Multiple Conclusions

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Abstract: I argue for the following four theses. (1) Denial is not to be analysed as the assertion of a negation. (2) Given the concepts of assertion and denial, we have the resources to analyse logical consequence as relating arguments with *multiple* premises and *multiple* conclusions. Gentzen's multiple conclusion calculus can be understood in a straightforward, motivated, non-question-begging way. (3) If a broadly anti-realist or inferentialist justification of a logical system works, it works just as well for *classical* logic as it does for *intuitionistic* logic. The special case for an anti-realist justification of intuitionistic logic over and above a justification of classical logic relies on an unjustified assumption about the shape of proofs. Finally, (4) this picture of logical consequence provides a relatively neutral shared vocabulary which can help us understand and adjudicate debates between proponents of classical and non-classical logics.

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Our topic is the notion of logical consequence: the link between premises and conclusions, the glue that holds together deductively valid argument. How can we understand this relation between premises and conclusions? It seems that any account begs questions. Painting with very broad brushtrokes, we can sketch the landscape of disagreement like this: "Realists" prefer an analysis of logical consequence in terms of the preservation of *truth* [29]. "Anti-realists" take this to be unhelpful and offer alternative analyses. Some, like Dummett, look to preservation of *warrant to assert* [9, 36]. Others, like Brandom [5], take inference as primitive, and analyse other notions in terms of it. There is plenty of disagreement on the "realist" side of the fence too. It is one thing to argue that logical consequence involves preservation of truth. It is another to explain how far truth must be preserved. Is the preservation essentially *modal* (in all circumstances [25]) or *analytic* (vouchsafed by

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the meanings of the terms involved) or *formal* (guaranteed by the logical structure of the premises and conclusions [28, 29]), or do we need a combination of these factors [12]? If there is to be some kind of privileged logical vocabulary, what is the principle of demarcation for that vocabulary [32]?

Even then, if we manage to find agreement on the significance and ground of logical consequence and the scope of logical vocabulary, there is scope for further disagreement. There are different accounts of the valid arguments, even those couched in the simple propositional connectives of conjunction, disjunction, the conditional and negation. Do we admit the law of the excluded middle as a truth of logic, or not [8, 10, 15, 37]? Is it legitimate to infer anything you like from a contradiction, or is this argument form invalid [1, 2, 27, 37]? Can one distribute conjunctions over disjunctions, or do quantum-mechanical experiments provide a counterexample to this inference [4, 13]? In the midst of all of this disagreement, is there *any* hope for finding a shared vocabulary between parties of these disagreements? In this paper I attempt to unearth some common ground where many have thought there is none, and to show how that from this common ground we can clarify some of these debates.

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First, I will argue that the speech-act of *denial* is best not analysed in terms of *assertion* and *negation* but rather, that denial is, in some sense, prior to negation. I will provide three different arguments for this position. The first involves the case of an agent with a limited logical vocabulary. The second argument, closely related to the first, involves the case of the proponent of a non-classical logic. The third will rely on general principles about the way logical consequence rationally constrains assertion and denial.

ARGUMENT ONE: Parents of small children are aware that the ability to *refuse*, *deny* and *reject* arrives very early in life. Considering whether or not something is the case – whether to accept that something is the case or to reject it – at least *appears* to be an ability children acquire quite readily. At face value, it seems that the ability to assert and to deny, to say *yes* or *no* to simple questions, arrives earlier than any ability the child has to form sentences featuring negation as an operator. It is one thing to consider whether or not A is the case, and it is another to take the *negation* $\sim A$ as a further item for consideration and reflection, to be combined with others, or to be supposed, questioned, addressed or refuted in its own right. The case of early development lends credence to the claim that the ability to deny can occur prior to the ability to form negations. If this is the case, the denial of A, in the mouth of a child, is perhaps best not analysed as the assertion of $\sim A$.

So, we might say that denial may be *acquisitionally prior* to negation. One can acquire the ability to deny before the ability to form negations.

ARGUMENT TWO: Consider a related case. Sometimes we are confronted with theories which propose non-standard accounts of negation, and sometimes we are confronted with people who endorse such theories. These will give us cases of people who appear to reject A without accepting $\sim A$, or who appear to accept $\sim A$ without rejecting A. If things are as they appear in these cases, then we have further reason to reject the analysis of rejection as the acceptance of a negation. I will consider just two cases.

Supervaluationism: The supervaluationist [10, 17, 37] account of truth-value gaps enjoins us to allow for claims which are not determinately true, and not determinately false. These claims are those which are true on some valuations and false on others. In the case of the supervaluational account of *vagueness*, borderline cases of vague terms are a good example. If Fred is a borderline case of baldness, then on some valuations "Fred is bald" is true, and on others, "Fred is bald" is false. So, "Fred is bald" is not true under the *super*valuation, and it is to be rejected. However, "Fred is not bald" is similarly true on some valuations and false on others. So, "Fred is not bald" is not true under the supervaluation, and it, too, is to be rejected. Truth value gaps provide examples where denial and the assertion of a negation come apart. The supervaluationist rejects A without accepting $\sim A$. When questioned, she will deny A, and she will *also* deny $\sim A$. She will not accept $\sim A$. The supervaluationist seems to be a counterexample to the analysis of denial as the assertion of a negation.

