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SIMPLIFIED SEMANTICS FOR RELEVANT LOGICS (AND SOME OF THEIR RIVALS)

ABSTRACT. This paper continues the work of Priest and Sylvan in *Simplified Semantics for Basic Relevant Logics*, a paper on the simplified semantics of relevant logics, such as $\mathbf{B}^+$ and **B**. We show that the simplified semantics can also be used for a large number of extensions of the positive base logic $\mathbf{B}^+$, and then add the dualising '*' operator to model negation. This semantics is then used to give conservative extension results for Boolean negation.

1. DEFINITIONS

The ternary relational semantics for basic relevant logics were greatly simplified by Priest and Sylvan in their recent paper *Simplified Semantics for Basic Relevant Logics* (hereafter 'SS'). But the story wasn't completed by that paper – the systems $\mathbf{B}^+$, **BD**, **BM** and **B** were each shown to have a simple semantics of some form or other, but the question of systems extending these was left unanswered. It was also undecided whether **B** had a simple four-valued semantics. The question of extensions of **B** was thought to be relatively simple, and by and large it is, but some of the completeness proofs are harder than one might expect. However, it is my pleasure to announce that a simplified semantics is obtainable for a large range of extensions of the basic relevant logic **B**, including the standard logics such as **R**, and **T** together with their contraction-free counterparts **RW** and **TW**. A notable omission is **E**, which escapes the simplified modelling.

1.1. *The System $\mathbf{B}^+$*

The first set of results deal with $\mathbf{B}^+$, a positive relevant propositional logic. To establish our terms, $\mathbf{B}^+$ is expressed in a language $\mathscr{L}$, which has the connectives $\wedge$, $\vee$ and $\rightarrow$, parentheses (and), and a stock of propositional variables $p, q, \ldots$ Formulae are defined recursively in the usual manner, and the standard scope conventions are in force; $\wedge$ and $\vee$ bind more strongly than $\rightarrow$. For example, $p \wedge q \rightarrow r$ is short

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for $(p \wedge q) \to r$. We will use $\alpha, \beta, \ldots$ to range over arbitrary formulae.

The system $\mathbf{B}^+$ can then be defined in terms of the following axioms and rules:

A1 $\alpha \to \alpha$,

A2 $\alpha \to \alpha \vee \beta, \beta \to \alpha \vee \beta$,

A3 $\alpha \wedge \beta \to \alpha, \alpha \wedge \beta \to \beta$,

A4 $\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee \gamma$,

A5 $(\alpha \to \beta) \wedge (\alpha \to \gamma) \to (\alpha \to \beta \wedge \gamma)$,

A6 $(\alpha \to \gamma) \wedge (\beta \to \gamma) \to (\alpha \vee \beta \to \gamma)$.

If $(\alpha_1 \ldots \alpha_n)/\beta$ is a rule, its *disjunctive form* is the rule $(\gamma \vee \alpha_1 \ldots \gamma \vee \alpha_n)/(\gamma \vee \beta)$. The rules for $\mathbf{B}^+$ are the following, along with their disjunctive forms:

R1 $\dfrac{\alpha, \alpha \to \beta}{\beta}$,

R2 $\dfrac{\alpha, \beta}{\alpha \wedge \beta}$,

R3 $\dfrac{\alpha \to \beta, \gamma \to \delta}{(\beta \to \gamma) \to (\alpha \to \delta)}$.

It is to be noted that the simplified semantics given can only model *disjunctive systems*. That is, systems such that the disjunctive form of every truth preserving rule is truth preserving. Not every logic satisfies this criterion – a notable candidate is **E**, for the disjunctive form of its characteristic rule, from α to $(\alpha \to \beta) \to \beta$, fails to be truth preserving. The reason for this is that $\alpha \vee \neg\alpha$ is a theorem of **E**, but $\neg\alpha \vee ((\alpha \to \beta) \to \beta)$ is a non-theorem. (See Brady's *Natural Deduction Systems for some Quantified Relevant Logics* for an introduction to the notion of a disjunctive system.)

1.2. *Semantics for* $\mathbf{B}^+$

The main construction in SS is the semantics given for $\mathbf{B}^+$. Their semantics is a simplified version of the original ternary relational semantics for relevant logics. The important definitions concerning the structure are collected here.

