ON PERMUTATION IN SIMPLIFIED SEMANTICS

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Abstract: This note explains an error in Restall's 'Simplified Semantics for Relevant Logics (and some of their rivals)' [2] concerning the modelling conditions for the axioms of *assertion* $A \to ((A \to B) \to B)$ (there called c6) and *permutation* $(A \to (B \to C)) \to (B \to (A \to C))$ (there called c7). We show that the modelling conditions for assertion and permutation proposed in 'Simplified Semantics' overgenerate. In fact, they overgenerate so badly that the proposed semantics for the relevant logic R validate the rule of disjunctive syllogism. The semantics provides for *no* models of R in which the "base point" is inconsistent.

This problem is not restricted to 'Simplified Semantics.' The techniques of that paper are used in Graham Priest's textbook *An Introduction to Non-Classical Logic* [1], which is in wide circulation: it is important to find a solution. In this article, we explain this result, diagnose the mistake in 'Simplified Semantics' and propose two different corrections.

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§1 THE PROBLEM.

The models for relevant logics considered in Restall's 'Simplified Semantics for Relevant Logics (and some of their rivals)' [2] take the form $\langle W, g, R, ^*, v\rangle$ where W is a set containing the element g, and R is a ternary relation on W, such that $Rgxy$ if and only if $x = y$. (This condition is the distinctive feature of the simplified semantics.) * is a function from W to W. Finally, v assigns to each atomic formula a truth value at each point.

The relation R is used for the interpretation of the conditional '$\to$'. $v_x(A \to B) = 1$ iff for each y, z where $Rxyz$, if $v_y(A) = 1$ then $v_z(B) = 1$. The designated point g, with its special condition on R (where $Rgxy$ iff $x = y$) ensures that for each model, $v_g(A \to B)$ iff A entails B in the model — if *every* A-point is a

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B-point. The * operator is used to model negation: $v_x(\neg A) = 1$ iff $v_{x^*}(A) = 0$. In this way, $v_x(A)$ and $v_{x^*}(A)$ may take the same value when x and x^* differ.

These models suffice for the basic relevant logic B. To model stronger logics, you need conditions on the accessibility relation R.

The conditions proposed in 'Simplified Semantics' of interest to us are as follows:

- D5 If $Rabc$ then for some $x \in W$, $Rabx$ and $Rxbc$. (This makes $(A \to (A \to B)) \to (A \to B)$ true at g.)
- D6 If $Rabc$ then $Rbac$. (This makes $A \to ((A \to B) \to B)$ true at g.)
- D7 If $Rabx$ and $Rxcd$ then for some $y \in W$, $Racy$ and $Rybd$. (This makes $(A \to (B \to C)) \to (B \to (A \to C))$ true at g.)
- D20 If $Rabc$ then Rac^*b^*. (This makes $(A \to B) \to (\neg B \to \neg A)$ true at g.)

Each of these conditions is wanted when it comes to modelling the relevant logic R, since each of the formulas made valid hold in that logic. However, there is a problem with the conditions as stated.

The basic property of g — that $Rgxy$ if and only if $x = y$ — gives us Rgg^*g^*. By D5, we have some x where Rgg^*x and Rxg^*g^*. From Rgg^*x it follows that $x = g^*$. This tells us that $Rg^*g^*g^*$. By D20 we have Rg^*gg, and by D6 we have Rgg^*g, which gives $g^* = g$.

This is a problem, since if $g^* = g$ then the base point g is consistent. This means that we leave out important models for the logic R. R is, by nature, *paraconsistent*. It allows for inconsistent but non-trivial theories. These models do not reflect that, as the argument from $A, \neg A$ to B is not relevantly valid. The completeness proof of 'Simplified Semantics' [2] claimed that for any invalid argument we have some model where the premises are true and conclusion untrue. So, the result stated there is incorrect.

We have seen the problem with D5, D6 and D20. The problem also occurs with D5, D7 and D20, since D7 entails D6. Suppose $Rbcd$. It follows that $Rgbb$ and $Rbcd$, so we may apply D7, to infer that there's a y where $Rgcy$ and $Rybd$. $Rgcy$ tells us that $c = y$, and so, we have $Rcbd$. So, we have moved from $Rbcd$ to $Rcbd$, which gives us D6.

This problem, unfortunately, is not restricted to 'Simplified Semantics.' The techniques of that paper also appear in Graham Priest's textbook *An Introduction to Non-Classical Logic* [1], so it is important to diagnose the problem and to find a solution.