Dialetheism: The dialetheist provides is the dual case [19, 20, 22, 23, 27]. A dialetheist allows for truth-value *gluts* instead of truth-value *gaps*. Dialetheists, on occasion, take it to be appropriate to assert both A and $\sim A$. A popular example is provided by the semantic paradoxes. Graham Priest's analysis of the liar paradox, for example, enjoins us to accept both the liar sentence and its negation, and to reject neither. In this case, it seems, the dialetheist accepts a negation $\sim A$ *without* rejecting A, the proposition negated. When questioned, he will assert A, and he will *also* assert $\sim A$. He will not reject $\sim A$. The dialetheist, too, seems to be a counterexample to the analysis of denial as the assertion of a negation.

In each case, we seem to have reason to take denial to be something other than the assertion of a negation, at least in the mouths of the supervaluationist and the dialetheist. This argument is not conclusive: the proponent of the analysis may well say that the supervaluationist and the dialetheist are confused about negation, and that their denials really *do* have the content of a negation, despite their protestations to the contrary. Although this is a possible response, there is no doubt that it does violence to the positions of both the supervaluationist and the dialetheist. We would do better to see if there is an understanding of the connections between assertion, denial, acceptance, rejection and negation which allows us to take these positions at something approaching face value. This example shows that denial may be *conceptually separated* from the assertion of a negation.

ARGUMENT THREE The third argument is more extensive than the other two. We will consider the relationship between logical consequence and assertion and denial. It is common ground that logical consequence, whatever it amounts to, has some kind of grip on assertion and denial, acceptance and rejection. It makes sense for us to analyse and to criticise or laud our own beliefs, or the beliefs of others, using canons of deductive consequence. But not only our beliefs fall under logic's gaze. So also our hypotheses, suppositions, stories and flights of fancy may also be evaluated using logical norms. We measure all such things for coherence or consistency. We look for consequences, for what leads *on* from what we have considered, and we look for premises, for what might lead *to* what we consider now. Logical notions are nothing if they have no applicability to regulate the cognitive states of agents like us, and the content of such states.

Consider, then, how logic might apply to the case of a cognitive agent, and consider the case of a simple deductively valid argument, with one premise A and one conclusion B. (We represent the validity thus: '$A \vdash B$.') What *grip* could this inference have on an agent? When could an agent fall *foul* of this inference, and when could an agent *comply* with it?

If an agent accepts A, then it is tempting to say that the agent also *ought* accept B, because B follows from A. But this is too strong a requirement to take seriously. Let's consider why not.

(1) The requirement as I have naïvely expressed it is ludicrous if read as it stands. Consider the circumstance in which an agent might accept A for no good reason. But the argument from A to A is valid, and the mere fact that the agent *happens* to accept A gives the agent no *reason* to accept A. So, the requirement that you ought to accept the consequences of your beliefs is altogether too strong as it stands, as we shall see.

This error in the requirement is corrected with a straightforward scope distinction. Instead of saying that if A entails B and if you accept A then you ought to accept B, we should perhaps say that if A entails B then it ought to be the case that if you accept A you accept B. But this, too, is altogether too strong, as the following considerations show.

(2) There are consequences of which we are unaware. As a result, logical consequence on its own provides no obligation to believe. Here is an example: I accept all of the axioms of Peano arithmetic (PA). I do not believe all of the consequences of those axioms. Goldbach's conjecture (GC) could well be a consequence of those axioms, but I am not aware of this if it is the case, and I do not accept GC. If GC is a consequence of PA, then there is a sense in which I have not lived up to some kind of standard if I fail to accept it. My beliefs are not as comprehensive as they could be. If I believed GC, then in some important sense I would not make any more mistakes than I have already made, because GC is a consequence of my prior beliefs. However, it is by no means clear that comprehensiveness of this kind is desirable.

(3) In fact, comprehensiveness is *undesirable* for limited agents like us. If the inference from A to $A \vee B$ is valid, and if our beliefs are always to be closed under logical consequence, then for any belief we must have infinitely many more. But consider a very long disjunction, in which *one* of the disjuncts we already accept.

In what sense is it desirable that we accept this? The belief may be too complex to even *consider*, let alone, to believe or accept or assert.

Notice that it is not a sufficient repair to demand that we merely accept the *immediate* logical consequences of our beliefs. It may well be true that logical consequence in general may be analysed in terms of chains of immediate inferences we all accept when they are presented to us. The problems we have seen hold for immediate consequence. The inference from the axioms of PA to Goldbach's conjecture might be decomposable into steps of immediate inferences. This would not make Goldbach's conjecture any more rationally obligatory, if we are unaware of that proof. If the inference from A to $A \vee B$ is an immediate inference, then logical closure licenses an infinite collection of (irrelevant) beliefs.[1]

(4) Furthermore, logical consequence is sometimes impossible to check. If I must accept the consequences of my beliefs, then I must accept all tautologies. If logical consequence is as complex as consequence in classical first-order logic, then the demand for closure under logical consequence can easily be *uncomputable*. For very many sets of statements, there is no algorithm to determine whether or not a given statement is a logical consequence of that set. Closure under logical consequence cannot be underwritten by algorithm, so demanding it goes beyond what we could rightly expect for an agent whose capacities are computationally bounded.