An *interpretation* for the language is a 4-tuple $\langle g, W, R, I \rangle$, where W is a set of worlds, $g \in W$ is the base world, R is a ternary relation on W, and I assigns to each pair (w, p) of worlds and propositional parameters a truth value, $I(w, p) \in \{0, 1\}$. Truth values at worlds are then assigned to formulae inductively as follows:

- $1 = I(w, \alpha \wedge \beta) \Leftrightarrow 1 = I(w, \alpha)$ and $1 = I(w, \beta)$,
- $1 = I(w, \alpha \vee \beta) \Leftrightarrow 1 = I(w, \alpha)$ or $1 = I(w, \beta)$,
- $1 = I(g, \alpha \rightarrow \beta) \Leftrightarrow$ for all $x \in W(1 = I(x, \alpha) \Rightarrow 1 = I(x, \beta))$,

and for $x \neq g$,

- $1 = I(x, \alpha \rightarrow \beta) \Leftrightarrow$ for all $y, z \in W(Rxyz \Rightarrow (1 = I(y, \alpha) \Rightarrow 1 = I(z, \beta)))$.

Then semantic consequence is defined in terms of truth preservation at g, the base world. In other words,

$$\Theta \models \alpha \Leftrightarrow \text{for all interpretations } \langle g, W, R, I \rangle$$
$$(1 = I(g, \beta) \text{ for all } \beta \in \Theta \Rightarrow 1 = I(g, \alpha)).$$

The soundness and completeness result for $\mathbf{B}^+$ can then be concisely stated as follows.

THEOREM 1. *If* $\Theta \cup \{\alpha\}$ *is a set of sentences, then*

$$\Theta \vdash \alpha \Leftrightarrow \Theta \models \alpha,$$

where $\vdash$ *is the provability relation of* $\mathbf{B}^+$.[1]

In this paper we will make a cosmetic alteration to the above definition of an interpretation. It is easily seen that the truth conditions for

'$\rightarrow$' can be made univocal if we define R to satisfy $Rgxy \Leftrightarrow x = y$. From now, we will use the following definition of an interpretation.

DEFINITION. An *interpretation* for the language is a 4-tuple $\langle g, W, R, I \rangle$, where W is a set of worlds, $g \in W$ is the base world, R is a ternary relation on W satisfying $Rgxy \Leftrightarrow x = y$, and I assigns to each pair (w, p) of worlds and propositional parameters a truth value, $I(w, p) \in \{0, 1\}$. Truth values at worlds are then assigned to formulae inductively as follows:

- $1 = I(w, \alpha \wedge \beta) \Leftrightarrow 1 = I(w, \alpha)$ and $1 = I(w, \beta)$,
- $1 = I(w, \alpha \vee \beta) \Leftrightarrow 1 = I(w, \alpha)$ or $1 = I(w, \beta)$,
- $1 = I(x, \alpha \rightarrow \beta) \Leftrightarrow$ for all $y, z \in W$ $(Rxyz \Rightarrow$ $(1 = I(y, \alpha) \Rightarrow 1 = I(z, \beta)))$.

Then semantic consequence is defined in terms of truth preservation at g, the base world. In other words,

$$\Theta \models \alpha \Leftrightarrow \text{for all interpretations } \langle g, W, R, I \rangle$$
$$(1 = I(g, \beta) \text{ for all } \beta \in \Theta \Rightarrow 1 = (g, \alpha)).$$

It is clear how one can translate between the two notions of an interpretation. The reason we use this altered definition is in the phrasing of conditions on R which give extensions of the logic $\mathbf{B}^+$. It is much less tedious to write '*Rabc* $\Rightarrow$ *Rbac*' than it is to write '*Rabc* $\Rightarrow$ *Rbac* for $a \neq g$ and *Rbgb* for each b', but these are equivalent definitions under each variety of interpretation.

