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FOUR-VALUED SEMANTICS FOR RELEVANT LOGICS (AND SOME OF THEIR RIVALS)

Abstract. This paper gives an outline of three different approaches to the four-valued semantics for relevant logics (and other non-classical logics in their vicinity). The first approach borrows from the 'Australian Plan' semantics, which uses a unary operator '$\star$' for the evaluation of negation. This approach can model anything that the two-valued account can, but at the cost of relying on insights from the Australian Plan. The second approach is natural, well motivated, independent of the Australian Plan, and it provides a semantics for the contraction-free relevant logic **C** (or **RW**). Unfortunately, its approach seems to model little else. The third approach seems to capture a wide range of formal systems, but at the time of writing, lacks a completeness proof.

1. DEFINITIONS

In their paper 'Simplified Semantics for Basic Relevant Logics' (hereafter 'SS1'), Priest and Sylvan gave a simplification of the usual ternary-relational frame semantics for relevant logics. The original semantics, due to Sylvan and Meyer, had to place many constraints on the ternary relation R, and it was quite unwieldy. In SS1, a simplification of the original semantics is defined, and the construction was used to model the logic $\mathbf{B}^+, \mathbf{BM}, \mathbf{BD}$ and $\mathbf{B}$, using two different approaches for negation — one using the dualising '$\star$' operator, and the other, using a four-valued evaluation. In a subsequent paper, 'Simplified Semantics for Relevant Logics (and some of their Rivals)' (hereafter 'SS2'), I extended the simplified semantics to deal with most of the well-known relevant propositional logics (and some others). The treatment of negation in SS2 was purely by way of the dualising '$\star$' operator, famous in the semantics for relevant logics. In this paper we consider the alternative approach to negation that involves a 4-valued evaluation.

While this paper is self contained with regard to the definitions and concepts involved, it sometimes defers to SS2 (which in turn, defers at times to SS1) for proofs of certain theorems. With that stated, it

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should be noted that it is quite possible to understand this paper independently of SS1 or SS2.

1.1. *The System* **BD**

The first set of results deal with **BD**, a weak relevant propositional logic. To establish our terms, **BD** is expressed in a language $\mathcal{L}$, which has the connectives $\wedge, \vee, \rightarrow$ and $\neg$, parentheses (and), and a stock of propositional variables $p, q, \ldots$. Formulae are defined recursively in the usual manner, and the standard scope conventions are in force; $\wedge$ and $\vee$ bind more strongly than $\rightarrow$. For example, $p \wedge q \rightarrow r$ is short for $(p \wedge q) \rightarrow r$. We will use $\alpha, \beta, \ldots$ to range over arbitrary formulae.

The system **BD** has the following axioms and rules:

A1. $\alpha \rightarrow \alpha$

A2. $\alpha \rightarrow \alpha \vee \beta, \ \beta \rightarrow \alpha \vee \beta,$

A3. $\alpha \wedge \beta \rightarrow \alpha, \ \alpha \wedge \beta \rightarrow \beta,$

A4. $\alpha \wedge (\beta \vee \gamma) \rightarrow (\alpha \wedge \beta) \vee \gamma,$

A5. $(\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta \wedge \gamma),$

A6. $(\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma),$

A7. $\neg(\alpha \vee \beta) \leftrightarrow \neg\alpha \wedge \neg\beta,$

A8. $\neg(\alpha \wedge \beta) \leftrightarrow \neg\alpha \vee \neg\beta,$

A9. $\alpha \leftrightarrow \neg\neg\alpha.$

If $(\alpha_1 \ldots \alpha_n)/\beta$ is a rule, its *disjunctive form* is the rule $(\gamma \vee \alpha_1 \ldots \gamma \vee \alpha_n)/(\gamma \vee \beta)$. The rules for $\mathbf{B}^+$ are the following, along with their dusjunctive forms:

R1. $\dfrac{\alpha, \alpha \rightarrow \beta}{\beta},$

R2. $\dfrac{\alpha, \beta}{\alpha \wedge \beta},$

R3. $\dfrac{\alpha \rightarrow \beta, \gamma \rightarrow \delta}{(\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \delta)}.$

It is to be noted that the simplified semantics given can only model *disjunctive systems*. That is, systems such that the disjunctive form of every truth preserving rule is truth preserving. Not every logic satisfies this criterion – a notable candidate is **E**, for the disjunctive form of its characteristic rule, from α to $(\alpha \to \beta) \to \beta$, fails to be truth preserving. The reason for this is that $\alpha \vee \neg\alpha$ is a theorem of **E**, but $\neg\alpha \vee ((\alpha \to \beta) \to \beta)$ is a non-theorem.

1.2. *Semantics for* **BD**

In SS1 a four-valued semantics was given for **BD**. Their semantics is a simplified version of the original ternary relational semantics for relevant logics. The important definitions concerning the structure are collected here.

An *interpretation* for the language is a 4-tuple $\langle g, W, R, I\rangle$, where W is a set of worlds, $g \in W$ is the base world, R is a ternary relation on W, and I assigns to each pair (w, p) of worlds and propositional parameters a truth value, $I(w,p) \subseteq \{0,1\}$. Truth values at worlds are then assigned to formulae inductively as follows:

- $1 \in I(w, \alpha \wedge \beta) \Leftrightarrow 1 \in I(w, \alpha)$ and $1 \in I(w, \beta)$,
- $0 \in I(w, \alpha \wedge \beta) \Leftrightarrow 0 \in I(w, \alpha)$ or $0 \in I(w, \beta)$,
- $1 \in I(w, \alpha \vee \beta) \Leftrightarrow 1 \in I(w, \alpha)$ or $1 \in I(w, \beta)$,
- $0 \in I(w, \alpha \vee \beta) \Leftrightarrow 0 \in I(w, \alpha)$ and $0 \in I(w, \beta)$,
- $1 \in I(g, \alpha \to \beta) \Leftrightarrow$ for all $x \in W(1 \in I(x, \alpha) \Rightarrow 1 \in I(x, \beta))$,

and for $x \neq g$,

- $1 \in I(x, \alpha \to \beta) \Leftrightarrow$ for all $y, z \in W(Rxyz \Rightarrow (1 \in I(y, \alpha) \Rightarrow 1 \in I(z, \beta)))$.

Note that the *falsity* of a conditional is arbitrary according to this definition of an evaluation. This is because **BD** is very weak with regards to negated conditionals. No negated conditionals are theorems, no conditionals of the form $\alpha \to \neg(\beta \to \gamma)$ are provable where α contains no negated conditionals, and so on. So **BD** tells us very little about the falsity of conditionals, and this is reflected in the semantics. The trouble that this causes for extending semantics to stronger systems will be made obvious later.

