ALWAYS MORE

Greg Restall*

School of Philosophy,
Anthropology and Social Inquiry
The University of Melbourne
restall@unimelb.edu.au

SEPTEMBER 18, 2009

A possible world is a *point* in logical space. It plays a dual role with respect to propositions.

(1) A possible world determines the truth value of every proposition. For each world w and proposition p, either at w, p is true, or at w, p is not true.

(2) Each set of possible worlds determines a proposition. If $S \subseteq W$ is a set of worlds, there is a proposition p true at exactly the worlds in S.

Perhaps such a proposition is not expressible in any language that you or I speak, but – so a familiar story goes – it is decided by each world, so it plays just the role that other propositions do, so it counts as a proposition in the same way. In fact, we can see just how it counts as a proposition: given all the worlds in S, our proposition p says that the world is one of the worlds in S. It describes a way the world is, even if we have no means of picking out the set S, so it is a proposition.[1]

But does this talk of possible worlds actually make sense?

Metaphysical worries about worlds are well known. These worries do not concern the role they play in the analysis of propositions: they call into question the 'otherness' of worlds, the profligacy of admitting locales where there are tailless kangaroos or blue swans. Worries of this sort can be assuaged by

*Thanks to the Logic Seminar at the University of Melbourne (especially Allen Hazen and Lloyd Humberstone), Shawn Standefer, and the audience at *Logica* 2009 for discussions on these matters. ¶ This research is supported by the Australian Research Council, through grant DP0343388, and Sam Phillips' *Don't Do Anything*.

[1] David Lewis' *On the Plurality of Worlds*, Chapter 1 [2], has a very good defence of this position on the relationship between worlds and propositions, but the view is not just his. The view is everywhere.

giving an account of worlds which takes them to be abstract, or fictions, or in some other way less real than the world you and I are thought to inhabit. Ontological profligacy is not so much of a concern if we have understood worlds in a metaphysically thin manner. The fact that this concern is so easily sidestepped shows that this concern is does not touch (1) and (2) – the properly logical notion of a possible world. In this paper, I will to consider the logical structure of commitment to (1) and (2). Do claims such as (1) and (2) have any unforeseen *logical* costs?

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It would seem like there is little reason to reject (1) and (2). To transpose talk of worlds into an algebraic key, structures satisfying (1) and (2) are well known. They are complete atomic Boolean algebras. In such algebras, atoms play the role of possible worlds: at each atom, the propositions *entailed* by that atom can be taken to be true, and the others are false. The fact that the algebra is complete means that every collection of atoms determines a proposition in the appropriate way: any set of atoms has a least upper bound, which is true at those and only those atoms. Atomicity gives us (1) and completeness gives us (2). If there are reasons to reject the combination of (1) and (2), then the construction of complete atomic Boolean algebras must somehow not apply.

In this paper, I will construct a logic, extending classical logic with a single unary operator, which has *no* complete Boolean algebras as models. If the family of *propositions* we are talking about in (1) and (2) has the kind of structure described in that logic, then (1) and (2) cannot jointly hold.

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The new operator, #, may be introduced in a straightforward manner. Here is the first cut at an account of #. Take a propositional language with the infinite supply of atoms $p_1, p_2, p_3, \ldots$ and define '$\#A$' to be the first propositional atom not occurring in the formula A.[2]

\# has interesting logical properties. Since $\#A$ is an atom not occurring in A, if A is satisfiable, so is $\#A \wedge A$, and so is $\neg\#A \wedge A$. In fact, if we can derive A from $\#A$, then A is a tautology. Similarly, if $\#A$ is derivable from A, then A itself is unsatisfiable. We have the following four principles[3]

[2] In other words, for now, $\#p_1$ *is* the atom p_2. So, $\#\#p_1 = \#p_2 = p_1$, even though there is a sense that 'p_1' does 'occur in' $\#\#p_1$. That is not the relevant sense here.

[3] Where, as usual, we take $X \vdash Y$ to hold if and only if there is no evaluation where each member X is true and each member of Y is not. So, $X \vdash$ when X cannot all be true together, and $\vdash Y$ when Y cannot all be false together.

$$\begin{array}{ll} \text{If } \#A \vdash A \text{ then } \vdash A & \text{If } \vdash A, \#A \text{ then } \vdash A. \\ \text{If } A \vdash \#A \text{ then } A \vdash. & \text{If } A, \#A \vdash \text{ then } A \vdash. \end{array} \tag{\#}$$

Now, as defined, #A is not anything like a connective: it is a syntactic device. It is not a congruence with respect to logical equivalence, since $\#p_1 = p_2$ but $\#(p_1 \wedge (p_1 \vee p_2)) = p_3$, even though p_1 is logically equivalent to $p_1 \wedge (p_1 \vee p_2)$.