So, these arguments show that logical *closure* is too strict a standard to demand, and failure to live up to it is no failure at all. Logical consequence must have some other grip on agents like us. But what could this grip be? Consider again the case of the valid argument from A to B, and suppose, as we did before, that an agent accepts A. What can we say about the agent's attitude to B? The one thing we can say about the agent's current attitude is that if she *rejects* B, she has made a mistake.

If an agent's cognitive state, in part, is measured in terms of those things she accepts and those she rejects, then valid arguments constrain those combinations of acceptance and rejection. As we have seen, a one-premise, one-conclusion argument from A to B constrains acceptance/rejection by ruling out accepting A and rejecting B. This explanation of the grip of valid argument has the advantage of symmetry. A valid argument from A to B does not, except by force of habit, have to be read as *establishing* the conclusion. If the conclusion is unbelievable, then it could just as well be read as *undermining* the premise. Reading the argument as constraining a pattern of acceptance and rejection gives this symmetry its rightful place.

It follows from this reflection that if there are reasoning and representing agents who do not have the concept of negation, and if it is still appropriate for us to analyse their reasoning using a notion of logical consequence, then we ought to take those agents as possessing the ability to *deny* without having the ability to *negate*. This seems plausible. As an agent accepts and rejects, it is filtering out

[1] This point is not new. Gilbert Harman,Harman, Gilbert for example, argues for it in *Change in View* [14].

information and ruling out possibilities. If the agent accepts A and B and also *rejects* the conjunction $A \wedge B$, then it has made a mistake, and this mistake can be explained without resorting to taking the agent to having a competence with manipulating *negations* as well as *conjunctions*.

What more can I say about the relationship between accepting and rejecting and the cognate speech-acts of assertion and denial? I leave some of the details to the next section, but here is some of what this picture involves. To accept A is to (in part) close off the possibility of rejecting A. To accept A and then to go on to *reject* A will result in a *revision* of your commitments, and not a mere *addition* to them. Similarly, to reject A is to (in part) close off the possibility of accepting A. To reject A and then to go on to *accept* A will result in a revision of your commitments, and not a mere addition to them.

I will close this section responding to the Fregean argument against the position I have just taken. Frege took it that denial is best analysed as the assertion of a negation because it seems that rejecting this analysis results in unnecessary proliferation of rules of inference.[2] I will use Dummett's example from his discussion of Frege's point [7, pp. 316–317]. Consider the argument from the premises 'If he is not a philosopher, he won't understand the question' and 'He is not a philosopher' to the conclusion 'He won't understand the question.' In this argument, the instance of 'he is not a philosopher' in the antecedent of the *conditional* premise is clearly a *negation*. A denial does not embed inside conditionals in this manner. However, it seems that the other premise, and the conclusion, may be treated as *denials* and not assertions of negations. If this is the case, then we must explain the connection between these denials and the *negation* found in the conditional premise. It seems better, and simpler, to treat the premises and the conclusion as assertions, for then the argument has the form of *modus ponens*, as it manifestly appears to be. Does this not pose a problem for any view which takes denial to be prior to negation?

There are a number of responses to this problem already available in the literature. Price's "Why 'Not'?" [21] proposes two-factor analysis of negation which allows an utterance of "he is not a philosopher" to be *both* an assertion of a negation and a denial. This would certainly dull the objection but it would not entirely defeat the nagging worry that any analysis of negation which utilises denial is committed to there being rather more arguments presented in Dummett's example than the simple *modus ponens* which appears on the surface.

Instead of a two-factor response, I propose an alternative picture of the situation. Arguments and argument forms do not, at the first instance, connect assertions or denials. Argument forms connect the content of these assertions and denials: propositions. The argument form of *modus ponens* connects two propositions as premises (A and $A \supset B$) and one conclusion (B). Those contents may be accepted or rejected (and asserted or denied), or we agnostic (or silent) about them. As we have seen, the validity of *modus ponens* tells us that the assertion of the premises A and $A \supset B$ together with the denial of the conclusion B is, in

[2] Allen Hazen informs me that Meinong's *assumptions* play the same role as Frege's *contents* [18].

some sense to be explained, a bad thing. But one can utilise the argument of *modus ponens* without asserting the premises, or while denying the conclusion.

What of Dummett's argument? It is an instance of *modus ponens*, pure and simple. The argument involves premises and a conclusion, and these include negations. Frege's point is a sharp one when wielded against the view that takes all outermost sentential negations to express denials (as in the view Frege targeted, of the orthodox Aristotelian logic of his day), but it has no effect on views which agree with his reading of the structure of the argument. On this view, there is but one argument there, but nonetheless, the argument in and of itself does not tell us whether to assert the premises (and thereby to rule out rejecting the conclusion) or to deny the conclusion (and thereby rule out accepting both premises).

In taking this view of the structure of arguments, it should be clear that I also distance myself from the superficially similar approach of Smiley's "Rejection" [35]. Smiley proposes an account of logical consequence where the unit of argument is not the proposition but the *judgement*: a proposition signed with a marker for acceptance or rejection. While there is a formal correspondence between this account of proof and the picture I prefer, Smiley's system seems to fall foul of the considerations entertained earlier in this section. Take an argument from a premise I accept to an impossibly complex conclusion which is a consequence of this argument. The system as it stands commends that if I accept the premise I ought to accept the conclusion. We have already seen that requiring this is altogether too strong. This is another reason to take arguments as connecting contents and not their assertions or denials.

