Technical Report TR-ARP-11-94

Automated Reasoning Project
Research School of Information Sciences and Engineering
and Centre for Information Science Research
Australian National University

November, 11, 1994

Displaying and Deciding Substructural Logics 1

Logics with Contraposition

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Abstract Many logics in the relevant family can be given a proof theory in the style of Belnap's display logic (Belnap 1982). However, as originally given, the proof theory is essentially *more expressive* than the logics they seek to model. In this paper, we consider a modified proof theory which more closely models relevant logics. In addition, we use this proof theory to provide decidability proofs for a large range of substructural logics.

Displaying and Deciding Substructural Logics 1
Logics with Contraposition

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There is rather a lot of interest these days in what have come to be called 'substructural logics.' The term picks out logics in which the standard complement of structural rules (in, say in a Gentzen proof theory or a natural deduction system) are not all present. While much of this interest is rather recent — arising since Girard's landmark "Linear Logic" (Girard 1987), some of it has quite a history; for example the last 35 years have seen a great deal of work on relevant logics (see the two volumes of *Entailment* (Anderson and Belnap 1975, Anderson et al. 1992) for a history of some of the work of this tradition). This paper is firmly in the latter tradition. However, much of what goes on here will be useful for the wider 'substructural' community, and a following paper ('Display and Decidability of Substructural Logics 2: The General Case') will generalise the results to apply to substructural logics in their generality.

Nuel Belnap's display logic (Belnap 1982) was originally conceived to provide a cut-free consecution calculus[1] for relevant logics, particularly, the relevant systems **R** and **E**. So, one would think, they would be ideally suited for providing a Gentzen-style proof theory for all logics in the 'relevant' family. Alas, this is not been seen to be the case. Display logic, as it stands, utilises boolean negation to do its work, and some logics in the relevant family cannot be conservatively extended with boolean negation. Some subtlety must be involved to get display logic to work in these cases. Belnap's method is to impose a form of apartheid on structures. Those not involving boolean negation are separated from those that do, and only the first-class structures (those which don't inovolve boolean negation) are allowed to take part in certain rules. The first major result of this paper is that this is not necessary. Another, simpler subtlety which will do the trick. We can treat all structures equally, and do without boolean negation. In the second part of the paper we will then use the calculus to prove decidability of a large class of substructural logics in the relevant family.

Before we can prove the results, or even explain the *problem* with boolean negation, we must first give an account of the logics we wish to study.

[1] I am following Anderson and Belnap 1975 in using 'consecution' in place of the more prevalent 'sequent.' Their reasons for doing so (given in *Entailment*) still hold, even if no-one else has taken up the usage. Further to their reasons I will add another. 'Sequent' carries the idea of premises or conclusions being listed. In 'consecution' the idea is muted. In the setting of display logic premises and conclusions can be bunched together in a more structured way than simply listing them. We co-opt 'consecution' to do duty for this kind of structured representation of premises and conclusions.

1 Ternary Frames and Models

Relevant logics can be modelled in ternary frames.

> **frame** A *frame* is a quadruple $\mathcal{F} = \langle W, N, R, ^* \rangle$ where W is a non-empty set (of *worlds*, or simply *points*) N is a distinguished subset of W (of *normal* points, points at which all of the theses of logic hold), $R \subseteq W^3$ is a ternary relation on W, of *relative accessibility* ($Rxyz$ if and only if relative to y, x is a possible antecedent for a consequent z). Finally, $^* : W \to W$ is a 'dualising' function to deal with negation. (The intended meaning is that $\sim A$ is true at x if and only if A is not true at x^*. So x^* is the maximal point consistent with x.) A ternary frame must satisfy a number of conditions.
>
> - Define $x \leq y$ to mean $(\exists z)(z \in N$ and $Rxzy)$.
> - N is 'closed upward.' If $x \in N$ and $x \leq y$ then $y \in N$ too.
> - We require that for each $x \in W$, $x \leq x$; and that for each $x, y, z \in W$, $x \leq y$ and $y \leq z$ entail that $x \leq z$.
> - Then R is appropriately 'tonic.' For each $x, y, z, x', y', z' \in W$, if $Rxyz$, $x' \leq x$, $y' \leq y$ and $z \leq z'$ then $Rx'y'z'$.
> - For each $x \in W$, $x^{**} = x$
> - For each $x, y, z \in W$, $Rxyz$ if and only if Rz^*yx^* (as a result, $x \leq y$ if and only if $y^* \leq x^*$).

These conditions are no doubt rather arcane.[2] However, they have some beautiful models. To fuel the imagination, I will present one interesting class of frames.

Take an arbitrary abelian group $\mathcal{A} = \langle A, e, \cdot, ^{-1} \rangle$. The corresponding quadruple $\mathcal{F}_{\mathcal{A}} = \langle A, \{e\}, R_{\mathcal{A}}, ^{-1} \rangle$ is a frame, where we define $R_{\mathcal{A}}abc$ as $a \cdot b = c$. To see this, note that $a \leq b$ is simply $a = b$. As $(\exists z)(z \in \{e\}$ and $R_{\mathcal{A}}azb)$ if and only if $R_{\mathcal{A}}aeb$ which is simply $a \cdot e = b$. So, $\leq$ is trivially a partial order and the tonicity requirement connecting $\leq$ and R is fulfilled rather easily. It is also true that $a^{-1-1} = a$ for any a. The contraposition condition rewrites

[2] And they are also notationally different to what you will find in a lot of the 'relevant' (and 'relevan*ce*') literature. What I take to be $Rxyz$ will often be written in much of the rest of the literature including works by my earlier self (Restall 1993, Restall 1995) as $Ryxz$. This is a mere notational difference. I choose this presentation in line with Mike Dunn's gaggle-theoretic work (Dunn 1991, Dunn 1993), and because it makes the 'direction' of implication match with the order of premise combination. Getting ahead of ourselves a little, we have $A \circ (A \to B) \to B$ as a theorem, instead of $(A \to B) \circ A \to B$, making it clear that $A \to B$ is the sort of thing that when given an A on the *left* produces a B, just as the direction of the arrow indicates. Of course, in the presence of the commutativity of fusion, or the commutativity of the first two places of R, it doesn't matter which order you choose.

as $a \cdot b = c$ if and only if $c^{-1} \cdot b = a^{-1}$. But that this holds in any abelian group (but not in all groups) can be easily checked.[3]

We use frames to model deduction in a language. To do that, we need our language.

> **propositional language** Given a countable collection Π of atomic propositions, we define our propositional language $\mathcal{L}(\Pi)$ by closing Π under the connectives
>
> $$\bot \quad t \quad \sim \quad \wedge \quad \vee \quad \circ \quad \rightarrow$$
>
> where $\bot$ and t are propositional constants, $\sim$ is a unary operator, and the rest are binary.

That gives us a language and structures in which to interpret the language. To make the story complete we need to define an interpretation.

> **UCLA propositions** In any frame $\mathcal{F}$, propositions are interpreted by suitable sets of points — namely those sets P which are closed upward (so, if $x \in P$ and $x \leq y$ then $y \in P$ too). Such a set of points is called a *UCLA proposition*,[4] and the set of all UCLA propositions is denoted 'Prop($\mathcal{F}$).'

> **model** Then a *model* $\mathcal{M}$ is a frame $\mathcal{F}$ together with a map $V : \Pi \rightarrow \text{Prop}(\mathcal{F})$, assigning a UCLA proposition to each atomic proposition in the language.
>
> In any model $\mathcal{M}$ we have a relation $\models$ between worlds and propositions, given by induction on the structure of propositions.
>
> - $x \models p$ iff $x \in V(p)$ for $p \in \Pi$.
> - $x \models \bot$ never.
> - $x \models t$ iff $x \in N$.
> - $x \models \sim A$ iff $x^* \not\models A$.
> - $x \models A \wedge B$ iff $x \models A$ and $x \models B$.
> - $x \models A \vee B$ iff $x \models A$ or $x \models B$.
> - $x \models A \rightarrow B$ iff for each y, z where $Ryxz$, if $y \models A$ then $z \models B$.
> - $x \models A \circ B$ iff for some y, z where $Ryzx$, $y \models A$ and $z \models B$.

[3]This interesting class of frames will be further discussed in my paper "Functional Frames for Substructural Logics" which will soon see the light of day.

[4]I'm indebted to Mike Dunn for this wonderful expression.

In cases where more than one model is under discussion, instead of $x \models A$ we will write $\mathcal{M}, x \models A$, to distinguish which supports relation is relevant.

Then we say $\mathcal{M} \models A$ if and only if $\mathcal{M}, x \models A$ for each $x \in N$, and $\mathcal{F} \models A$ if and only if $\mathcal{M} \models A$ for each model $\mathcal{M}$ on the frame $\mathcal{F}$. Finally, for a class $\mathfrak{C}$ of frames, $\mathfrak{C} \models A$ if and only if $\mathcal{F} \models A$ for each $\mathfrak{C} \in \mathcal{F}$. The logic **DW** is determined by the class of *all* frames. (The more basic logic **B**, sometimes discussed in the relevant literature, is given by replacing the condition that $Rxyz \Rightarrow Rz^*xy^*$ by the weaker $x \leq y \Rightarrow y^* \leq x^*$.)

A simple result about models is that in every model, every proposition is interpreted as a UCLA proposition in the frame. That is, we have the following *persistence* property.

Lemma 1 (Persistence) *For every model $\mathcal{M}$, for all points x, y and for every formula A, if $\mathcal{M}, x \models A$ and $x \leq y$ then $\mathcal{M}, y \models A$.*

In what follows, we will write $||A||_{\mathcal{M}}$ for the set of all points in $\mathcal{M}$ at which the proposition A is supported.

Theorem 2 *Every model $\mathcal{M}$ supports the following propositions*

$$\begin{gathered} A \wedge B \to A \quad A \wedge B \to B \quad A \to A \vee B \quad B \to A \vee B \\ A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C) \quad (A \to B) \wedge (A \to C) \to (A \to B \wedge C) \\ (A \to C) \wedge (B \to C) \to (A \vee B \to C) \quad (A \to B) \to (\sim B \to \sim A) \\ \sim\sim A \to A \end{gathered}$$

In addition, the propositions supported in a model are closed under the following rules

$$\begin{aligned} A \to B, A &\;\Rightarrow\; B \\ A, B &\;\Rightarrow\; A \wedge B \\ A \to B, C \to D &\;\Rightarrow\; (B \to C) \to (A \to D) \\ A &\;\Leftrightarrow\; t \to A \\ A \circ B \to C &\;\Leftrightarrow\; B \to (A \to C) \end{aligned}$$

(Where 'closure under $A, B \Rightarrow C$' means, if $\mathcal{M} \models A$ and $\mathcal{M} \models B$ then $\mathcal{M} \models C$, and closure under $A \Leftrightarrow B$ is simply closure under $A \Rightarrow B$ and closure under $B \Rightarrow A$)

Proving this result is a simple exercise in interpreting the conditions for being a model.

Note that we do not have $\mathcal{M} \models A \vee B$ if and only if $\mathcal{M} \models A$ or $\mathcal{M} \models B$ (because the normal points N in a model could agree on a disjunction but disagree as to which disjunct supports the disjunction). However, we can argue from $\mathcal{M} \models A \vee B$ to $\mathcal{M} \models A$ or $\mathcal{M} \models B$ if the frame is *reduced*, meaning that $N = \{x : e \leq x\}$ for some point e. Then one can reason as follows: $\mathcal{M} \models A \vee B$ if and only if $\mathcal{M}, e \models A \vee B$ if and only if $\mathcal{M}, e \models A$ or $\mathcal{M}, e \models B$, if and only if $\mathcal{M} \models A$ or $\mathcal{M} \models B$.

Theorem 3 *The theses and rules in Theorem 2 form a Hilbert axiomatisation of exactly the theses supported in all* **DW** *models. In other words, if a proposition does not follow from those theses, by the application of those rules, then there is a* **DW** *frame in which it is not supported.*

There is no need to repeat the proof of this theorem here. Techniques available in Routley and Meyer's original paper (Routley and Meyer 1973) or in the standard references (Anderson et al. 1992, Dunn 1986) suffice. We construct a canonical model out of the non-trivial prime theories in **DW**, and we can show that it refutes all non-theorems.

2 The Display Calculus

Gentzen-style proof theories deal with *consecutions*. A consecution is usually a statement like $A, B, C \vdash D, E$, which is interpreted as: If each of A, B and C are true, then (at least) one of D and E follow as a matter of logic. Then the behaviour of the logical connectives are defined in terms of their interaction with the means of combining premises or conclusions (here the comma) and the turnstile.