2. SOUNDNESS

THEOREM 2. *For each row n in the list below, the logic* $\mathbf{B}^+$ *with the axiom (or rule)* **C***n added is sound with respect to the class of* $\mathbf{B}^+$ *interpretations* $\langle g, W, R, I \rangle$ *where R satisfies condition* **D***n.*

C1 $\alpha \wedge (\alpha \rightarrow \beta) \rightarrow \beta$

C2 $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)$

C3 $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

C4 $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$

C5 $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$

C6 $\alpha \to ((\alpha \to \beta) \to \beta)$

C7 $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$

C8 $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

C9 $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

C10 $\dfrac{\alpha}{(\alpha \to \beta) \to \beta}$, and its disjunctive forms

D1 $Raaa$

D2 $Rabc \Rightarrow R^2a(ab)c$

D3 $R^2abcd \Rightarrow R^2b(ac)d$

D4 $R^2abcd \Rightarrow R^2a(bc)d$

D5 $Rabc \Rightarrow R^2abbc$

D6 $Rabc \Rightarrow Rbac$

D7 $R^2abcd \Rightarrow R^2acbd$

D8 $R^2abcd \Rightarrow R^3ac(bc)d$

D9 $R^2abcd \Rightarrow R^3bc(ac)d$

D10 $Raga$

Where we have defined:

$$R^2abcd = (\exists x)(Rabx \wedge Rxcd),$$

$$R^2a(bc)d = (\exists x)(Rbcx \wedge Raxd),$$

$$R^3ab(cd)e = (\exists x)(R^2abxe \wedge Rcdx).$$

Proof. To prove soundness in each case, we take an arbitrary $\mathbf{B}^+$ interpretation $\langle g, W, R, I \rangle$ and assume that the relation R satisfies condition **D***n*. Then it suffices to demonstrate that the value of a formula of the form of **C***n* must always be 1 at g. (Or that the corresponding rule is always truth preserving at g.) The soundness results are simple and mechanical.

For those who like a quick proof, it suffices to note that any $\mathbf{B}^+$ interpretation $\langle g, W, R, I \rangle$ is also an old-style interpretation – with the proviso that $Rgxy \Leftrightarrow x = y$. Hence, we can shamelessly appropriate the soundness proofs from *Relevant Logics and Their Rivals* (hereafter RLR), pp. 304–305.

Or, we can work the proofs independently. We give two examples to show how it is done.

1. Assume that R satisfies *Raaa* for each $a \in W$. Then,

$$I(g, \alpha \wedge (\alpha \to \beta) \to \beta) = 0$$

if and only if there is a w where $I(w, \alpha \wedge (\alpha \to \beta)) = 1$ and $I(w, \beta) = 0$. Then, $I(w, \alpha) = 1$ and $I(w, \alpha \to \beta) = 1$, so *Raaa* gives $I(w, \beta) = 1$, contrary to our assumption. Thus **C1** is true at g.

10. Assume that *Raga* for each a, and that

$$I(g, \alpha \vee \gamma) = 1, \text{ and } I(g, ((\alpha \to \beta) \to \beta) \vee \gamma) = 0.$$

Then we must have a w where $I(w, \alpha \to \beta) = 1$ and $I(w, \beta) = 0$, and that $I(g, \gamma) = 0$. So, $I(g, \alpha) = 1$, as $I(g, \alpha \vee \gamma) = 1$. By assumption $Rwgw$, and hence $I(w, \beta) = 1$, contrary to our assumption. Hence **C10** is truth preserving at g.

3. RESULTS CONCERNING PRIME THEORIES

The completeness result for $\mathbf{B}^+$ relies on a standard model construction, where the worlds are prime theories. We will need certain definitions and facts about prime theories to prove this result for logics extending $\mathbf{B}^+$.

- If Π is a set of $\mathscr{L}$-sentences, $\Pi_{\to}$ is the set of all members of Π of the form $\alpha \to \beta$.
- $\Sigma \vdash_\Pi \alpha \Leftrightarrow \Sigma \cup \Pi_{\to} \vdash \alpha$.
- Σ is a Π-*theory* $\Leftrightarrow$:
  1. $\alpha, \beta \in \Sigma \Rightarrow \alpha \wedge \beta \in \Sigma$,
  2. $\vdash_\Pi \alpha \to \beta \Rightarrow (\alpha \in \Sigma \Rightarrow \beta \in \Sigma)$.
- Σ is *prime* $\Leftrightarrow (\alpha \vee \beta \in \Sigma \Rightarrow \alpha \in \Sigma$ or $\beta \in \Sigma)$.
- If X is any set of sets of formulae, the ternary relation R on X is defined thus:

$$R\Sigma\Gamma\Delta \Leftrightarrow (\gamma \to \delta \in \Sigma \Rightarrow (\gamma \in \Gamma \Rightarrow \delta \in \Delta)).$$

