Constructive Logic, Truth and Warranted Assertibility

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Abstract

Shapiro and Taschek [7] have argued that simply using intuitionistic logic and its Heyting semantics, one can show that there are no gaps in warranted assertibility. That is, given that a discourse is faithfully modelled using Heyting's semantics for the logical constants, that if a statement S is not warrantedly assertible, then its negation ~S is. Tennant [8] has argued for this conclusion on similar grounds. I show that these arguments fail, albeit in illuminating ways. I will show that an appeal to constructive logic does not commit one to this strong epistemological thesis, but that appeals to semantics of intuitionistic logic nonetheless *do* give us certain conclusions about the connections between warranted assertibility and truth.

1 Truth

Truth value gaps are hard to come by. Given these three simple principles

- T-*introduction*: $A \supset T\langle A\rangle$;
- *Contraposition*: $A \supset B \vdash {\sim}B \supset {\sim}A$;
- *Transitivity*: $A \supset B, B \supset C \vdash A \supset C$.

we can show that there are no gaps in truth in the following sense: If *S* is not true, then *S*'s negation, ~*S*, is true.[1]

$$\dfrac{\dfrac{\dfrac{}{A \supset T\langle A\rangle}\,T\text{-}\mathit{intro.}}{{\sim}T\langle A\rangle \supset {\sim}A}\,\mathit{Contrap.} \qquad \dfrac{}{{\sim}A \supset T\langle {\sim}A\rangle}\,T\text{-}\mathit{intro.}}{{\sim}T\langle A\rangle \supset T\langle {\sim}A\rangle}\,\mathit{Trans.}$$

So, using very little infererential machinery we have shown that there are no truth value gaps. If a statement is not true, its negation is true. There are no statements S such that neither S nor ~S are true, for if S is not true, then its negation, ~S, is true.

Of course, this does not mean that either S is true or its negation is true. For that, we need more. In particular, we need an inference such as

$${\sim}A \supset B \vdash A \vee B$$

to convert ${\sim}T\langle A\rangle \supset T\langle {\sim}A\rangle$ to $T\langle A\rangle \vee T\langle {\sim}A\rangle$. But *this* inference is not given by the principles endorsed so far. It is neither constructively valid, nor valid in

[1] Here '⊃' is a conditional, 'T' is a truth predicate, '⟨–⟩' is an enquotation device, and '⊢' represents validity.

Łukaseiwicz's many valued logics. Still, we have an interesting consequence. In any logic with these three principles, there are no *gaps* in truth. If S is not true, then its negation is true.

This is not a new result, and neither is it surprising. However, it does bear repeating. Any proponent of a logic endorsing these three simple principles ought not endorse a *failure* of the law of the excluded middle, for if $A \vee {\sim}A$ is not true, then its negation is true: we have $T\langle{\sim}(A \vee {\sim}A)\rangle$. From here it is (in Łukaseiwicz's logic and in intuitionistic logic, at least) a very short move to $T\langle A \wedge {\sim}A\rangle$, the truth of a contradiction.

Or topic, however, is *warranted assertibility*. To what extent do the same principles apply in this case?

2 Warranted Assertibility

Gaps in warranted assertibility, at least *seem* to be easy to come by. There seem, at least for those of us who are inclined to realism for some discourse or other, to be claims S such that neither S nor ~S are warrantedly assertible [9]. If warranted assertibility and truth have the same extension then we could conclude that there must be no gaps in warranted assertibility. Shapiro and Taschek call this claim (N).

(N) For any statement S, if S is not warrantedly assertible, then ~S is warrantedly assertible. [7, page 79]

As we have seen with truth, any argument for such a thesis is sensitive to the logical principles allowed. Shapiro and Taschek endeavour to show that (N) is *provable* provided that it is read in a manner acceptable for a constructive reasoner. They argue that we ought read (N) in the following way, given an intuitionistic semantics for negation and the conditional.