§2 THE DIAGNOSIS.

The mistake in the paper is not difficult to isolate. As we indicated above, the completeness proof breaks down.

The "proof" constructs a canonical model for refuting an invalid argument from Σ to A. You expand Σ into a set Π of sentences that is prime ($B \lor C \in \Pi$ iff $B \in \Pi$ or $C \in \Pi$) and a Π-theory (not only is Π closed under *entailment*, but also Π-entailment, so whenever $B \to C \in \Pi$, if $B \in \Pi$ then $C \in \Pi$) which does not contain A. Then the model consists of the collection of all prime Π-theories, together with a special point $\underline{\Pi}$. This special point will be the base point in the model. For the relation R, we specify $R\underline{\Pi}\Gamma\Delta$ iff $\Gamma = \Delta$ for each $\Gamma, \Delta \in W$, and otherwise, $R\Sigma\Gamma\Delta$ iff whenever $B \to C \in \Sigma$ and $B \in \Gamma$, we also have $C \in \Delta$, for each prime Π-theory Σ, Γ and Δ. In general, in the canonical model the theory Σ makes true just the formulas in Σ. The special point $\underline{\Pi}$ makes true exactly the same formulas as the point Π. Π and $\underline{\Pi}$ differ only with regard to the relation R.

Given a logic extending the basic relevant logic B, the canonical models for B provide counterexamples to invalid arguments. So far, so good. The mistake in the completeness "proof" occurs elsewhere. In 'Simplified Semantics' Restall noted that we needed to do extra work to ensure that in the canonical model we can verify D6 and and D7, even when the logic validates the corresponding axioms. The difficulty arises when we permute a base point $\underline{\Pi}$ away from the first position in R. We have $R\underline{\Pi}\Sigma\Sigma$ but—as the definition of R stands—we do not have $R\Sigma\underline{\Pi}\Sigma$. $\underline{\Pi}$ enters into R-relations only in the first position except for the case $R\underline{\Pi}\,\underline{\Pi}\,\underline{\Pi}$.[1] In 'Simplified Semantics' Restall proposed a re-definition of the behaviour of R to take account of this fact.

Unfortunately, Restall did not check that R, so redefined, would also satisfy *other* frame conditions (such as D5 or D20) in the presence of their corresponding axioms. The redefinition that Restall proposed does not work. The modified canonical models do not satisfy each of the frame conditions required for modelling the relevant logic R.

§3 THE FIRST OPTION: ONE SPECIAL POINT.

Looking at the failed completeness proof, it seems that there are two options for repair. The first might be to correct the modification: show that *some* redefinition of the accessibility relation R allows us to define a model in which the modelling conditions are satisfied. The second might be to modify the frame conditions in the list D5–D20. The result of the first section shows that the first option cannot work. The problem is not merely with the completeness construction, but with the model conditions themselves. *Any* model satisfying D5, D6 and D20 has a consistent base point. No fancy footwork with a canoncial model construction lets us escape this fact.

So, we must modify at least one of the frame conditions. Given the diagnosis,

[1] This choice is not essential to the construction. We could take, for example, $R\Sigma\underline{\Pi}\Delta$ if and only if $R\Sigma\Pi\Delta$, but then the problem would obtain in reverse. Under the condition C6, we have $R\Sigma\Pi\Delta$ whenever $\Sigma \subseteq \Delta$ (see *Case 2b* earlier), so choose some $\Sigma \subset \Delta$. We have $R\Sigma\Pi\Delta$, and then $R\Sigma\underline{\Pi}\Delta$ and by D6, $R\underline{\Pi}\Sigma\Delta$ when $\Sigma \neq \Delta$, contradicting our condition for the base point.

the conditions that permute points from the first place in R into another place in the relation are suspect. In our list of conditions, D6 and D7 are suspect.

In the canonical model construction, $R\underline{\Pi}\Sigma\Sigma$ is true, but $R\Sigma\underline{\Pi}\Sigma$ is false. So D6 is false in the canonical model. We must find a condition *true* in the canonical model which is strong enough to ensure that in any model in which it holds, the axiom C6 (that is, $A \to ((A \to B) \to B))$ is true at g.