It has been known for years that for relevant logics, and others in their near vicinity, you need to use at least *two* kinds of premise combination to adequately model the logic (Dunn 1974 and independently, Minc 1972). One is the standard $\wedge$, which we shall call *extensional conjunction*, and the other, the more exotic $\circ$, which we call *intensional conjunction*, or more succinctly, *fusion*. We need both sorts of ways of putting premises together in order to draw the distinctions relevant logics need.

The central point of the Belnap's display calculus is to find a version of cut that is both *valid* and *eliminable* for relevant logics. In most cases 'simple cut' (sometimes called 'pure transitivity')

$$\frac{X \vdash A \quad A \vdash Y}{X \vdash Y}\,(\mathrm{Cut})$$

is valid — in the sense that whenever $X \vdash A$ and $A \vdash Y$ are true under interpretation, so is $X \vdash Y$ — but it is not eliminable. We can't in general eliminate this kind of cut rule because in the usual cut elimination proofs we often have to perform a cut on a formula *inside* a structure, in the process of pushing cuts back in a proof. Stronger forms of the rule, like versions of the Mix rule not generally valid. (Most forms of Mix build in forms of contraction or weakening that are not valid in our context.) The innovation of display logic is to enrich the calculus in such a way as to ensure that simple cuts are enough. If we ever wish to perform a cut on a formula inside a structure, we can transform that consecution into an equivalent one in which the formula of choice is either the entire antecedent or the entire consequent of new consecution. This is called 'displaying' the chosen structure. We achieve this by adding rules which allow structure to be moved around. Belnap's rules are given as follows:[5] in the figure below, consecutions in the same row are said to be *immediately display equivalent*. *Display equivalence* is the transitive closure of immediate display equivalence.

$$\begin{array}{ccc} X;Y \Vdash Z & Y \Vdash *X;Z & \\ X \Vdash Y;Z & *Z;X \Vdash Y & *Y;X \Vdash Z \\ X \Vdash Y & *Y \Vdash *X & **X \Vdash Y \end{array} \tag{1}$$

In this presentation, I have used the semicolon for premise combination rather than the '$\circ$' of the rest of the display logic literature,[6] simply because the relevant logic literature has prior claim on '$\circ$' as an object-language connective. In addition, I follow Kracht and Goré in making $*$ a prefixing structural operator instead of the postfix of Belnap and Wansing, because I wish to keep it as distinct as possible from the superscript * of the model theory. We also use $\Vdash$ to form consecutions instead of the more traditional $\vdash$, which we keep for provability.

Note that the first row of equivalences is in some sense 'incomplete,' because we do not have a consecution equivalent to $X;Y \Vdash Z$ in which X is the sole antecedent. This is not an oversight, because such a consecution is given by the other rules. To see this, note first that $X \Vdash Y;Z$ is display equivalent to $X \Vdash Z;Y$. And using this, we can show that $X;Y \Vdash Z$ is display equivalent to $X \Vdash *(*Z;*Y)$, so completing the first row.

These equivalences nicely model valid rules in our structures, when we interpret the semicolon on the left to be fusion, and the semicolon on the right to be *fission*, defined by setting $A + B$, the fission of A and B as $\sim A \to B$. You can check for yourself that the following stand or fall together in every **DW** model.

$$\begin{array}{ccc} A \circ B \to C & B \to (\sim A + C) & \\ A \to (B + C) & \sim C \circ A \to B & \sim B \circ A \to C \\ A \to B & \sim B \to \sim A & \sim\sim A \to B \end{array}$$

This method works for 'unwrapping' intensional structure. But what about extensional structure? Belnap's solution is to invoke a parallel collection of display postulates, connecting extensional conjunction on the left, with extensional disjunction on the right, and with *boolean negation* tying them

[5] Modulo, of course, our preference for swapping the order of intensional conjunction.

[6] Which is not, as yet, as large as it deserves to be. Extant display logic works are, in addition to the Belnap's original paper, there is the follow-on (Belnap 1990). Then Wansing's paper (Wansing 1993) modifies the display logic framework in order to model classical modal systems with a unary structural operation instead of a binary one. Kracht 1994 pins down the modal systems which can be modelled using structural extensions of Wansing's methods, and he shows that the mere presence of a Cut-free display proof theory does not entail decidability. Finally, Goré 1994a uses the Wansing-style modal interpretation to simplify Belnap's original treatment of intuitionistic logic.

together. The problem with that suggestion is that in some of the systems we are interested in modelling, there is no way to extend the system with boolean negation. Suppose we have a model $\mathcal{M}$. To define boolean negation we must require that $\mathcal{M}, x \models \Leftrightarrow A$ if and only if $\mathcal{M}, x \not\models A$. So, $||\Leftrightarrow A||_{\mathcal{M}} = W \backslash ||A||_{\mathcal{M}}$. But this is not, in general, a UCLA proposition. The only way to ensure that UCLA propositions are closed under complement is to collapse the containment relationship $\leq$ to identity. This is a hefty price to pay.[7] Belnap discusses this possibility himself, and provides a way around it by imposing a form of apartheid on structures. We will consider that method at the end of Section 5. For now, we will examine a way of displaying extensional structure *without* the use of boolean negation.

Granted that we must use extensional conjunction in an antecedent positions, we need a way to display both X and Y in $X, Y \vdash Z$. But this is easy. Recall the deduction theorem. $A \wedge B \vdash C$ if and only if $A \vdash B \supset C$, if and only if $B \vdash A \supset C$, where the $\supset$ is at least an intuitionistic conditional. This is no problem — we can interpret an intuitionistic conditional in our frames without difficulty. $\mathcal{M}, x \models A \supset B$ if and only if for each $y \geq x$, if $\mathcal{M}, y \models A$ then $\mathcal{M}, y \models B$. The resulting set $||A \supset B||_{\mathcal{M}}$ is always a UCLA proposition, as is easily verified. We do not gain any new propositions by enriching our expressive powers to include intuitionistic implication. We can get the display properties we need by deeming the following consecutions to be display equivalent.

$$X, Y \vdash Z \qquad X \vdash Y, Z \qquad Y \vdash X, Z \tag{2}$$

The novelty is this: comma is interpreted as extensional conjunction in antecedent position, and the intuitionistic conditional in consequent position. This is a deviation from the normal behaviour of the comma in Gentzen systems, in which it behaves disjunctively in consequent position.[8] However, the innovation is appropriate. In the standard Gentzen system for intuitionistic logic, the comma does not appear in the consequent position at all.[9] We are free to define how it ought to behave once it gets there.

Enough of motivation. Now for some definitions. First, we formally

[7]It is quite simple to show that **CK** (a logic we will meet later in this paper) is not conservatively extended by boolean negation. Ed Mares has recently (Mares 1994) shown that **E** is not conservatively extended by boolean negation either.

[8]Belnap mentions the possibility of a structural connective behaving as a conditional in the 'Further Developments' section of his original paper. It has taken until now for any further development to take place.

[9]It does, however in some nonstandard systems. There is a 'semi-traditional; Gentzen system for **J** with a multiple consequent, treating the comma as disjunction, provided that we are careful in our statements of the connective rules. Perhaps more interesting is my colleague Goré's "Yet Another Way to Display Intuitionistic Logic" (Goré 1994b) which uses this interpretation of consequent bunching to give a simple display calculus for intuitionistic logic, and then presenting classical logic and intermediate logics as a supersystems of intuitionistic logic given by adding only structural rules.

define the notion of a structure, where we add to what we have seen the atomic structures **1** and **I**, which shall be important in what follows.

> **structure** Any of **1**, **I**, or a formula is a structure, and if X and Y are structures, then so X, Y is a structure, $X; Y$ is a structure and so is $*X$.
>
> **consecution** If X and Y are structures, then $X \vdash Y$ is a consecution.
>
> **display equivalence** Two consecutions are said to be display equivalent if and only if one can be transformed into the other by way of the equivalences displayed in (1) and (2).

Now we can show that any substructure in a consecution can be displayed. We need the definition of antecedent and consequent parts.

> **antecedent and consequent parts** In $X \vdash Y$, we say X is an antecedent part, and Y is a consequent part. Then, antecedent and consequent parts of substructures of antecedent or consequent parts are defined recursively as follows.
>
> - If V, W is an antecedent part of a consecution, then so are V and W. If V, W is a consequent part of a consecution, V is an antecedent part of that consecution, and W is a consequent part.
> - If $V; W$ is an antecedent part of a consecution, then so are V and W. If $V; W$ is a consequent part of a consecution, then so are V and W.
> - If $*V$ is an antecedent part, then V is a consequent part. If $*V$ is a consequent part, then V is an antecedent part.

So, for example, in the consecution $*(X, *Y) \vdash *Z$, these are the antecedent parts

$$*(X, *Y) \qquad Z \qquad X \qquad Y$$

and these are the consequent parts

$$*Z \qquad X, *Y \qquad *Y$$

To prove that any structure can be displayed, it is helpful to have the notion of a *context*.

> **context** A *context* is a structure with an arbitrary substructure replaced by the Void (written '$\Leftrightarrow$'). Given a context f and a structure X, $f(X)$ is given by replacing the Void in f by X.

So, for example $f = X;(\Leftrightarrow, *Y)$ is a context. $f(Z;Y) = X;((Z;Y), *Y)$, and $f(*Y) = X;(*Y, *Y)$.

Note that in this example, when f is in an antecedent position, its Void is also in antecedent position. But when it is in consequent position, instead of its Void being in consequent position, it is still in an antecedent position. This motivates the following definition.

> **antecedent/consequent positive/negative** A context f is said to be *antecedent positive* if the indicated X is an antecedent part of $f(X) \vdash Z$, and it is *antecedent negative* if that X is a consequent part of $f(X) \vdash Z$. It is *consequent positive* if that X is a consequent part of $Z \vdash f(X)$ and *consequent negative* if that X is an antecedent part of $Z \vdash f(X)$.

So, $*\Leftrightarrow$ is both antecedent and consequent negative. $X, \Leftrightarrow$ is both antecedent and consequent positive, while $\Leftrightarrow, X$ is antecedent positive but consequent negative. (Note that every context is either antecedent positive or antecedent negative, but not both, and similarly, either consequent positive or consequent negative, but not both.)

With all of that done, we have a general 'displayability' result.

Lemma 4 *Any context f can be 'unravelled' as follows. If f is placed in antecedent position, then there is a corresponding context f^a which is given by displaying the Void in f. If f is antecedent positive, then $f(X) \vdash Y$ is display equivalent to $X \vdash f^a(Y)$. If f is antecedent negative, then $f(X) \vdash Y$ is display equivalent to $f^a(Y) \vdash X$. Similarly, if f is placed in consequent position, then there is a corresponding context f^c, such that $Y \vdash f(X)$ is display equivalent to $f^c(Y) \vdash X$ if f is consequent positive, or $X \vdash f^c(Y)$ if it is consequent negative.*

That lemma is more difficult to state than to prove. Its proof is a simple induction on the complexity of contexts.

Proof. If $f = \Leftrightarrow$ then f is antecedent positive and consequent positive, and $f^a = f^c = \Leftrightarrow$.

Suppose $f = *g$ (so $f(X) = *g(X)$ for some context g). Then $f^a = g^c(*\Leftrightarrow)$, and $f^c = g^a(*\Leftrightarrow)$. The reason is as follows. $f^a(X) \vdash Y$ is simply $*g(X) \vdash Y$ and this is display equivalent to $*Y \vdash g(X)$. This is display equivalent to either of $g^c(*Y) \vdash X$ or $X \vdash g^c(*Y)$ depending on whether g is consequent positive or negative. In either case, f^a turns out to be $g^c(*\Leftrightarrow)$ as desired. Simlar reasoning gives $f^c = g^a(*\Leftrightarrow)$.

Suppose $f = g, Z$ for some structure Z. (So, $f(X) = g(X), Z$.) Then $f(X) \vdash Y$ is simply $g(X), Z \vdash Y$, which is display equivalent to $g(X) \vdash$

Z, Y, which is equivalent to either of $X \vdash g^a(Z, Y)$ or $g^a(Z, Y) \vdash X$ depending on X. So f^a is $g^a(Z, \Leftrightarrow)$. On the other hand, $Y \vdash f(X)$ is $Y \vdash g(X), Z$, which is display equivalent to $g(X) \vdash Y, Z$, so f^c is $g^a(\Leftrightarrow, Z)$.