(N′) For any statement S, there is a procedure to transform any warrant for the assertion that there is no warrant for S into a warrant for ~S. [7, page 79]

But given that (N) just *means* (N′), we have a surprising consequence. Shapiro and Taschek put it like this:

> Now, according to Heyting semantics, a warrant for the assertion that there is no warrant for S simply *is* a warrant for ~S. (N) interpreted as (N′) not only does not fail, it is a tautology. [7, page 80]

This is a serious issue, and not just for anti-realists (such as Tennant) who espouse a form of constructive logic. It is also an issue for *logical pluralists* who endorse constructive reasoning [1, 2]. As a pluralist, I endorse constructive reasoning, *and* the Heyting semantics Shapiro and Taschek use to derive (N′), but I do not want to be forced from this endorsement of constructivity to what seems to be a very strong conclusion — denying al gaps in warranted assertibility. In fact, a *motivation* for pluralists concerning logical consequence to endorse constructive reasoning is partly that there is a notion of construction or warrant which *is* incomplete, and hence is best modelled by structures containing such incompleteness. Has the semantics become self defeating, undercutting its own explanatory power? The rest of this paper will examine Shapiro and Taschek's argument in some detail, and another argument due to Neil Tennant [8] to indicate what lessons it might teach those who endorse constructive reasoning and Heyting's semantics.

3 Semantics

To start, we must consider semantics for intuitionistic logic. A sensible place to start is the connection the role of proof or *construction* in intuitionistic logic. Constructions obey the following laws:

- A construction of $A \wedge B$ is a construction of A together with a construction of B.
- A construction of $A \vee B$ is a construction of A or a construction of B.
- A construction of $A \supset B$ is a technique for converting constructions of A into constructions of B.
- There is no construction of $\perp$.
- A construction of $\sim A$ is a technique for converting constructions of A into constructions of $\perp$.

This is the Brouwer, Heyting and Kolmogorov (BHK) interpretation of the intuitionistic connectives [3, 4, 5], and it plays an important role especially in the formalisation of mathematical theories where the notion of a proof of a proposition or of a construction of an object can be rigourously defined. In more general settings, the notion of proof or construction is perhaps better replaced by the notion of *warrant* — this indeed brings us closer to our aim of considering warranted assertibility. This is best modelled in structures such as Kripke models, which can be seen as collections of *warrants* ordered by relative strength. Formally, a Kripke model consists of a set W of *warrants*, partially ordered by inclusion (written '$\sqsubseteq$') where $v \sqsubseteq w$ whenever w is *at least as strong as* than v. Then we have the following conditions on the connectives, where '$w \Vdash A$' is to be read as 'w warrants A.'

- $w \Vdash A \wedge B$ if and only if $w \Vdash A$ and $w \Vdash B$.
- $w \Vdash A \vee B$ if and only if $w \Vdash A$ or $w \Vdash B$.
- $w \Vdash A \supset B$ if and only if for any $v \sqsupseteq w$, if $v \Vdash A$ then $v \Vdash B$.
- $w \Vdash \sim A$ if and only if for each $v \sqsupseteq w$, $v \nVdash A$.

The points in a Kripke structure for intuitionistic logic do a good job of modelling *warrants* ordered by a notion of relative strength. The clauses for conjunction and disjunction are straightforward, but the rules implication and negation deserve a little discussion. A body of information warrants $A \supset B$ if and only if when *combined with* any warrant for A you have a warrant for B. The assumption guiding Kripke models is that a warrant for $A \supset B$ combined with one for A will be a *stronger* warrant. So, $A \supset B$ warranted by w if and only if any stronger warrant for A is also a warrant for B. Negation is similar: w is a warrant for $\sim A$ just when w cannot be *extended* to warrant A.[2]

[2] I take it that Kripke models are more suitable in this extended context where warrants are bodies of information, for it seems that a body of information may well be a warrant for a conditional without being a *function*. The BHK interpretation takes any construction of a conditional to be a *function*. Kripke models have the generality we require.

