# ASSERTION, DENIAL, COMMITMENT, ENTITLEMENT, AND INCOMPATIBILITY (AND SOME CONSEQUENCE)

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*Abstract*: In this short paper, I compare and contrast the kind of *symmetric* treatment of negation favoured in different ways by Huw Price (in "Why 'Not'?") and by me (in "Multiple Conclusions") with Robert Brandom's analysis of scorekeeping in terms of commitment, entitlement and incompatibility.

Both kinds of account are what Brandom calls a *normative pragmatics*. They are both semantic anti-realist accounts of meaning in the significance of vocabulary is explained in terms of our rule-governed (*normative*) practice (*pragmatics*). These accounts differ from intuitionist semantic anti-realism by providing a way to distinguish the inferential significance of "A" and "A is warranted." Although proof plays a central role, in neither accont is verification the primary bearer of meaning. Our accounts make these distinctions in terms of a subtle analysis of our practices. On the one hand according to Price and me, we assert as well as deny; on the other, Brandom distingushes downstream commitments from upstream entitlements and the notion of incompatibility definable in terms of these. In this paper I will examine a number connections between these different approaches, and end with a discussion of the kind of account of *proof* that might emerge from these considerations.

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This paper consists of a series of vignettes exploring the issue of the structure of inferential relationships between premises and conclusions. The sections are connected in two ways. (1) Each takes the dual role of *assertion* and *denial* to be central in characterising logical consequence. (2) As a whole, they form a plea for philosophers and logicians to take a *liberal* view of the kinds of structure of proof. This paper is a part of a larger project of using insights

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from proof theory to understand the normative significance of logical concepts [10, 11].

Semantic anti-realists have a number of different options for their explanatory primitives, when it comes to articulating the behaviour of logical consequence, incompatibility, and related notions. I will explore some of these options (in particular, choices for how to connect consequence and incompatibility), and I will defend a set of tools for looking at these connections.

## 1 CONSEQUENCE FROM INCOMPATIBILITY

Consider this quote from Robert Brandom, taken from an essay on Hegel and objective idealism.

> ...relations of determinate negation allow the definition of *consequence relations* that are modally robust in the sense of supporting counterfactual inferences ... The proposition or property $p$ entails $q$ just in case everything incompatible with (ruled out or excluded by) $q$ is incompatible with (ruled out or excluded by) $p$.
>
> "*Holism & Idealism in Hegel's* Phenomenology," p. 180

There is something compelling in this picture. To make a claim is to rule something out. It seems *relatively* clear that once one has incompatibility between claims, one can define something like a kind consequence between claims. Let's see what we can do with this. If we have a relation $\perp$ of incompatibility, define $A \vdash_I B$ (*incompatibility consequence*) as follows:

$$\forall C(\text{if } \perp B, C \text{ then } \perp A, C)$$

This is a plausible constraint way to connect consequence and incompatibility. Nonetheless, it is not available for the friend of intuitionistic logic, if $\vdash_I$ is intuitionist logical consequence, and $\perp$ is intuitionist incompatibility.

Recall that for intuitionists $\sim\sim A \nvdash A$, and $\perp B, C$ iff $C \vdash \sim B$. We will show that $\sim\sim A \vdash_I A$. Suppose $\perp A, C$. Then $C \vdash \sim A$. Since $\sim A \vdash \sim\sim\sim A$, $C \vdash \sim\sim\sim A$. It follows that $\perp\sim\sim A, C$. Therefore, $\sim\sim A \vdash_I A$.

The upshot is straightforward: For friends of intuitionistic logic, content and consequence cannot be characterised by incompatibility, because $A$ and $\sim\sim A$ are incompatible with exactly the same statements.

