonism. For fictionalism, mathematical theories are made up, and we never need go to check that there are objects that the theories are talking about—beyond assuring ourselves that the theory is consistent [8, 9]. So far, we agree with fictionalism. However, instead of taking the posited theory to be *fictional*, we can take them to be true. There is no need to take mathematicians to be *mistaken* except for an overactive sense of ontological economy.

3.6. Structuralism

This view is *structuralist* [26] because the only general assurance that a mathematical theory is a conservative addition (if consistent) is when the new vocabulary is completely disjoint from our old vocabulary. A mathematical theory cannot be *about* cows or tables or chairs or whatever else we are talking about when we are not doing mathematics. It may, on the other hand, be *applied* to such things, by taking deductions and conclusions couched in mathematical vocabulary and applying them elsewhere. (Arithmetical facts are applied when we count cows or pay the bills, topological facts may be used in discussions of the large-scale structure of the universe, and there are many other applications besides.) Mathematical facts (the kinds of things to which we are committed in *using* mathematical vocabulary) are structural because they are reapplicable. In one sense, any *arithmetical* claim, because it is founded on a very simple base (i.e. the rules of arithmetic) may be re-applied to any structure on which those rules may be reinterpreted. In this sense, we have structuralism without having to give an account of what a mathematical structure is, since we have an alternative explanation of the meaning of mathematical vocabulary.

4. Miscellaneous Concluding Remarks

4.1. Using and Mentioning

The distinction between *adopting* a mathematical theory and *exploring* a mathematical theory plays an important role. We may not want to adopt *all* mathematical theories as they come up, in the same kind of way as I urged you to adopt arithmetic. We can, for example, adopt something rather strong such as ZFC and then *interpret* claims about, say non-wellfounded sets as claims about *graphs*, which themselves are thought of as particular sets in ZFC (ordered pairs consisting of a carrier set and a relation on that set) [1]. You learn ZFA (ZF with the *anti*-foundation axiom) by translating it into your native tongue. You *can* do this, but you may find that when you do so, you begin to speak ZFA like a native, and cease to translate it.

On the other hand, there is much to be said for keeping your mathematical vocabulary (in essence) very small by adopting a set theory such as ZFC and translating other vocabulary into it as necessary. This ensures that the addition of new vocabulary (if interpretable within your set theory) will not come at the cost of consistency given that you have already paid the price of adopting your favoured theory of sets.

Similarly, you can do mathematics of particular structures without adopting the vocabulary of that discourse at all. You can do it by *mentioning* the vocabulary and not *using* it. You could (using the language of *formulas* and *proofs*) consider whether or not a particular sentence follows from some set of axioms. You could, if you wish, say that "$2 + 2 = 4$ is a theorem of PA" without taking it that $2 + 2 = 4$. (This may be the attitude of mathematicians exploring set theoretical axioms that they do not take to be plausible [16, 17].)

4.2. Ontology and Epistemology

It answers the ontological question of the existence of mathematical objects in two ways. Firstly, given the vocabulary that we *use*, the internal question has a straightforward answer. There are numbers. There are sets. They exist necessarily and independently of us. (That last claim is not a part of the mathematical theory. It will follow from a decent theory of modality and dependence.) However, we do not need to explain mathematical knowledge by means of "contact" with the realm of mathematical objects. The *general* question (what about new kinds of mathematical objects that we haven't considered?) can only be answered piecemeal. It seems that whatever language we adopt, we can add *more*. (Reflection principles seem to ensure that whatever mathematical theory we adopt, it may be extended. CON(T) adds new sets/structures over T.)

The epistemic question of how we come to *know* mathematical truths also has a two-track answer. Given particular mathematical concepts we may draw consequences on the basis of traditional deductive argument. The more interesting question is why we use concepts such as the ones that we have in the ways that we do. For this, different kinds of answers are available. A *pragmatic* answer will explain the choice of some vocabulary rather than another. This seems to do justice to the kinds of discussions set theorists have concerning open questions such as the continuum hypothesis.

Consider the position of the mathematician exploring the theory of sets. The best theory commits us to GCH $\vee$ $\neg$GCH, but it seems that it leaves open which disjunct is true. Contemporary set theory is a complicated affair in which the search is on for different considerations in favour of GCH or $\neg$GCH. The set theorist is attempting, in these circumstances, to articulate and sharpen our account of the concept "set" in ways that satisfy sensible desiderata, such as the goal to MAXIMISE the set-theoretical universe [16, 17, 18]. This is quite sensible, given the set-theoretical goal of finding a large home (or 'vocabulary') in which to interpret or translate all different kinds of mathematics. The kind of *freedom* involved in this exercise explains both the appeal and the coherence of staggeringly large cardinal axioms.

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