ŁUKASIEWICZ, SUPERVALUATIONS, AND THE FUTURE

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ABSTRACT: In this paper I consider an interpretation of future contingents which motivates a unification of a Łukasiewicz-style logic with the more classical supervaluational semantics. This in turn motivates a new non-classical logic modelling what is "made true by history up until now." I give a simple Hilbert-style proof theory, and a soundness and completeness argument for the proof theory with respect to the intended models.

KEYWORDS: non-determinism, open future, Łukasiewicz's three-valued logic, supervaluations.

Will there be a sea battle tomorrow? If we take indeterminism seriously, we might agree that there is—as yet—no fact of the matter concerning a sea battle tomorrow. It is neither settled now that there will be a battle tomorrow, nor settled now that there won't be a battle tomorrow.

Once we agree on this basic point, there are at least two major ways to develop this idea in a formal system. The first is due to Łukasiewicz. He thought that future contingents motivated a three-valued logic [3]. Statements which are true now are evaluated as T, those which are false now are evaluated as F, and those which are now neither true nor false are evaluated as N. Truth values of truth-functionally compound statements are evaluated using the truth tables of Łukasiewicz's three-valued logic.

Another approach to future contingents uses van Fraassen's technique of *supervaluations* [6, 7]. A history (a completed series of moments) decides absolutely every statement one way or another, and a statement is evaluated as T at a moment when it is true at all histories passing through that moment, it is evaluated as F at a moment when it is false at all histories passing through that moment, and otherwise it is indeterminate.

These two approaches differ in their evaluation of truth-functionally compound statements. For Łukasiewicz's approach, if A is neither true now nor false now, so is $A \wedge {\sim}A$. Evaluating $A \wedge {\sim}A$ with supervaluations, however, you get a different result. Since $A \wedge {\sim}A$ is false in every history (they are complete and consistent two-valued evaluations) it is also false at every moment—even if that moment has not decided between A and ${\sim}A$. Furthermore, for a supervaluationist truth-at-a-moment is *persistent*. An earlier moment is included in *more* histories than a later moment, so anything true at the later earlier moment is true at the later moment. On Łukasiewicz's approach, persistence fails. If two statements receive the value N at a moment, then a conditional with one as the antecedent and the other as the consequent is T. If these two statements are then resolved, one as T and the other as F, then a conditional with the true antecedent and false consequent becomes *false* after earlier being true, and persistence fails.

In the rest of this paper I will develop both Łukasiewicz's approach and the supervaluational approach a little more, and show that contrary to these appearances, they need not be seen as rivals. With some small modifications to the way that we understand the guiding intuitions behind Łukasiewicz's three-valued logic, we may see that supervaluations and a properly three-valued system may live together quite happily.

1 A ŁUKASIEWICZ-STYLE APPROACH

To facilitate comparison with the supervaluational approach with branching time, we will consider a slight revision of Łukasiewicz's logic. First, we will have a *frame* $\mathcal{F}$ of *moments*. That is, we have a collection F of moments, ordered by a reflexive, transitive and antisymmetric relation $\leqslant$, of 'earlier than or equal to'. Statements in our language are tenseless (instead of "there will be a sea battle tomorrow" they are of the kind "there is a sea battle on October 21, 2005"), and consist of a class of atomic statements, then closed under the connectives $\wedge$, $\sim$ and $\rightarrow$ in the usual fashion.

A *model* consists of a frame and a pair of relations $\vDash^+$ and $\vDash^-$ between moments and statements satisfying a number of conditions. Firstly, atomic statements are *persistent*. That is, if $m \leqslant n$ and $m \vDash^+ p$ (the world up to m makes p true) then $n \vDash^+ p$ too, and similarly, if $m \vDash^- p$ (the world up to m makes p false) then $n \vDash^- p$ too.

Secondly, conjunction and negation interact with $\vDash^+$ and $\vDash^-$ as follows:

- $m \vDash^+ A \wedge B$ iff $m \vDash^+ A$ and $m \vDash^+ B$;
- $m \vDash^- A \wedge B$ iff $m \vDash^- A$ or $m \vDash^- B$.
- $m \vDash^+ {\sim}A$ iff $m \vDash^- A$;
- $m \vDash^- {\sim}A$ iff $m \vDash^+ A$.

The interesting clause we need is that for implication. Łukasiewicz would evaluate implication as follows:

- $m \vDash^+ A \to B$ iff if $m \vDash^+ A$ then $m \vDash^+ B$ and if $m \vDash^- B$ then $m \vDash^- A$;
- $m \vDash^- A \to B$ iff $m \vDash^+ A$ and $m \vDash^- B$.