For all the oddity with conditionals, the extensional connectives and negation have the standard semantic clauses for a four-valued evaluation (each formula gets one of the four values are $\varnothing, \{0\}, \{1\}$ and $\{0,1\}$). This kind of evaluation for connectives dates back at least Dunn's work published in 1976 and Belnap's in 1977. A three valued version due to Priest emerged in 1979, and it has been rediscovered many times since. Because this way of evaluating formulae gained a lot of currency with relevant logicians in the United States, it is called the *American Plan* in contrast to the antipodean approach using the Routley–Meyer star, the *Australian Plan.*

Then semantic consequence is defined in terms of truth preservation at g, the base world. In other words,

$$\Theta \vDash \alpha \Leftrightarrow \text{ for all interpretations } \langle g, W, R, I\rangle (1 = I(g,\beta) \\ \text{for all } \beta \in \Theta \Rightarrow 1 = I(g,\alpha)).$$

The soundness and completeness result for **BD** can then be consisely stated as follows.

THEOREM 1. *If* $\Theta \cup \{\alpha\}$ *is a set of sentences, then*

$$\Theta \vdash \alpha \Leftrightarrow \Theta \vDash \alpha,$$

where $\vdash$ *is the provability relation of* **BD**.[1]

As in SS2 we will make a cosmetic alteration to the above definition of an interpretation. It is easily seen that the truth conditions for '$\rightarrow$' can be made univocal if we define R to satisfy $Rgxy \Leftrightarrow x = y$. From now, we will use the definition of an interpretation with an appropriately modified relation R.

It was shown in SS1 that this semantics is sound and complete with respect to **BD**. Some of the *positive* extensions of this semantics trivially extend to the four-valued interpretation — the proofs in SS2 translate easily into the four-valued context. To be precise, in the table below, the semantics with condition **D**n added is sound and complete for the logic $\mathbf{BD} + \mathbf{C}n$.

C1.	$\alpha \wedge (\alpha \to \beta) \to \beta$	**D1**.	$Raaa$
C2.	$(\alpha \to \beta) \wedge (\beta \to \gamma) \to (\alpha \to \gamma)$	**D2**.	$Rabc \Rightarrow R^2a(ab)c$
C3.	$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$	**D3**.	$R^2abcd \Rightarrow R^2b(ac)d$
C4.	$(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$	**D4**.	$R^2abcd \Rightarrow R^2a(bc)d$
C5.	$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$	**D5**.	$Rabc \Rightarrow R^2abbc$
C6.	$\alpha \to ((\alpha \to \beta) \to \beta)$	**D6**.	$Rabc \Rightarrow Rbac$
C7.	$(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$	**D7**.	$R^2abcd \Rightarrow R^2acbd$
C8.	$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$	**D8**.	$R^2abcd \Rightarrow R^3ac(bc)d$
C9.	$(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$	**D9**.	$R^2abcd \Rightarrow R^3bc(ac)d$
C10.	$\dfrac{\alpha}{(\alpha \to \beta) \to \beta}$, and its disjunctive forms	**D10**.	$Raga$

Where we have defined:

$$R^2abcd = (\exists x)(Rabx \wedge Rxcd),$$
$$R^2a(bc)d = (\exists x)(Rbcx \wedge Raxd),$$
$$R^3ab(cd)e = (\exists x)(R^2abxe \wedge Rcdx).$$

For the proof of this fact, we defer to SS2, where the proofs are given for the two-valued interpretation. In the four-valued context, the proofs are identical, except for the cosmetic alteration of '1 =' to '1 $\in$', and we will consider them done.

In SS2, an addition is made to the notion of an interpretation, to deal with more extensions to the basic logics. A *containment relation* on W was defined to be a relation satisfying the following conditions:

$$a \leqslant b \Rightarrow \begin{cases} (I(a,p) = 1 \Rightarrow I(b,p) = 1) & \text{for each propositional variable } p, \\ Rbcd \Rightarrow Racd & \text{if } a \neq g, \\ Rbcd \Rightarrow c \leqslant d & \text{if } a = g. \end{cases}$$

Then could be shown (by induction on the complexity of α) that it satisfied $a \leqslant b \Rightarrow (I(a,\alpha) = 1 \Rightarrow I(b,\alpha) = 1)$ for any formula α. This is the relevant property of a containment relation. To have an analogous relation in the four-valued context, we need both that $a \leqslant b \Rightarrow (1 \in I(a,\alpha) \Rightarrow 1 \in I(b,\alpha))$, and $a \leqslant b \Rightarrow (0 \in I(a,\alpha) \Rightarrow 0 \in I(b,\alpha))$, to ensure that the induction step for negation goes through. Unfortunately, as the falsity condition for a conditional is arbitrary, there is no way of ensuring that the latter condition is satisfied in the case where α is a conditional. So, the extensions that need a containment relation cannot be modelled in this manner. Extensions involving negation are likewise ruled out, for they explicitly use the dualising '$\star$' operator, which is unavailable in this context. So, we are left in a sorry state. There are two alternatives available. Firstly, we will unashamedly steal from the $\star$-semantics to give a four-valued interpretation for (almost) anything that has a $\star$-interpretation, and secondly, we will add negation conditions for implication, to give a smoother semantics for a number of systems. Before we can do that, however, we need to give an overview of the $\star$-semantics, and of the canonical model construction used to prove completeness results.

1.3. *The '$\star$' Semantics*

A $\star$-interpretation for our language is a 5-tuple $\langle g, W, R, I, \star\rangle$, where g, W and R are as before (so $Rgxy \Leftrightarrow x = y$), $\star$ is a function from W into itself. I has been altered to be a function from pairs of propositional parameters and worlds to the set $\{0, 1\}$, instead of to the *power set* of $\{0, 1\}$. Then I is extended inductively as follows:

- $I(w, \alpha \wedge \beta) = 1 \Leftrightarrow I(w, \alpha) = 1$ and $I(w, \beta) = 1$,
- $I(w, \alpha \vee \beta) = 1 \Leftrightarrow I(w, \alpha) = 1$ or $I(w, \beta) = 1$,
- $I(w, \neg\alpha) = 1 \Leftrightarrow I(w^\star, \alpha) = 0$,
- $I(x, \alpha \rightarrow \beta) = 1 \Leftrightarrow$ for all $y, z \in W$ $(Rxyz \Rightarrow (I(y, \alpha) = 1 \Rightarrow I(z, \beta) = 1))$.

These semantics are sound and complete with respect to the logic **BM**, which is **BD** + **R4** – **A9**, where we have

R4. $$\frac{\alpha \rightarrow \beta}{\neg\beta \rightarrow \neg\alpha}.$$

To model **B**, which is **BM** + **R4** and **BD** + **R4** we require that for all $w, w^{\star\star} = w$.