We have one consideration, then, on purely anti-realist grounds, to reject intuitionist logic. (These aren't particularly *strong* grounds, of course.) Nonetheless, if you take incompatibility to be one of the basic materials for the construction of your theory of concepts (as the "Hegelian" Brandom of *Holism & Idealism* does), and if you take consequence to be related to incompatibility in the way we've seen, then intuitionistic logic is not for you. This does not mean, of course, that *classical* logic is for you, but it does constrain the kinds of logical options available to you. One way to read the constraint is as a kind of *separability* condition. If $A \nvdash B$, then there is some $C$ where $\not\perp A, C$ (so $C$ is compatible with $A$) but $\perp B, C$ (so $C$ is incompatible with $B$).[1]

## 2 INCOMPATIBILITY FROM CONSEQUENCE

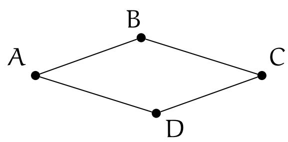
Now let us see what happens when we connect incompatibility and consequence in a *different* way. We'll try to define incompatibility in terms of consequence. We will assume that we have a consequence relation $\vdash$, at our disposal, relating a premises to a conclusion. (So, it makes sense to say that $X \vdash A$, where $X$ is a set of premises and $A$ is a single conclusion.) Consider options for defining $\perp X$ in terms of $\vdash$.

### 2.1 ENTAILING EVERYTHING

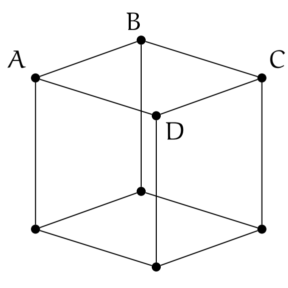
Here is a straightforward attempt at defining incompatibility from consequence:

ATTEMPT 1 $\perp X$ if and only if $X \vdash C$ for *every* $C$. (Incompatibility is *Post-inconsistency*.)

As a *static* analysis, this has something going for it. However, if we are interested in the case where a language is *augmented* by new material, it fails. Consider a language with just four primitive statements, $A$, $B$, $C$ and $D$, ordered by $\vdash$ as follows:




In this structure, $X \vdash E$ holds iff the greatest lower bound of the elements of $X$ to be less than or equal to the object $E$. It follows, then, that $A, C$ entails everything, and hence, $\perp A, C$. However, we can *enlarge* the structure, to get a *new* structure:




[1] Yes, this *does* rewrite the negated universal claim $\sim\forall C(\text{if } \perp B, C \text{ then } \perp A, C)$ as an existentially quantified $\exists C(\not\perp A, C \text{ and } \perp B.C)$, which is not accepted by intuitionists. However, since we have already seen that this approach is not favoured by intuitionists, it is hard to see that the quibble is worth any worry.

in which the facts about logical consequence are preserved (for all choices from the old structure, $X \vdash E$ in the old structure, if and only if $X \vdash E$ in the new one) but incompatibility has changed. Now $A$ and $C$ are ***no longer*** incompatible, since $A$ and $C$ together do not entail any of the new elements below them.

I take that to be a consideration against ATTEMPT 1. Even though consequence is preserved from the old structure into the new structure, incompatibility as defined by ATTEMPT 1 is not. If incompatibility is one of the 'inferential' features of a structure, then the inferential features must be more than what is recorded in the validities of the from $X \vdash A$ with multiple premises and a single conclusion — at least if we are ever concerned with the prospect of transitions from one structure to another, larger one.

## 2.2 ENTAILING f

Now for another attempt at defining incompatibility: Instead of defining $\perp$ in terms of $\vdash$ alone, we bring in a special statement $f$.

ATTEMPT 2 $\perp X$ iff $X \vdash f$.

Now *this* is preserved when we go from one structure to a larger one, as whatever entails $f$ in the small structure entails $f$ in any extension, as $f$ retains its place. But I don't think this is the *heart* of the matter either, because it only makes sense as an analysis in contexts where we have this special statement $f$. It seems that we can *define* incompatibility in *more* cases than those.

For example, consider a 'language' with two statements $A$ and $B$ where we have $A \nvdash B$ and $B \nvdash A$, and $\perp A, B$. This *seems* to be a coherent structure. But, according to ATTEMPT 2, $A$ and $B$ are not incompatible since there is no statement that both entail. In this structure there is no candidate $f$. (Of course, this 'language' is, deficient in some sense, because it cannot express the conjunction of $A$ and $B$. You might want to say that the statement $f$ is *implicit*, rather than *explicit*.)