A similar argument is needed for $f = Z, g$, $f = Z; g$ and $f = g; Z$. We leave them to the willing reader. $\dashv$

Note that with the display rules we have so far, $X, Y \vdash Z$ is display equivalent to $Y, X \vdash Z$. So, extensional conjunction is commutative. This is obviously desirable if extensional conjunction in the display calculus is to mimic extensional conjunction in our frames. However, we want more. We need extensional conjunction to be associative, idempotent, and to allow weakening. In addition we use the structural constant $\mathbf{I}$ which acts as the absurdly false proposition $\perp$ in consequent position. The structural rules we posit are as follows.

$$\text{(eW)}\ \frac{X, X \vdash W}{X \vdash W} \qquad \text{(eK)}\ \frac{X \vdash W}{Y, X \vdash W}$$

$$\text{(eB)}\ \frac{(X, Y), Z \vdash W}{X, (Y, Z) \vdash W} \qquad (\mathbf{I}\text{E})\ \frac{X \vdash \mathbf{I}}{X \vdash Y}$$

In addition we need structural rules to do duty for the constant t. It is associated with *intensional* structure, by way of the equivalence $A \circ t \leftrightarrow A$. The corresponding rules are these.

$$(\mathbf{1}\text{I})\ \frac{X \vdash Y}{X; \mathbf{1} \vdash Y} \qquad (\mathbf{1}\text{E})\ \frac{X; \mathbf{1} \vdash Y}{X \vdash Y}$$

Finally, as in all good proof theories, we need a way to introduce connectives on either side of the turnstile. The display system makes this quite simple because we can postulate rules in which the formula introduced is either all of the antecedent of the consecution, or all of the consequent of the consecution, because the display property ensures that this is no loss of generality. So, we present the connective rules in Figure 1. Note that intensional connectives (like $\rightarrow$, $\circ$ and t and $\sim$) are paired with intensional structure, while extensional connectives (like $\wedge$, $\vee$ and $\perp$) are either structure free, or paired with extensional structure. Given all of these rules we can define the notion of a proof.

> **inference** An inference is a pair of a set of consecutions (the premises) and a consecution (the conclusion). In the case where the set of premises is empty, the inference is said to be an *axiom*.

$$(\mathrm{Ax})\ p \vdash p$$

$$(\bot \vdash)\quad \bot \vdash \mathbf{I} \qquad\qquad (\vdash \bot)\quad \frac{X \vdash \mathbf{I}}{X \vdash \bot}$$

$$(t \vdash)\quad \frac{\mathbf{1} \vdash X}{t \vdash X} \qquad\qquad (\vdash t)\quad \mathbf{1} \vdash t$$

$$(\wedge \vdash)\quad \frac{A, B \vdash X}{A \wedge B \vdash X} \qquad\qquad (\vdash \wedge)\quad \frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B}$$

$$(\vee \vdash)\quad \frac{A \vdash X \quad B \vdash X}{A \vee B \vdash X} \qquad (\vdash \vee 1)\quad \frac{X \vdash A}{X \vdash A \vee B} \qquad (\vdash \vee 2)\quad \frac{X \vdash B}{X \vdash A \vee B}$$

$$(\to \vdash)\quad \frac{X \vdash A \quad B \vdash Y}{A \to B \vdash *X; Y} \qquad\qquad (\vdash \to)\quad \frac{A; X \vdash B}{X \vdash A \to B}$$

$$(\circ \vdash)\quad \frac{A; B \vdash X}{A \circ B \vdash X} \qquad\qquad (\vdash \circ)\quad \frac{X \vdash A \quad Y \vdash B}{X; Y \vdash A \circ B}$$

$$(\sim \vdash)\quad \frac{*A \vdash X}{\sim A \vdash X} \qquad\qquad (\vdash \sim)\quad \frac{X \vdash *A}{X \vdash \sim A}$$

Figure 1: Connective Rules

rule A rule is a set of inferences. For our purposes, the rules are *display equivalence* (the set of all inferences where the (only) premise is display equivalent to the conclusion), identity (which is an axiom), (eW), (eK), (eB), (**I**E), (**1**I), (**1**E) and the connective rules. For the latter cases, each rule is simply the set of all inferences of the form we have displayed. We will later consider what happens when we add new rules to the calculus.

proof A proof of a consecution $X \vdash Y$ is a tree, with $X \vdash Y$ as the root, and in which each node follows from all of its immediate predecessors by some rule or other. A consecution is said to be *provable with cut* if it has a proof in which we are have perhaps used the rule Cut.

Lemma 5 *For every formula A, the consecution $A \vdash A$ has a proof.*

Proof. The proof is an induction on the complexity of the formula A. The base case is immediate. Here are the inductive steps for implication and disjunction.

$$\dfrac{\dfrac{\dfrac{B \vdash B \quad A \vdash A}{A \to B \vdash *A; B}}{A; A \to B \vdash B}}{A \to B \vdash A \to B} \qquad \dfrac{\dfrac{A \vdash A}{A \vdash A \vee B} \quad \dfrac{B \vdash B}{B \vdash A \vee B}}{A \vee B \vdash A \vee B}$$

The rest of the steps are left to the committed reader. ⊣

That completes the presentation of the display proof theory for **DW**. In the next three sections we will show that, under a suitable interpretation, it models validity in **DW** frames exactly.

3 The Completeness Proof

In this section we will show that with Cut we can prove anything that is valid in all **DW** frames. We will show that for any formula A, valid in all frames, $\mathbf{1} \vdash A$ is provable in the display calculus, using Cut. Then we will indicate how this can be extended to arbitrary consecutions.[10] Firstly we will show that each of the rules in the Hilbert calculus preserve provability. To start we need a simple lemma.

[10]This proof will proceed by way of the Hilbert Calculus for **DW**. This is a little unsatisfactory, methodologically. A preferable method is to go directly from the display calculus to frames (if $X \nvdash Y$ then there is a point in a frame in which X obtains but Y doesn't). But that procedure is a little subtle. Another way to proceed is via an algebraic representation, in terms of propositional structures, but to take that route would lead us too far afield. So, in this section we will take the tried and true path via Hilbert systems, and show that the results for theoremhood extend to arbitrary consecutions.

Lemma 6 *We can prove* $\mathbf{1} \vdash A \to B$ *(using Cut) if and only if there is a proof of* $A \vdash B$ *(perhaps also using Cut).*

Proof. From left to right we can reason as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B \quad A \vdash A}{A \to B \vdash *A; B} \quad \mathbf{1} \vdash A \to B}{\mathbf{1} \vdash *A; B}\,(\text{Cut})}{A; \mathbf{1} \vdash B}}{A \vdash B}$$

The other direction is a simple application of ($\mathbf{1}$I) and ($\vdash\to$). ⊣

As a result, with Cut we can show that *modus ponens* is admissible.

Corollary 7 *If* $\mathbf{1} \vdash A \to B$ *and* $\mathbf{1} \vdash A$ *are provable then* $\mathbf{1} \vdash B$ *is too (using Cut).*

Similarly, adjunction is admissible.

Lemma 8 *If* $\mathbf{1} \vdash A$ *and* $\mathbf{1} \vdash B$ *are provable then* $\mathbf{1} \vdash A \wedge B$ *is provable also.*

The proof of that fact is a trivial result of contraction for comma. The more difficult rule is the transitivity fact.

Lemma 9 *If* $\mathbf{1} \vdash A \to B$ *and* $\mathbf{1} \vdash C \to D$ *are provable then* $\mathbf{1} \vdash (B \to C) \to (A \to D)$ *is provable (perhaps using cut).*

Proof. We can assume that we have proofs of $A \vdash B$ and $C \vdash D$ by Lemma 6, and and then the proof of $\mathbf{1} \vdash (A \to B) \to (C \to D)$ goes as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash B \quad C \vdash D}{B \to C \vdash *A; D}}{A; (B \to C) \vdash D}}{B \to C \vdash A \to D}}{\mathbf{1} \vdash (A \to B) \to (C \to D)}$$

⊣

Finally, we need to validate the rules for t and for fusion. The t rule is provable quite simply. (Take it as an exercise.)

Lemma 10 $\mathbf{1} \vdash t \to A$ *is provable if and only if* $\mathbf{1} \vdash A$ *is, using Cut.*

The fusion rules take a little more work.

Lemma 11 *There is a proof of* $\mathbf{1} \vdash A \circ B \to C$ *if and only if* $\mathbf{1} \vdash B \to (A \to C)$ *is provable (using Cut).*

Proof. Firstly, suppose that there is a proof of $\mathbf{1} \vdash B \to (A \to C)$, or equivalently, of $B \vdash A \to C$ by Lemma 6. Then this is a proof of $A \circ B \to C$, and can be expanded to a proof of $\mathbf{1} \vdash A \circ B \to C$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad C \vdash C}{A \to C \vdash *A; C} \quad B \vdash A \to C}{B \vdash *A; C}\,(\text{Cut})}{A; B \vdash C}}{A \circ B \vdash C}$$

The other direction is similar.

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A; B \vdash A \circ B} \quad A \circ B \vdash C}{A; B \vdash C}\,(\text{Cut})}{B \vdash A \to C}$$

So, each of the rules of the Hilbert system given in Theorem 2 preserve provability. To complete the proof of our theorem we need to show that each of the axioms are provable. They are (for proofs, check the appendix). So, we announce our theorem.

Theorem 12 *If* $\mathcal{F} \models A$ *for each frame* $\mathcal{F}$ *then* $\mathbf{1} \models A$ *is provable in the display calculus, using Cut.*

To extend the result to arbitrary consecutions, we need to show that if $X \vdash Y$ is valid in all frames (however we define this) then $X \vdash Y$ is provable. So, in the rest of the section we provide an interpretation of $X \vdash Y$ so that it makes sense to speak of it holding in a frame. For that we provide a translation from the language of consecutions to the language $\mathcal{L}_{\supset}(\Pi)$, which extends $\mathcal{L}(\Pi)$ by adding the binary connective $\supset$. We translate a consecution into a formula as follows. Firstly, $t(X \vdash Y) = a(X) \to c(Y)$. Then we define a and c recursively in the obvious way.

$$\begin{array}{rcl@{\qquad}rcl}
a(A) & = & A & c(A) & = & A \\
a(\mathbf{1}) & = & t & c(\mathbf{1}) & = & \sim t \\
a(\mathbf{I}) & = & \sim \bot & c(\mathbf{1}) & = & \bot \\
a(X;Y) & = & a(X) \circ a(Y) & c(X;Y) & = & \sim c(X) \to c(Y) \\
a(X,Y) & = & a(X) \wedge a(Y) & c(X,Y) & = & c(X) \supset c(Y) \\
a(*X) & = & \sim c(X) & c(*X) & = & \sim a(X)
\end{array}$$

We then say that $X \vdash Y$ holds in a model $\mathcal{M}$ when $\mathcal{M} \models t(X \vdash Y)$. It is a simple (but tedious) exercise to show that (using Cut) if we can prove $\mathbf{1} \vdash t(X \vdash Y)$ in the display calculus (when extended with the appropriate rule for $\supset$), then we can also prove $X \vdash Y$ in the original display calculus (without the use of the rule for $\supset$). I leave the details of this to the interested reader. Instead of pursuing the niceties of this extension of the completeness result, we will show that Cut is eliminable from the display calculus.

4 Cut Elimination

The proofs of the previous section make essential appeals to the Cut rule in a number of places. It is our job in this section to show that any consecution provable with Cut also has a Cut-free proof.

The proof Belnap gives is general and simple. I will sketch it here, but refer the reader to his paper (Belnap 1982) for the details. For his proof we need the concept of parametric structures in inferences.

> **parameters** In an inference *Inf* falling under some rule, a collection of instances of a structure X is said to be a *family of parameters* if and only if for all structures Y, the inference *Inf*(Y), given by replacing all of those instances of X in *Inf* by Y, is also an instance of that rule. A structure X occurring in a family of parameters in an inference is said to be a parameter.

For example, any step of the form

$$(\vdash \circ) \quad \frac{X \vdash A \quad Y \vdash B}{X; Y \vdash A \circ B}$$

both occurrences of X and its substructures are parameters, as are both occurrences of Y and its substructures, but $X; Y$ is not a parameter, and neither is A, B or $A \circ B$.

> **congruence** Two structures in an inference are said to be congruent if they are members of some parametric family in that inference.

So, in that rule instance above, both instances of X are congruent, as are both instances of Y. If $X = (*Y; Z)$, then the instances of Y in X are congruent to each other (and themselves) but they are not congruent to the other instances of Y.

This definition differs from Belnap's account of parameters and congruence, in that Belnap's appeals to the presentation of rules as schemas. The way is open on this account to consider rules which cannot be presented as schemas — in fact, one can, on the present account, take the calculus to have only one rule, given as the union of all the original rules. That sort of presentation is not particularly perspicuous in practice, but it may have theoretical advantages for an account of parameters. The point being that an inference can have its parameters substituted for other structures keeping *validity.* That need not mean that the result, after substitution, is an instance of the very same rule as the original inference; for our purposes we only need that the result is an instance of some rule or other. If there is only one rule (encompassing all the original rules) then our account of being a parameter is as liberal as it can be while still performing the task we will need of it in the proof that follows.