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In this section I will explain how this perspective on agents motivates the structural rules of the classical multiple premise, multiple conclusion sequent calculus of Gentzen. But before we get to the formal details of how one might understand the particular logical connectives, we need to spend a little more time considering the behaviour of assertion and denial, and the corresponding states of acceptance and rejection.

In what follows, we will use the notion of a STATE. Given a particular language – which may be rich, containing many different notions, including logical constants, but which may also be completely devoid of any logical constants at all – a STATE expressed in that language is a pair of sets of statements expressed in that language. We will use the notation '$[X : Y]$' to represent states, where X and Y are sets of statements. A state might be used to represent the *outlook* of an agent which we take to *accept* each statement in X and *reject* each statement in Y. We might also use a state to represent the *context* in some dialogue or discourse at which each statement X is *asserted* and each statement in Y is *denied*.

We will avail ourselves of the usual notational shorthand of proof theory, by taking $[A : B]$ to be the state consisting of the singleton set $\{A\}$ accepted and the singleton set $\{B\}$ rejected. Similarly, if $[X : Y]$ is some state, we will take

$[X, A : B, Y]$ to be the state which adds the statement A to the left set X and adds the statement B to the right set Y. Furthermore, we will simply use *nothing* to denote the empty set of statements, so $[X : \ ]$ is a state in which nothing is denied (or rejected) and $[\ : Y]$ is a state in which nothing is asserted (or accepted). It follows that $[\ : \ ]$ is the minimal state which accepts nothing and rejects nothing.

With the notion of a state at hand we may begin to consider how we might *evaluate* states. Even with this thin notion of state as the focus of our discussion, we can lay down some criteria for evaluating states. Not all states are on a par, for some states are self-defeating. In particular, if a state contains a statement in both the left set and the right set, then this state undermines itself. If the state represents the cognitive architecture of an agent, then this agent both *accepts* and *rejects* some statement. If the state represents the state of play in some dialogue or discourse, then some statement has both been *asserted* and *denied*. The state is undermined.

We must take care in expressing this feature of states, if we are to keep the discussion relatively neutral. This requirement is not the same as the requirement of *consistency* or *non-contradiction* rejected by the dialetheist. The dialetheist recommends that we accept both a statement A and its negation $\sim A$, not that we simultaneously accept and reject A. Nothing in this requirement need be seen as inimical to the friend of contradictions. Priest's own account of the relationship between assertion and denial indicates that a denial expresses a refusal to accept, not the acceptance of a negation [24, 26].

Similarly, nothing in this requirement need be seen as inimical to the anti-realist, or to the quasi-realist who might prefer that we explain our primitive notions without appealing to a prior notion of *truth*. We do not explain the consistency requirement in terms of the impossibility of A being both *true* and *false* at the same time. While we might wish to explain the coherence or incoherence of a state in terms of truth, this is by no means required at this early stage of the discussion.

The fact that a state where the left- and right-sets overlap is self-defeating is the first of a number of observations about how states can undercut themselves. Instead of continuing to call these states self-defeating, we will call them *incoherent* because we will also talk about states which are not self-defeating, and seems more pleasing to call these states *coherent* than to call them *non-self-defeating*. We will also use a suggestive notation for calling states incoherent. If $[X : Y]$ is incoherent, we will write '$X \vdash Y$.'[3]

None of this discussion should suggest that given a particular language there is only one notion of coherence or one notion of logical consequence. There may be different criteria for measuring the coherence of combinations of assertions and denials [3]. In the considerations that follow, we are examining the features of *any* notion of coherence. Here are features one might plausibly take to be constitutive of a relation of coherence. We start with the consistency requirement we have already discussed.

[3]Note that once one reads this turnstile as a form of *consequence* from X to Y, one must read X and Y differently—it is the *conjunction* of all X which entails the *disjunction* of all Y.

CONSISTENCY: The state $[A : A]$ is incoherent. In other words, $A \vdash A$.

The next requirement trades on the features of collections. If there is an incoherence in the state $[X : Y]$ then that incoherence remains no matter what we *add* to the left- and right-sets. The only way to transform the incoherent $[X : Y]$ into a coherent state is to remove something from X or something from Y.

SUBSTATE: If $[X : Y]$ is coherent, and if $X' \subseteq X$ and $Y' \subseteq Y$, then $[X' : Y']$ is also coherent. In other words (and contrapositively), if $X \vdash Y$, $X \subseteq X'$ and $Y \subseteq Y'$, then $X' \vdash Y'$.

Those familiar with substructural logics [30] will be aware that the substate requirement is equivalent, on this reading, with the structural rule of *weakening*: If $X \vdash Y$ then $X, A \vdash Y$. A form of this rule is rejected in standard relevant logics such as R, on grounds of relevance. If we can infer from X to Y we need not use A in an inference from X, A to Y. The conflict here is merely apparent. Accepting *our* form of weakening does not mean accepting *all* forms of weakening. Nothing said here counts against the existence of a form of premise combination for which weakening is unacceptable.[4]

The next requirement is potentially more controversial. If we have a coherent state $[X : Y]$ then either its extension to assert A or its extension to deny A is coherent.