How can we respect this possibility? One way to do this is to allow $f$ to be outside the language. ($f$ is, if you like, an 'ideal element.') Now, consider what '$A, B \vdash f$' might mean. If '$A, B \vdash C$' is the trace of a deduction starting at $A$ and $B$ and ending at $C$, then '$A, B \vdash f$' is the trace of a deduction from $A$ and $B$ and ending at … *what*?

A deduction for $A, B \vdash f$ (which we might call a *refutation* of $A, B$) is a proof starting at $A$ and $B$ and *without a concluding formula*. (Consider proofs that end "Contradiction!" You can think of them as stepping from a particular contradiction to *nowhere*.)

The idea of a proof with no *conclusion* should not be foreign to you. We are familar with the idea of a proof with no *premises*. Strictly speaking, this does not mean that the proof started nowhere, or that it featured no premises during the construction of that proof. All it means is that the premises have been *discharged*. Can something similar not be the case for conclusions? It seems that the stage in a proof where we have just performed a *reductio* on a hypothesis, inferring an absurdity from the premises $X$, could be such a case. We have a *refutation* of the collection of premises when we turn our attention away from the particular contradictory conclusion, to say *that's absurd*, and *then* go on to conclude the negation of one of the premises, or whatever else we do. It seems that it is not much of a jump to think of $A, B \vdash f$ as $A, B \vdash$, which records a proof with *no* conclusion, but which refutes its collection of premises.

This brings us to the first tentatve *finding* of our investigations:

TENTATIVE FINDING 1: If you want to think of incompatibility as defined by consequence, then think of consequence as not only relating premises not just to a single conclusion ($X \vdash A$) but to also allow *refutations* ($X \vdash$), which allow premises without a conclusion.

But if you allow (1) deductions with many premises and a single conclusion and (2) deductions with many premises and *no* conclusion, then what is stopping us from considering (3) deductions with many premises and many conclusions? One concern is the worry that multiple conclusions don't make any sense. I want to assuage that concern in what follows.

## 3 ASSERTION AND DENIAL, COMMITMENT AND ENTITLEMENT

Opponents of multiple-premise–multiple-conclusion consequence reject the idea for a number of different reasons. The most substantial is that it is hard to read $X \vdash Y$ in terms of preservation of warrant to assert.

### 3.1 WARRANT

If $X \vdash A$, then if you have warrant to assert each member of $X$, you have warrant to assert $A$. If $X \vdash$, then you don't have warrant to assert each member of $X$. If $X \vdash Y$ (for example, $A \vee B \vdash A, B$) then if you have warrant to assert each member of $X$ then you have warrant to do *what*?

Well, the one thing you *don't* have warrant for is to assert $A \vee B$ and to *deny* both of $A$ and $B$. In general, $X \vdash Y$ iff there could be no warrant to assert each member of $X$ and deny each member of $Y$. For this to work, you need a few principles connecting assertion and denial. (In particular, denying $A$ will not necessarily be asserting $\sim A$, at least for the friend of truth-value gaps or gluts.) See my paper "Multiple Conclusions" for more on this [9]. I think that these theses are defensible on purely anti-realist grounds.

### 3.2 BILATERALISM, COMMITMENT AND ENTITLEMENT

Keeping assertion and denial as important theoretical primitives is one way to be *bilateral*. (cf. Price's "Why 'Not'?" [7] and "'Not' Again" [8]) There are other ways to introduce "bilateral" elements into one's account of concepts. The Brandom of *Making it Explicit* [1] and *Articulating Reasons* [2] does so in terms of *commitment* and *entitlement*. For Brandom, an agent's commitments and entitlements help constitute the dialectical *score* in the game of giving and asking for reasons. Incompatibility is then defined in terms of commitment and entitlement: $\perp A, B$ iff *commitment to* $A$ *precludes entitlement to* $B$.

I find this account suggestive, but obscure. (1) It is hard to get to grips with the formal properties of commitment and entitlement, and (2) *preclusion* is also underspecified.