The beauty of Belnap's cut elimination argument is the way in which he isolates the requirements that a calculus must meet in order for Cut to be shown to be eliminable by his proof. The requirements are as follows. Along the way we will see how they are met in the current calculus.

C2 *Congruent parameters are occurrences of the same structure.* This is satisfied by the way we defined congruence.

C3 *Each parameter is congruent to at most one constituent of the conclusion of Inf.* This may be verified by eye in the presentation of all of the rules. As Belnap points out, the 'Mingle' rule

$$\frac{X \vdash Y}{X; X \vdash Y}$$

does not satisfy this condition. We could not even salvage the rule by replacing the one occurrence of X in the premise with X, X, under the condition that the first X in the premise be congruent to the first X in the conclusion (and the second to the second), since under any account of congruence all X's are congruent to one another, because you cannot change one without changing the others, while keeping validity. (Unless we have other rules present, of course.)

C4 *Congruent parameters are either all antecedent parts or all consequent parts of Inf.* This may be verified by eye in all of the rules we have. It only ensures that we don't have really strange rules, in which structures may swap from being in antecedent position to consequent.

C5 *If a formula is non-parametric in the conclusion of Inf, it is either the entire antecedent or the entire consequent of that conclusion.* Non-parametric formulae are said to be the *principal constituents* of *Inf.* This may be verified by eye in our rules. Note that the requirement is only that non-parametric *formulas* are entire antecedents or consequents. There are non-parametric structures which are not either entire antecedents or entire consequents (see $*X$ in the conclusion of ($\rightarrow\vdash$) for example).

C6/7 *Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulae.* This fact is given by our definition of congruence.[11]

C8 *If there are inferences* Inf_1 *and* Inf_2 *with respective conclusions* $X \vdash A$ *and* $A \vdash Y$ *with* A *principal in both instances, then either* $X \vdash Y$ *is identical to one of* $X \vdash A$ *or* $A \vdash B$ *or it is possible to pass from the premises of* Inf_1 *and* Inf_2 *to* $X \vdash Y$ *by means of inferences falling under the rules together with instances of Cut in which the cut formula is a subformula of* A*.* This is the only condition which requires a significant amount of checking. The only cases in which we have $X \vdash A$ and $A \vdash Y$ with A principal in both consecutions is where one of the consecutions is the axiom $p \vdash p$, but in that case both are, and $X \vdash Y$ is identical to both the earlier consecutions, or the case in which both consecutions come by matching connective rules. I will provide the details for conjunction, because it involves our different treatment of extensional structure. The rest are proved in a similar vein.

$$\dfrac{\dfrac{X_1 \vdash B \quad X_2 \vdash C}{X_1, X_2 \vdash B \wedge C} \qquad \dfrac{B, C \vdash Y}{B \wedge C \vdash Y}}{X_1, X_2 \vdash Y}\ (\mathrm{Cut})$$

This can be transformed to the following proof, with Cuts on B and C.

$$\dfrac{\dfrac{X_2 \vdash C \qquad \dfrac{\dfrac{X_1 \vdash B \quad \dfrac{B, C \vdash Y}{B \vdash C, Y}}{X_1 \vdash C, Y}\ (\mathrm{Cut})}{C \vdash X_1, Y}}{X_2 \vdash X_1, Y}\ (\mathrm{Cut})}{X_1, X_2 \vdash Y}$$

This family of conditions is enough to ensure that any consecution which can be proven with the aid of Cut can also be proven without it. This result can be approached in two stages.

[11] In what is otherwise an excellent paper, Kracht 1994 complains needlessly about this condition. He says that 'it makes no sense to substitute structures for formulas' (footnote 1, page 9). This is mistaken. Formulae are structures, so replacing a formula with a structure is a restricted case of replacing a structure with another structure. No category mistake is involved.

Lemma 13 Parametric Stage *If for each of $X \vdash M$ and $M \vdash Y$ there are derivations ending in inferences in which the respective displayed M's are not parametric, and for all proper subformulae M' of M, $X' \vdash Y'$ is derivable if $X' \vdash M'$ and $M' \vdash Y$, then $X \vdash Y$ is derivable too.*

Proof. This is a simple application of the condition **C8**. ⊣

Lemma 14 Principal Stage *Part 1. Suppose that $X \vdash M$ is derivable, and that for all X', if there is a derivation of $X' \vdash M$ ending in an inference in which the displayed M is not parametric, then $X' \vdash Y$ is derivable. Then, $X \vdash Y$ is derivable.*

Part 2. Suppose that $M \vdash Y$ is derivable, and that for all Y', if there is a derivation of $M \vdash Y'$ ending in an inference in which the displayed M is not parametric, then $X \vdash Y'$ is derivable. Then, $X' \vdash Y$ is derivable.

We refer the reader to Belnap 1982 and the later Belnap 1990 for the proofs of this and the following theorem.

Theorem 15 *In any display calculus satisfying* **C2** *to* **C8**, *Cut is eliminable.*

This is the core of the cut elimination argument, applying the principle and parametric stages, by induction on the complexity of the cut formula.

In addition to **C2** to **C8**, Belnap provides another condition.

C1 Each formula which is a constituent of some premise of an inference is a subformula of some formula in the conclusion of that inference.

This is verified by eye in each inference we have presented. As a result, we have the subformula theorem.

Theorem 16 *Any display calculus satisfying* **C1** *has the subformula property. That is, any Cut-free proof of the consecution $X \vdash Y$ contains only (structures made up entirely of) subformulae of the formulae in X and Y.*

So, as a result, the display calculus as we first presented it is a complete proof procedure for the logic of **DW** frames. The remaining task is to show that it is also sound.

5 The Soundness Proof

We must show that any consecution provable in the display calculus is valid when interpreted in structures. For every consecution of the form $X \vdash Y$, we can interpret it as a statement $l(X) \subseteq r(Y)$ about UCLA propositions $l(X)$ and $r(Y)$ in an appropriate model.

We must then show that each display postulate is valid under a suitable interpretation in the models. As stated above, we will interpret consecutions as statements about UCLA propositions. The interpretation must vary with respect to whether the structure appears in an antecedent or consequent position. The definition is recursive, and it goes as follows:

$$\begin{array}{rlrl} l(A) &= ||A||_{\mathcal{M}} & r(A) &= ||A||_{\mathcal{M}} \\ l(\mathbf{I}) &= W & r(\mathbf{I}) &= \emptyset \\ l(\mathbf{1}) &= N & r(\mathbf{1}) &= \{x : x^* \not\in N\} = \hat{N} \\ l(X,Y) &= l(X) \cap l(Y) & r(X,Y) &= l(X) \supset r(Y) \\ l(X;Y) &= l(X) \circ l(Y) & r(X;Y) &= r(X) + r(Y) \\ l(*X) &= \widehat{r(X)} & r(*X) &= \widehat{l(X)} \end{array}$$

In the table above we use these definitions: $\sigma \circ \gamma = \{z : \exists x \in \sigma, \exists y \in \gamma \text{ where } Rxyz\}$, $\sigma + \gamma = \{z : \forall x, y \text{ where } Rxzy,\ (x^* \not\in \sigma \Rightarrow y \in \gamma)\}$, $\sigma \supset \gamma = \{z : \forall y \geq z(y \in \sigma \Rightarrow y \in \gamma)\}$ and $\hat{\sigma} = \{x : x^* \not\in \sigma\}$, for all UCLA propositions σ, γ.

Lemma 17 *For any* UCLA *propositions σ, γ, δ and ϵ, the following equivalences hold.*

$$\begin{gathered} \sigma \cap \gamma \subseteq \delta \Longleftrightarrow \sigma \subseteq \gamma \supset \delta \Longleftrightarrow \gamma \subseteq \sigma \supset \delta \\ \sigma \circ \gamma \subseteq \delta \Longleftrightarrow \gamma \subseteq \hat{\sigma} + \delta \\ \sigma \subseteq \gamma + \delta \Longleftrightarrow \hat{\delta} \circ \sigma \subseteq \gamma \Longleftrightarrow \hat{\gamma} \circ \sigma \subseteq \delta \\ \sigma \subseteq \gamma \Longleftrightarrow \hat{\gamma} \subseteq \hat{\sigma} \Longleftrightarrow \hat{\hat{\sigma}} \subseteq \gamma \\ \sigma \circ N = \sigma \end{gathered}$$

To prove this we first take a detour into the semantic corollary of contraposition.

Lemma 18 *For all σ and γ, $\sigma + \gamma = \gamma + \sigma$.*

Proof. Take $z \in \sigma + \gamma$. Then we can reason as follows

$$\begin{aligned} z \in \sigma + \gamma &\Longleftrightarrow \forall x, y(Rxzy \Rightarrow (x^* \not\in \sigma \Rightarrow y \in \gamma)) \\ &\Longleftrightarrow \forall x, y(Rxzy \Rightarrow (y^{**} \not\in \gamma \Rightarrow x^* \in \sigma)) \\ &\Longleftrightarrow \forall x^*, y^*(Ry^*zx^* \Rightarrow (y^{**} \not\in \gamma \Rightarrow x^* \in \sigma)) \\ &\Longleftrightarrow \forall x, y(Rxzy \Rightarrow (x^* \not\in \gamma \Rightarrow y \in \sigma)) \\ &\Longleftrightarrow z \in \gamma + \sigma \end{aligned}$$

So, $\sigma + \gamma \subseteq \gamma + \sigma$, which is sufficient for our result. ⊣

Using this speeds up our proof of the previous lemma.

Proof. Take $\sigma \circ \gamma \subseteq \delta$. We wish to show that $\gamma \subseteq \hat{\sigma} + \delta$. So, take $z \in \gamma$. If $Rxzy$ and $x^* \not\in \hat{\sigma}$ then by the definition of $\hat{}$, $x \in \sigma$, and $y \in \delta$ follows from our assumption that $\sigma \circ \gamma \subseteq \delta$. So, $z \in \hat{\sigma} + \delta$ as desired. The converse argument is similar.

Now take $\sigma \subseteq \gamma + \delta$. We wish to show that $\hat{\delta} \circ \sigma \subseteq \gamma$. Take a $y \in \hat{\delta} \circ \sigma$. This means that there are $x \in \hat{\delta}$, $z \in \sigma$ where $Rxzy$. But this means that

$x^* \notin \delta$. Now, $\sigma \subseteq \gamma + \delta$, so we have $z \in \gamma + \delta$, and as a result, for any x and y (and in particular, the ones we have before us) $Rxzy \Rightarrow (x^* \notin \sigma \Rightarrow y \in \gamma)$. The two antecedent conditions are satisfied here, so $y \in \gamma$ as we desired.

For the converse, suppose $\hat{\hat{\delta}} \circ \sigma \subseteq \gamma$. Take $z \in \sigma$, in order to show that $z \in \gamma + \delta$. Suppose that $z \notin \gamma + \delta$, for the sake of argument. That means $z \notin \delta + \gamma$, by our previous lemma, and hence, that there are x, y where $Rxzy$, $x^* \notin \delta$ and $y \notin \gamma$. But this means that $x \in \hat{\hat{\delta}}$, and then $z \in \sigma$ with $Rxzy$ gives $y \in \hat{\hat{\delta}} \circ \sigma \subseteq \gamma$, contradicting $y \in \gamma$. This means we must have had $z \in \gamma + \delta$ as desired.

The other condition, to the effect that $\sigma \subseteq \gamma + \delta \iff \hat{\gamma} \circ \sigma \subseteq \delta$ follows from this condition by the commutativity of $+$.

The final conditions are trivial. Since $x^{**} = x$, we have $\hat{\hat{\sigma}} = \sigma$ for any set σ and $\sigma \subseteq \gamma \iff \hat{\gamma} \subseteq \hat{\sigma}$. Finally, by the constraints on N we must have $\sigma \circ N = \sigma$. $\dashv$

> **validity in $\mathcal{M}$** A consecution $X \vdash Y$ is valid in $\mathcal{M}$ just when $l(X) \subseteq r(Y)$.

So, by Lemma 17 we have the following result.

Corollary 19 *For any model $\mathcal{M}$, display equivalences and structural rules preserve validity in $\mathcal{M}$.*

The only results not given by Lemma 17 are the extensional structural rules, but these follow immediately from the interpretation of comma in antecedent position by intersection, and **I** in consequent position by $\emptyset$.