But this would contradict persistence, as we have seen: suppose A and B are both neither true nor false at m. Then by this condition, $A \to B$ is true at m. But if A becomes true at a later n, and B becomes false at that n, then $A \to B$ becomes false at that n, contradicting the condition that if something is true it remains true.

Slaney, Surendonk and Girle noticed this [5], and argued for the slight revision of Łukasiewicz's logic by evaluating $A \to B$ as follows:

- $m \vDash^+ A \to B$ iff for every $n \geqslant m$ if $n \vDash^+ A$ then $n \vDash^+ B$ and if $n \vDash^- B$ then $n \vDash^- A$;
- $m \vDash^- A \to B$ iff $m \vDash^+ A$ and $m \vDash^- B$.

Once we adopt this condition (and so, modify Łukasiewicz's logic just enough to satisfy persistence for all formulas) we will find the accommodation with a supervaluational account within our grasp. The resulting system is a well-behaved logic, a little weaker than Łukasiewicz's three-valued logic. A Hilbert style axiomatisation is simple.

$$
\begin{gathered}
A \to (B \to A \wedge B) \\
(A \to B) \to \big((C \to A) \to (C \to B)\big) \\
A \to \big((A \to B) \to B\big) \\
A \circ B \to (A \circ (B \circ B)) \vee ((A \circ A) \circ B) \\
A \wedge B \to A \quad A \wedge B \to B \\
A \wedge (B \vee C) \to (A \wedge B) \vee C \\
{\sim}{\sim}A \to A \\
(A \to {\sim}B) \to (B \to {\sim}A) \\
A, A \to B \vdash B
\end{gathered}
$$

where $A \circ B$ is ${\sim}(A \to {\sim}B)$, and $A \vee B$ is ${\sim}({\sim}A \wedge {\sim}B)$.[1] We can also introduce $\top$ as $p \to p$ for some arbitrarily chosen p, and $\bot$ as ${\sim}\top$, noting that these satisfy $m \vDash^+ \top$ and $m \nvDash^- \top$ always, and $m \nvDash^+ \bot$ and $m \vDash^- \bot$ in every model respectively, so it does not matter which p we chose to begin with. Similarly, we can set $A \supset B$ to be $A \to (A \to B)$, and it follows that this satisfies the intuitionistic evaluation condition: $m \vDash^+ A \supset B$ iff for every $n \geqslant m$, if $n \vDash^+ A$ then $n \vDash^+ B$ (and it also satisfies $m \vDash^- A \supset B$ iff $m \vDash^+ A$ and $m \vDash^- B$.) It follows that the $\wedge, {\sim}, \supset$ fragment of our logic is exactly that of Nelson's constructive negation [4, 8]. And conversely, we can define

[1]The axiom $A \circ B \to (A \circ (B \circ B)) \vee ((A \circ A) \circ B)$ ensures that this logic is not a sub-logic of Łukasiewicz's infinitely valued logic (or even the four-valued logic). It is false in these logics. Assign A and B the values $\frac{2}{3}$. Then $A \circ B$ has the value $\frac{1}{3}$, while $A \circ (B \circ B)$ and $(A \circ A) \circ B$ both have value 0.

$A \to B$ in Slaney, Surendonk and Girle's system F** [5] by setting $A \to B$ to be $(A \supset B) \wedge ({\sim}B \supset {\sim}A)$.

In this system, we can define entailment between a set of statements Σ and another statement A as follows. $\Sigma \vdash A$ holds whenever for any model and for every moment m in that model, if $m \vDash^{+} B$ for each $B \in \Sigma$, then $m \vDash^{+} A$ as well. It is not difficult to show (and Slaney, Surendonk and Girle show this) that $\Sigma \vdash A$ holds iff there is a proof of A from Σ.

In this formalisation, $\vDash^{+}$ and $\vDash^{-}$ encode the notions of 'makes true' and 'makes false'. We'll leave discussion of how well they do this for the third section. Now we will sketch the supervaluational approach.

2 SUPERVALUATIONS

For the supervaluational approach we still deal with frames of moments ordered in a tree [1, 2, 6]. However, we have only one relation $\vDash$ between moments and statements, and this relation is parasitic on another relation between *histories* and statements. A *history* is a maximal set of totally ordered moments in a frame. A *supervaluational model* on a frame $\mathcal{F}$ is a relation $\vDash$ between the histories in that frame and statements, satisfying the usual boolean constraints.