- $\Sigma \vdash_\Pi \Delta \Leftrightarrow$ for some $\delta_1, \ldots, \delta_n \in \Delta$, we have $\Sigma \vdash_\Pi \delta_1 \vee \ldots \vee \delta_n$.
- $\vdash_\Pi \Sigma \rightarrow \Delta \Leftrightarrow$ for some $\sigma_1, \ldots, \sigma_n \in \Sigma$ and $\delta_1, \ldots, \delta_n \in \Delta$ we have $\vdash_\Pi \sigma_1 \wedge \ldots \wedge \sigma_n \rightarrow \delta_1 \vee \ldots \vee \delta_n$.
- Σ is Π-*deductively closed* $\Leftrightarrow (\Sigma \vdash_\Pi \alpha \Rightarrow \alpha \in \Sigma)$.
- Where L is the set of all $\mathscr{L}$-sentences, $\langle \Sigma, \Delta \rangle$ is a Π-*partition* if and only if:
  1. $\Sigma \cup \Delta = L$,
  2. $\nvdash_\Pi \Sigma \rightarrow \Delta$.

In all of the above definitions, if Π is the emptyset, the prefix 'Π-' is omitted; so a $\varnothing$-theory is simply a theory, and so on. The following results are proved in SS:

LEMMA 3. • *If* $\langle \Sigma, \Delta \rangle$ *is a* Π-*partition then* Σ *is a prime* Π-*theory.*

- *If* $\nvdash_\Pi \Sigma \rightarrow \Delta$ *then there are* $\Sigma' \supseteq \Sigma$ *and* $\Delta' \supseteq \Delta$ *such that* $\langle \Sigma', \Delta' \rangle$ *is a* Π-*partition.*
- *If* Σ *is a* Π-*theory, and* Δ *is closed under disjunction, with* $\Sigma \cap \Delta = \varnothing$, *then there is a* $\Sigma' \supseteq \Sigma$ *such that* $\Sigma' \cap \Delta = \varnothing$ *and* Σ' *is a prime* Π-*theory.*
- *If* Π *is a prime* Π-*theory, is* Π-*deductively closed, and* $\alpha \rightarrow \beta \notin \Pi$, *then there is a prime* Π-*theory,* Γ, *such that* $\alpha \in \Gamma$ *and* $\beta \notin \Gamma$.
- *If* Σ *is a prime* Π-*theory with* $\gamma \rightarrow \delta \notin \Sigma$, *then there are prime* Π-*theories,* Γ *and* Δ *such that* $R\Sigma\Gamma\Delta$, $\gamma \in \Gamma$ *and* $\delta \notin \Delta$.

Given some Π-theory Σ with various properties, we are interested in finding a *prime* Π-theory $\Sigma' \supseteq \Sigma$ that retains those properties, because our canonical model structures requires the worlds to be prime. The following lemmas do this, and hence they are called *priming lemmas.* The content of these is contained in either SS or RLR, but not in the form we need; we repeat them here for completeness' sake.

LEMMA 4. *If* Σ, Γ *and* Δ *are* Π-*theories, such that* $R\Sigma\Gamma\Delta$ *and* Δ *is prime, then there is a prime* $\Gamma' \supseteq \Gamma$ *where* $R\Sigma\Gamma'\Delta$.

Proof. We construct Γ' by defining $\Theta = \{\alpha : (\exists \beta \notin \Delta)(\alpha \rightarrow \beta \in \Sigma)\}$. Θ is closed under disjunction, for suppose $\alpha_1, \alpha_2 \in \Theta$. Then there are $\beta_1, \beta_2 \notin \Delta$ such that $\alpha_1 \rightarrow \beta_1, \alpha_2 \rightarrow \beta_2 \in \Sigma$. As Δ is prime, $\beta_1 \vee \beta_2 \notin \Delta$. It is easy to show that $\alpha_1 \vee \alpha_2 \rightarrow \beta_1 \vee \beta_2 \in \Sigma$, as Σ is a Π-theory. So we have $\alpha_1 \vee \alpha_2 \in \Theta$. Moreover, $\Gamma \cap \Theta = \varnothing$. For, suppose that $\alpha \in \Gamma \cap \Theta$. Then there is some $\beta \notin \Delta$ where $\alpha \rightarrow \beta \in \Sigma$, contradicting $R\Sigma\Gamma\Delta$.