Theorem 20 *If a consecution $X \vdash Y$ is provable in the display calculus, then it is valid in any model $\mathcal{M}$.*

Proof. By induction on the length of the proof. The axioms obviously hold in $\mathcal{M}$, as $l(p) \subseteq r(p)$, $l(\mathbf{1}) \subseteq r(t)$ and $l(\bot) \subseteq r(\mathbf{I})$ always. In our previous lemma we have shown that display equivalences, and intensional or extensional structural rules preserve validity in a model. The connective rules remain. Consider $(\rightarrow\vdash)$. Suppose that $l(X) \subseteq r(A)$ and $l(B) \subseteq r(Y)$. Then $l(A \rightarrow B) = ||A \rightarrow B||_{\mathcal{M}} = \{x : \forall y, z \text{ where } Ryxz(y \in ||A||_{\mathcal{M}} \Rightarrow z \in ||B||_{\mathcal{M}})\}$. But this is a subset of $\{x : \forall y, z \text{ where } Ryxz(y \in l(X) \Rightarrow z \in r(Y))\}$ since $||A||_{\mathcal{M}} = r(A)$ and $||B||_{\mathcal{M}} = l(B)$. And this is easily seen to be $\widehat{l(X)} + r(Y) = r(*X; Y)$, as desired. The results for the other connective postulates are similar, and are a good exercise in manipulation of UCLA propositions in frames. $\dashv$

As a result, the display calculus is sound. The axioms are valid in models, and all rules preserve validity. As a corollory of this result, if $\mathbf{1} \vdash A$ is provable in the display calculus, then for any model $\mathcal{M}$, $\mathcal{M} \models A$.

This semantic proof differs from the usual proof-theoretic methods in that we interpret consecutions directly in models instead of giving every consecution a corresponding formula. There is a point to this. Firstly, it heightens the awareness that display logic and frames are kissing cousins (an awareness which will be further heightened in the following section). Display logic proofs are ways of reasoning about frames. Secondly, it is important because it gives us a soundness result for any fragment of the language under consideration. We need not have $\circ$ in our propositional language in order to show that any display logic proofs involving semicolon are valid. Fusion is simply an operation on UCLA propositions (which may or may not be expressible in the language we are interpreting), and as an operation, it must satisfy the conditions laid down in the display logic rules. Similarly, **1** models a UCLA proposition. In the antecedent it is interpreted as N, and in the consequent, $\hat{N}$. We may have a way of talking about N in our language (here, t) but we need not. In the logic **DW** which does not have t, we can still show that any **DW** proof is sound, because frames for that logic must have such a set N anyway, and it must satisfy the properties we require of it. Similarly in logics without the intuitionistic-style $\supset$, the frame must have a containment relation $\leq$, and $\supset$ does feature as a function on UCLA propositions, even if it is not featured in the language of the logic under discussion. All of the display logic constructs (here we have semicolon, comma, $*$, **1** and **I**) have analogues in frames, independently of whether they occur in the language we use to reason *in* frames.

This is *not* the case with Belnap's original use of boolean negation. In most frames, boolean negation is not an operation on UCLA propositions. It is an operation on arbitrary sets of points in a model, and in Belnap's formulation of display logic, structures correspond to arbitrary sets of points, not only UCLA propositions. This is why he needs to invoke apartheid to deal with logics which are not conservatively extended by boolean negation (such as intuitionistic logic, **E**, and **CK**). While I agree with Belnap's comment[12] that there is no problem in principle with our calculus using concepts which cannot be expressed in the language we originally considered, there is an important distinction to be made. Either the new concepts can be added conservatively (such as backward looking modalities in conventional modal logic, or fusion and $\supset$ in relevant logics) or they cannot (such as boolean negation in intuitionistic logic and in **E** and **CK**). In the latter case, the extra concepts cannot simply be added to the logic without cost. Of course, you can get a *different* logic with the new concept (extending intuitionistic logic with boolean negation on intuitionistic frames to get what is essentially **S4**) which contains the old logic as a special class of formulae (say, the necessitated formulae). But then the expressive power of the language is essentially increased; the new operation does not keep you within the bounds

[12] In endnote 13 (Belnap 1982).

of the old class of UCLA propositions. That can be a Bad Thing. Proof search becomes more complicated because there is essentially more available to be proved.[13]

6 Division of Labour

The display calculus proof theory and frames-based model theory bear similar marks. The rules for connectives are one thing, and then underneath them, you can modify the structure in order to get stronger logics. In the display calculus, you add structural rules. With frames, you add conditions on the ternary relation. The structural constants comma, semicolon and star are tied intimately to the features $\leq$, R and * in ternary frames. In adding structural rules, or adding conditions to frames, you do not change the behaviour of the connectives, rather, you change the behaviour of the 'substrate' with which the connectives interact.

Most work on either the semantics or the proof theory of relevant logics has only examined a small number of extensions to the basic **DW**, showing how they are all modellable with corresponding conditions on the ternary relation R or additions to the structural rules of the proof theory, or whatever. That is truly the way of the past. If we have the means for truly *general* results, we should make use of them. I, for one, do not know which logics in the relevant family will turn out to be interesting or useful in future research, so we will consider a truly *large* class of extensions of **DW**. We will consider extensions of the display calculus which add any number of structural rules each of the form

$$\frac{X \vdash Z}{Y \vdash Z}$$

Where X and Y are structures containing only formulae, $\mathbf{I}$, $\mathbf{1}$, comma and semicolon. We restrict ourselves in this way for two reasons. Firstly, it is much simpler to not worry about the effects of $*$. Secondly, most of the popular extensions to **DW** are given by adding of rules of this form. For example, the following are examples of such structural rules.

$$\frac{X;(Y;Z) \vdash W}{(X;Y);Z \vdash W}\,(\mathrm{B}) \qquad \frac{X;(Y;Z) \vdash W}{(X;Z);Y \vdash W}\,(\mathrm{B}') \qquad \frac{X \vdash Z}{\mathbf{1};X \vdash Z}\,(\mathrm{CII})$$

$$\frac{(X;Y);Z \vdash W}{(Y;X);Z \vdash W}\,(\mathrm{C}) \qquad \frac{X \vdash Z}{Y;X \vdash Z}\,(\mathrm{K}) \qquad \frac{X;X \vdash Z}{X \vdash Z}\,(\mathrm{W})$$

[13] It is fairly clear — at least to me — that these considerations have more than a little connection with the in-house relevant logic dispute about the propriety of boolean negation, and the broader dispute about the validity of disjunctive syllogism. Following through these considerations must, however, be left to another place and another time.

It is easy to see that adding a structural rule of this form to the display calculus is equivalent to adding an axiom of the form $A \to B$ to the Hilbert system, where A and B are *confusions.*[14]

> **confusion** t is a confusion, $\top$ is a confusion, any propositional variable is a confusion, and if A and B are confusions, then so are $A \circ B$ and $A \wedge B$.

So, axioms of this form include

$$(p \circ q) \circ r \to p \circ (q \circ r) \qquad (p \circ q) \circ r \to p \circ (r \circ q) \qquad t \circ p \to p$$

$$(p \circ q) \circ r \to (q \circ p) \circ r \qquad q \circ p \to p \qquad p \to p \circ p$$

and many, many others.[15] Given a confusion A, there is a corresponding condition $\mathcal{R}^A$ which encodes what the ternary relation must be like in order for the fusion to be true at a point. The condition is defined by induction on the complexity of R.

> **The condition** $\mathcal{R}^A(x)$ Given a confusion A, with atomic propositions among $p_1, \ldots, p_n$, the corresponding condition $\mathcal{R}^A(x)$ is defined recursively.

$$\begin{aligned} \mathcal{R}^t(x) &= x \in N \\ \mathcal{R}^\top(x) &= \texttt{true} \\ \mathcal{R}^{p_i}(x) &= x_i \leq x \\ \mathcal{R}^{B \circ C}(x) &= \exists z_1 \exists z_2 (\mathcal{R}^B(z_1) \wedge \mathcal{R}^C(z_2) \wedge R z_1 z_2 x) \\ \mathcal{R}^{B \wedge C}(x) &= \mathcal{R}^B(x) \wedge \mathcal{R}^C(x) \end{aligned}$$

So, $\mathcal{R}^A(x)$ means, given that $\mathcal{M}, x_i \models p_i$ for each i, then $\mathcal{M}, x \models A$. We have the following result.

Lemma 21 *In any model $\mathcal{M}$, for any confusion A with atomic constituents $p_1, \ldots p_n$, $\mathcal{M}, x \models A$ if and only if there are $x_1, \ldots, x_n \in W$ such that for each i, $\mathcal{M}, x_i \models p_i$ and $\mathcal{R}^A(x)$.*

Proof. A simple induction on the complexity of A. $\mathcal{M}, x \models p_i$ if and only if there is some x_i where $\mathcal{M}, x_i \models p_i$ and $x_i \leq x$. $\mathcal{M}, x \models t$ if and only if $x \in N$, $\mathcal{M}, x \models \top$ always. This deals with the base cases. For the

[14] This terminology stems from Meyer and Slaney, but I expand the notion to include t and $\top$ which are obviously the empty fusion and the empty conjunction respectively. As all close readers of (Slaney to appear) will note, this is parallel to Slaney's nuanced distinction between the null multiset (in which every element occurs zero times) and the null set (which simply has no elements).

[15] To see *how* many others, note that $A \circ \top$ and $\top \circ A$ act as backward looking modalities $\blacklozenge_1 A$ and $\blacklozenge_2 A$, with accessibility relations $yR_1z = (\exists x)Ryxz$ and $yR_2z = (\exists x)Rxyz$. With two independent modalities there are lots of things to play with. Wait for my forthcoming "Functional Frames for Substructural Logics" for more discussion of this point.

induction steps, note that $\mathcal{M}, x \models B \circ C$ if and only if there are z_1, z_2 where $\mathcal{M}, z_1 \models B$, $\mathcal{M}, z_2 \models C$, and Rz_1z_2x. But by hypothesis, $\mathcal{M}, z_1 \models B$ if and only if $\mathcal{R}^B(z_1)$ and $\mathcal{M}, z_2 \models C$ if and only if $\mathcal{R}^C(z_2)$, giving us our condition. The case for conjunction is trivial. ⊣

So, in any model we have conditions under which a confusion is supported at a point. We can strengthen this result as follows, giving us a condition under which the relation $\mathcal{R}^A$ holds of a point x in a frame.

Lemma 22 *Given a model $\mathcal{M}$, and a confusion A with constituents $p_1, \ldots p_n$, such that there are points x_i where $\mathcal{M}, x \models p_i$ if and only if $x_i \leq x$, then $\mathcal{M}, x \models A$ if and only if $\mathcal{R}^A(x)$.*

Proof. For $\mathcal{R}^A(x) \Rightarrow \mathcal{M}, x \models A$, use Lemma 21. For the other direction, another induction on the complexity of A suffices. Clearly if $\mathcal{M}, x \models t$ if and only if $x \in N$, $\mathcal{M}, x \models \top$ always, and $\mathcal{M}, x \models p_i$ if and only if $x_i \leq x$. For the induction steps, note that $\mathcal{M}, x \models B \circ C$ if and only if for some z_1, z_2 where Rz_1z_2x, $\mathcal{R}^B(z_1)$ and $\mathcal{R}^C(z_2)$ (by hypothesis), but this is exactly $\mathcal{R}^{B\circ C}(x)$. The case for conjunction is similar. ⊣

This lemma is enough to give us a strong correspondence result.

Theorem 23 *For any confusions A and B with constituents $p_1, \ldots, p_n$, a frame $\mathcal{F}$ satisfies $(\forall x, x_1, \ldots x_n)(\mathcal{R}^A(x) \Rightarrow \mathcal{R}^B(x))$ if and only if $\mathcal{F} \models A \to B$*

Proof. For left to right it is a simple exercise of proving that $\mathcal{F} \models A \to B$, given the condition. Suppose the frame satisfies $(\forall x)(\mathcal{R}^A(x) \Rightarrow \mathcal{R}^B(x))$. Then suppose $\mathcal{M}, x \models A$, for some model $\mathcal{M}$ on $\mathcal{F}$. Then, there must be $x_1, \ldots, x_n$ such that $\mathcal{M}, x_i \models p_i$ for each i, and $\mathcal{R}^A(x)$ (by Lemma 21). But this means that $\mathcal{R}^B(x)$, and by Lemma 21 we have $\mathcal{M}, x \models B$ as desired.

Now for right to left, we use Lemma 22 instead. Suppose that $\mathcal{F} \models A \to B$. Then given $x_1, \ldots, x_n$, construct a model $\mathcal{M}$ in which $V(p_i) = \{x : x_i \leq x\}$ for each i. So, $\mathcal{M}, x \models p_i$ if and only if $x_i \leq x$. Suppose $\mathcal{R}^A(x)$. Then by Lemma 22, $\mathcal{M}, x \models A$. So by the condition that $\mathcal{F} \models A \to B$ we have $\mathcal{M}, x \models B$ too, and Lemma 22 gives us $\mathcal{R}^B(x)$ as desired. ⊣

This last theorem gives us a soundness result ensuring that any extension of **DW** by an axiom of the form $A \to B$, where A and B are confusions, is sound with respect to the class of frames satisfying the condition corresponding to $A \to B$. It also gives us a correspondence result. Any *frame* validating $A \to B$ must also satisfy the corresponding condition. It does not yet give us completeness. We have yet to show that the *canonical model* for a logic including $A \to B$ satisfies corresponding condition corresponding to $A \to B$. We cannot do this simply by way of the results we have so far, because we cannot apply Lemma 22, since it appeals to an evaluation which makes a proposition true only at x_i and those points containing x_i. For many points in the canonical model, there are no such 'witnessing formulae.'[16] Rather, we must argue directly that the canonical model satisfies the corresponding condition. For this we need a little more argument.