1.4. *Canonical Models*

Our canonical model structure will use certain kinds of theories (sets of formulae) as worlds, and we will define a relationship R between them. A sentence is true in a world just in case it is *in* the world and false in a world if its negation is in the world. The constructions and results we need are below:

- If Π is a set of $\mathcal{L}$-sentences, $\Pi_{\rightarrow}$ is the set of all members of Π of the form $\alpha \rightarrow \beta$.
- $\Sigma \vdash_{\Pi} \alpha \Leftrightarrow \Sigma \cup \Pi_{\rightarrow} \vdash \alpha$.
- Σ is a Π-*theory* if and only if $\alpha, \beta \in \Sigma \Rightarrow \alpha \wedge \beta \in \Sigma$, and, $\vdash_{\Pi} \alpha \rightarrow \beta \Rightarrow (\alpha \in \Sigma \Rightarrow \beta \in \Sigma)$.
- Σ is *prime* $\Leftrightarrow (\alpha \vee \beta \in \Sigma \Rightarrow \alpha \in \Sigma$ or $\beta \in \Sigma)$.
- If X is any set of Π-theories, the ternary relation R on X is defined thus:

$$R\Pi\Gamma\Delta \Leftrightarrow \Gamma = \Delta$$

$$R\Sigma\Gamma\Delta \Leftrightarrow (\gamma \rightarrow \delta \in \Sigma \Rightarrow (\gamma \in \Gamma \Rightarrow \delta \in \Delta)) \text{ if } \Sigma \neq \Pi.$$

- Σ is Π-*deductively closed* $\Leftrightarrow (\Sigma \vdash_{\Pi} \alpha \Rightarrow \alpha \in \Sigma)$.
- If Σ is a set of formulae, $\Sigma^{\star} = \{\alpha\colon \neg\alpha \notin \Sigma\}$.

In all of the above definitions, if Π is the empty set, the prefix 'Π-' is omitted; so a $\varnothing$-theory is simply a theory, and so on. The following results are proved in SS1:

LEMMA 2 (*Priming*).

- *If Σ is a prime Π-theory with $\gamma \rightarrow \delta \notin \Sigma$, then there are prime Π-theories, Γ and Δ such that $R\Sigma\Gamma\Delta$, $\gamma \in \Gamma$ and $\delta \notin \Delta$.*
- *If Σ, Γ and Δ are Π-theories, such that $R\Sigma\Gamma\Delta$ and Δ is prime, then there is a prime $\Gamma' \supseteq \Gamma$ where $R\Sigma\Gamma'\Delta$.*
- *If Σ, Γ and Δ are Π-theories, such that $R\Sigma\Gamma\Delta$ and Δ is prime, then there is a prime $\Sigma' \supseteq \Sigma$ where $R\Sigma'\Gamma\Delta$.*
- *If Σ, Γ and Δ are Π-theories, such that $R\Sigma\Gamma\Delta$ and $\delta \notin \Delta$, then there are prime Π-theories Γ' and Δ' such that $\Gamma \subseteq \Gamma'$, $\delta \notin \Delta'$, and $\Delta \subseteq \Delta'$.*

The completeness of the simplified semantics is then demonstrated in the following way. Given a set of formulae $\Theta \cup \{\alpha\}$, such that $\Theta \nvdash \alpha$, we construct an interpretation in which Θ holds at the base world, bur α doesn't. Firstly, note that there is a prime theory Π such that $\Pi \supseteq \Theta$, but $\alpha \notin \Pi$, by Lemma 2. The worlds of the interpretation are the Π-theories, except that Π and $\Pi^\star$ are duplicated, so Π' and $\Pi'^\star$ act like ordinary worlds (and not the base world), in that $R\Pi'\Gamma\Delta$ iff for each $\alpha \to \beta \in \Pi'$, if $\alpha \in \Gamma$, $\beta \in \Delta$. g is Π itself and R is as defined above. Then we determine I, by assigning $I(\Sigma, p) = 1 \Leftrightarrow p \in \Sigma$ for each propositional variable p and Π-theory Σ. It can then be proved that $I(\Sigma, \beta) = 1 \Leftrightarrow \beta \in \Sigma$ for each formula β, so we have that Θ holds at Π, the base world, and α does not. This construction is called the *almost canonical* interpretation, for it differs from the canonical interpretation given in SS1, by way of the duplication of the base world.

2. FOUR VALUES BEYOND **BD**

The matter of finding a four-valued semantics for **B** is not as simple as giving it a $\star$-semantics. We will show that such a semantics does exist, and demonstrate soundness and completeness. First, however, we need a preliminary result, and some terminology. A two-valued interpretation with a '$\star$' operator will henceforth be called a *$\star$-interpretation*. The first result establishes the connection between these $\star$-interpretations and four-valued interpretations.[2]

THEOREM 3. *Any $\star$-interpretation that models* **B** *generates a four-valued interpretation on the same set of worlds, with exactly the same truths in each world.*

Proof. Let the $\star$-interpretation be $\langle g, W, R, I_\star, \star\rangle$. We define a four-valued interpretation $\langle g, w, R, I_4\rangle$ by requiring

$$1 \in I_4(w, \alpha) \text{ if and only if } I_\star(w, \alpha) = 1,$$
$$0 \in I_4(w, \alpha) \text{ if and only if } I_\star(w^\star, \alpha) = 0$$

for each world w and sentence α, and $\star$ is the world 'inversion' map of the $\star$-system. In a picture, this is:

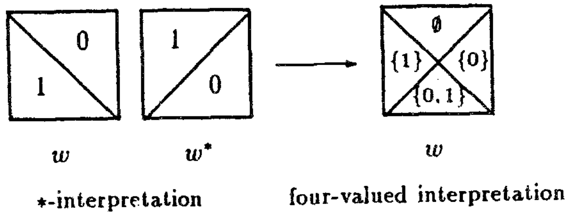

Fig. 1.

It is clear that the four-valued interpretation has exactly the same truths in each world as the $\star$-interpretation. All we need to do is to show that I_4 actually gives a four-valued interpretation. This is the case, as the inductive definitions of a four-valued interpretation are satisfied.

$$\begin{aligned}
1 \in I_4(w, \alpha \wedge \beta) &\Leftrightarrow 1 = I_\star(w, \alpha \wedge \beta) \\
&\Leftrightarrow 1 = I_\star(w, \alpha) \text{ and } 1 = I_\star(w, \beta) \\
&\Leftrightarrow 1 \in I_4(w, \alpha) \text{ and } 1 \in I_4(w, \beta) \\
0 \in I_4(w, \alpha \wedge \beta) &\Leftrightarrow 0 = I_\star(w^\star, \alpha \wedge \beta) \\
&\Leftrightarrow 0 = I_\star(w^\star, \alpha) \text{ or } 0 = I_\star(w^\star, \beta) \\
&\Leftrightarrow 0 \in I_4(w, \alpha) \text{ or } 0 \in I_4(w, \beta) \\
1 \in I_4(w, \neg\alpha) &\Leftrightarrow 1 = I_\star(w, \neg\alpha) \\
&\Leftrightarrow 0 = I_\star(w^\star, \alpha) \Leftrightarrow 0 \in I_4(w, \alpha) \\
0 \in I_4(w, \neg\alpha) &\Leftrightarrow 0 = I_\star(w^\star, \neg\alpha) \\
&\Leftrightarrow 1 = I_\star(w^{\star\star}, \alpha) \Leftrightarrow 1 = I_\star(w, \alpha) \\
&\Leftrightarrow 1 \in I_4(w, \alpha) \\
1 \in I_4(w, \alpha \to \beta) &\Leftrightarrow 1 = I_\star(w, \alpha \to \beta)
\end{aligned}$$

$$\Leftrightarrow \begin{cases} (\forall w')(I_\star(w', \alpha) = 1 \Rightarrow I_\star(w', \beta) = 1) & \\ \Leftrightarrow (\forall w')(1 \in I_4(w', \alpha) \Rightarrow 1 \in I_4(w', \beta)) & \text{if } w = g, \\ (\forall x, y) \text{ where } Rwxy \ (I_\star(x, \alpha) = 1 \Rightarrow I_\star(y, \beta) = 1) & \\ \Leftrightarrow (\forall x, y) \text{ where } Rwxy \ (1 \in I_4(x, \alpha) \Rightarrow 1 \in I_4(y, \beta)) & \text{otherwise.} \end{cases}$$