EXTENSIBILITY: If $[X : Y]$ is coherent, then so is one of $[X, A : Y]$ and $[X : A, Y]$. In other words (and contrapositively) if $X \vdash A, Y$ and $X, A \vdash Y$ then $X \vdash Y$.

This may appear controversial because it appears to endorse a form of the law of the excluded middle. It tells us that if A is *undeniable* in the context of the state $[X : Y]$ then it is coherent to assert A, provided that was $[X : Y]$ is already coherent. However, this does not rule out truth-value gaps and it does not implicitly endorse the law of the excluded middle.[5] On the contrary, this requirement follows from the intuitive picture of the connection between assertion and denial. To deny A is to place it out of further consideration. To go on and to accept A is to change one's mind. Dually, to accept A is to place its *denial* out of further consideration. To go on to deny A is to change one's mind. If one *cannot* coherently assert A, in the context of a coherent state $[X : Y]$, then it must at least be *coherent* to place it out of further consideration, for the inference relation itself has already, in effect, done so. Any move to accept A must take a step back by withdrawing some of the background state $[X : Y]$.

EXTENSIBILITY underwrites the transitivity of entailment. If $A \vdash B$ and $B \vdash C$,

[4] The importance of allowing different forms of premise combination is clearly explained in Slaney's "A General Logic" [34].

[5] As we will see later, all of this may be used to present the proof-theory of intuitionistic logic. The position is compatible with an anti-realist account of intuitionistic logic. The reading of this account of assertion and denial is a subtle one, for the intuitionist. Our sense of denial is not as strong as the intuitionist's assertion of a negation, but not as weak as the intuitionist's mere failure to assert. The requirement is that to deny, in our sense, is to *refuse* to accept. A statement is rejected if any move to accept it would be a change of mind, and not merely a supplementation with new information.

then by the SUBSTATE condition, we have $A \vdash B, C$ and $A, B \vdash C$. By EXTENSIBILITY, then, it follows that $A \vdash C$.

One might consider yet another structural feature for coherence.

LOCALITY: If $[X : Y]$ is incoherent, then there are finite $X' \subseteq X$ and $Y' \subseteq Y$ such that $[X' : Y']$ is incoherent.

According to LOCALITY, incoherence never requires an infinite body of assertions and denials. Just as EXTENSIBILITY is the coherence version of the *cut* rule, CONSISTENCY is *identity* and SUBSTATE is *thinning*, the rule of LOCALITY corresponds to *compactness*. Although locality is an important feature of a logicality, it will not play any role in the discussion that follows.

We have just motivated all of the structural rules of a standard multiple conclusion consequence relation as rules for the constraint of assertion and denial (or accepting and rejecting). (None of this, of course, counts against logics with *different* collections of structural rules [30]. The only consequence for these logics is that premise or conclusion combination is not to be read as joint assertion or joint denial.) Before going on to consider the significance of this for the choice of a logical system, and for the evaluation of different rules for each connective, we would do well to linger a while to see what can be expressed in this vocabulary.

If $X \vdash Y$ then it is incoherent to assert all of X and deny all of Y. This has a number of special cases worth spelling out:

» If $A \vdash$ then it is incoherent to assert A.

» If $A, B \vdash$ then it is incoherent to assert both A and B.

» If $\vdash B$ then it is incoherent to deny B.

» If $\vdash A, B$ then it is incoherent to deny both A and B.

» If $A \vdash B$ then it is incoherent to assert A and deny B.

Notice that the multiple-premise, multiple-conclusion structure enables us to represent both notions of *inference* ($A \vdash B$), *incompatibility* or *contrariety* ($A, B \vdash$) and *sub-contrariety* ($\vdash A, B$). On this picture there is no need for separate fundamental abilities to infer and to register incompatibility. These are all species of the larger phenomenon of regulating patterns of acceptings and rejectings.

Now notice that '$A \vdash$' does not commit us to rejecting A. It just rules out (on pain of incoherence) accepting A. Similarly, $\vdash B$ does not commit us to accepting B. It just rules out (on pain of incoherence) rejecting B. Consider, then, the sense in which '$A \vdash B$' tells us that B follows *from* A. If all it does is rule out the case in which we assert A and deny B, there seems to be little room for *consequence*.

Appearances are deceptive, in this case. If $A \vdash B$ and we accept A, then given the choice between accepting or rejecting B (and keeping our attitude to A fixed) we must accept B if we are to maintain coherence. If we reject B, then we fall into incoherence. The feature undergirding the *consequence* behind '$A \vdash B$' is the consideration of B. Once B is up for consideration, we can consider what our present commitments bring to bear. If it is the case that $A \vdash B$ and we already accept A then rejecting B is out of the question, unless we revise our opinion of A. As it is coherent to either accept B or to reject it (provided that our current state is coherent) then we may accept B at no further cost to coherence. EXTENSIBILITY tells us that any incoherence in $[X, A, B : Y]$ is already present in $[X, A : Y]$, provided that $A \vdash B$. Adding new consequences of already accepted items maintains coherence, no matter what the background assumptions might be. Similarly, if $A \vdash B$ and we have rejected B, then accepting A is not an option (if I am to continue to reject B) but rejecting A comes at no cost to coherence, no matter what the background assumptions might be. Provided that we may work with the notions of *assertion* and *denial*, we may express a relation of logical consequence relating multiple premises and multiple conclusions.