Recall that the canonical structure is made up of particular sorts of *theories*, namely non-trivial prime theories.[17]

> **theory** A theory is a set of formulae a closed under conjunction (if $A \in a$ and $B \in a$ then $A \wedge B \in a$) and entailment (if $A \vdash B$ is provable and $A \in a$ then $B \in a$). A theory a is said to be *prime* if in addition, whenever $A \vee B \in a$ either $A \in a$ or $B \in a$. A theory a is said to be *trivial* if $\bot \in a$ or $a = \emptyset$.

Given any theories a, b, c, we define ab to be $\{C : \exists A \in a, B \in b \text{ where } \vdash A \circ B \to C\}$ and *Rabc* if and only if $ab \subseteq c$. Then we can set N to be the set of all non-trivial prime theories containing t. The resulting structure is said the canonical frame. It is simple to show that $a \leq b$ if and only if $a \subseteq b$. The canonical model is given by assigning $V(p) = \{a : p \in a\}$. The completeness result involves showing that this structure satisfies the condition $\mathcal{M}, a \models A$ if and only if $A \in a$. A crucial step in the proof is the squeezing lemma, a folkloric result in relevant logics. See the standard references (Anderson et al. 1992, Dunn 1986) for the proof.

Lemma 24 *If c is a non-trivial prime theory, and if $ab \subseteq c$ then there are non-trivial prime theories $a' \supseteq a$ and $b' \supseteq b$ where $a'b' \subseteq c$.*

In order to prove completeness we need to show that the canonical frame satisfies the condition $\mathcal{R}^A \Rightarrow \mathcal{R}^B$, if we have $A \to B$ (and all its substitution instances) as a theorem (or equivalently if the display calculus has the corresponding structural rule). To do that, we will need to show that for any non-trivial prime theory x, and any theories $x_1, \ldots, x_n$, if $\mathcal{R}^A(x, x_1, \ldots, x_n)$ then $\mathcal{R}^B(x, x_1, \ldots, x_n)$. So, we will take first an arbitrary collection $x_1, \ldots, x_n$ of theories in the canonical frame, and given them, we will define theories which will match the process of building up theories referred to in the conditions $\mathcal{R}^A$ and $\mathcal{R}^B$. The definitions we need are as follows.

> **theory corresponding to a formula** Given a collection $x_1, \ldots, x_n$ of theories, and a confusion A with atomic constituents $p_1, \ldots, p_n$, we define the theory $A(x_1, \ldots, x_n)$ corresponding to A (as a function of $x_1, \ldots, x_n$) as follows.
>
> - $p_i(x_1, \ldots, x_n)$ is x_i.
> - $t(x_1, \ldots, x_n) = \{A : \vdash t \to A\}$.

[16] If there are κ formulae, there are 2^κ points in the canonical model.

[17] For those that don't recall this, there are many ways the memory can be refreshed (Anderson et al. 1992, Routley and Meyer 1973, Dunn 1986, Restall 1993).

- $\top(x_1, \ldots, x_n) = \emptyset$ (the empty theory).
- $(A \circ B)(x_1, \ldots, x_n) = A(x_1, \ldots, x_n)B(x_1, \ldots, x_n)$ (using the definition of the fusion of two theories given before).
- $(A \wedge B)(x_1, \ldots, x_n) = \{C :$ for some $C_1 \in A(x_1, \ldots, x_n)$ and $C_2 \in B(x_1, \ldots, x_n), \vdash C_1 \wedge C_2 \to C\}$.

Note that the theory corresponding to $A \wedge B$ is *larger* than the theories corresponding to A and B respectively (specifically, it is the closure of their union). This may seem counter-intuitive, given the connection between conjunction and intersection, but it is the right decision. The theory corresponding to A is simply this: given that the atomic propositions p_i in A are interpreted as being true only at points in the canonical model containing the theory x_i, then A is true at a point in the canonical model if and only if that point contains the theory $A(x_1, \ldots, x_n)$. The theory corresponding to a conjunction is a *bigger* theory (as it contains everything in the theories of either conjunct) so it is true at fewer points, as one would expect.

Lemma 25 *Where x_1, ... , x_n, and x are points in the canonical model, $\mathcal{R}^A(x)$ in the canonical model if and only if $A(x_1, \ldots, x_n) \subseteq x$.*

Proof. By induction on the construction of A. Clearly the result holds for $A = t$, $\top$ or p_i. Consider the case for conjunction $(B \wedge C)(x_1, \ldots, x_n) \subseteq x$ if and only if $B(x_1, \ldots, x_n) \subseteq x$ and $C(x_1, \ldots, x_n) \subseteq x$. So, we have our induction step. For fusion, $(B \circ C)(x_1, \ldots, x_n) \subseteq x$ if and only if $B(x_1, \ldots, x_n)C(x_1, \ldots, x_n) \subseteq x$, and by the squeezing lemma this means that there are prime theories $z_1 \supseteq B(x_1, \ldots, x_n)$ and $z_2 \supseteq C(x_1, \ldots, x_n)$ where $z_1 z_2 \subseteq x$ (that is, Rz_1z_2x) and this is enough for this case of the induction step. So, the lemma is proved. ⊣

This is almost enough for our completeness result. We need only one more lemma.

Lemma 26 *If $\vdash A \to B$, then for any theories $x_1, \ldots, x_n$, $B(x_1, \ldots, x_n) \subseteq A(x_1, \ldots, x_n)$.*

Proof. By induction on the complexity of A it is simple to show that $C \in A(x_1, \ldots, x_n)$ if and only if there are some $E_i \in x_i$ such that $\vdash A(E_1, \ldots, E_n) \to C$. (Where $A(E_1, \ldots, E_n)$ is the formula resulting from substituting E_i for p_i in A.) But then we can reason as follows. If $C \in B(x_1, \ldots, x_n)$ then there are $E_i \in x_i$ such that $\vdash B(E_1, \ldots, E_n) \to C$. Since $\vdash A(E_1, \ldots, E_n) \to B(E_1, \ldots, E_n)$ (by uniform substitution on $\vdash A \to B$), we have $\vdash A(E_1, \ldots, E_n) \to C$. But this means that $C \in A(x_1, \ldots, x_n)$ as desired. ⊣

Theorem 27 *If $\vdash A \to B$, then the canonical model structure satisfies the corresponding condition, $(\forall x)(\mathcal{R}^A(x) \Rightarrow \mathcal{R}^B(x))$.*

Proof. Suppose $\vdash A \to B$. Then for any non-trivial prime theory x, and any theories $x_1, \ldots, x_n$, if $A(x_1, \ldots, x_n) \subseteq x$ then $B(x_1, \ldots, x_n) \subseteq x$, by Lemma 26. But $A(x_1, \ldots, x_n) \subseteq x$ if and only if $\mathcal{R}^A(x)$ and similarly $B(x_1, \ldots, x_n) \subseteq x$ if and only if $\mathcal{R}^B(x)$, so we have for any x, if $\mathcal{R}^A(x)$ then $\mathcal{R}^B(x)$ as desired. ⊣

So, we have a large family of extensions to **DW**. Any logic extending **DW** by adding a collection of axioms of the form $A \to B$, where A and B are confusions, has a corresponding display calculus rule, and a condition on **DW** frames, together with a soundness, completeness, and correspondence result. These extensions include the famous ones B, B′, C, CII, K and W. This gives us the familiar class of relevant logics

Logic	Extra Rules
TW	B, B′
T	B, B′, W
EW	B, B′, CII
E	B, B′, CII, W
C	B, B′, C
R	B, B′, C, W
CK	B, B′, C, K

But the resources we have available permit many other extensions too.

We must be careful: Not any extension of **DW** by a class of rules of the form

$$\frac{X \vdash Z}{Y \vdash Z}$$

where X and Y are confusions will preserve Cut elimination or the subformula property. An axiom of the form $A \to A \circ B$ will not preserve the subformula property, for example. For its rule will be

$$\frac{X ; Y \vdash Z}{X \vdash Z}$$

which does not satisfy condition **C1**. We need a smaller class of rules. In addition, the 'Mingle' rule does not satisfy condition **C3**. However, these are the only two conditions we need consider. Every other condition is automatically satisfied by these extensions.

> **proper axioms and rules** An axiom of the form $A \to B$, where A and B are confusions with atomic propositions $p_1, \ldots, p_n$ is said to be proper if every p_i in B appears somewhere in A and no p_i appears more than once in B. The rules corresponding to proper axioms are said to be proper rules.

Note that each of B, B′, C, CII, K and W are proper.

Theorem 28 *Every extension of* **DW** *by proper rules has a Cut-free display calculus with the subformula property.*

Proof. A simple matter of checking each of **C1** to **C8**. That the rules are proper means **C1** and **C3** are taken care of. For the rest, note **C2** and **C6/7** are satisfied by definition, **C4** is trivial because the rules modify only antecedent parts, and the constituents of confusions in an antecedent are still antecedent parts. For **C5** there are *no* non-parametric formulae — these are purely structural rules. Similarly, the rules do not affect **C8** which deals with connective rules or axioms. ⊣

7 Decidability

Even though display logic is much more expressive than traditional Gentzen systems (we have means of moving structures around and nesting antecedents inside consequents, inside antecedents, and so on) it still yields decidability results in certain cases. The key idea is to show that in proof search we do not need to backtrack through all possible consecutions from which a consecution may have come (because there are infinitely many; consider the display rules, (**1**E) or (eW), each of which can destroy material). Rather, we show that if any consecution has a proof, then a terminating search procedure will find a proof, and if it does not have proof, the procedure will terminate. To do this, we need to gain control of contraction, and the intensional display rules. The intensional display rules are the simplest to take care of.

> **$*$-reduced structures** A structure is $*$-reduced if and only if it has no substructures of the form $**X$. Clearly any structure is display equivalent to a $*$-reduced structure. Given a structure X, its reduced asterisked form $\overline{X}$ is defined as $*X$ if X is not of the form $*Y$, and Y if X is of the form $*Y$. Clearly if X is $*$-reduced, so is $\overline{X}$. Given a consecution S, its $*$-reduction is given by replacing all substructures of the form $**X$ by X, until the result is $*$-reduced.

Clearly we should have no need to use non-$*$-reduced structures in any proof a $*$-reduced consecution. The only reason we might have to introduce a structure of the form $**X$ in a proof is in the connective rule ($\to\vdash$). But once we replace that by its equivalent $*$-reduced cousin

$$(\to\vdash)' \quad \frac{X \vdash A \quad B \vdash Y}{A \to B \vdash \overline{X}; Y}$$

we can get away without any stacked asterisks in our proofs.

Lemma 29 *Any $*$-reduced consecution has a $*$-reduced proof. That is, it has a proof in which every structure that appears is $*$-reduced.*

Proof. Take any proof of a $*$-reduced consecution. Consider the tree you obtain by replacing every consecution in the proof with its $*$-reduction. The resulting tree is a proof, as can easily be checked by examining each rule. The only difficulty could be caused by ($\rightarrow\vdash$), but that has been dealt with. The only other rules which introduce asterisks do so on formulae which are either the entire antecedent or the entire consequent (so no stacking can occur there) or display rules (but if S_1 and S_2 are display equivalent, so are their $*$-reductions). So, the result remains a proof. $\dashv$

Dealing with ($\mathbf{1}$E) and (eW) is more difficult. For that, we need more notions of 'needless structure,' and a broader notion of equivalence.

> **equivalence** Two consecutions $X \vdash Y$ and $Z \vdash W$ are said to be *equivalent* if one can be transformed into the other by means of display equivalences or (eB).

Note that two eqiuvalent consecutions convey exactly the same proof theoretic information. They are interderivable, and in addition, they contain the same amount of structure. No structure is introduced or destroyed by these rules, unlike (eW), (eK) and the **1** rules.

> **nearness** In any consecution $X \vdash Y$, the substructures Z_1 and Z_2 are *near* if $X \vdash Y$ is equivalent to $Z_1, Z_2 \vdash W$, or $W \vdash *(*Z_1, *Z_2)$ (those instances of Z_1 and Z_2 being indicated).