So, every $\star$-interpretation that models **B** gives a four-valued interpretation on the same set of worlds and with the same truths in the same worlds. □

We say that a four-valued interpretation with a corresponding $\star$-interpretation is *closed under duality*, because it is easy to show that a four-valued interpretation has a corresponding $\star$-interpretation if and only if for every world w, there is another world $w^\star$ such that

$$\begin{aligned} I(w,\alpha) = \{1\} &\Leftrightarrow I(w^\star,\alpha) = \{1\}, \\ I(w,\alpha) = \{0\} &\Leftrightarrow I(w^\star,\alpha) = \{0\}, \\ I(w,\alpha) = \varnothing &\Leftrightarrow I(w^\star,\alpha) = \{0,1\}, \\ I(w,\alpha) = \{0,1\} &\Leftrightarrow I(w^\star,\alpha) = \varnothing. \end{aligned}$$

for each sentence α. The world $w^\star$ is said to be the *dual* of the world w — so, a four-valued interpretation is closed under duality if and only if the dual of every world in the interpretation is also a world in the interpretation. This condition is too difficult to check when presented with a four-valued model. An equivalent condition that is simpler to check is provided in the following theorem:

THEOREM 4. *A four-valued interpretation $\langle g, W, R, I\rangle$ that models* **B** *is closed under duality if and only if there is an involution $\star$: $W \to W$ such that*

- $0 \in I(w,p)$ *if and only if* $1 \notin I(w^\star,p)$
- $0 \in I(w,\alpha \to \beta)$ *if and only if* $1 \notin I(w^\star,\alpha \to \beta)$

for each world w, propositional variable p, and formulae α and β.

Proof. The 'only if' part is immediate, as these conditons are a subset of the conditions we have closure under duality. The other part is a matter of proving that $0 \in I(w,\alpha)$ if and only if $1 \notin I(w^\star,\alpha)$ for each w and α. The other half of the conditions follow from the fact that $w^{\star\star} = w$.

To show that $0 \in I(w,\alpha)$ if and only if $1 \notin I(w^\star,\alpha)$, we use induction on the complexity of formulae. The base case, and the case for $\to$ are given. The case for conjunction is as follows — $0 \in I(w,\alpha \wedge \beta)$ if and only if $0 \in I(w,\alpha)$ or $0 \in I(w,\beta)$ if and only if $1 \notin I(w^\star,\alpha)$ or $1 \notin I(w^\star,\alpha)$ if and only if $1 \notin I(w^\star,\alpha \wedge \beta)$. The cases for disjunction and negation are similar, and are left as an exercise. □

This condition is simple enough to check at the level of propositional parameters, and the remaining portion of the condition deals with falsity conditions for entailment, which have to be explicity specified in a four-valued model, in any case. While this is saving the four-valued interpretation by an explicit use of '$\star$', which the four-valued interpretation is designed to avoid, there does not seem to be any way of avoiding it, if the truth conditions of entailments are to be kept as they are, as some kind of duality operator is the natural way to model rule-contraposition, which is the characteristic rule of **B**. In any case the construction we have just given is enough to prove the following theorem:

THEOREM 5. *The collection of four-valued interpretations closed under duality is sound and complete with respect to* **B**.

Proof. It is an immediate corollary of the fact that the $\star$-interpretations that satisfy $w^{\star\star} = w$ are sound and complete with respect to **B**, but an independent proof is possible. We need to show that the rule $(\alpha \to \beta)/(\neg\beta \to \neg\alpha)$ holds in all four-valued models closed under duality. To see that this is the case, assume that for all $w, 1 \in I(w, \alpha) \Rightarrow 1 \in I(w, \beta)$. So, if $1 \in I(w, \neg\beta)$, we must have $0 \in I(w, \beta)$, and so, $1 \notin I(w^\star, \beta)$. This means that $1 \notin I(w^\star, \alpha)$ by our assumption, and hence $0 \in I(w, \alpha)$. This results in $1 \in I(w, \neg\alpha)$, which is what we wanted. This gives us soundness.

Completeness can be obtained by the standard canonical model construction; it suffices to show that in the interpretation consisting of the prime Π-theories, the dual of every world is also a world. To see that this is the case, consider Σ, a prime Π-theory. Set $\Sigma^\star = \{\alpha\colon \neg\alpha \notin \Sigma\}$. $\Sigma^\star$ is a Π-theory, as if $\alpha, \beta \in \Sigma^\star$, then $\neg\alpha, \neg\beta \notin \Sigma$, so $\neg\alpha \vee \neg\beta \notin \Sigma$ as Σ is prime, and as $\vdash_\Pi \neg\alpha \vee \neg\beta \leftrightarrow \neg(\alpha \wedge \beta)$, we have $\neg(\alpha \wedge \beta) \notin \Sigma$, giving $\alpha \wedge \beta \in \Sigma^\star$. If $\alpha \in \Sigma^\star$ and $\vdash_\Pi \alpha \to \beta$, then if $\beta \notin \Sigma^\star$ we have $\neg\beta \in \Sigma$, and **R4** gives $\vdash_\Pi \neg\beta \to \neg\alpha$, so $\neg\alpha \in \Sigma$, contradicting $\alpha \in \Sigma^\star$. $\Sigma^\star$ is prime, as $\alpha \vee \beta \in \Sigma^\star$ gives $\neg(\alpha \vee \beta) \notin \Sigma$, and so $\neg\alpha \wedge \neg\beta \in \Sigma$. This gives $\neg\alpha \notin \Sigma$ or $\neg\beta \notin \Sigma$, that is, $\alpha \in \Sigma^\star$ or $\beta \in \Sigma^\star$.