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Not everyone is happy with multiple conclusion presentations of logical consequence. Here is a representative critical passage, from Tennant's *The Taming of the True* [36].

> … the classical logician has to treat of sequents of the form $X : Y$ where the succedent Y may in general contain more than one sentence. In general, this smuggles in non-constructivity through the back door. For provable sequents are supposed to represent acceptable arguments. In normal practice, arguments take one from premisses to a single conclusion. There is no acceptable interpretation of the 'validity' of a sequent $X : Q_1, \ldots, Q_n$ in terms of preservation of warrant to assert when X contains only sentences involving no disjunctions. If one is told that $X : Q_1, \ldots, Q_n$ is 'valid' in the extended sense for multiple-conclusion arguments, the intuitionist can demand to know precisely which disjunct Q_i, then, proves to be derivable from X. No answer to such a question can be provided in general with the multiple-conclusion sequent calculus of the classical logician. It behooves us, then, to stay with a natural deduction system, and to present it in sequent form only if we observe the requirement that sequents should not have multiple conclusions. [36, page 320]

There are two criticisms of multiple conclusion consequence in this passage. The first, implicit, criticism concerns 'normal practice.' According to Tennant, in normal practice an argument has multiple premises and a single conclusion, and sequents are to be used to represent the structure of such arguments. A sequent $X : A$ represents the periphery of an argument, with premises X at the leaves of a tree, and A at its conclusion, the root.

This point about the structure of everyday arguments and proofs is not straightforward. Everyday arguments are most often not *explicitly* presented in tree form, but linearly. Just as we might find upward branching implicit in a tree (with multiple premises and single conclusion) we *might* also find downward branching present in the structure of linear proofs. The standard classical multiple conclusion proof of the intuitionistically invalid sequent $\forall x(Fx \vee Gx) \vdash \forall xFx \vee \exists xGx$ is a case in point.

> Suppose everyone is either *happy* or *tired*. Choose a person. It follows that this person is either happy or tired. There are two cases. Case (i) this person is happy. Case (ii) this person is tired, and as a result someone is tired. As a result, either this person is happy or *someone* (namely that person) is tired. But the person we chose was arbitrary, so either *everyone* is happy or someone is tired.

Case-based reasoning, like this, can be represented in a multiple-conclusion sequent calculus. Here is a straightforward multiple-conclusion sequent proof of the target sequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x(Fx \vee Gx) \vdash \forall x(Fx \vee Gx)}{\forall x(Fx \vee Gx) \vdash Fa \vee Ga}}{\forall x(Fx \vee Gx) \vdash Fa, Ga}}{\forall x(Fx \vee Gx) \vdash Fa, \exists xGx}}{\forall x(Fx \vee Gx) \vdash \forall xFx, \exists xGx}}{\forall x(Fx \vee Gx) \vdash \forall xFx \vee \exists xGx}$$

The sequent $\forall x(Fx \vee Gx) \vdash Fa, \exists xGx$ in this proof represents the stage in the English-language proof where we have two cases active, one concluding in Fa and the other, in $\exists xGx$. This demonstration appears to keep two conclusions (Fa and Ga) active at the one time, and it makes available a proof of the constructively invalid distribution principle. However, Tennant's complaint that this is "smuggles in non-constructivity through the back door" is at the very least too swift. You *could* complain that this proof is somehow non-constructive, but that point does not count uniquely against the proof structure we have chosen. Constructivity can equally be restored by restricting the application of the universal quantifier introduction rule to sequents with only one formula on the right. (This will be discussed further, below.) The structure of proof itself is does not dictate what rules one employs for connectives using this structure. It certainly makes non-constructive proof *available*, if the vocabulary allows it, but it does not *mandate* it.

So, the argument from the structure of everyday argument is not conclusive. Shoesmith and Smiley's classic *Multiple-Conclusion Logic* contains much more discussion of this issue [33], and I refer the reader there for more details. It certainly *appears* that we can use multiple conclusion reasoning to represent certain structures in everyday proof, but this point is not conclusive.

Tennant's second argument is more important for our purposes. He claims that we cannot explain the validity of multiple conclusion sequents in an anti-realistically acceptable fashion.[6] Tennant is right that the criterion of a preservation of warrant to assert will not do to explain the validity of a multiple conclusion sequent. But as we have seen, this is not the only option for reading such sequents. Preservation of warrant to assert is no more acceptable in reading multiple conclusion sequents than is converse preservation of warrant to deny for multiple premise sequents. Given a valid sequent $A, B : C$, and given that we have warrant to deny C, the reasoner (intuitionist or not) can demand to know precisely which conjunct A or B, then, proves to be refutable from C. The reasoner can demand this as much as she wishes, but no answer will be forthcoming. The valid sequent $A, B : C$ does not wear on its face which premise A or B ought be denied, and neither does the valid sequent $A : B, C$ wear on its face which conclusion ought be asserted.

So, the presence of multiple conclusions, on their own, does not make the reading of sequents any less acceptable to the anti-realist. The explanation of the significance of these sequents is given purely in terms of the norms governing assertion and denial, and nothing we have seen so far entails that these norms must be explained in the terms of truth, reference or correspondence. So, the picture of inferential relations as governing a norm of combinations of assertions and denials makes available a reading of sequents which does not lean upon truth or reference in the first instance.