Warrants may well be *incomplete* and hence should not be expected to warrant, for every claim A, either it or its negation ~A. There are many Kripke models in which there are points w and statements A such that $w \nVdash A$, and $w \nVdash \sim A$. This jointly ensures that $A \vee \sim A$ need not be warranted by a body of information.

Of course, for a pluralist, it does not follow that $A \vee \sim A$ is *not true*, or even, not *necessarily* true. It is consistent to maintain that all of the theorems of classical logic are indeed true, and that all of the arguments of classical logic are *valid* while all along retaining the canons of constructive mathematical reasoning, and the rejection of certain classical inferences. The crucial fact which makes this position consistent is the shift in context. Classical inferences are valid, *classically*. They are not *constructively* valid. If we use a classical inference step, say the inference then we have not (we think) moved from truth to falsity, and we cannot move from truth to falsity. It is impossible for the premises of a classically valid argument to be true if the conclusion is false. However (according to this semantics) such an inference *can* take one from a truth warranted by a body of information, to one not warranted by that body of information. So, the inference, despite being classically valid, can be rejected on the grounds of non-constructivity. A further argument must be given that these inferences fail to preserve truth.[3]

This *pluralist* account of constructive inference is not a view that will be shared by those constructivists who wholeheartedly reject the use of classical inference. Nonetheless, it is a 'classical' view which makes sense of constructive logic. The issue for such a pluralist then, is whether this commits you to anti-realism in the form of (N) or its cousin (N′).

4 Warranted Assertibility

The *simplest* argument for (N) is in fact not not through its reading (N′), given by Shapiro and Taschek, but to argue for (N) directly. We have seen that the no-gaps thesis for truth is given by T-*introduction*, *Contraposition* and *Transitivity*. To argue for (N) we can simply replace all instances of T in the proof by the predicate W of warranted assertibility. For this, we need just the following principle:

- *W-introduction*: $A \supset W\langle A \rangle$.

What reason, given the semantics of the previous section, might we have for endorsing *W-introduction*? One answer, due to Neil Tennant [8], is straightforward. Given the semantics of the conditional we have what seems to be a compelling justification for *W-introduction*. We have a warrant w for $A \supset B$ just when any warrant v (extending w) for A is also a warrant for B. Now consider *any* warrant w for A at all. Is this a warrant for $W\langle A \rangle$? Clearly if $w \Vdash A$ then A has a warrant (namely w). Why not take w as a warrant for $W\langle A \rangle$ too? What *else* is needed to warrant $W\langle A \rangle$ other than the warrant for A?

As convincing as this argument might appear, it is merely a sketch. The rhetorical questions need answers, and in the general context of warranted as-

[3]And this argument, if given, must be presented with a great degree of subtlety, as the first section shows us. It would not do to say that, for example, in the constructively invalid inference, from ~~A to A, the premise may be true while the conclusion is not, for then ~~A would be true and ~A would also be true, a contradiction.

sertibility, these answers have not been given.[4] Perhaps indeed w might warrant A, but w alone may well not be enough information to warrant $W\langle \mathsf{A}\rangle$. Perhaps *more* information is needed to warrant $W\langle \mathsf{A}\rangle$.

These considerations are reminiscent of intuitions behind the failure of the KK thesis in epistemic logic: Perhaps a proposition is known without being known that it is known. However the "*WW*" thesis for warrant is even *less* compelling than the KK thesis. The point at issue here is *not* that A might be warranted without $W\langle \mathsf{A}\rangle$ being warranted. (Note, for the intuitionistic semantics for the conditional $w \nVdash \mathsf{A} \supset \mathsf{B}$ not simply when $w \Vdash \mathsf{A}$ and $w \nVdash \mathsf{B}$, but rather, when there is a counterexample at some *stronger* warrant.) The point is merely that A might be warranted without *that* warrant being sufficient to warrant $W\langle \mathsf{A}\rangle$. Perhaps whenever A has a warrant we merely have a guarantee that *however things turn out* $W\langle \mathsf{A}\rangle$ is also warranted. What this motivates, then, is the weaker claim

- $\sim\sim W$-*introduction*: $\mathsf{A} \supset \sim\sim W\langle \mathsf{A}\rangle$.