Thus, applying a part of Lemma 3 there is a prime Π-theory $\Gamma' \supseteq \Gamma$ where $\Gamma' \cap \Theta = \varnothing$. To see that $R\Sigma\Gamma'\Delta$, let $\alpha \to \beta \in \Sigma$ and $\alpha \in \Gamma'$. Then $\alpha \notin \Theta$, so we have $\beta \in \Delta$. ■

LEMMA 5. *If* Σ, Γ *and* Δ *are* Π*-theories, such that* $R\Sigma\Gamma\Delta$ *and* Δ *is prime, then there is a prime* $\Sigma' \supseteq \Sigma$ *where* $R\Sigma'\Gamma\Delta$.

Proof. This time let $\Theta = \{\alpha : (\exists \beta, \gamma)$ where $\vdash_\Pi \alpha \to (\beta \to \gamma)$, $\beta \in \Gamma$ and $\gamma \notin \Delta\}$. Θ is closed under disjunction, for if $\alpha_1, \alpha_2 \in \Theta$, then for some $\beta_1, \beta_2 \in \Gamma$ and $\gamma_1, \gamma_2 \notin \Delta$, $\vdash_\Pi \alpha_1 \to (\beta_1 \to \gamma_1)$ and $\vdash_\Pi \alpha_2 \to (\beta_2 \to \gamma_2)$. Hence $\beta_1 \wedge \beta_2 \in \Gamma$ (as Γ is a Π-theory) and $\gamma_1 \vee \gamma_2 \notin \Delta$ (as Δ is prime). It is straightforward to show that then $\vdash_\Pi \alpha_1 \vee \alpha_2 \to (\beta_1 \wedge \beta_2 \to \gamma_1 \vee \gamma_2)$, because we have $\vdash_\Pi \alpha_1 \vee \alpha_2 \to ((\beta_1 \to \gamma_1) \vee (\beta_2 \to \gamma_2))$. So we have $\alpha_1 \vee \alpha_2 \in \Theta$.

$\Sigma \cap \Theta = \varnothing$. For if $\delta \in \Sigma \cap \Theta$, then there are $\beta \in \Gamma$ and $\gamma \notin \Delta$ where $\vdash_\Pi \delta \to (\beta \to \gamma)$. Then as $\delta \in \Sigma$, and as Σ is a Π-theory we have $\beta \to \gamma \in \Sigma$, contradicting $R\Sigma\Gamma\Delta$.

Lemma 3 then gives a prime $\Sigma' \supseteq \Sigma$, disjoint from Θ. $R\Sigma'\Gamma\Delta$ obtains, as if $\beta \to \gamma \in \Sigma$, $\beta \to \gamma \notin \Theta$, so for all β' and γ' where $\vdash_\Pi (\beta \to \gamma) \to (\beta' \to \gamma')$, if $\beta' \in \Gamma$, then $\gamma' \in \Delta$. But $\vdash_\Pi (\beta \to \gamma) \to (\beta \to \gamma)$, so we have our result. ■

LEMMA 6. *If* Σ, Γ *and* Δ *are* Π*-theories, such that* $R\Sigma\Gamma\Delta$ *and* Δ *is prime, then there are prime* $\Sigma' \supseteq \Sigma$ *and* $\Sigma' \supseteq \Gamma$ *where* $R\Sigma'\Gamma'\Delta$.

Proof. Apply both of the lemmas above. ■

LEMMA 7. *If* Σ, Γ *and* Δ *are* Π*-theories, such that* $R\Sigma\Gamma\Delta$ *and* $\delta \notin \Delta$, *then there are prime* Π*-theories* Σ' *and* Δ' *such that* $\Gamma \supseteq \Gamma'$, $\delta \notin \Delta'$, *and* $\Delta \supseteq \Delta'$.

Proof. First construct Δ'. Take Θ_1 to be the closure of $\{\delta\}$ under disjunction. As $\vdash_\Pi \delta \vee \ldots \vee \delta \to \delta$ and Δ is a Π-theory, $\Delta \cap \Theta_1 = \varnothing$. By a part of Lemma 3, there is a prime Π-theory $\Delta' \supseteq \Delta$ where $\Delta' \cap \Theta_1 = \varnothing$, and it follows that $R\Sigma\Gamma\Delta'$, and that $\delta \notin \Delta'$.