We can remedy this by setting #A to be defined as the first propositional atom which is not in *some* formula equivalent to A. Then this satisfies substitutivity of equivalents. Now, $\#(p_1 \wedge (p_1 \vee p_2)) = p_2$, since there is some formula equivalent to $p_1 \wedge (p_1 \vee p_2)$ (namely, p_1) in which p_2 doesn't occur, but there is *no* formula equivalent to $p_1 \wedge (p_1 \vee p_2)$ in which p_1 doesn't occur. It is straightforward to verify that # so defined still satisfies the four conditions given in (#).

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Now, consider the logic extending classical propositional logic with an operator # satisfying the four (#) conditions.[4] A logic of this form can have well-defined models. We have seen one, with # defined syntactically. Logics extending classical logic with # make sense, and are coherent. There is nothing inconsistent or incoherent in the logic of #.

However, the logic is still *odd*. While a logic like this can have a Boolean algebra as a model — the Lindenbaum algebra of equivalence classes of provably equivalent formulas will do as an example — they have no atomic Boolean algebras as models. Recall: a is an *atom* in a Boolean algebra if for every element x either $a \wedge x = 0$ (the bottom element in the algebra) or $a \wedge x = a$. There are no elements between 0 and a. An algebra is said to be *atomic* if every element is the join of some collection of atoms.

Now, since every finite Boolean algebra is atomic, every model for the logic will be *infinite*. But not every infinite Boolean algebra will work, either. The algebra of all subsets of some infinite set — ordered by inclusion, and with the usual Boolean operators of intersection, union and complementation — will not do either, since each singleton set is an atom.

Here is why no algebra for # is atomic. Take a Boolean algebra with an atom a. Consider $\#a$. By the conditions (#), since the atom a is neither 1 nor 0 (it

[4] I will not call them *rules* for they are not inference rules *defining* the connective #. A language can contain two independent operators $\#_1$ and $\#_2$ both satisfying the conditions (#). In fact, one way to understand Kaplan's paradox over the size of the collection of possible worlds is to think of 'I believe that p' as #p. For it seems that whether I believe p or not is genuinely logically independent from p, at least when p is logically contingent [1]. I owe this observation to Allen Hazen.

is neither a tautology nor a contradiction) then neither $a \wedge \#a$ nor $a \vee \#a$ are tautologies nor contradictions. But this is inconsistent with a's being an atom, for $a \wedge \#a$ entails a but is not the bottom element of the algebra. So, the algebra is not only not atomic, but it cannot contain *any* atoms.

This talk about *algebras* and *atoms* has consequences for theories of *worlds*. (1) and (2) commit us to taking the collection of propositions to be an atomic Boolean algebra. If each proposition is modelled by the set of worlds in which it is true, and if *every* set of worlds models a proposition, then each *singleton* set of worlds is an atom. It is true *somewhere*, but there is no non-trivial proposition stronger than it. This rules out #, or it rules out taking (1) and (2) to jointly hold.

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Perhaps # is a mere syntactic device. Is it an artefact of the presentation of a language with infinitely many atomic sentences? Can we specify a model for a # satisfying the (#) conditions in which the language of sentences plays no special role? I will explain here how we can construct just such models, to provide a language-invariant structure in which the Boolean connectives and # may be interpreted. Here's how.

Let's think in terms of worlds, to start. Take the set W of *worlds* to be the irrational numbers in the Real Line. The propositions at *Level* n are the unions of the any selection of irrational intervals of length $\frac{1}{2^n}$: $(\frac{z}{2^n}, \frac{z+1}{2^n})$ where z is an integer. These are closed under union (the union of any collection of intervals is a collection of intervals), intersection (there is no worry about endpoints of abutting intervals, as these don't reach their endpoints, which are rational) and complement (the complement of some collection of intervals is the collection of the other intervals: since the endpoints are rational, they don't occur in either a set or its complement). The propositions at each level are *finer* classifications of points than at any of the previous level.