Here the idea is that an instance of a structure X is superfluous if it occurs nearby another instance of the same structure. For clearly, we can weaken in a duplicate of X at will, and contraction will allow us to eliminate it.

> **superfluous 1s** In $X \vdash Y$, a **1** occurring as a substructure is said to be superfluous if the structure is display equivalent to $Z; \mathbf{1} \vdash W$. (That **1** being indicated.)

The idea here is the same. If I get rid of superfluous **1**s, I lose no information, because the **1** rules allow us to add them or get rid of them as our hearts desire.

Note that we are relying on a principle of re-identification of structures though rules. This can be formally defined, but we will not pause to do so here. Rather, we will consider when it is possible to delete structure from a consecution.

> **deletion** A substructure Z can be deleted from a $*$-reduced consecution $X \vdash Y$ if and only if it is a substructure of some structure in $X \vdash Y$, containing a comma or a semicolon. (In

other words, it is not any of X, Y, $\overline{X}$ and $\overline{Y}$.) The result of deleting Z from $X \vdash Y$ is given by replacing the structure in which Z appears by a structure without Z, by way of the following rules. Replace any of $W \star Z$, $Z \star W$, $W \star *Z$ or $*Z \star W$ by W, where $\star$ is either comma or semicolon.

For example, we can delete Z in $X; (Y; *Z) \vdash X, Y$ to get $X; Y \vdash X, Y$. We cannot delete X in $*(Y; *Z) \vdash *X$ because the resulting consecution would have no consequent.

Using this notion of deletion, we can define the *reduction* of a consecution in terms of deleting superfluous structures.

reduction For any $*$-reduced consecution $X \vdash Y$, define its *reduction* $r(X \vdash Y)$ as follows. For any instances of Z nearby to another Z, delete the first such Z (from left to right) which can be deleted (it is simple to see that at least one such Z can be deleted, lest they fail to be nearby each other). For any deletable superfluous $\mathbf{1}$s, delete them. If the superfluous $\mathbf{1}$ is the entire antecedent, then the consecution is of the form $\mathbf{1} \vdash Y; Z$. Replace this consecution by $\overline{Y} \vdash Z$. If the superfluous $\mathbf{1}$ is the entire consequent (starred) then the consecution is of the form $*(Z; W) \vdash *\mathbf{1}$. Replace this by $\overline{Z} \vdash W$.

A consecution is reduced if and only if $X \vdash Y = r(X \vdash Y)$. Clearly a consecution is reduced if and only if it has no superfluous $\mathbf{1}$s and no structures nearby another instance of the same structure.

semi-reduced A consecution is semi-reduced just when either it is reduced, or it has a superfluous $\mathbf{1}$, such that upon its deletion, the consecution is reduced, or it has two instances of a structure X nearby one another, such that upon the deletion of one, the consecution is reduced. A proof is semi-reduced iff all of its constituent consecutions is.

To accommodate our desire to use only reduced consecutions, or perhaps semi-reduced consecutions in a proof, we need modify the extensional connective rules a little, in order to incorporate (eW) into them. We replace $(\vdash \wedge)$ and $(\vdash \vee[1,2])$ by the following rules.

$$(\vdash \wedge)' \quad \frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B} \qquad (\vdash \vee)' \quad \frac{X \vdash *(*A, *B)}{X \vdash A \vee B}$$

It is clear that any proof using the old rules can be made into a proof using the new rules by inserting appropriate weakenings, contractions and display equivalences. Finally, we need to ensure that proofs do not loop back on themselves.

irredundant A proof is said to be irredundant just when no consecution appears twice in any of its branches.

Lemma 30 *Any provable reduced consecution has an irredundant semi-reduced proof.*

Proof. Take a proof of $X \vdash Y$, and apply r to every node. The result is almost a proof. Clearly all its leaves are axioms, because for every axiom C, $r(C)$ is also an axiom. All we need do is add some semi-reduced consecutions between premises and conclusions of 'reduced' rule applications to make them fall under the rules as we have presented them. Clearly display equivalences, and extensional structural rules survive unscathed, except for ($\mathbf{1}$E), ($\mathbf{1}$I) and (eW) which collapse into the identity rule (in which the premise is the same consecution as the conclusion). For connective rules we need more work. We will do the work for conjunction, implication and negation, and leave the rest for the willing reader.

Firstly the conjunction rules. How can we make $r(A \wedge B \vdash X)$ follow from $r(A, B \vdash X)$? If $r(A, B \vdash X)$ is $A, B \vdash X'$ for some X', we can go to $A \wedge B \vdash X'$, but this may not be $r(A \wedge B \vdash X)$ for two reasons. Either X' may contain a matching $A \wedge B$ which we need to eliminate (by an application of (eW)), or X might have contained matching instances of A and B matching the A and B in the antecedent of $A, B \vdash X$. So we must use (eK) (and display rules) to weaken them back. Then the result is $r(A, B \vdash X)$, as required. In this proof the only threat to reduction would be the two appearances of $A \wedge B$ in a consecution, appearing for a little while until we could delete one. But this is still within the bounds of semi-reduction.

If $r(A, B \vdash X)$ is *not* of the form $A, B \vdash X'$, we must do a little more work. This is either because one of A or B was deleted, either because it matched an A or B in X or they matched each other. In either case, we can go from $r(A, B \vdash X)$ to a semi-reduced consecution of the form $A, B \vdash X'$ (semi-reduced if $A = B$) by judicious applications of display rules and (eW) and then to $A \wedge B \vdash X'$, in which we can weaken in any required instances of A and B, and use (eW) to eliminate any matching instance of $A \wedge B$. The intervening steps are all semi-reduced.

For $(\vdash \wedge)'$ the reasoning is simpler. $r(X \vdash A)$ must be of the form $X' \vdash A$, and $r(X \vdash B)$ must be of the form $X'' \vdash B$. X' and X'' can only differ because in X' an A has been deleted, and in X'' a B has been deleted. (Note that had we used the original $(\vdash \wedge)$ rule we would have been in trouble, because in the move from $X \vdash A$ and $Y \vdash B$ to $X, Y \vdash A \wedge B$ there is no way to keep semi-reduction in general.) Use (eW) to add A and B if necessary (keeping semi-reduction) to produce X''', and then we can infer $X''' \vdash A \wedge B$, which becomes $r(X \vdash A \wedge B)$ upon the deletion of any matching $A \wedge B$ in X. The intermediate consecutions are all semi-reduced.

For $(\to\vdash)'$, $r(X \vdash A)$ and $r(B \vdash Y)$ must be of the form $X' \vdash A$ and $B \vdash Y'$, so we can immediately infer $A \to B \vdash \overline{X}; Y$. Then X or Y might be a superfluous $\mathbf{1}$, in which case we must delete one. Then we have $r(A \to B \vdash \overline{X}; Y)$, as no other reductions are possible.

For $(\vdash\to)$, $r(A; X \vdash B)$ is either $A; X' \vdash B$ or $A \vdash B$. In the first case, deduce $X' \vdash A \to B$, and delete any matching $A \to B$ in X', to get $r(X \vdash A \to B)$. In the other case, use ($\mathbf{1}$I) to deduce $A; \mathbf{1} \vdash B$ (semi-reduced) and then $\mathbf{1} \vdash A \to B$ (reduced).

For $(\sim\vdash)$, $r(*A \vdash X)$ is of the form $*A \vdash X'$, so we can deduce $\sim A \vdash X'$. In this consecution we delete any matching $\sim A$ to get $r(\sim A \vdash X)$ as desired. The case for $(\vdash\sim)$ is totally dual.

If this proof is not irredundant, simply excise the branch sections between repetitions of a consecution. The result remains a proof. $\dashv$

We are nearly home. To complete the proof of decidability it is simply a matter of finding a notion of complexity which will not increase in a proof from root to branch, and then to put the pieces together. Because of the flexibility of the display rules, we have to be quite subtle in our notion of complexity.[18]

complexity The complexity $com(X \vdash Y)$ of the consecution $X \vdash Y$ is determined by induction on its construction. Firstly we define the complexity of a structure appearing within a consecution.

- $com(p) = com(\bot) = com(t) = 1$.
- $com(A \wedge B) = com(A \vee B) = \max(com(A), com(B))$.
- $com(\sim A) = com(A) + 1$.
- $com(A \to B) = com(A \circ B) = com(A) + com(B) + 1$.
- $com(\mathbf{I}) = 1$.
- $com(\mathbf{1}) = 1$, unless that $\mathbf{1}$ is superfluous, in which case its complexity is zero.
- $com(*X) = com(X)$.
- $com(X; Y) = com(X) + com(Y) + 1$.
- $com(X, Y) = com(X) + com(Y) + 1$ if X, Y is in consequent position.
- $com(X, Y) = \max(com(X), com(Y))$ if X, Y is in antecedent position.

[18] This notion is essentially due to Giambone 1985, modified for the setting of display logic.

Then we set $com(X \vdash Y)$ to be the least value of $com(X') + com(Y')$ where $X' \vdash Y'$ is equivalent to $X \vdash Y$. We need this subtlety because we wish $p \vdash p, q$ to have the same complexity as $p, p \vdash q$, since we can pass from one to the other freely. For our purposes, display equivalent consecutions are informationally identical, so we need to keep their complexity the same.

Lemma 31 *In any proof, complexity decreases from root to leaves.*

Proof. Inspect each of the rules. Clearly display equivalences and (eB) keep complexity invariant, by construction. Of the structural rules only (**1**E) and (eW) add new structure, but these do not change complexity.

The connective rules must be examined case by case. We will work through the conjunction, and implication and negation rules, leaving the rest for the reader. Take $(\wedge \vdash)$. The way we have defined it, the complexity of $A, B \vdash X$ will be identical to the complexity of $A \wedge B \vdash X$, as A, B in antecedent position has the same complexity as $A \wedge B$, and these are near the same structures.

For $(\vdash \wedge)'$, it is clear that the complexity of $X \vdash A$ and $X \vdash B$ is no greater than that of $X \vdash A \wedge B$.

Now consider $(\rightarrow\vdash)'$. The indicated $A \rightarrow B$ in the conclusion is not superfluous (there is nothing nearby). Similarly, nothing in $\overline{X}$ could be nearby anything in Y, so the complexity of $X \vdash A$ and that of $B \vdash X$ are strictly less than that of $A \rightarrow B \vdash \overline{X}; Y$, since we have traded in a connective, unless the X in the conclusion is a superfluous **1**. Then the complexity still cannot increase because the complexity of $X \vdash A$ is simply complexity of A, plus one. The complexity of $A \rightarrow B \vdash \overline{X}; Y$ must at least match this.

Now take $(\vdash\rightarrow)$. Complexity does not increase from conclusion to premise, because the X cannot be a superfluous **1**, and any matching within the X in the conclusion must be duplicated in the X occurring in the premise. The only fancy work is when the $A \rightarrow B$ matches an $A \rightarrow B$ occurring in X. In that case, the complexity is at most constant from conclusion to premise, because we retain the matching $A \rightarrow B$ in the X in the premise, while we trade the other $A \rightarrow B$, for an A, a B and a semicolon, while the X may become a superfluous **1**.

Finally, negation. In both rules we trade in $\sim A$ for $*A$, thereby reducing the complexity of the structures. As these structures occupy the same place in their respective consecutions, complexity cannot increase. It can only decrease because there are more display equivalences of the upper consecution because of the increased freedom given by the asterisk in place of the negation. $\dashv$

Lemma 32 *Given a consecution $X \vdash Y$, and a natural number m there are a finite number of semi-reduced consecutions built up from subformulae of formulae in X and Y, and with complexity no greater m, and only a finite number of semi-reduced structures of complexity no greater than $m \Leftrightarrow 1$.*

Proof. By induction on m. The result holds for $m = 2$. Structures of complexity 1 are formulae involving only atoms, conjunction and disjunction. There are finitely many of these in $X \vdash Y$. These can only be put together by the comma in the antecedent, or asterisk, and there are only finitely many ways to do this, keeping semi-reduction. Finally, we may have a single superfluous **1**. Again, only finitely many possibilities there. So, there are only finitely many possibilities for consecutions of complexity 2.

Now suppose the result for $n < m$, and consider consecutions of complexity m. Any such consecution must contain only the following sorts of structures.

- A subformula of A (finitely many of these) or the star of a subformula of A (finitely many of these).
- $X;Y$, or X,Y in consequent position, where X and Y have complexity less than m, made up of subformulae of A, and by hypothesis, there are finitely many of these, or the star of such a structure (and there are finitely many of these).
- Or we could add a superfluous **1** without addition of complexity, but we can add *only one* in the whole consecution. So this does not increase the number greatly.
- An X,Y in consequent position, where each of X and Y have complexity less than m. Each of X and Y can themsleves be of the form Z,W, but eventually the process must stop. These structures must have complexity less than m and not of the form Z,Y, and hence there must be finitely many of them. The structure X,Y may only repeat one such structure, lest the consecution fail to be semi-reduced. So, we must have finitely many of these.