To construct Γ', take Θ_2 to be $\{\alpha : \exists \beta \notin \Delta'$ where $\alpha \to \beta \in \Sigma\}$. Θ_2 is closed under disjunction and $\Gamma \cap \Theta_2 = \varnothing$. (See SS for a proof, or take it as an exercise.) Lemma 3 shows that there is a prime Π-theory $\Gamma' \supseteq \Gamma$ where $\Gamma' \cap \Theta_2 = \varnothing$. To show that $R\Sigma\Gamma'\Delta'$, take $\alpha \to \beta \in \Sigma$ and $\alpha \in \Gamma'$. Then $\alpha \notin \Theta_2$, and hence $\beta \in \Delta'$. ■

The completeness of the simplified semantics for $\mathbf{B}^+$ is then demonstrated in the following way. Given a set of formulae $\Theta \cup \{\alpha\}$, such that $\Theta \nvdash \alpha$, we construct an interpretation in which Θ holds at the base world, but α doesn't. Firstly, note that there is a prime theory Π such that $\Pi \supseteq \Theta$, but $\alpha \notin \Pi$, by Lemma 3. The worlds of the interpretation are the Π-theories, g is Π itself and R is as defined above, except that $R\Pi\Gamma\Delta$ if and only if $\Gamma = \Delta$.[2] Then we determine I, be assigning $I(\Sigma, p) = 1 \Leftrightarrow p \in \Sigma$ for each propositional variable p and Π-theory Σ. It can then be proved that $I(\Sigma, \beta) = 1 \Leftrightarrow \beta \in \Sigma$ for each formula β, so we have that Θ holds at Π, the base world, and α does not.

4. COMPLETENESS

To show completeness for the extensions, it is usual to show that any canonical model of the logic in question satisfies the conditions corresponding to the logic. So for **C1**, we show that any canonical model formed from the logic $\mathbf{B}^+ + \mathbf{C1}$ satisfies the condition *Raaa* for each *a*. The completeness result then follows immediately as described in the last section.

Unfortunately for us, the results of this form appear to break down when we extend the logic too far beyond $\mathbf{B}^+$. A simple example is given by the logic $\mathbf{B}^+ + \mathbf{C5}$. It is not at all clear that the canonical interpretation of this logic satisfies the condition $Rabc \Rightarrow R^2abbc$. The reason is as follows: Suppose that Σ, Γ and Δ are prime Π-theories such that $R\Sigma\Gamma\Delta$. We wish to find a prime Π-theory Ω such that $R\Sigma\Gamma\Omega$ and $R\Omega\Gamma\Delta$. The general approach is to let Ω be the smallest set satisfying the first condition – it will turn out to be a Π-theory, and a priming lemma gives us a corresponding prime theory Ω' – and then we demonstrate that Ω satisfies the second condition. (And a priming lemma ensures that Ω' will also. The details are given when we get to the proof. They are sketched here to motivate what follows.) This proof goes through, except for the case when Ω turns out to be Π. In that case $R\Omega\Gamma\Delta$ if and only if $\Gamma = \Delta$, and this does not seem to follow from what we have assumed. It is at this step that many of the completeness arguments fail.

So instead of using the original canonical interpretation, we will use another, in which the standard arguments work. We note that the

difficulty with the standard argument arises when a Π-theory (say Ω) constructed to satisfy $R\Omega\Gamma\Delta$, turns out to be Π itself. What would solve the problem is some *other* Π-theory which has exactly the same truths as Π, but which has 'orthodox' R-relations with other Π-theories. In other words, we wish to have a Π-theory Π', which satisfies $R\Pi'\Gamma\Delta$ if and only if $\alpha \to \beta \in \Pi' \Rightarrow (\alpha \in \Gamma \Rightarrow \beta \in \Delta)$, instead of the more restrictive condition of $\Gamma = \Delta$. Then this world will take the place of Π, whenever we need it in the first place of an R-relation. This is only by way of motivation, and does not constitute a proof. We will formally explicate this model structure, and prove the completeness theorems with it:

DEFINITION. Given that Π is a prime Π-theory of a logic L extending $\mathbf{B}^+$, an *almost-canonical interpretation* for L is a 4-tuple $\langle\langle\Pi, 1\rangle, W, R, I\rangle$, where