Here's one story connecting multiple-premise–multiple-conclusion sequents with the language of commitment and entitlement. Any pair $[X : Y]$ of sets of statements constitutes a POSITION. In a position $[X : Y]$ the elements of $X$ are explicitly ASSERTED and the elements of $Y$ are explicitly DENIED. (Or you can say that $X$ is ENDORSED and $Y$ is CHALLENGED.) A position $[X : Y]$ is *coherent* iff $X \nvdash Y$.

Now for commitment and entitlement: we can use consequence to *define* them, relative to positions.

- A position $[X : Y]$ is *committed* to $A$ iff $X \vdash A, Y$.
- A position $[X : Y]$ is *entitled* to $A$ iff $X, A \nvdash Y$.

The commitments (relative to a position) cannot be denied (in that position), at the risk of incoherence. The entitlements *can* be coherently asserted. This notion of 'entitlement' is very weak—they are entitled as assertions only in the mild sense that they are not absolutely ruled out on grounds of coherence.

Brandom's analysis of incompatibility in terms of coherence and entitlement is consistent with this account of commitment and entitlement. If $\vdash A, B$ (that is, $A, B \vdash$) then if $[X : Y]$ is committed to $A$ (that is, $X \vdash A, Y$) then by *cut*, $X, B \vdash Y$ (that is $[X : Y]$ is not entitled to $B$). Conversely, if commitment to $A$ doesn't preclude entitlement to $B$, there is some coherent position $[X : Y]$ at which $A$ is a commitment and $B$ is not an entitlement. If follows that $X \vdash A, Y$ and $X, B \vdash Y$. So, if $A$ and $B$ *are* incompatible, then $A, B \vdash$, and the cut rule would give $X \vdash Y$, contrary to the coherence of $[X : Y]$.

It is one thing to have a bilateral account of consequence: this is enough to connect some kind of notion of consequence to normative status concerning both assertions and denials. This is, it seems to me, unproblematic. However, it is another thing entirely to connect this notion of consequence to a notion of *proof*. Sequent systems are useful in giving an account of inferential relations between collections of statements. However, it has seemed to many that a statement of consequence is a record of a *proof*, a proof is a way to get from premises to conclusions. If that is the case, then what kind of proof might be recorded in a multiple-premise, multiple-conclusion sequent? Can the picture be as *attractive* as the compelling picture of intuitionistic natural deduction? Can it satisfy the kinds of theoretical constraints (normalisation, separability, etc.) appropriate for natural deduction systems?

Before sketching the kind of proof structure for which a multiple-premise multiple-conclusion sequent is most naturally suited, it is worth seeing why it does not fit so well with traditional natural deduction proofs, such as those made popular by Fitch [3], Lemmon [5] or Prawitz [6]. The crucial feature of proofs as defined in each of these popular systems is that they can have any number of premises at any one time, but they always have a single *conclusion*. at any stage of the development of the proof, the *perimeter* (the undischarged premises and current conclusion) form a sequent of the form $X \vdash A$. The proof from premises $X$ to conclusion $A$ is a license to infer $A$ from $X$. Steps in developing a proof (say, discharging the assumption of $A$ in a proof to the conclusion $B$ in order to get the new conclusion $A \to B$) can be seen as taking one from one sequent (the proof is first from $X, A$ to $B$) to another (the new proof takes one from $X$ to $A \to B$).

The step to proofs with *no* conclusion can be incorporated without much fuss: instead of thinking of a proof to a reductio to be a proof to this or that special conclusion, we could introduce a new punctuation mark (say '#') to mark *refutations* of premises. Instead of the usual inference rule indicating that some marked out *contradiction* is the conclusion from the two premises $A$ and ${\sim}A$, we could have a special proof structure, of the form

$$\frac{A \qquad {\sim}A}{\#}$$

in which there is officially *no* conclusion, but is merely a refutation of the premise collection $\{A, {\sim}A\}$. (In fact, this is the official account of natural deduction in Neil Tennant's *Natural Logic* [12].)