To validate C6 at g, it suffices to show that for each $a \in W$, whenever A is true at a, so is $(A \to B) \to B$. To check $(A \to B) \to B$ at a, suppose that $Rabc$ and $A \to B$ is true at b. We need to verify that B is true at c. Now, we have A at a, $A \to B$ and b and we want B at c. The most straightforward route is to have $Rbac$, that is, to use condition D6. How can we get the same effect without applying D6? In the canonical model we may have $R\underline{\Pi}\Sigma\Sigma$ but not $R\Sigma\underline{\Pi}\Sigma$. However, we *do* have $R\Sigma\Pi\Sigma$, as the 'surrogate' base point Π enters into normal R relations. Π and $\underline{\Pi}$ are in a special relationship. Anything true at $\underline{\Pi}$ is true at Π, and *vice versa*. In general, we could validate C6 if we could argue that when $Rabc$, and A is true at a, $A \to B$ is true at b, then there is an a' where $Rba'c$, and where A is also true at a'. For then, we could still conclude that B is true at c. But how are a and a' related? How can we ensure that the A true at a is also true at a'?

Fortunately, the solution is at hand. In Section 5 of 'Simplified Semantics' Restall discusses the *inclusion* relation among worlds. This is a reflexive, transitive relation $\leqslant$ on W satisfying this condition:

$$a \leqslant b \Rightarrow \begin{cases} Rbcd \Rightarrow Racd & \text{if } a \neq g \\ Rbcd \Rightarrow c \leqslant d & \text{if } a = g \\ b^* \leqslant a^* & \text{always} \end{cases}$$

Given that truth-at-a-point is preserved along inclusion for atomic formulas (if $a \leqslant b$ and $v_a(p) = 1$ then $v_b(p) = 1$) then it is preserved for *all* formulas in the language, by an easy inductive argument. We can use $\leqslant$ to relate b and b' from the previous discussion.

In the canonical model, the inclusion relation is the subset relation between prime Π-theories, and for the special point $\underline{\Pi}$, we fix $\underline{\Pi} \leqslant \Gamma$ if and only if $\Pi \leqslant \Gamma$ and $\Sigma \leqslant \underline{\Pi}$ if and only if $\Sigma \leqslant \Pi$. So Π and $\underline{\Pi}$ are indistinguishable as far as $\leqslant$ is concerned. It is straightforward to verify that $\leqslant$ so defined is an inclusion relation.

Replace D6 by the following clause:

D6′ If $Rabc$ then for some $a' \geqslant a$, $Rba'c$

We have seen that any model in which D6′ is satisfied makes $A \to ((A \to B) \to B)$ true at g.

We must verify that our canonical model satisfies D6′ when the logic validates $A \to ((A \to B) \to B)$, in order to patch the completeness proof. This is a straightforward argument with two cases. Suppose we have $R\Sigma\Gamma\Delta$. We wish to find a $\Sigma' \geqslant \Sigma$ such that $R\Gamma\Sigma'\Delta$. *Case 1*: $\Sigma \neq \underline{\Pi}$. In this case $\Gamma, \Delta \neq \underline{\Pi}$ too (as $\underline{\Pi}$ only appears in the second or third place of an R-fact if also appears in the first place) and we may choose $\Sigma' = \Sigma$, and the standard argument to the effect that $R\Gamma\Sigma\Delta$ works.[2] *Case 2*: $\Sigma = \underline{\Pi}$. In this case the R-fact is $R\underline{\Pi}\Gamma\Gamma$. There are two subases. *Case 2a*: $\Gamma = \underline{\Pi}$. Permuting $R\underline{\Pi}\,\underline{\Pi}\,\underline{\Pi}$ we have nothing to prove. *Case 2b*: $\Gamma \neq \underline{\Pi}$. Now we do not have $R\Gamma\underline{\Pi}\Gamma$, but we do have $R\Gamma\Pi\Gamma$ (since if $A \to B \in \Gamma$ and $A \in \Pi$ we can deduce that $(A \to B) \to B \in \Pi$ (since $A \to ((A \to B) \to B)$ is valid), and since Γ is closed under Π-entailment, we have $B \in \Gamma$, as desired) and $\Pi \leqslant \underline{\Pi}$. So we have verified D6′ in the canonical model.

The clause D7 faces the same problems. Even though it does not *look* like there is any permutation from first place into second place (as there is in D6), setting $a = g$ or $x = g$ may have this effect. We modify the clause as follows, to find something that both validates $(A \to (B \to C)) \to (B \to (A \to C))$ and is satisfied in the canonical model.

D7′ If $Rabx$ and $Rxcd$ then for some y, and some $b' \geqslant b$, $Racy$ and $Ryb'd$.