- $W = \{\langle\Sigma, 0\rangle: \Sigma \text{ is a prime } \Pi\text{-theory}\} \cup \{\langle\Pi, 1\rangle\}$,
- R is defined on W^3 to satisfy:
  - $R\langle\Pi, 1\rangle xy$ if and only if $x = y$,
  - $R\langle\Sigma, 0\rangle\langle\Gamma, i\rangle\langle\Delta, j\rangle$ if and only if for each α and β, $\alpha \to \beta \in \Sigma \Rightarrow (\alpha \in \Gamma \Rightarrow \beta \in \Delta)$, for $i, j \in \{0, 1\}$,
- $I(\langle\Sigma, i\rangle, \alpha) = 1$ if and only if $\alpha \in \Sigma$, for $i \in \{0, 1\}$.

In a moment, we will demonstrate that this actually is an interpretation (by showing that I satisfies the inductive properties needed for an interpretation), but first, we will simplify our notation. It is clear that the almost-canonical interpretation is simply the canonical interpretation with another world with the same truths as the base world, but entering into different R-relations (when it appears in the first place of R). Other than that it is identical, so we will ignore the ordered-pair notation, and simply write Π for what was $\langle\Pi, 1\rangle$, the base world; Π' for $\langle\Pi, 0\rangle$, its double; and for each other world $\langle\Sigma, 0\rangle$, we will simply use Σ. Further, instead of writing $I(\langle\Sigma, i\rangle, \alpha) = 1$, we simply write $\alpha \in \Sigma$. In this way, we cut down on notation, and the parallel with the canonical interpretation is made clear. In fact, you can ignore the whole business with ordered-pairs, and simply imagine W to be the set of all prime Π-theories, each of which is painted blue, and a single set with the same elements as Π,

which is painted red. The red one is the base world, and has R defined on it in its own peculiar way, and the blue ones have R defined on them as normal. However you think of it, seeing it in use will (hopefully) make it clear. The first proof demonstrates that it is actually an interpretation.

THEOREM 8. *The almost-canonical interpretation is worthy of its name; that is, it is an interpretation.*

Proof. Define I by requiring that:

$$I(\Sigma, p) = 1 \text{ if and only if } p \in \Sigma,$$
$$\text{for } p \text{ a propositional parameter,}$$

and that it satisfy the usual inductive definitions of an interpretation. We simply need to show that $I(\Sigma, \alpha) = 1$ if and only if $\alpha \in \Sigma$ for *every* formula α. We do this by induction on the complexity of the formulae.

- It works by stipulation on the base case.
- $I(\Sigma, \alpha \wedge \beta) = 1$
  if and only if $I(\Sigma, \alpha) = 1$ and $I(\Sigma, \beta) = 1$ (by the inductive definition of I),
  if and only if $\alpha \in \Sigma$ and $\beta \in \Sigma$ (by the inductive hypothesis),
  if and only if $\alpha \wedge \beta \in \Sigma$ (as Σ is a Π-theory).
- $I(\Sigma, \alpha \vee \beta) = 1$
  if and ony if $I(\Sigma, \alpha) = 1$ or $I(\Sigma, \beta) = 1$ (by the inductive definition of I),
  if and only if $\alpha \in \Sigma$ or $\beta \in \Sigma$ (by the inductive hypothesis),
  if and only if $\alpha \vee \beta \in \Sigma$ (as Σ is a *prime* Π-theory).
- $I(\Sigma, \alpha \to \beta) = 1$
  if and only if for each Γ, Δ where $R\Sigma\Gamma\Delta$ $(I(\Gamma, \alpha) = 1 \Rightarrow I(\Delta, \beta) = 1)$,
  if and only if for each Γ, Δ where $R\Sigma\Gamma\Delta$ $(\alpha \in \Gamma \Rightarrow \beta \in \Delta)$.

We desire to show that this last condition obtains if and only if $\alpha \to \beta \in \Sigma$. We take this in two cases — firstly when Σ is the base world.

Then, $I(\Pi, \alpha \to \beta) = 1$
if and only if for each Γ, $(\alpha \in \Gamma \Rightarrow \beta \in \Gamma)$,
if and only if $\alpha \to \beta \in \Pi$, as Γ is a Π-theory, and by the fourth part of Lemma 3.