$\Sigma^\star$ is a dual of Σ, as $1 \in I(\Sigma, \alpha) \Leftrightarrow \alpha \in \Sigma \Leftrightarrow \neg\neg\alpha \in \Sigma \Leftrightarrow \neg\alpha \notin \Sigma^\star \Leftrightarrow 1 \notin I(\Sigma^\star, \neg\alpha) \Leftrightarrow 0 \notin I(\Sigma^\star, \alpha)$. And $0 \in I(\Sigma, \alpha) \Leftrightarrow 1 \in I(\Sigma, \neg\alpha) \Leftrightarrow \neg\alpha \in \Sigma \Leftrightarrow \alpha \notin \Sigma^\star \Leftrightarrow 1 \notin I(\Sigma^\star, \alpha)$. □

In this way, any four-valued model of **B** can be converted to a two-valued model, and conversely. So the results for axioms and rules extending **B** also hold for the four-valued semantics, when an appropriate dualising operation $\star$ is made explicit. This means that we can use the definition below for the containment relation:

$$a \leqslant b \Rightarrow \begin{cases} (1 \in I(a,p) \Rightarrow 1 \in I(b,p)) & \text{for every propositional variable } p, \\ Rbcd \Rightarrow Racd & \text{if } a \neq g \\ Rbcd \Rightarrow c \leqslant d & \text{if } a = g \\ b^\star \leqslant a^\star & \end{cases}$$

and it will satisfy the condition that for all α, if $a \leqslant b$, then $1 \in I(a,\alpha) \Rightarrow 1 \in I(b,\alpha)$. So, the results of SS2 show us that we can extend **B** by adding any axiom from among the following, by using the appropriate rules:

C11. $\alpha \to (\beta \to \beta)$	**D11.** $Rabc \Rightarrow b \leqslant c$
C12. $\beta \to (\alpha \to \beta)$	**D12.** $Rabc \Rightarrow a \leqslant c$
C13. $\alpha \to (\beta \to (\gamma \to \alpha))$	**D13.** $R^2abcd \Rightarrow a \leqslant d$
C14. $\alpha \to (\beta \to \alpha \wedge \beta)$	**D14.** $Rabc \Rightarrow a \leqslant c$ and $b \leqslant c$
C15. $(\alpha \wedge \beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$	**D15.** $R^2abcd \Rightarrow$ for some x $b \leqslant x$, $c \leqslant x$ and $Raxd$
C16. $(\alpha \to \beta) \vee (\beta \to \alpha)$	**D16.** $a \leqslant b$ or $b \leqslant a$
C17. $\alpha \to (\alpha \to \alpha)$	**D17.** $Rabc \Rightarrow a \leqslant c$ or $b \leqslant c$
C18. $(\alpha \wedge \beta \to \gamma) \to ((\alpha \to \gamma) \vee (\beta \to \gamma))$	**D18.** $Rabc$ and $Rade \Rightarrow$ for some x $b \leqslant x, d \leqslant x$ and ($Raxc$ or $Raxe$)
C19. $(\alpha \to \neg\alpha) \to \neg\alpha$	**D19.** $Raa^\star a$ for $a \neq g$, and $g^\star \leqslant g$
C20. $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$	**D20.** $Rabc \Rightarrow Rac^\star b^\star$
C21. $\alpha \vee \neg\alpha$	**D21.** $g^\star \leqslant g$

The four-valued semantics given here is not particularly exciting — the reason for the four-valued semantics is to get *away* from dualising operators, and to give negation a more pleasing modelling. One way of doing this is to introduce *another* ternary relation S, to deal with false conditionals. This is the original American plan, and could be followed in the simplified case. However, there is a smoother possibility that uses neither $\star$ nor a ternary S, and which will capture the relevant logic **C** (but not much else, it seems).

3. NATURAL FOUR-VALUED SEMANTICS

It was noted at the end of SS1 that the thing that makes the four-valued semantics difficult is contraposition. One way to address this is to rewrite the truth conditions for the conditional as follows, 'wiring in' the validity of contraposition:

$$\begin{aligned}&1 \in I(w, \alpha \to \beta) \text{ if and only if for each } x, y \text{ where } Rwxy,\\ &1 \in I(x, \alpha) \Rightarrow 1 \in I(y, \beta) \text{ and } 0 \in I(x, \beta) \Rightarrow 0 \in I(y, \alpha).\end{aligned}$$

In this case, the rule form of contraposition,

$$\frac{\alpha \to \beta}{\neg\beta \to \neg\alpha}$$

is given, but some of the theorems and rules of $\mathbf{B}^+$ fail. What is needed is a policy for the falsity of conditionals. If this is left arbitrary, there is no guarantee that an axiom such as $(\alpha \to \beta) \wedge (\alpha \to \gamma) \to (\alpha \to \beta \wedge \gamma)$ will come out as true, for the falsity of the consequent will *not* deliver the falsity of the antecedent. One policy for the falsity of conditionals is motivated by the result from strong relevant logics, to the effect that where $\alpha \circ \beta$ (the 'fusion' of α and β) is defined as $\neg(\alpha \to \neg\beta)$, we have that:

$$\vdash (\alpha \circ \beta \to \gamma) \leftrightarrow (\alpha \to (\beta \to \gamma)).$$

This gives a connection between a negated conditional — $\alpha \circ \beta$ — and a purely positive formula. The corresponding condition on the conditional would then be:

$$\begin{aligned}&0 \in I(w, \alpha \to \beta) \text{ if and only if there are } x, y,\\ &\text{where } Rxyw,\ 1 \in I(x, \alpha) \text{ and } 0 \in I(y, \beta).\end{aligned}$$

Given this, the contraposition *axiom* becomes logically true, so the semantics is sound for **B** + **C20** (more commonly known as **DW**), but is not complete. If we add the condition **D6**, that $Rabc \Rightarrow Rbac$ for each a, b, c, we have completeness for **DW** + **C6**, which we will call **DWA** (for 'asserting **DW**', as the implication satisfies **C6**, which is commonly called 'assertion'). It is important that assertion hold, for it is only in its presence that the biconditional connecting the fusion and the nested implication. We call this kind of semantics a *natural four-valued semantics*.

THEOREM 6. *The natural four-valued semantics is sound with respect to* **DWA**.

Proof. We need to show that if $\langle g, W, R, I\rangle$ is an interpretation, then the axioms of **DWA** hold at g, and the rules are truth-preserving at g. It is not *entirely* trivial, so we will work the details for **A6**, **R1**, **R3** and assertion, and leave the rest for the reader.

For **A6**, suppose that $1 \in I(w, (\alpha \to \gamma) \wedge (\beta \to \gamma))$. We wish to show that $1 \in I(w, \alpha \vee \beta \to \gamma)$. To that end, assume that for some x, y where $Rwxy$, $1 \in I(x, \alpha \vee \beta)$. Without loss of generality, it is α that is true — that is, $1 \in I(x, \alpha)$. $1 \in I(w, \alpha \to \gamma)$ gives us $1 \in I(y, \gamma)$ as desired. Now suppose that for some x, y where $Rwxy$, $0 \in I(y, \gamma)$. In that case, $1 \in I(w, \alpha \to \gamma)$ gives $0 \in I(x, \alpha)$ and similarly, $1 \in I(w, \beta \to \gamma)$ gives $0 \in I(x, \beta)$. This gives $0 \in I(x, \alpha \vee \beta)$ as desired.

On the other hand, suppose that $0 \in I(w, \alpha \vee \beta \to \gamma)$. That means that for some x, y where $Rxyw$, $1 \in I(x, \alpha \vee \beta)$ and $0 \in I(y, \gamma)$. In this case, without loss of generality, $1 \in I(x, \beta)$. But then we have that $0 \in I(w, \beta \to \gamma)$ and hence $0 \in I(w, (\alpha \to \gamma) \wedge (\beta \to \gamma))$ as desired. This gives us **A6**.

For **R1**, suppose that $1 \in I(g, \gamma \vee \alpha)$ and $1 \in I(g, \gamma \vee (\alpha \to \beta))$. If $1 \in I(g, \gamma)$ then $1 \in I(g, \gamma \vee \beta)$. Otherwise, $1 \in I(g, \alpha)$, $1 \in I(g, \alpha \to \beta)$ and $Rggg$ give $1 \in I(g, \beta)$ and hence, $1 \in I(g, \gamma \vee \beta)$, as desired.