This clause validates $(A \to (B \to C)) \to (B \to (A \to C))$ is can be checked by the usual argument. Suppose $A \to (B \to C)$ is true at a, to verify $B \to (A \to C)$. For this, suppose that $Rabx$ and B is true at b. Is $A \to C$ true at x? Suppose that $Rxcd$, and A is true at c. We want C true at d. By our condition, we have a $y \in W$ and a $b' \geqslant b$ where $Racy$ and $Ryb'd$. We have A true at c and $A \to (B \to C)$ at a gives $B \to C$ at y. Since $b' \geqslant b$ we have B true at b' and $Ryb'd$ gives C true at d as desired.

Clause D7′ is satisfied in the canonical model, as can be checked in a number of cases. *Case 1*: $a = \underline{\Pi}$. We have $R\underline{\Pi}bb$ ($x = b$) and $Rbcd$. We wish to show that $R\underline{\Pi}cy$ and $Ryb'd$ for some y and some $b' \geqslant b$. $R\underline{\Pi}cy$ forces $y = c$, so we need $Rcb'd$ for some $b' \geqslant b$, given the condition that $Rbcd$. *Case 1a*: $b = \underline{\Pi}$, $c = \underline{\Pi}$. Here, we have $d = \underline{\Pi}$ and choose $b' = \underline{\Pi}$ and we are done. *Case 1b*: $b = \underline{\Pi}$, $c \neq \underline{\Pi}$. Then we have $R\underline{\Pi}cc$ ($c = d$), and we choose $b' = \Pi$, since we have $Rc\Pi c$ by the standard argument (this is the first point in the argument where we need to choose a value for b' distinct than b).

Case 2: $x = \underline{\Pi}$. Here, since $Rab\underline{Pi}$ we must have $a = b = \underline{\Pi}$. Given $Rxcd$, i.e., $R\underline{\Pi}cd$, we have $c = d$ also. Therefore, to find a $b' \geqslant b$ and a y where $Racy$ and $Ryb'd$, we need $b' \geqslant \underline{\Pi}$ and y where $R\underline{\Pi}cy$ and $Ryb'c$. For $R\underline{\Pi}cy$ we need $c = y$. For $Ryb'c$ we need, then $Rcb'c$. For this, if $c \neq \underline{\Pi}$ we may choose $b' = \Pi \geqslant \underline{\Pi}$ for the usual argument for $Rz\Pi z$ succeeds (this is the first point in the argument where we need to choose a value for b' distinct than b). If $c = \underline{\Pi}$, then we may

[2] If $A \to B \in \Gamma$ then if $A \in \Sigma$ by the validity of $A \to ((A \to B) \to B)$ in the logic we have $(A \to B) \to B \in \Sigma$ and $R\Sigma\Gamma\Delta$ gives us $B \in \Delta$.

choose $b' = b = \underline{\Pi}$, since $R\underline{\Pi}\underline{\Pi}\underline{\Pi}$.

Case 3: $a, x \neq \underline{\Pi}$. In this case $b, c, d \neq \underline{\Pi}$, we may choose $b = b'$ and $c = c'$ and the standard construction works.

To interpret negation in the canonical model with a single extre point we need one small extra patch. To validate $A \to \neg\neg A$ it suffices to enforce $a \leqslant a^{**}$. To validate $\neg\neg A \to A$, it suffices to enforce $a^{**} \leqslant a$. To validate the equivalence of A and $\neg\neg A$ it has been traditional to impose the (natural) condition $a = a^{**}$. In our case, this will not be satisfied in the canonical model if we interpret * in the simplest fashion.

- $\Sigma^* = \{A : \neg A \notin \Sigma\}$ for each prime Π-theory, Σ.
- $\underline{\Pi}^* = \Pi^*$.

(So, according to *, Π and $\underline{\Pi}$ are indistinguishable. Again, it is only R that can distinguish Π and $\underline{\Pi}$.) In this case, in the presence of $A \leftrightarrow \neg\neg A$, we have $\underline{\Pi} \neq \underline{\Pi}^{**} = \Pi$. To model the $A \leftrightarrow \neg\neg A$ in this semantics, it suffices to admit that $a \leqslant a^{**}$ and $a^{**} \leqslant a$, without requiring identity in the case of the base point.

There is no doubt that admitting $a \leqslant a^{**}$ and $a^{**} \leqslant a$ without enforcing $a = a^{**}$ is not attractive. The alternative is to allow an *extra* extra point to stand as the 'star' of $\underline{\Pi}$. This approach has different complications: we need to define the behaviour of this new point in R and see that all of the conditions are satisfied for your target logics. There are many target logics and many conditions to check. This approach is the topic for the next section.