Indeed, the reading of coherence of states that we have given thus far does not lean upon any specific notion of *warrant*. However, it could if we wished to use warrant as a guide to coherence. It is be open to us to define the coherence for states in the following way. Coherent states are those underwritten by a possible warrant. So, on this picture, $[X : Y]$ is coherent if and only if it is possible for there to be a warrant to assert all of X and deny all of Y. This will validate CONSISTENCY and SUBSTATE trivially. The only wrinkle is the verification that assertion, denial and warrant are connected in such a way as to validate EXTENSIBILITY. For this, we need to show that if there is some warrant to assert X and deny Y then there is some warrant to either assert X, A and deny Y or to assert X and deny A, Y. If there is no possible warrant to assert X, A and deny Y then this looks suspiciously like there is good reason for anyone committed to accepting X and denying Y to deny A.[7] Instead of continuing to develop this point here, we will now proceed to sketch what this might say for the choice of one's principles governing logical connectives.

* * *

[6] This objection is not restricted Tennant's writing. Dummett has similar critical comments in the *Logical Basis of Metaphysics* [9, page 187].

[7] Here I demonstrate the disjunction $A \vee B$ by assuming $\sim A$ and deriving B. Constructivists might quibble with this argument form, but this alone is no reason to reject this instance of that form. All valid arguments are instances of invalid argument forms. If this argument is really invalid, it has counterexample. I leave it to constructivists to provide a plausible counterexample to the inference. Suppose there is a possible warrant for $[X : Y]$. Exactly *how* can the claim that either there is a possible warrant for $[X, A : Y]$ or there is a possible warrant for $[X : A, Y]$ fail?

We have defended the priority of denial in an analysis of the inferential properties of negation. Given this move, the obvious next step is to notice that we can *define* the behaviour of negation like this: An assertion of $\sim A$ has the same significance as the denial of A, and a denial of $\sim A$ has the same significance as the assertion of A. End of story [21, 35]. This is not our approach, for we have seen to many examples of inferential practices where this identification is rejected. Supervaluationists and some intuitionists are happy to deny both A and $\sim A$. Dialetheists at the very least appear to assert both A and $\sim A$. Do we have a means of *explaining* the divergences in practice in such a way as to clarify what is at stake in the disagreement between these parties?

The classical rules for negation take the following form.

$$\frac{X \vdash A, Y}{X, \sim A \vdash Y}\,(\sim L) \qquad \frac{X, A \vdash Y}{X \vdash \sim A, Y}\,(\sim R)$$

These rules tell us that in the $[X : Y]$, asserting $\sim A$ has the same effect as a denial of A and denying $\sim A$ has the same effect as the assertion of A. Clearly these rules are not acceptable to all parties. The intuitionist and supervaluationist both reject $(\sim R)$ in the case where Y is non-empty, because it licenses the following derivation:

$$\frac{A \vdash A}{\vdash A, \sim A}\,(\sim R)$$

For the intuitionist and supervaluationist, sometimes it is appropriate to deny both A and $\sim A$, so they take themselves to have a counterexample to $(\sim R)$ as it stands. The dialetheist, similarly, rejects the inference $(\sim L)$ because it licenses the following inference

$$\frac{A \vdash A}{A, \sim A \vdash}\,(\sim L)$$

But for the dialetheist, sometimes it is appropriate to assert both A and $\sim A$, so $(\sim L)$ cannot be accepted in its full generality.

If we restrict the $(\sim R)$ rule to the case of an empty consequent Y, (and do the same for the $(\supset R)$ rule, which we will not consider here) but leave other rules as they are, we have a system for intuitionstic logic. This system has a feature. If we add *another* negation connective (we will write it like this: '$-$') satisfying the classical $(-L)$ and $(-R)$ rules, then the two negations collapse: $\sim$ inherits its classical brother's features.

$$\frac{\dfrac{A \vdash A}{A, \sim A \vdash}\,(\sim L)}{\sim A \vdash -A}\,(-R) \qquad \frac{\dfrac{A \vdash A}{A, -A \vdash}\,(-L)}{-A \vdash \sim A}\,(\sim R)$$

So, the *restricted* intuitionistic rules maintain their distinctive features only when other connectives are barred from entry into the system.

Dummett notices this feature in another context. One can consider a restriction to the disjunction and conjunction rules, as follows:

$$\frac{A \vdash Y \quad B \vdash Y}{A \lor B \vdash Y}\,(\lor L') \qquad \frac{X \vdash A \quad Y \vdash B}{X \vdash A \land B}\,(\land R')$$

In the absence of a left context X in $(\lor L')$ and a right context Y in $(\land R')$, the distribution of conjunction over disjunction $(A \land (B \lor C) \vdash (A \land B) \lor (A \land C))$ cannot be proved. Yet if you add a *new* disjunction with the traditional $(\lor L)$ rule, the two disjunctions collapse and distribution can be proved [9, page 290].[8] Dummett takes this to be a failing for the rules for non-distributing conjunction and disjunction. The rules do not serve to *fix* the interpretation of conjunction or disjunction, because given the addition of new rules, the behaviour of the old connectives change. If this were to be a valid criticism in the case of non-distributive disjunction and conjunction, it would be a criticism in the case of intuitionistic negation too, for we have seen the cases to be completely parallel.