For **R3**, suppose that $1 \in I(g, \epsilon \vee (\alpha \to \beta))$ and $1 \in I(g, \epsilon \vee (\gamma \to \delta))$. We wish to show that $1 \in I(g, \epsilon \vee ((\beta \to \gamma) \to (\alpha \to \delta)))$. If $1 \in I(g, \epsilon)$, we have our result. Otherwise we have that $1 \in I(g, \alpha \to \beta)$ and $1 \in I(g, \gamma \to \delta)$.

Suppose that $1 \in I(w, \beta \to \gamma)$. Then if for some x, y where $Rwxy$, $1 \in I(x, \alpha)$ we have $1 \in I(x, \beta)$, and so $1 \in I(y, \gamma)$, giving $1 \in I(y, \delta)$, so $1 \in I(w, \alpha \to \beta)$. Conversely, if $0 \in I(y, \delta)$, we have $0 \in I(y, \gamma)$, and so $0 \in I(x, \beta)$, giving $0 \in I(x, \alpha)$, and so we have $1 \in I(w, \alpha \to \delta)$.

If $0 \in I(w, \alpha \to \delta)$, then there are x, y where $Rxyw$, $1 \in I(x, \alpha)$ and $0 \in I(y, \delta)$. Our assumptions give us $1 \in I(x, \beta)$ and $0 \in I(y, \gamma)$, and so $0 \in I(w, \beta \to \gamma)$. These all give us that $1 \in I(g, (\beta \to \gamma) \to (\alpha \to \delta))$ as desired.

For assertion, suppose that $1 \in I(w, \alpha)$. We wish to show that $1 \in I(w, (\alpha \to \beta) \to \beta)$. If $Rwxy$ and $1 \in I(x, \alpha \to \beta)$, $Rxwy$ gives us $1 \in I(w, \beta)$. If $Rwxy$ and $0 \in I(x, \beta)$, we have $0 \in I(y, \alpha \to \beta)$ as $1 \in I(w, \alpha)$. This gives us $1 \in I(w, (\alpha \to \beta) \to \beta)$.

If $0 \in I(w, (\alpha \to \beta) \to \beta)$, we have x, y where $Rxyw$, $1 \in I(x, \alpha \to \beta)$ and $0 \in I(y, \beta)$. But this gives us $0 \in I(w, \alpha)$ as desired. So, $1 \in I(g, \alpha \to ((\alpha \to \beta) \to \beta))$.

So soundness is proved. □

We need a small lemma for the proof of completeness.

LEMMA 7. *A logic extending* **DW** *contains assertion iff it contains the rule*

$$\alpha \to (\beta \to \gamma) \vdash \beta \to (\alpha \to \gamma)$$

Proof. To obtain assertion, apply the rule to $\vdash (\alpha \to \beta) \to (\alpha \to \beta)$. To obtain the rule, suppose that we have $\Theta \vdash \alpha \to (\beta \to \gamma)$, for some set of sentences Θ. Then, $\Theta \vdash ((\beta \to \gamma) \to \gamma) \to (\alpha \to \gamma)$, by **R3**, and as $\vdash \beta \to ((\beta \to \gamma) \to \gamma)$, we have $\Theta \vdash \beta \to (\alpha \to \gamma)$, as desired. □

THEOREM 8. *The natural four-valued semantics is complete with respect to* **DWA**.

Proof. We will use the almost canonical model, modified to satisfy permutation. In other words, we will show that the collection of all prime Π-theories (with Π duplicated in the usual fashion), with R' defined as

$$R'\Sigma\Gamma\Delta \text{ if and only if } \forall\alpha, \beta, \alpha \to \beta \in \Sigma \Rightarrow (\alpha \in \Gamma \Rightarrow \beta \in \Delta) \text{ for } \Sigma, \Gamma \neq \Pi.$$

(Note that if $R'\Sigma\Gamma\Delta$, and $\alpha \to \beta \in \Sigma$, then $\neg\beta \to \neg\alpha \in \Sigma$ by contraposition, and hence, if $\neg\beta \in \Gamma$, $\neg\alpha \in \Delta$, so our definition parallels the truth-conditions for conditionals, albeit, not explicitly.)

$$R'\Pi\Gamma\Delta \text{ if and only if } R'\Gamma\Pi\Delta \text{ if and only if } \Gamma = \Delta.$$

Define I by requiring that:

$$1 \in I(\Sigma, p) \text{ if and only if } p \in \Sigma,$$
$$\text{for } p \text{ a propositional parameter,}$$
$$0 \in I(\Sigma, p) \text{ if and only if } \neg p \in \Sigma,$$
$$\text{for } p \text{ a propositional parameter,}$$

and that it satisfy the usual inductive definitions of an interpretation. We just need to show that $1 \in I(\Sigma, \alpha)$ if and only if $\alpha \in \Sigma$, and $0 \in I(\Sigma, \alpha)$ if and only if $\neg\alpha \in \Sigma$ for *every* formula α. We do this by the usual induction on the complexity of the formulae.

- It works by stipulation on the base case.
- $1 \in I(\Sigma, \alpha \wedge \beta)$
  if and only if $1 \in I(\Sigma, \alpha)$ and $1 \in I(\Sigma, \beta)$ (by the inductive definition of I),
  if and only if $\alpha \in \Sigma$ and $\beta \in \Sigma$ (by the inductive hypothesis),
  if and only if $\alpha \wedge \beta \in \Sigma$ (as Σ is a Π-theory).
  $0 \in I(\Sigma, \alpha \wedge \beta)$
  if and only if $0 \in I(\Sigma, \alpha)$ or $0 \in I(\Sigma, \beta)$ (by the inductive definition of I),
  if and only if $\neg\alpha \in \Sigma$ or $\neg\beta \in \Sigma$ (by the inductive hypothesis),
  if and only if $\neg\alpha \vee \neg\beta \in \Sigma$ (as Σ is a *prime* Π-theory),
  if and only if $\neg(\alpha \wedge \beta) \in \Sigma$ (by **A8**),
- The case for disjunction is dual, and is left as an exercise.
- $1 \in I(\Sigma, \neg\alpha)$
  if and only if $\neg\alpha \in \Sigma$ (by hypothesis),
  if and only if $0 \in I(\Sigma, \alpha)$.
  The other case for negation is dual.
- $1 \in I(\Sigma, \alpha \to \beta)$
  if and only if for each Γ, Δ where $R'\Sigma\Gamma\Delta$ ($1 \in I(\Gamma, \alpha) \Rightarrow 1 \in I(\Delta, \beta)$).
  If $\alpha \to \beta \in \Sigma$, then we have for each Γ, Δ where $R'\Sigma\Gamma\Delta$ ($1 \in I(\Gamma, \alpha) \Rightarrow 1 \in I(\Delta, \beta)$) by the definition of R'.
  If $\alpha \to \beta \notin \Sigma$, then we have Γ, Δ where $R\Sigma\Gamma\Delta, \alpha \in \Gamma$ and $\beta \notin \Delta$, (even in the case $\Sigma = \Pi$), by a part of Lemma 2. To show that $R'\Sigma\Gamma\Delta$, note that if $\Gamma = \Pi$, we can take $\Gamma = \Pi'$.
- $0 \in I(\Sigma, \alpha \to \beta)$
  if and only if there are Γ and Δ where $R'\Gamma\Delta\Sigma$, $\alpha \in \Gamma$ and $\neg\beta \in \Delta$.
  Firstly, if $R'\Gamma\Delta\Sigma$, $\alpha \in \Gamma$ and $\neg\beta \in \Delta$, then we have that $(\alpha \to \beta) \to \beta \in \Gamma$ by assertion, which contraposed gives $\neg\beta \to \neg(\alpha \to \beta) \in \Gamma$, so $R'\Gamma\Delta\Sigma$ gives $\neg(\alpha \to \beta) \in \Sigma$.
  Conversely, if $\neg(\alpha \to \beta) \in \Sigma$, set $\Gamma = \{\gamma\colon \vdash_\Pi \alpha \to \gamma\}$ and $\Delta = \{\gamma\colon \vdash_\Pi \neg\beta \to \gamma\}$. Then $\alpha \in \Gamma$ and $\neg\beta \in \Delta$. To show