§4 THE SECOND OPTION: TWO SPECIAL POINTS.

So, a second simplified semantics for logics stronger than B can be read off a different canonical model construction, in which points are prime Π-theories, with the addition of *two* special points, $\underline{\Pi}$ (which has the same members as Π) and $\underline{\underline{\Pi}}$ (which has the same members as Π^*).[3] To define R in this canonical model, we now need *three* clauses, for prime Π-theories, for $\underline{\Pi}$, and for $\underline{\underline{\Pi}}$.

- $R\Sigma\Gamma\Delta$ iff whenever $A \to B \in \Sigma$ and $A \in \Gamma$ then $B \in \Delta$.
- $R\underline{\Pi}\Gamma\Delta$ iff $\Gamma = \Delta$.
- $R\underline{\underline{\Pi}}\Gamma\Delta$ iff whenever $A \to B \in \underline{\underline{\Pi}}$ (that is, when $A \to B \in \Pi^*$) and $A \in \Gamma$ then $B \in \Delta$.

To define the function * on this structure, we have for prime Π-theories $\Sigma^* = \{A : \neg A \notin \Sigma\}$ as before; and we set $\underline{\Pi}^* = \underline{\underline{\Pi}}$ and $\underline{\underline{\Pi}}^* = \underline{\Pi}$. For the inclusion relation

[3] As before, this means that officially $\underline{\Pi}$ and $\underline{\underline{\Pi}}$ are not sets. Exactly how they are to be distinguished from the prime Π-theories is unimportant. We will abuse notation to consider the 'members' of $\underline{\underline{\Pi}}$, which are no more and no less than the members of Π^*.

$\leqslant$, $\underline{\underline{\Pi}}$ acts in just the same way as Π^* (just as $\underline{\underline{\Pi}}$ and Π are indistinguishable by $\leqslant$). The rest of the definition of the canonical model is kept from before.

In this structure, then, $x = x^{**}$ holds universally, if A is equivalent to $\neg\neg A$ in the underlying logic.

Now we must consider the axioms C6 and C7 to see how corresponding conditions fare in this new canonical model. Recall the modified frame conditions.

D6′ If $Rabc$ then for some $a' \geqslant a$, $Rba'c$

D7′ If $Rabx$ and $Rxcd$ then for some y, and some $b' \geqslant b$, $Racy$ and $Ryb'd$.

In the presence of C6: $A \to ((A \to B) \to B)$, D6′ holds in the new canonical model. The reasoning from before applies, with the addition of a new case, *Case 3*: $\Sigma = \underline{\underline{\Pi}}$. In this case, if $R\underline{\underline{\Pi}}\Gamma\Delta$, then $R\Pi^*\Gamma\Delta$ and we have $R\Gamma\Pi^*\Delta$ and $\Pi^* \geqslant \underline{\underline{\Pi}}$, so the condition is satisfied.

For C7, the condition remains sound for identical reasons.

For completeness, we may follow the reasoning of §3·11 must be modified slightly, in to deal with the different behaviour of R on the extra points. The argument for completeness goes as follows: Suppose $R\Sigma\Gamma\Delta$ and $R\Delta\Theta\Xi$; we want Ω and $\Gamma' \supset \Gamma$ such that $R\Sigma\Theta\Omega$ and $R\Omega\Gamma'\Xi$.

Case 1: $\Sigma = \underline{\underline{\Pi}}$. Then $\Gamma = \Delta$; set $\Omega = \Theta$. Then we have $R\underline{\underline{\Pi}}\Gamma\Gamma$ and $R\Gamma\Theta\Xi$; and need $R\underline{\underline{\Pi}}\Theta\Theta$ and $R\Theta\Gamma'\Xi$. The first is automatic by construction. (a) $\Gamma = \Theta = \underline{\underline{\Pi}}$; then $\Xi = \underline{\underline{\Pi}}$; set $\Gamma' = \underline{\underline{\Pi}}$; we need $R\underline{\underline{\Pi}}\,\underline{\underline{\Pi}}\,\underline{\underline{\Pi}}$; and this is automatic by construction. (b) $\Theta = \underline{\underline{\Pi}}$, $\Gamma \neq \underline{\underline{\Pi}}$; then we have $R\Gamma\underline{\underline{\Pi}}\Xi$; let $\Gamma' = \Xi$; then $R\Theta\Xi\Xi$ is automatic from construction, and we need to see that $\Xi \supseteq \Gamma$: Suppose $A \in \Gamma$; from $\vdash [A \to (B \to C)] \to [B \to (A \to C)]$ it follows that $\vdash A \to [(A \to B) \to B]$ and so that $\vdash A \to [(A \to A) \to A]$; so $(A \to A) \to A \in \Gamma$; but $\vdash A \to A$; so $A \to A \in \underline{\underline{\Pi}}$; so with $R\Gamma\underline{\underline{\Pi}}\Xi$, $A \in \Xi$; so $\Xi \supseteq \Gamma$. (c) Otherwise set set $\Gamma' = \Gamma$; suppose $A \to B \in \Theta$ and $A \in \Gamma$; then since $\vdash A \to [(A \to B) \to B$, $(A \to B) \to B \in \Gamma$; so with $R\Gamma\Theta\Xi$, $B \in \Xi$; so $R\Theta\Gamma\Xi$.