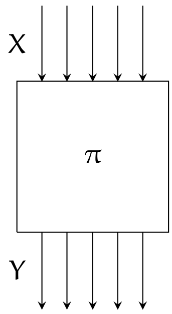
However, matters get more complex when it comes to sequents with multiple conclusions. We wish to have proofs of the structure $A \vee B \vdash A, B$, in which we have a single premise and *two* conclusions. If a proof is a tree structure with one formula at the root, then there seems to be no position, other than the root, at which to place a formula as a conclusion.[2] One option is to use *negation*, and to move from $X \vdash A, B$ to $X, {\sim}B \vdash A$. This is artificial, for it provides more than one way to represent a sequent, depending on which conclusion formula is actually honoured with the position in the conclusion of the proof.

The most direct way to represent proofs which stand to $X \vdash Y$ sequents as natural deduction proofs stand to $X \vdash A$ sequents is to look a these sequents, and to devise structures that parallel them exactly. This is our task in the next section.

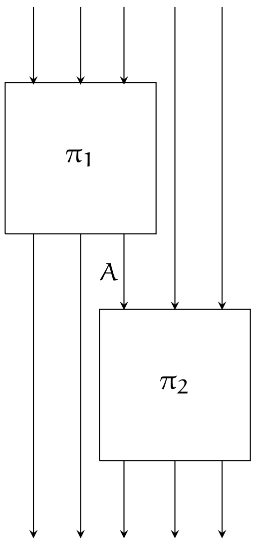
[2]There are very interesting proposals to modify tree proofs to allow this, for example by means of a 'restart' rule [4]. A proof from $X$ to $A$ can *restart* by dropping the conclusion $A$ and adding a new conclusion $B$ (in effect, stepping from $X \vdash A$ to $X \vdash A, B$).

## 4 PROOF STRUCTURE

In a multiple-premise, multiple-concluson logic, a *proof* for $X \vdash Y$ has each element of $X$ as an *input* and each element of $Y$ as an *output*. It will be simplest to represent proofs as directed graphs, where formulas label *edges* and rules label *nodes*. We can depict such a strucure like this:

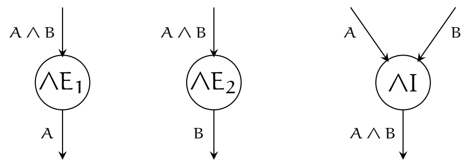



The simplest proof is the *identity* proof with the single premise and conclusion $A$. It is a naked edge labelled with $A$. The *cut* rule gives an account of how proofs may be chained together:

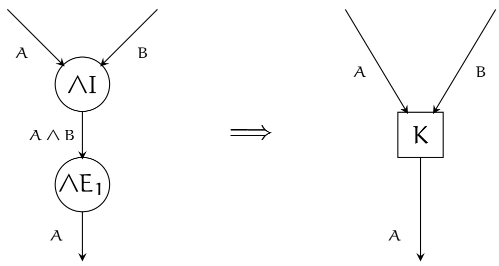



Given two proofs: $\pi_1$ and $\pi_2$, where $A$ is an output of $\pi_1$ and an input of $\pi_2$, we get a new proof chaining $\pi_1$ to $\pi_2$. The *inputs* of the new proof are all of the inputs of $\pi_1$, together with the inputs of $\pi_2$ other than the $A$ input we singled out, and the *outputs* are the outputs of $\pi_2$ together with all of the outputs of $\pi_1$ other than the $A$ output. This is the *cut* rule: If $X \vdash Y, A$ and $A, X' \vdash Y'$ then $X, X' \vdash Y, Y'$.

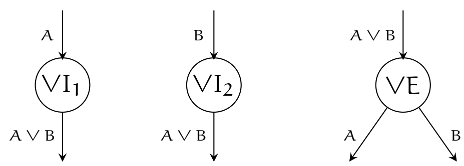
The conjunction rules are straightforward, and should be familiar. We write them using the "arrow and node" notation, though the effect is identical to the traditional rules for conjuction:

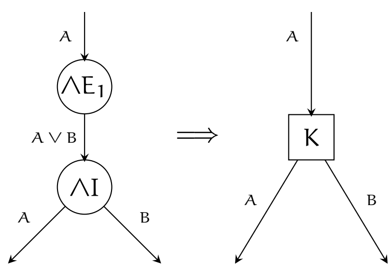



Pairs of ∧I/∧E links can be removed in a *normalisation* process.