This is justified as follows: if $w \Vdash \mathsf{A}$ then indeed there will never be a $v \sqsupseteq w$ such that for each $u \sqsupseteq v$, $u \nVdash W\langle \mathsf{A}\rangle$. On the contrary, for each $v \sqsupseteq w$, there is a $u \sqsupseteq v$ where $u \Vdash W\langle \mathsf{A}\rangle$. If w warrants A, then any warrant extending w is itself extended by some warrant sufficient for $W\langle \mathsf{A}\rangle$. This is indeed justifiable on the grounds of the Kripke semantics. Tennant's argument is not sufficient to give us a constructive justification of $\mathsf{A} \supset W\langle \mathsf{A}\rangle$. However these considerations do seem to suffice for $\mathsf{A} \supset \sim\sim W\langle \mathsf{A}\rangle$, and while this is not *everything* the intuitionist might desire, it does tell us something interesting about warrant. We can use the weaker thesis $\sim\sim W$-*introduction* together with the intuitionistically acceptable inference of $\sim\sim$-*introduction* ($\mathsf{A} \vdash \sim\sim\mathsf{A}$) to reason as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\sim\sim W\textit{-intro.}}{\mathsf{A} \supset \sim\sim W\langle \mathsf{A}\rangle}}{\sim\sim\sim W\langle \mathsf{A}\rangle \supset \sim\mathsf{A}}\,\textit{Contrap.}}{\sim W\langle \mathsf{A}\rangle \supset \sim\mathsf{A}}\,\sim\sim\textit{-intro.} \qquad \dfrac{\sim\sim W\textit{-intro.}}{\sim\mathsf{A} \supset \sim\sim W\langle \sim\mathsf{A}\rangle}}{\sim W\langle \mathsf{A}\rangle \supset \sim\sim W\langle \sim\mathsf{A}\rangle}\,\textit{Trans.}$$

This indeed tells us something interesting about warrant in Kripke models. If we have warrant for $\sim W\langle \mathsf{A}\rangle$ then this very warrant is also warrant for $\sim\sim W\langle \sim\mathsf{A}\rangle$. So we are sure that *eventually* $W\langle \sim\mathsf{A}\rangle$ will be warranted, and that $W\langle \sim\mathsf{A}\rangle$ is not *unwarranted* given this body of information.

If you are happy to fudge double negations (as a *classical* reasoner will be happy to do, if the aim is simply to preserve truth) then indeed given $\mathsf{A} \supset \sim\sim W\langle \mathsf{A}\rangle$, we know that if $\sim W\langle \mathsf{A}\rangle$ is true, then so is $W\langle \sim\mathsf{A}\rangle$. The claim (N) for no gaps in warrant indeed is forced on the *classical* reasoner who endorses Heyting semantics (as a pluralist may well do) but it isn't motivated for the purely constructive reasoner, who is left with $\sim W\langle \mathsf{A}\rangle \supset \sim\sim W\langle \sim\mathsf{A}\rangle$. This is a surprising result, which is enough to motivate a *weaker* no-gaps thesis for warrant. There are still no gaps, in the sense that $\sim(\sim W\langle \mathsf{A}\rangle \wedge \sim W\langle \sim\mathsf{A}\rangle)$ is true: It is not the case that A is not warranted and neither is $\sim\mathsf{A}$. Perhaps this is

[4] In the metamathematics of constructive mathematics some work has been done on this question [3]. Justifications have been given there for the claim that a construction for A ought also be a construction of that construction actually being a construction for A.

enough to make the pluralist uneasy. I will consider what this means in the last section of the paper.