Case 2: $\Delta = \underline{\underline{\Pi}}$. Then $\Theta = \Xi$; we have $R\Sigma\Gamma\underline{\underline{\Pi}}$ and $R\underline{\underline{\Pi}}\Theta\Theta$; set $\Gamma' = \Gamma$. Then we need Ω such that $R\Sigma\Theta\Omega$ and $R\Omega\Gamma\Theta$. By the construction from SS2 there is Ω such that $R\Sigma\Theta\Omega$; suppose $A \to B \in \Omega$ and $A \in \Gamma$; then by the construction there is $C \in \Theta$ and $C \to (A \to B) \in \Sigma$; so with C7, $A \to (C \to B) \in \Sigma$; so with $R\Sigma\Gamma\underline{\underline{\Pi}}$, $C \to B \in \underline{\underline{\Pi}}$; so $B \in \Theta$; so $R\Omega\Gamma\Theta$.

Case 3: $\Sigma, \Delta \neq \underline{\underline{\Pi}}$. Set $\Gamma' = \Gamma$; then by the previous construction, there is an Ω such that $R\Sigma\Theta\Omega$; suppose $A \to B \in \Omega$ and $A \in \Gamma$; then by the construction there is a $C \in \Theta$ and $C \to (A \to B) \in \Sigma$; so with C7, $A \to (C \to B) \in \Sigma$; so with $R\Sigma\Gamma\Delta$, $C \to B \in \Delta$; and with $R\Delta\Theta\Xi$, $B \in \Xi$; so $R\Omega\Gamma\Xi$.

This completes the difficult cases of the completeness proof for the second patched simplified semantics. The clauses for negation are straightforward, and we may pronounce the completeness proof finished. We have two different fixes for the mistake in Restall's 'Simplified Semantics'.

§5 INCLUSION ELSEWHERE.

It is striking that this difficulty for the simplified semantics occurs at the very place where the four-valued approach seems to run aground in Routley's 'The American Plan Completed' [3]. There, a reasonably natural four-valued semantics for relevant implication meets difficulty with the verification of our condition D6 [3, page 154].

Routley's project was to model negation by allowing four values in the evaluation — true-only, false-only, both-true-and-false and neither-true-nor-false, and to have two ternary accessibility relations governing the conditional, one for truth and the other governing falisty. Alas, once we impose a contraposition axiom such as $(A \to B) \to (\neg B \to \neg A)$, we require connections between the two accessibility relations that seem to make it difficult to impose the required constraints on those relations when it comes to strong conditions such as D6. Routley's own favoured solution is to mimic the $*$ operator directly in the four-valued semantics (thereby undermining the distinctiveness of the American plan).

It seems to us that the approach here gives us hope that a different kind of solution might be found. Once we admit four values in the semantics, then there seems to be more than one possible inclusion relation worth modelling. Not only the obvious relations $\leqslant$ for preserving truth, and $\leqslant^{\bullet}$ for preserving non-falsity (or, if you prefer, preserving falsity in the converse direction) but also the relations $\leqslant^{*}$ (where $a \leqslant^{*} b$ whenever what is true at a is not false at b) and $\leqslant^{\sharp}$ (where $a \leqslant^{\sharp} b$ whenever what is not false at a is true at b). Relations like these are definable in the canonical model, and it seems to us that admitting them will provide enough control over the the interaction between truth and falsity to open up new modelling conditions for axioms such as C6. However, we must leave working through these details for another place and time.

§6 REFERENCES.

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