- $h \vDash A \wedge B$ iff $h \vDash A$ and $h \vDash B$.
- $h \vDash {\sim}A$ iff $h \nvDash A$.
- $h \vDash A \to B$ iff $h \nvDash A$ or $h \vDash B$.

These conditions enconde the constraint that histories decide every statement one way or the other. They are consistent and complete. Then we can have a derived notion of a statement being true at a moment by setting

- $m \vDash_s^{+} A$ iff for every history h where $m \in h$, $h \vDash A$.

and we may equally well define the notion of a statement being false at a moment by setting

- $m \vDash_s^{-} A$ iff for every history h where $m \in h$, $h \nvDash A$.

and the classical nature of truth-at-a-history shows us that $m \vDash^{-} A$ if and only if $m \vDash_s^{+} \neg A$.

Using truth-at-a-moment we have another notion of entailment $\Sigma \vdash' A$ iff for every model, and every moment m in that model, if $m \vDash_s^{+} B$ for each $B \in \Sigma$, then $m \vDash_s^{+} A$ too. Or using the definition of $\vDash_s^{+}$ in terms of truth-in-a-history, we see that $\Sigma \vdash' A$ if and only if for every history, if $h \vDash B$ for each $B \in \Sigma$, then $h \vDash A$ too.

3 A SYNTHESIS

The two approaches we have seen differ in their evaluation of formulae. We may have $m \not\models^+ A \vee {\sim}A$, while we must always have $m \models^+_s A \vee {\sim}A$. People usually conclude from this that the two approaches are invariably opposed to one another. You must either evaluate formulae with respect to one scheme or another. Either the law of the excluded middle fails (and we use a Łukasiewicz style evaluation of formulae) or it doesn't (and we use a supervaluational approach). But this is to ignore the possibility that the two evaluations of statements are complementary. It is this which drives the *synthesis* of the two approaches, which I will examine in the rest of the paper.

The guiding idea of the synthesis is that the two formalisations are giving an account of different things. Firstly, an F** evaluation gives an account of *what is made true/false by history up until some moment*. That is, if $m \models^+ A$, then there is something in history up until m in virtue of which A is true. (And correspondingly, if $m \models^- A$, then there is something in history up until m in virtue of which A is false). The supervaluational approach models *what is be true, given that the history of the world passes through this moment*. Both are important notions, but they are distinct. If there is a sea battle tomorrow or not, then this is true in virtue of something which happens *tomorrow*, not some part of history up until now. So, the two notions disagree on the evaluation of that statement.

How can a proper synthesis of the two approaches work? We might think that we could reason as follows. A history h makes A true iff $m \models^+ A$ for some $m \in h$. But this would be wrong. Consider an infinite history of coin tosses. It's reasonable to assume that the history as a whole would make "there are either an infinite number of heads or an infinite number of tails tossed" without that being true in virtue of any moment in that history. So a history might make something true without any particular moment making that true. (Of course, if a moment makes A true, then so will any history of which that moment is a part. But the converse need not hold, in general.)

The constraint we need however, is that each history in a model can be *consistently completed*. That is, given a history h in some model, the set of formulae $H_h = \{A : m \models^+ A \text{ for some } m \in h\}$ can be extended to a consistent, complete F** theory. After all, the things made true by the moments of a history are all true (in that history), so they ought to be consistently part of a world — and a world, we assume, decides all statements as true or false.[2]

This (somewhat surprisingly) cuts down on the number of F** models we can use. We can find a model in which $(A \supset \bot) \wedge ({\sim}A \supset \bot) \supset \bot$ is invalid. This is a model in which at some point m, at no future point does A get either

[2] This does not count against the 'temporal' motivations of Łukasiewicz's logic. For we can agree that failures of $A \vee {\sim}A$ are possible in the sense of the world *not yet* deciding between A and ${\sim}A$. However, to say that an entire *history* doesn't decide between A and ${\sim}A$ is to have some other motivation for the failure of excluded middle.

affirmed or denied. As a result, $A \supset \bot$ and $\sim A \supset \bot$ are both true at m, but $\bot$ fails at m, so our formula is false at m. This means that any history h passing through m cannot be consistently completed — since $A \supset \bot$ and $\sim A \supset \bot$ are in H_h, we cannot have either A or $\sim A$ in a consistent extension of H_h.

So, for histories to be consistently completed, we need to ensure that our models validate $(A \supset \bot) \wedge (\sim A \supset \bot) \supset \bot$. It turns out that this is all we need to ensure, as our next two theorems show.