Before that, let us consider what Shapiro and Taschek had to say about the no-gap thesis for warrant, (N). They held that given a constructive semantics for implication and negation (N) should be read as follows:

(N′) For any statement S, there is a procedure to transform any warrant for the assertion that there is no warrant for S into a warrant for ~S. [7, page 79]

To say that (N) and (N′) are equivalent is altogether too swift. The Kripke semantics for negation tells us that warrant for ~A is a warrant which cannot be extended to a warrant for A. This does *not* mean that the assertion of ~A *means* that there is no warrant for A. To do this is to conflate A and $W\langle A\rangle$. All that the semantics for the connectives tells us is the following. The statement (N) is

(N) For any statement S, if S is not warrantedly assertible, then ~S is warrantedly assertible. [7, page 79]

This is warranted by w if for every $v \sqsupseteq w$, if $v \Vdash {\sim}W\langle A\rangle$ then $v \Vdash W\langle {\sim}A\rangle$. In words, we have

(N″) For any statement S, any warrant for the claim that S is not warrantedly assertible is also warrant for the claim that ~S is warrantedly assertible.

Shapiro and Taschek have slipped from warrant for the claim that ~S is warrantedly assertible to warrant for ~S. This is too swift. To justify (N) on intuitionistic grounds we need a justification of (N″), and for this we need a coherent story about the warrants for claims of the form $W\langle A\rangle$. None has yet been given.

5 What can we Say?

Both Tennant's argument, and that of Shapiro and Taschek have failed, but we are left with a milder thesis about warrant which has some bite.

- ${\sim}W\langle A\rangle \supset {\sim}{\sim}W\langle {\sim}A\rangle$

This too tells us that there is no case in which a both statement S and its negation ~S are not warranted, for if S is not warranted, it is not the case that its negation is not warranted. This genuinely seems to be a form of anti-realism. Truth and warranted assertibility may not coincide, but they do not diverge *very far*. Warranted assertibility, on this account, still suffers no gaps.

Consider the motivation of both the BHK interpretation of intuitionistic logic and the Kripke semantics. The crucial idea was that warrant may be incomplete and is prone to extension. A body of information may not warrant a claim, but it may be extended to a larger body which may warrant that claim. In fact, according to the story, if a body of information *cannot* be extended to warrant a statement, then that body of information itself is a warrant for the negation of that statement. So, indeed, in a weak sense, all truths are warranted.[5]

Nonetheless, we will have bodies of information w for which $w \not\Vdash A$ and $w \not\Vdash {\sim}A$. If w is our current state of information, can we not say that neither A nor ${\sim}A$ are *warranted*? The answer here is both *yes* and *no*. We have already seen an interpretation of $W\langle A\rangle$ which bars us from asserting ${\sim}W\langle A\rangle \wedge {\sim}W\langle{\sim}A\rangle$. But this is not the only way to talk of warrant, and it is perhaps not the *best* way to talk of warrant. We need *some* way to understand and interpret the claim that both $w \not\Vdash A$ and $w \not\Vdash {\sim}A$. (If we cannot say this, then all pretense of using the Kripke semantics to enlighten us about the behaviour of the connectives is gone.)

Clearly, we *can* interpret this claim in a straightforward fashion. We can interpret W as a predicate indexed to a body of information. If w is a warrant, then $W_w\langle A\rangle$ can assert that w warrants A. The semantics of $W_w\langle A\rangle$ is not obvious, except for the following desiderata which seem straightforward.

- If $v \Vdash W_w\langle A\rangle$ then $w \Vdash A$;
- If $v \Vdash {\sim}W_w\langle A\rangle$ then $w \not\Vdash A$.

That is, $W_w\langle A\rangle$ is only warranted if w indeed warrants A, and ${\sim}W_w\langle A\rangle$ is only warranted if w indeed *doesn't* warrant A. If this is the case, then indeed ${\sim}W_w\langle A\rangle \supset W_w\langle{\sim}A\rangle$ fails, if $w \not\Vdash A$ and $w \not\Vdash {\sim}A$, for for any warrant v we have $v \Vdash {\sim}W_w\langle A\rangle$ (since nothing can warrant $W_w\langle A\rangle$, given that $w \not\Vdash A$) and $v \not\Vdash W_w\langle{\sim}A\rangle$ (since nothing can warrant $W_w\langle{\sim}A\rangle$, as $w \not\Vdash {\sim}A$). So, given *this* understanding of warrant, the 'no gaps for warrant' thesis fails.