that $R\Gamma\Delta\Sigma$, assume that $\gamma \to \delta \in \Gamma$ and $\gamma \in \Delta$. Then $\vdash_\Pi \alpha \to (\gamma \to \delta)$ and $\vdash_\Pi \neg\beta \to \gamma$. Contraposing gives $\vdash_\Pi \alpha \to (\neg\delta \to \neg\gamma)$ and $\vdash_\Pi \neg\gamma \to \beta$, which together give $\vdash_\Pi \alpha \to (\neg\delta \to \beta)$. Assertion, with Lemma 7, yields $\vdash_\Pi \neg\delta \to (\alpha \to \beta)$, and contraposing finally gives $\vdash_\Pi \neg(\alpha \to \beta) \to \delta$, which assures us that $\delta \in \Sigma$, and hence that $R\Gamma\Delta\sigma$. Applying the last part of Lemma 2 gives prime theories $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$ where $R\Gamma'\Delta'\Sigma$. To ensure that $R'\Gamma'\Delta'\Sigma$, if either of Γ' or Δ' are equal to Π when considered as sets, identify them with Π'. This gives us the result, and completes the proof. □

Now **DWA** is not a logic of much interest. Some of its extensions are. *One* of the usual extensions still work with this semantics, and this will give us a natural semantics for **C** (called **RW**, or **R–W** by some) which is **DWA**+**C3**

THEOREM 9. *The condition* **D3** *is sound and complete with respect to the axiom* **C3**, *in the natural four-valued semantics.*

Proof. Completeness is easy — it involves showing that R' in the almost canonical structure satisfies the condition **D3** under the assumption that **C3**. The proof in SS2 can be used for this, and the reader is referred there.

Soundness is an order of magnitude more difficult, and so we will work the details.

We have that $R^2abcd \Rightarrow R^2a(ab)c$. Assume that $1 \in I(w, \alpha \to \beta)$, we wish to show that $1 \in I(w, (\beta \to \gamma) \to (\alpha \to \gamma))$. To do this, assume that $Rwxy$, and that $1 \in I(x, \beta \to \gamma)$. We wish to show that $1 \in I(y, \alpha \to \gamma)$. Assume that $Ryzt$ and that $1 \in I(z, \alpha)$. $Rwxy$ and $Ryzt$ gives a y' where $Rxy't$ and $Rwzy'$. It follows that $1 \in I(y', \beta)$, and hence $1 \in I(t, \gamma)$. Further, if $Ryzt$ and $0 \in I(z, \gamma)$, we still have a y' where $Rxy't$ and $Rwzy'$. So, $0 \in I(y', \beta)$ and hence $0 \in I(t, \alpha)$ as desired. Thus, $1 \in I(y, \alpha \to \gamma)$.

Now assume that $Rwxy$, $0 \in I(x, \alpha \to \gamma)$, and we wish to show that $0 \in I(y, \beta \to \gamma)$. We have $Rztx$ where $1 \in I(z, \alpha)$ and $0 \in I(t, \gamma)$. This gives $Rztx$ and $Rxwy$, and hence there is an x' where $Rtx'y$ and $Rzwx'$, and hence $Rx'ty$ and $Rwzx'$. $Rwzx'$ and $1 \in I(z, \alpha)$ gives

$1 \in I(x', \beta)$, and $0 \in I(t, \gamma)$ gives $0 \in I(y, \beta \to \gamma)$, as desired. This gives us that $1 \in I(w, (\beta \to \gamma) \to (\alpha \to \gamma))$.

Now assume $0 \in I(w, (\beta \to \gamma) \to (\alpha \to \gamma))$, and we wish to show that $0 \in I(w, \alpha \to \beta)$. So we have x, y where $Rxyw$ and $1 \in I(x, \beta \to \gamma)$ and $0 \in I(y, \alpha \to \gamma)$. This in turn ensures that there are z, r where $Rzty$, $1 \in I(z, \alpha)$ and $0 \in I(t, \gamma)$. $Rxyw$ and $Rxty$ give $Rtzy$ and $Ryxw$, so we have a y' where $Rzy'w, Rtxy'$, which in turn gives $Rxty'$. So, $0 \in I(y', \beta)$ and hence $Rzy'w$ gives $0 \in I(w, \alpha \to \beta)$, as desired.

This shows that $1 \in I(g, (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)))$, as we wished. □

CK and **R** are interestingly dual systems (see John Slaney's 'Finite Models for Non-Classical Logics' to see *some* examples of the duality between them). The natural semantics gives another setting in which their duality can be exposed — for the expected way of restricting R to model these systems both fail, for 'dual' reasons.

To perform the extension to **CK**, we would need to expand the definition of a containment relation:

$$a \leqslant b \Rightarrow \begin{cases} (1 \in I(a,p) \Rightarrow 1 \in I(b,p)) & \text{for each propositional variable } p, \\ (0 \in I(a,p) \Rightarrow 0 \in I(b,p)) & \text{for each propositional variable } p, \\ Rbcd \Rightarrow Racd & \text{if } a \neq g, \\ Rbcd \Rightarrow c \leqslant d & \text{if } a = g. \\ Rcda \Rightarrow Rcdb & \end{cases}$$

and then show that it did what we wished of it. An easy induction on the length of formulae shows that if $\leqslant$ is a containment relation on $\langle g, W, R, I \rangle$ then $a \leqslant b$, then $a \leqslant b \Rightarrow (1 \in I(a, \alpha) \Rightarrow 1 \in I(b, \alpha))$ and $(0 \in I(a, \alpha) \Rightarrow 0 \in I(b, \alpha))$ for every formula α.