Theorem 1 *Each of the following classes of models validate exactly the same formulae.*

1. *Models which validate* $(A \supset \bot) \wedge (\sim A \supset \bot) \supset \bot$.
2. *Models in which for every history* h*, the set of formulas* H_h *has a complete, consistent extension.*
3. *Models in which every history has an endpoint which is complete. (That is, every history* h *has a last moment* m_h*, and for every atom* p, $m_h \vDash^+ p$ *or* $m_h \vDash^- p$.)
4. *Finite models in which the endpoint of every history is complete.*

Proof We need only show that any formula invalidated in a model validating $(A \supset \bot) \wedge (\sim A \supset \bot) \supset \bot$ is also invalidated in a finite model in which each endpoint is complete. (For this model is also one in which every H_h has a complete consistent extension.) This is achieved by a simple filtration argument. Given a formula B, consider a frame with an evaluation which invalidates B while validating every instance of $(A \supset \bot) \wedge (\sim A \supset \bot) \supset \bot$. We will perform a filtration by identifying points in the model which agree on every subformula of B, and each $(p \supset \bot) \wedge (\sim p \supset \bot) \supset \bot$ where p is an atom in B (call this set of formulae $\mathcal{B}$, for convenience). Let this equivalence relation be denoted by '$\sim$', and let $[m]$ be the equivalence class of m under $\sim$. Then we let $[m] \leqslant [n]$ iff $m' \leqslant n'$ for some $m' \sim m$ and $n' \sim n$. This is clearly a partial order on the class of equivalence classes. There are a finite number of equivalence classes, so this is a finite frame.

We set $[m] \vDash^+ p$ (for p an atom in B) iff $m \vDash^+ p$, and $[m] \vDash^- p$ iff $m \vDash^- p$. (The evaluation of atoms not in B does not matter, for now.) It is a simple induction on the complexity of formulae to show that these points agree with the original model on formulae in $\mathcal{B}$. The only interesting induction step is the $\supset$ one.

Let's show that $[m] \vDash^+ C_1 \supset C_2$ iff $m' \vDash^+ C_1 \supset C_2$ for any $m' \sim m$, assuming the equivalence for C_1 and C_2. Firtstly, if $[m] \vDash^+ C_1 \supset C_2$ iff for every $[n] \geqslant [m]$, if $[n] \vDash^+ C_1$ then $[n] \vDash^+ C_2$. Now, if $n \geqslant m$, then $[n] \geqslant [m]$, and if $n \vDash^+ C_1$, then $[n] \vDash^+ C_1$ (by induction hypothesis) so $[n] \vDash^+ C_2$, and hence $n \vDash^+ C_2$ (by induction hypothesis), giving $m \vDash^+ C_1 \supset C_2$ as desired.

Now assuming that $m \vDash^+ C_1 \supset C_2$, we wish to show that $[m] \vDash^+ C_1 \supset C_2$. Here, take $[n] \geqslant [m]$, where $[n] \vDash^+ C_1$. We want $[n] \vDash^+ C_2$. Well, as

$[n] \geqslant [m]$, there's some $n' \sim n$ and $m' \sim m$ where $n' \geqslant m'$. By the filtration construction, m' and m agree on all formulae in $\mathcal{B}$, so since $m \vDash^+ C_1 \supset C_2$, we have $m' \vDash^+ C_1 \supset C_2$ too. And since $[n] \vDash^+ C_1$ we have $n' \vDash^+ C_1$ by hypothesis, and so $n' \vDash^+ C_2$, giving the result we wished.

We also wish to show that $m \vDash^- C_1 \supset C_2$ iff $[m] \vDash^- C_1 \supset C_2$, but this is simple. We reason as follows: $m \vDash^- C_1 \supset C_2$ iff $m \vDash^+ C_1$ and $m \vDash^- C_2$ iff $[m] \vDash^+ C_1$ and $[m] \vDash^- C_2$ iff $[m] \vDash^- C_1 \supset C_2$.

Now, the filtered frame is finite, it invalidates B somewhere, and every point validates $(p \supset \bot) \wedge ({\sim}p \supset \bot) \supset \bot$ for atoms p in B.

One final wrinkle involves ensuring the antisymmetry of $\leqslant$. If $[m] \leqslant [n]$ and $[n] \leqslant [m]$, then $[m]$ and $[n]$ must agree on all formuale in $\mathcal{B}$, so they must be the same equivalence class under $\sim$. So we have antisymmetry.