This process is simple and *local* if you add 'weakening' links, which take two inputs, and return one of the inputs as an output.) As you can see, the disjunction rules are *exactly* dual to the conjunction rules. Normalisation works as before.




The 'traditional' disjunction elimination rule:

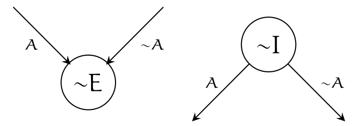
$$\dfrac{A \vee B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}$$

can be seen as a 'reworking' of these rules to satisfy the single conclusion constraint. The negation rules are the *most* radical departure from traditional natural deduction, and they take the largest advantage of the multiple-premise/multiple-conclusion structure of inference. These rules take their cue from the classical

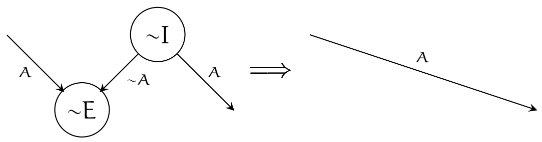
Gentzen rules

$$\frac{X \vdash A, Y}{X, {\sim}A \vdash Y} \qquad \frac{X, A \vdash Y}{X \vdash {\sim}A, Y}$$

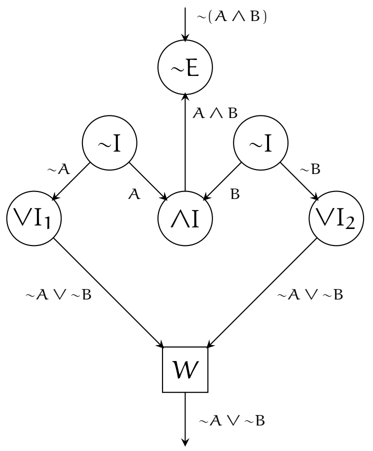
Our reading of these rules is that an output $A$ can be traded in for an input ${\sim}A$, and an input $A$ can traded in for an output ${\sim}A$.




(Think of these nodes in use. For $({\sim}E)$, you can plug an output $A$ of a proof $\pi$ into the $({\sim}E)$ node, which uses up that output, and gives the proof a new $A$ input. Similarly for $({\sim}I)$.) Notice that normalisation is trivial (an introduction and elimination step for ${\sim}A$ is rewritten by an $A$ arrow).




Here is a proof and a corresponding sequent derivation, for the intuitionistically unprovable de Morgan law ${\sim}(A \land B) \vdash {\sim}A \lor {\sim}B$.




$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \land B}}{{\sim}(A \land B), A, B \vdash}}{{\sim}(A \land B), A \vdash {\sim}B}}{{\sim}(A \land B) \vdash {\sim}A, {\sim}B}}{{\sim}(A \land B) \vdash {\sim}A, {\sim}A \lor {\sim}B}}{{\sim}(A \land B) \vdash {\sim}A \lor {\sim}B, {\sim}A \lor {\sim}B}}{{\sim}(A \land B) \vdash {\sim}A \lor {\sim}B}$$

As you can see, the structures of these circuits, or *proofnets*, mirrors the multiple-premise, multiple-concluson nature of sequent derivations closely. Although these circuits are not as *familiar* as trees as structures for proofs, they are enough like proofs to reassure us that classical logic has an inferential basis in the same manner as intuitionistic logic.

Elsewhere, these proof structures have been used to give a normative pragmatic account not only of classical logic, but also of the modal operators of S5 [10].

So, let me conclude by adding two further findings to the first.

TENTATIVE FINDING 2: Multiple-premise, multiple-conclusion consequence relations are not only defensible on inferentialist grounds, they also have a proof theory with good properties, appropriate for the dualities inherent in classical two-valued logic. (You don't have to put up with *ad hoc* proof theory for classical logic.)

TENTATIVE FINDING 3: Any justification of a logical system on proof-theoretic grounds depends *crucially* on assumptions made about the *structure* of proof. (All the better, then, to be *explicit* about those assumptions.) Semantically anti-realist considerations only drive you toward intuitionistic logic if you make certain *assumptions* about the structure of proofs.

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