This semantics is crude, in that the language explicitly picks out a point in a Kripke model. We can proceed more subtly as follows. Take a *pointed* model to be a triple $\langle W, \sqsubseteq, w_@\rangle$ where $w_@ \in W$ is thought of as the *actual* body of information available. Interpret $W_@\langle A\rangle$ as saying that A is *actually* warranted, where our conditions for $W_@\langle A\rangle$ are *at least* the following:

- If $v \Vdash W_@\langle A\rangle$ then $w_@ \Vdash A$;
- If $v \Vdash {\sim}W_@\langle A\rangle$ then $w_@ \not\Vdash A$.

Then as before, if $w_@$ is an incomplete body of information, such that $w_@ \not\Vdash A$ and $w_@ \not\Vdash {\sim}A$, then indeed the 'no gaps for *actual* warrant' thesis will fail. We need not have ${\sim}W_@\langle A\rangle \supset W_@\langle{\sim}A\rangle$.[6] (The reasoning is just as in the case of W_w.)

It is instructive to see how $W_@$ differs from our original warrant predicate W in the case of the *$W_@$-introduction* rule. Do we have reason to endorse $A \supset W_@\langle A\rangle$? In particular, do we have any reason to think that $w_@ \Vdash A \supset W\langle A\rangle$ in

[5]And, of course, in a weak sense, some *untruths* are warranted too. There are claims A such that $w \Vdash A$ and $v \Vdash {\sim}A$, for different warrants w and v. These warrants are those which have no co-descendants (no warrants u where $w, v \sqsubseteq u$). If we add to our conditions each pair of warrants has a co-descendant, then indeed all warrants will be consistent. But then we extend the propositional logic to include ${\sim}A \vee {\sim\sim}A$ as a theorem, and as a result, the *primeness* property of the logic (if $A \vee B$ is a theorem, then either A is a theorem or B is a theorem) is lost.

[6]The reader may note a similarity between $W_@$ and Fine's semantics for D ("definitely") in his "Vagueness, Truth and Logic" [6]. This is not an accident. Both "warrantedly" and "definitely" seem to be "point-shift" operators of the same kind, on a par with "actually" in modal logic. Any differences between them seem to come down to the differences between the points involved in the semantics: warrants, specifications and worlds, respectively.

our Kripke models? Does our *actual* state of information warrant the claim that if A is true it is (actually) warranted? The answer here is a clear *no*. Provided that $w_@ \nVdash A$ and $w_@ \nVdash {\sim}A$ (which, recall, is a condition without which our Kripke models make very little sense) then we have a $v \sqsupseteq w_@$ such that $v \Vdash A$, and since $v \nVdash W_@\langle A\rangle$, we have $w_@ \nVdash A \supset W\langle A\rangle$. In words, if $w_@$ is incomplete with respect to A, then if it is *extended* by a warrant for A, that warrant does not assure us that A is *actually* warranted. No, A is not *actually* warranted. What we do know is that there is state of information extending the actual one which does suffice as a warrant for A, but this does not itself give us a warrant for A. (If it did, then we would have a warrant for A now, contradicting what we have assumed.)

There are *two* ways to understand the claim that a statement has a warrant, using the semantics for intuitionistic logic. The first way leaves the body of information used give the warrant to vary and there is no surprise that this gives us a notion of warrant *very* close to the notion of truth. (Though as we have seen, the idea that coincides in extension to truth is less motivated, for an intuitionist, that it seems at first blush.) If this is the only way to talk of warranted assertibility we find ourselves unable to say all that we would like to say in motivating and presenting the semantics itself. If we use the second form, which takes warrant to be indexed to a state of information — perhaps the *actual* information available to us — then we are indeed able to claim that warrant has gaps, without fear of contradiction.[7]

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