Since the consecution must be made from these structures, there are only finitely many such consecutions. $\dashv$

Theorem 33 *Whether $X \vdash Y$ is provable or not is decidable.*

Proof. Generate a *proof search tree* of $r(X \vdash Y)$ as follows. Enter $r(X \vdash Y)$ as the bottom node. Above each consection C occurring at height k in the tree, enter nothing if C is an axiom, or branch and add every semi-reduced C' where C is the conclusion of some rule for which C is the conclusion and such that the tree is irredundant.

This procedure is effective, and it has the finite fork property (there are only a finite number of consecutions to add at each step) and the finite branch property (there are only a finite number of consecutions which can appear in such a tree, and they cannot repeat along a branch. So, the tree is finite. Clearly, however, if $r(X \vdash Y)$ has a proof, it has an irredundant, semi-reduced proof, so it will appear as a subtree of the search tree. $\dashv$

This decidability result extends quite simply to extensions of **DW** too. All we need is the finite branch and finite fork properties to be maintained, and the generation procedure to be effective.

> **flatness** A proper rule is said to be *flat* if for every instance the complexity of the premise is no greater than the complexity of the conclusion.

Theorem 34 *Any logic extending* **DW** *with a decidable set of flat structural rules is decidable.*

Proof. Clearly the proof procedure is no more complex. We add a decidable set of flat structural rules. We have effectiveness because of the decidability of the class of rules, we have the finite fork property because at any fork we add only structures made of subformulae of the original consecution, and of no greater complexity (by the flatness of the rules), and we have the finite branch property because complexity still decreases — the rules are flat. $\dashv$

Since B, B′, CII, C, and K are flat (but notoriously, W is not), we have the following corollary.

Corollary 35 *Each of* **TW**, **EW**, **C**, *and* **CK** *are decidable.*

There are many other flat, proper rules, such as

$$\frac{X \vdash Y}{[X;\mathbf{I}]^n \vdash Y}\,(\blacklozenge_1^n) \qquad \frac{X \vdash Y}{{}^n[\mathbf{I};X] \vdash Y}\,(\blacklozenge_2^n)$$

for each n, where ${}^0[\mathbf{I};X] = X$, and ${}^{n+1}[\mathbf{I};X] = \mathbf{I};{}^n[\mathbf{I};X]$ and similarly, $[X;\mathbf{I}]^0 = X$ and $[X;\mathbf{I}]^{n+1} = [X;\mathbf{I}]^n;\mathbf{I}$. This gives us many other decidable logics extending **DW** to work with. The rules $(\blacklozenge_1^n)$ and $(\blacklozenge_2^n)$ correspond to the axioms $\blacklozenge_1^n A \to A$ and $\blacklozenge_2^n A \to A$, where, you will recall, we have defined $\blacklozenge_1 A$ to be $A \circ \top$ and $\blacklozenge_2 A$ to be $\top \circ A$.

This fact should be kept in tension with Kracht's result that not only are there undecidable displayable logics (which relevant logicians knew since Urquhart showed that **R** and **E** were undecidable) but that in fact it is undecidable in general whether a displayable logic is decidable (Kracht 1994). This is true; display logic is very expressive, and when you have the power to talk about arbitrary semigroups, you know you're looking for trouble when it comes to decidability. But this ought not be a case for mourning. Two facts should be kept in mind. Firstly, the world is a complex place, and any logical formalisms which do it justice in terms of expressive power are simply going to be undecidable. This is not a problem with the formalism, but a simple fact about the world. Display logic has just this expressive power. Secondly, just as there are decidable subsystems of classical first-order logic, so there are decidable displayable logics. And in fact, many of these are interesting systems, such as the relevant logics without contraction we have seen. A large class of displayable logics — those with flat structural rules — are decidable, and of independent interest.

8 Comparisons with Other Methods, and Future Work

The chief alternative to this presentation of the decidability of relevant logics has been the traditional Dunn-Minc systems, using the decidability proof of Giambone 1985, modified by Brady 1991 to deal with negation. Brady's innovation is important: he replaces formulae with *signed formulae* (instead of A you have TA or FA), thereby encoding enough symmetry between truth and falsity to model negation. However, it does not have the expressive power of display logic. For one thing, the system as it stands cannot be extended to model quantification. The problem is the distribution of the universal quantifier over disjunction. The display logic framework can easily prove this, since it in effect allows repetition in consequent position (by way of $*$, which is as you would expect of an intuitionistically underivable proposition). The proof is left as an exercise for the committed reader.

Display logic is a smoother framework than Brady's in a number other respects. Not only does our system give us a decidability result for **EW** (as our system uses a structural constant **1** to model t, so we can have a rule for CII without breaking the subformula property, and we can independently handle the complexity of rules involving **1**), but it more closely mirrors the frame semantics of our logics. We have exactly three structural operations — semicolon, comma and $*$ corresponding with the notions of R, $\leq$ and * on frames. This makes the framework a natural home for a proof theory of relevant logics.

The other standard way of proving decidability of logics with a worlds semantics is by filtration and the finite model property. This method works for **DW**, but the standard proofs fail for logics with axioms like B and B′. It is as yet unknown whether these logics have the finite model property. As a result only a proof theoretical argument, like ours, has so far been found for this range of logics. Clearly there is opportunity for more work to be done here, in reading proof search as a process of model generation. We must leave this for another day.

There is also scope for extending the results to rules which fail to be flat. Not every decidable logic has flat structural rules. In either of the original Belnap representation or the simpler Wansing representation of classical modal logics, a simple logic like **S4** has rules in which the modal structure increases from conclusion to premises. The logic is still decidable. In this case, some kind of control over the structure needs to be maintained, just as we have gained control over the purely extensional structure of the comma and the asterisk. This seems quite possible, after all, only a finite number of 'modal prefixes' differ in proof theoretical sense, so a normal form result should be rather easily obtained. Another result worth keeping in mind is the work done on decision procedures for combinators. It is well known that indentity in the free term algebra generated by application among the combinators B, C and K is decidable. This is the algebraic analogue of the flatness of the rules B, C and K. It is less known that some 'increasing' combinators, like L (given by the rule $\mathsf{L}ab = a(bb)$) are also decidable (Statman 1989, Sprenger and Wymann-Böni 1993). It remains to be seen how results like these can be applied to our setting.

Further work needs also to be done on correspondence between conditions on frames, axioms, and display rules. We have done the simple case for confusions (which as far as I can tell is new with this paper). Much more needs doing for general correspondence results. In the absence of boolean negation this is not easy, for we have a large degree of freedom in constructing points in the canonical model structure. You need only look at the current state of intuitionistic modal logic to see that general correspondence results are much harder to find in the absence of boolean negation.

We also ought consider logics *without* negation, or with a negation without all of the properties we have assumed here. For example, the logic **P** of Peirce monoids, studied in Restall 1994, cannot be conservatively extended with a contraposing negation of the kind we've discussed. A contraposing negation adds B′ to any logic with B. (That's an exercise. Derive B′ from B, given the display rules.) But some logics, like **P** brook no form of commutativity, even the restricted commutativity in B′. So, **P** resists the kind of treatment we have seen. However, not all is lost. We could treat intensional conjunction like extensional conjunction, and display it by simple residuation. In the case of **P**, we need two residuals, for left and right residuation. For the logic **P** itself, this is not particularly useful — Restall 1994 already gives us a cut-free Gentzen proof theory yielding decidability — the gain will be found in the study of supersystems of **P**, such as that given by the addition of boolean negation.[19] Then we have a system for which the extensional structure has the original Belnap display rules, and the intensional structure

[19]**P** is useful in theoretical linguistics. A string x has type $A \to B$ if whenever x has type A, the concatenation yx has type B, x has type $A \wedge B$ if and only if it has types A and B, and it has type $A \vee B$ if and only if it has either type A or type B. It seems sensible to consider the type $\neg A$, which a string has if and only if it doesn't have type A. That is boolean negation. The logic $\mathbf{P}^{\neg}$ is the resulting system. Not much is known about it.

has the negation-free residuation display rules. For a detailed treatment of this and other topics in the region the reader must wait for this paper's sequel.

Finally, there is work to be done in implementing the decision procedure. At first sight this seems tremendously complex. Keeping track of matching pairs and superfluous **1**s is a computationally expensive job. As is testing for reduction and semi-reduction. However, we need not implement the proof search in the way set out in the decidability proof. Rather, it seems more sane to work with reduced sequents througout, making the rules of the implemented system the 'reduced' versions of our original rules. This is analagous to the formulation of a consecution calculus in which antecedent and consequent are sets of formulae, resulting in contraction being implicit in the statements of the rules. Proof search need not be completely naïve, either. We may distinguish between invertible and non-invertible rules, applying the former before the latter, and making a sanity check before embarking on a non-invertible rule.[20] By utilising such simple insights as these, we should be able to implement a display logic theorem prover for substructural logics without too much pain. It could also serve as a proof assistant in cases when the logic fails to be decidable, such as **R**.

With these new vistas opened up before us, as a result of the groundwork of this paper, the time has come to finish the presentation of the results we have so far, and to explore the landscapes ahead.[21]

9 Appendix

Here are the promised proofs of the rest of the axioms and rules of **DW**.

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A} \qquad \dfrac{\dfrac{B \vdash B}{A, B \vdash B}}{A \wedge B \vdash B} \qquad \dfrac{A \vdash A}{A \vdash A \vee B} \qquad \dfrac{B \vdash B}{B \vdash A \vee B} \qquad \dfrac{\mathbf{I} \vdash A}{\perp \vdash A}$$

[20]Such as in Slaney 1994's `minlog`, a theorem prover for intuitionistic and minimal logic, which checks consecutions given by non-invertible rules for classical provability before attempting to prove them.

[21]Thanks to Rajeev Goré for many very fruitful conversations, and to Nuel Belnap, for the wonderful ideas of Display Logic. Thanks are also due to John Slaney, who helped me see the possibility of the decidability argument, Mike Dunn, whose work on gaggle theory has shaped my thinking in more ways than I can tell, and Steve Giambrone, whose ground-breaking decidability results amaze me more and more after working though all of the details. Finally, an audience at the ARP, especially Errol Martin, Bob Meyer, and John Slaney, helped me sort out the interpretation and presentation of these results.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}}{A, B \vdash (A \wedge B) \vee (A \wedge C)}}{B \vdash A, (A \wedge B) \vee (A \wedge C)} \quad \dfrac{\dfrac{\dfrac{A \vdash A \quad C \vdash C}{A, C \vdash A \wedge C}}{A, C \vdash (A \wedge B) \vee (A \wedge C)}}{C \vdash A, (A \wedge B) \vee (A \wedge C)}}{B \vee C \vdash A, (A \wedge B) \vee (A \wedge C)}}{A, B \vee C \vdash (A \wedge B) \vee (A \wedge C)}}{A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A \to B \vdash *A; B}}{A \to B, A \to C \vdash *A; B}}{A; (A \to B, A \to C) \vdash B} \quad \dfrac{\dfrac{\dfrac{A \vdash A \quad C \vdash C}{A \to C \vdash *A; C}}{A \to B, A \to C \vdash *A; C}}{A; (A \to B, A \to C) \vdash C}}{(A; (A \to B, A \to C)), (A; (A \to B, A \to C)) \vdash B \wedge C}}{A; (A \to B, A \to C) \vdash B \wedge C}}{A \to B, A \to C \vdash A \to B \wedge C}}{(A \to B) \wedge (A \to C) \vdash A \to B \wedge C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad C \vdash C}{A \to C \vdash *A; C}}{A \to C, B \to C \vdash *A; C}}{A \vdash *(A \to C, B \to C); C} \quad \dfrac{\dfrac{\dfrac{B \vdash B \quad C \vdash C}{B \to C \vdash *B; C}}{A \to C, B \to C \vdash *B; C}}{B \vdash *(A \to C, B \to C); C}}{A \vee B \vdash *(A \to C, B \to C); C}}{A \vee B; (A \to C, B \to C) \vdash C}}{\dfrac{A \to C, B \to C \vdash A \vee B \to C}{(A \to C) \wedge (B \to C) \vdash A \vee B \to C}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad \dfrac{\dfrac{B \vdash B}{*B \vdash *B}}{\sim B \vdash *B}}{A \to \sim B \vdash *A; *B}}{A \to \sim B \vdash *B; *A}}{B; A \to \sim B \vdash *A}}{B; A \to \sim B \vdash \sim A}}{A \to \sim B \vdash B \to \sim A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{*A \vdash *A}}{*A \vdash \sim A}}{* \sim A \vdash A}}{\sim \sim A \vdash A}$$

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