Propositions at Level 0 are collections of intervals such as $(-2, -1)$, $(0, 1)$, $(3, 4)$, etc. Propositions at Level 1 are collections of *finer* intervals $(-1.5, -1)$, $(1.5, 2)$, $(3, 3.5)$, etc., and so on, throughout each finite level.[5]

Let's interpret sentences in the language of propositional logic—enhanced with the operator '#'—as propositions at *some* level or other. If A and B are interpreted as propositions at Level n, then $\neg A$, $A \wedge B$ and $A \vee B$ are also interpreted as propositions at Level n, since the union, intersection or complement of propositions at Level n are also at Level n.

[5] We could just as easily do this with regions on a grid in irrational 2-space, or cubes in 3-space, etc. We have all the generality we need in one dimension, however.

To interpret #A, where A is interpreted as a proposition at Level n, we will choose a proposition at Level $n+1$. In particular, we will choose an *alternating* proposition at Level $n+1$: the proposition consisting of all of the intervals $(\frac{z}{2^n}, \frac{z+1}{2^n})$ where z is even integer. The alternating proposition at Level 0 is

$$\cdots(-4,-3)\cup(-2,-1)\cup(0,1)\cup(2,3)\cup(4,5)\cdots$$

the alternating proposition at Level 1 is

$$\cdots(-2,-1.5)\cup(-1,-0.5)\cup(0,0.5)\cup(1,1.5)\cup(2,2.5)\cdots$$

and so on. This choice for #A satisfies the four conditions given in #. Let A be interpreted as a proposition at Level n. If A is not true everywhere, then it is false at some interval $(\frac{z}{2^n}, \frac{z+1}{2^n})$. Now consider #A. It is true at $(\frac{2z}{2^{n+1}}, \frac{2z+1}{2^{n+1}})$, where A is not true. And #A is not true at $(\frac{2z+1}{2^{n+1}}, \frac{2z+2}{2^{n+1}})$, where A is not true. Similarly, if A is not false everywhere, it is true at some interval $(\frac{z}{2^n}, \frac{z+1}{2^n})$. #A is true at $(\frac{2z}{2^{n+1}}, \frac{2z+1}{2^{n+1}})$, where A is true. And #A is not true at $(\frac{2z+1}{2^{n+1}}, \frac{2z+2}{2^{n+1}})$, where A is true. In other words, #A is truly independent of A. If A is true somewhere, at some such places, #A is true, and at others, #A is false. If A is false somewhere, at some such places, #A is true, and at others, #A is false.

This fact is completely general. We for any proposition A we have found another proposition #A. #A is more finely grained than A, and the four rules of extensibility are satisfied. In models like these, it makes sense to think of '#' as an operator on propositions, and not merely a syntactic device for constructing sentences from other sentences. The language may now be finite, or indeed it may have *no* non-logical constants![6]

So, we have a syntax-free model in which the four conditions (#) hold, so we must have either of (1) and (2) failing if we are to think of these points as *worlds*. It is easy to see which. Not every set of worlds is a proposition. Only some sets of points — those at some Level or other — count as a proposition. Others are not.

However, we do not need to think of this model in that way. We could, instead, take (1) to fail, if we wish to avoid commitment to worlds altogether. The appeal to worlds in these models is not essential: we could instead refrain from all talk of worlds and appeal instead to *regions* in a formal topological space. The definition of propositions in terms of sets of points—irrational numbers in our case—is not essential. The construction gives us an atomless Boolean algebra, and these are well known algebraic structures. The value of the relatively

[6] If there are no non-logical constants, but the logical constants $\bot$ and $\top$, then we can still construct the alternating propositions at each level, and a whole host of other propositions. Consider, for example, what $\#\top \wedge \#\#\top$ and $\#\top \wedge \neg\#\#\top$ are, to get a feel for what propositions may be constructed.

concrete construction here is the manner in which extensibility corresponds to propositions being more and more finely grained, without that ever coming to an end. The model shows that the idea of indefinite extensibility of propositions is coherent: and operators like # are one way to give formal structure to the intuitive idea that the collection of propositions is indefinitely extensible. Wherever we find ourselves in the collection of propositions, we haven't exhausted its depths. For any proposition at all, there is always more.

REFERENCES

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[2] DAVID K. LEWIS. *On the Plurality of Worlds*. Blackwell, Oxford, 1986.