So, this means that all histories have endpoints (by antisymmetry, and the finitude of the frame), and since these endpoints validate $(p \supset \bot) \wedge ({\sim}p \supset \bot) \supset \bot$ for atoms p in B, they must be complete with respect to these atoms. (And we can make the other atoms true everywhere, and false nowhere, for simplicity). This is enough to construct the desired model, and so prove the theorem. □

This theorem is quite strong. We can strengthen it to full completeness of deducibility with respect to finite frames with complete endpoints (and hence for frames with complete endpoints) by noting that consequence is compact, by our axiomatisation. This gives us our next theorem.

Theorem 2 *The logic given by adding* $(A \supset \bot) \wedge ({\sim}A \supset \bot) \supset \bot$ *to* F** *is sound and complete with respect to the following classes of models:*

1. *Models in which for every history* h*, the set of formulas* H_h *has a complete, consistent extension.*

2. *Models in which every history has a complete endpoint.*

3. *Finite models in which the endpoint of every history is complete.*

Furthermore, if I have a model in which for every history h, the set of formulas H_h has a consistent and complete extension, then I can perform some *surgery* on the frame by adding a complete endpoint m_h to each history h (if the history doesn't already have one), without disturbing the evaluation of formulae on the original moments. So, without any loss of generality, we can restrict our attention to models with complete endpoints. These represent 'history as a whole'. Once we have them, we can define the supervaluational evaluation of formulae at moments $m \vDash^+_s A$ by setting it to be equivalent to $(\forall m \leqslant h)(h \vDash^+ A)$. It's simple to then show that if $m \vDash^+ A$ then $m \vDash^+_s A$. The converse doesn't hold, of course.

So, we have a synthesis of the supervaluational and Łukasiewicz-style of evaluations of statements in a temporal structure. This has motivated a small modification of F** to ensure that histories can be consistently completed.

4 MORALS OF THE STORY

There are a number of things we can learn from this story.

First, that syntheses of non-classical with classical insights are possible, and that this can refine both our classical and non-classical stories. One example is the conclusion that F** is incomplete as it stands.

Second, with different notions of truth at a point in a model come different consequence relations. On a model we can say that A is a classical consequence of Σ if for every moment m where $m \models^+_s B$ for each $B \in \Sigma$, then $m \models^+_s A$ too. (Or equivalently, for every history h where $h \models B$ for each $B \in \Sigma$, then $h \models A$ too). This is a classical notion of consequence.[3] A more finely grained notion of consequence can be defined in terms of $\models^+$. On this notion, A is a consequence of Σ just when for every moment m where $m \models^+ B$ for each $B \in \Sigma$, $m \models^+ A$ too. This is a more discerning notion of consequence — the notion encoded by F^{**} together with $(A \supset \bot) \wedge ({\sim}A \supset \bot) \supset \bot$.

On our synthesis, two classical tautologies, like $A \vee {\sim}A$ and $B \vee {\sim}B$ are completely indistinguishable as far as $\models$ and our first consequence relation goes, as classical tautologies are true in all histories. However, $\models^+$ and $\models^-$ can distinguish classical tautologies. Even though they are both true in all histories, the *part* of a history which makes them true can differ.[4]

References

[1] Nuel Belnap and Mitchell Green. Indeterminism and the thin red line. *Philosophical Perspectives (Logic and Language)*, 8:365–388, 1994.

[2] John Burgess. The unreal future. *Theoria*, 44:157–174, 1978.

[3] Jan Łukasiewicz. On determinism. In L. Borkowski, editor, *Selected Works*. North Holland, Amsterdam, 1970.

[4] D. Nelson. Constructible falsity. *Journal of Symbolic Logic*, 14:16–26, 1949.

[5] John K. Slaney, Timothy Surendonk, and Roderic Girle. Time, truth and logic. Technical Report TR–ARP–11–89, Automated Reasoning Project, Australian National University, 1994.

[6] Richmond Thomason. Indeterminist time and truth-value gaps. *Theoria*, 3:264–281, 1970.

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[3] It is *a* classical notion of consequence, but as an anonymous referee reminded me, the classicality of this relation is restricted to single-conclusion consequence relations. It seems natural to define a multiple-conclusion consequence relation using supervaluations such that $A \vee B$ does not entail the *set* A, B, since we may have supervaluations that render the disjunction true without making either disjunct true.

[4] Thanks to an audience at the *Łukasiewicz in Dublin conference* for their helpful comments on a much earlier version of this paper. Thanks, too, to two anonymous referees for comments that helped improve the paper.

[7] Bas van Fraassen. Singular terms, truth-value gaps, and free logic. *Journal of Philosophy*, 63:481–495, 1966.

[8] Heinrich Wansing. *The Logic of Information Structures*. Number 681 in Lecture Notes in Artificial Intelligence. Springer-Verlag, 1993.