Then, to extend using **C12**, note what happens when you attempt to show soundness using the standard condition **D12**. Assume that the condition $Rabc \Rightarrow a \leqslant c$ holds in an interpretation. We wish to show that $1 \in I(g, \beta \to (\alpha \to \beta))$. To do that, we need to show that (among other things) if $1 \in I(w, \beta)$, then $1 \in I(w, \alpha \to \beta)$, and to do that, we need to show that if $Rwxy$ and $0 \in I(x, \beta)$, we have $0 \in I(y, \alpha)$. This does not seem to be ensured. We have that $Rxwy$ by the condition for assertion, and so, $x \leqslant y$ gives $0 \in I(y, \beta)$. But $Rwxy$ gives $w \leqslant y$ and so $1 \in I(y, \beta)$. So, if we are sure that y is *consistent* (that is, for no β is $I(y, \beta) = \{0, 1\}$), we can ensure that our condition holds (if vacuously). What we would need to do is show that in the presence of **C12**, a consistency assumption on worlds could be made. However, once this is done, more than **CK** is captured. So it seems that **CK** escapes this modelling.

Dual problems beset the introduction of anything that will take **C** up to its contraction-added (and more famous) cousin **R**. The obvious candidate to add to **C** to get **R** is **C5** (which is the inference of contraction, in axiom form), but any of **C1, 2, 8, 9** or **19** would do as well. (The same problems beset all of them, because in the context of the conditions on R that we have, each condition can be transformed into the others.) In the context of **DWA, C5** is equivalent to $(\alpha \circ \alpha \to \beta) \to (\alpha \to \beta)$ which in turn is equivalent to $\alpha \to \alpha \circ \alpha$. Now the truth and falsity conditions for fusion can be deduced from those for implication. We wish to show that $\alpha \to \alpha \circ \alpha$ hold at g, under the condition that $Rabc \Rightarrow Rabbc$. (Which gives, when $a = g$, that $Rbbb$.)

To get the conditional to hold, we need that for all w, if $0 \in I(w, \alpha \circ \alpha)$, then $0 \in I(w, \alpha)$. Now $0 \in I(w, \alpha \circ \alpha)$ iff $1 \in I(w, \alpha \to \neg\alpha)$, which is simply that for all x, y where $Rwxy$, if $1 \in I(x, \alpha)$, $0 \in I(y, \alpha)$. In the context of **R5**, all we have to go on is that $Rwww$, so we have that if $1 \in I(w, \alpha)$, $0 \in I(w, \alpha)$. This is *not* enough to show that $0 \in I(w, \alpha)$, for $I(w, \alpha)$ might simply be empty. The soundness proof grinds to a halt at this point. Of course, if the *completeness* of each world is ensured, the soundness proof goes through, but again, more than **R** is captured if this line is followed.

So, the natural four-valued semantics is incredibly discerning — it will only brook a modification to give **C**, and none of the other standard deviant systems (like **R**) can be modelled in this way. Clearly, other possibilities ought to be examined.

4. ANOTHER POSSIBILITY?

Another possibility for the negation condition for a conditional comes from considering the conditions for the *truth* of a conditional. If for $\alpha \to \beta$ to be true at w we need for each x, y where $Rwxy$, if α is true in x then β is true in y, and if β is false in x then α is false in y, then for $\alpha \to \beta$ to be false at w we should have a counterexample to this. Provided that we construe a counterexample as a situation in which the antecedent is true and the consequent false. A natural way of formulating this is as follows.

$$\begin{aligned}&0 \in I(w, \alpha \to \beta) \text{ iff for some } x, y \text{ where}\\ &Rwxy, \text{ either } 1 \in I(x, \alpha) \text{ and } 0 \in I(y, \beta) \text{ or } 1 \in I(y, \alpha)\\ &\text{and } 0 \in I(x, \beta)\end{aligned}$$

The semantics with this condition gives more than **DW**. One thing we get is the rule

$$\frac{\alpha \wedge \neg\beta}{\neg(\alpha \to \beta)}.$$

Another extra is the axiom

$$\neg(\alpha \to \gamma) \wedge (\alpha \to \neg\beta) \to \neg(\alpha \to \beta)$$

which holds in all of the model structures. This is a theorem of classical logic, but it is not a theorem of **R** or of **CK**,[3] and so, it is not a theorem of any systems weaker than these. It follows that again, systems like **R** and **CK** cannot be modelled along these lines — furthermore, nothing weaker than **R** or **CK** can be modelled with this semantics. Whatever they capture, the systems will be a new family, outside the standard relevant systems.

If we take $\perp$ to be the propositional constant that satisfies $\vdash \perp \to \alpha$ for each α ($\perp$ is false only in each world) and if we use '$-$' to designate a strong negation defined by $-\alpha =_{\text{df}} \alpha \to \perp$, our axiom can be recast as

$$\neg - \alpha \wedge (\alpha \to \neg\beta) \to \neg(\alpha \to \beta)$$

or dually, as

$$(\alpha \to \beta) \to -\alpha \vee \neg(\alpha \to \neg\beta).$$

This has a certain plausibility about it. If you read '$-\alpha$' as "α is absurd", then it is read as "If α is not absurd, and if α then $\neg\beta$, then it is not the case that if α then β" or dually, "If (if α then β), then either α is absurd, or it is not the case that if α then $\neg\beta$."

The new axiom and rule *seem* to characterise the new conditions in the semantics, but completeness has not been proved. We will sketch the difficulty in proving it, so others can have a crack at it. Taking the almost canonical model structure as before, we must ensure that if $R\Sigma\Gamma\Delta$, then $\gamma \in \Gamma$ and $\neg\delta \in \Delta$ ensures that $\neg(\gamma \to \delta) \in \Sigma$. This does not seem to follow from our original definition of R, so we must add it as another restriction on R. This is no problem in itself. The difficulty arises with the proofs of the priming lemmas. We must have that if $\alpha \to \beta \notin \Sigma$, then there are Δ and Γ where $\alpha \in \Delta$ and $\beta \notin \Delta$ and $R\Sigma\Gamma\Delta$. Proving that the modified R relation holds between the theories constructed with the old method seems impossible. The usual completeness proof does not work, and as none have yet been found, completeness for the new semantics must be left as an open problem.

NOTES

[1] It should be noted that the probability relation '$\vdash$' used here is distinct from the '$\vdash$' that appears in other sections of the relevant logic literature. In our case, $\Theta \vdash \alpha$ iff there is a proof of α that uses premises from among the elements of Θ. In 'A General Logic' by Slaney, for example, $\Theta \vdash \alpha$ iff there is a proof of $\beta_1 \wedge \ldots \wedge \beta_n \to \alpha$ for some $\beta_i \in \Theta$. These notions are distinct. In the notion Slaney uses, it turns out that $\Theta \vdash \alpha$ iff or every theory in which the elements of Θ are true, so is α. In our notion, the theories in question are restricted to those that are *regular* (or *detatched* — meaning that if $\alpha \to \beta$ and α are in the theory, so is β) and *normal* (containing all the theorems).

[2] The connection between the two- and four-valued systems has a long history, as can be seen in RLR section 3.2. I make no claim to originality in this section, other than noting that the connection between the two kinds of system still holds in the simplified case.

[3] It is simple to check its failure in the matrices **RM3** and **Ł3**.

[4] Thanks to Graham Priest for helpful discussion on many of the issues covered in this paper.

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