Dummett has a possible response to this issue in the case of intuitionistic negation: he can attempt to argue that the new rules added (for Boolean negation, '$-$') are illegitimate. He must argue that the manipulation of multiple formulas on the right violates some kind of constraint on the structure of proof. However, if this succeeds in the case of intuitionistic logic, it leaves the way open for a *parallel* case in for non-distributive conjunction and disjunction. Perhaps there are *other* constraints on the structure proof, other than the multiple-premise single-conclusion constraint assumed by the intuitionist. We have known all along that the behaviour of connectives is supremely sensitive to the structural rules available for the manipulation of sequents [30]. This is another case where this sensitivity arises.

If no convincing case can be found for the restriction of formulas in contexts in the statements of rules, then it is tempting to conclude that we should accept the completely unrestricted classical inference principles, because these rules are perfectly general, simple and can be conservatively added to any non-logical vocabulary. The discussion of conservative extension here is quite complicated, because we have already seen examples of where it fails. If we have weak logical principles already in our vocabulary, the addition of strong logical principles can result in a non-conservative extension. However, there is a sense that if we have a purely non-logical vocabulary, the addition of classical inference principles can be totally conservative over the old language, as can be shown by a traditional cut-elimination or normalisation proof. It would be very comforting to think, then, that logical vocabulary can be always conservatively added over some discourse and that it can, in Brandom's suggestive phrase, be purely *expressive* of the inferential commitments already endorsed in the base vocabulary without adding any new commitments in the old vocabulary [5, 6].

[8] The case is identical if we lift the restriction on the *conjunction right* rule too, but Dummett does not consider this case, presumably because he does not take there to be any restriction in the rule $\land R'$.

Unfortunately, the matter is not so simple. There seem to be properly incompatible extensions to a single base language and consequence relation. The dialetheist's motivating examples provide us with a pertinent case. Consider adding to one's vocabulary a predicate $\in$ and variable-binding term forming operator $\{\ :\ \}$ satisfying the following rules, which are a form of Frege's BASIC LAW (V) for class membership.

$$\frac{X, \phi(a) \vdash Y}{X, a \in \{x : \phi(x)\} \vdash Y}\,(\in L) \qquad \frac{X \vdash \phi(a), Y}{X \vdash a \in \{x : \phi(x)\}, Y}\,(\in R)$$

It is trivial to show that this addition to the language is coherent if the language contains only predicates and names and no other logical vocabulary.

Given *this* collection of rules, it is impossible to add Boolean negation and preserve the transitivity of inference and consistency, as the Russell paradox shows. Let r be the term $\{x : \sim(x \in x)\}$: We have the following two proofs.

$$\dfrac{\dfrac{r \in r \vdash r \in r}{r \in r \vdash \sim(r \in r)}\,(\in R)}{\vdash \sim(r \in r)}\,(\sim R) \qquad \dfrac{\dfrac{r \in r \vdash r \in r}{\sim(r \in r) \vdash r \in r}\,(\in L)}{\sim(r \in r) \vdash}\,(\sim L)$$

Given the transitivity of entailment, we have the empty sequent '$\ \vdash\ $', and by weakening, triviality results. It follows that conservative extension criteria, on their own, do not suffice to choose one logical system over another. One's starting point matters.[9]

* * *

Here are some concluding points.

¶ Everyone has an opportunity to account for the meaning of their logical vocabulary in terms the way it constrains assertion and denial. This justification is independent of issues of realism or anti-realism. If you like, you can explain these constraints on assertion and denial in terms of what is true or what can possibly be true. If you like, you can explain these constraints in terms of what is warranted or possibly warranted. Or you could give some other account. Or you can be agnostic. Whatever approach you choose for the explication of deductively valid inference, common ground between positions is found in the way that logical consequence constrains assertion and denial.

¶ The multiple-conclusion calculus, long thought to be formally useful as an account of the preservation of *truth* and a formal setting for classical consequence [11, 31] can also be understood and appropriated by those who take the right semantic theory to be expressed in the anti-realist vocabulary of norms of assertion and denial rather than the realist talk of truth, correspondence and reference.

[9]This is not to say, of course, that there are no *other* reasons to favour one approach to another. We might reject the $\in$ rules because they seem to be existentially committing, for example.

¶ Non-classical logicians, such as dialetheists, supervaluationsts and intuitionists cannot be charged with incoherence. There seems to be enough shared vocabulary to understand and evaluate their positions. These different theories are different proposals for the logic of negation, and thereby, for the way that negation constrains assertion and denial.

¶ It is often thought that intuitionistic logic fares especially well when it comes to proof-theoretic justification of logical consequence. The justification here is seriously incomplete. The intuitionist has to do more work to explain the priority of assertion over denial and the resulting restriction of connective rules on the right. This restriction might be plausible and defensible, but if that kind of explanation is possible for the intuitionist, it might also be possible for others.

¶ The debates over the paradoxes of self-reference can be seen disagreement over what vocabulary is more important to save, and what kind of inferential machinery is most important. *No-one* can have it all. Triviality results from the combination of ($\in$L), ($\in$R), ($\sim$L) and ($\sim$R). Something must go.

¶ We have not answered all of the questions, but we have at least managed to see how different people are playing on the same field, with something like the same rules.

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