If Σ is not the base world, then $R\Sigma\Gamma\Delta$ if and only if $R'\Sigma\Gamma\Delta$, where R' is the relation on Π-theories defined univocally as $R'\Sigma\Gamma\Delta$ if and only if $(\forall \alpha \to \beta \in \Sigma)(\alpha \in \Gamma \Rightarrow \beta \in \Delta)$.

Then we have: $I(\Sigma, \alpha \to \beta) = 1$,
if and only if for each Γ, Δ where $R\Sigma\Gamma\Delta$, $(\alpha \in \Gamma \Rightarrow \beta \in \Delta)$,
if and only if $\alpha \to \beta \in \Sigma$, by the definition of R, and by the fifth part of Lemma 3.

So for *any* world Σ, $I(\Sigma, \alpha \to \beta) = 1$ if and only if $\alpha \to \beta \in \Sigma$. This completes the inductive proof. ■

We now have enough results to prove completeness.

THEOREM 9. *For each row n in the list given in Theorem 2, the logic* $\mathbf{B}^+$ *with the axiom (or rule)* $\mathbf{C}n$ *added is complete with respect to the class of* $\mathbf{B}^+$ *interpretations* $\langle g, W, R, I \rangle$ *where R satisfies condition* $\mathbf{D}n$.

Proof. We will demonstrate these individually, showing that the canonical model of any logic satisfying $\mathbf{B}^+$ and axiom (or rule) $\mathbf{C}n$ must satisfy restriction $\mathbf{D}n$.

1. We wish to show that $R\Sigma\Sigma\Sigma$ for each prime Π-theory Σ, under the assumption of **C1**. Consider $\alpha \to \beta \in \Sigma$, and $\alpha \in \Sigma$. Thus $\alpha \wedge (\alpha \to \beta) \in \Sigma$, and $\vdash_\Pi \alpha \wedge (\alpha \to \beta) \to \beta$ gives $\beta \in \Sigma$, and so, if $\Sigma \neq \Pi$, $R\Sigma\Sigma\Sigma$. If $\Sigma = \Pi$, the result follows immediately.
2. We wish to show that for all prime Π-theories Σ, Γ and Δ where $R\Sigma\Gamma\Delta$, there is a prime Π-theory Θ' where $R\Sigma\Gamma\Theta'$ and $R\Sigma\Theta'\Delta$. Let $\Theta = \{\beta : (\exists \alpha)(\alpha \to \beta \in \Sigma) \wedge (\alpha \in \Gamma)\}$. Θ is a Π-theory because:
   - $\beta_1, \beta_2 \in \Theta$ means that there are $\alpha_1, \alpha_2 \in \Gamma$ where $\alpha_1 \to \beta_1$, $\alpha_2 \to \beta_2 \in \Sigma$. So, $\alpha_1 \wedge \alpha_2 \in \Gamma$ and $(\alpha_1 \to \beta_1) \wedge (\alpha_2 \to \beta_2) \in \Sigma$. But this gives $\alpha_1 \wedge \alpha_2 \to \beta_1 \wedge \beta_2 \in \Sigma$. (Because $\vdash_\Pi \alpha_1 \wedge \alpha_2 \to \alpha_1$, so $\vdash_\Pi (\alpha_1 \to \beta_1) \to (\alpha_1 \wedge \alpha_2 \to \beta_1)$ by **R3** and $\vdash_\Pi (\beta_1 \to \beta_1)$. Similarly, we have $\vdash_\Pi (\alpha_1 \to \beta_1) \to (\alpha_1 \wedge \alpha_2 \to \beta_1)$, so **A5** gives us $\vdash_\Pi (\alpha_1 \to \beta_1) \wedge (\alpha_2 \to \beta_2) \to (\alpha_1 \wedge \alpha_2 \to \beta_1 \wedge \beta_2)$.) This ensures that $\beta_1 \wedge \beta_2 \in \Theta$.
   - If $\vdash_\Pi \alpha \to \beta$ and $\alpha \in \Theta$, there is a $\gamma \in \Gamma$ where $\gamma \to \alpha \in \Sigma$, and because we have $\vdash_\Pi (\gamma \to \alpha) \to (\gamma \to \beta)$ by prefixing, we then have $\gamma \to \beta \in \Sigma$. This gives $\beta \in \Theta'$, as desired.

   Now $R\Sigma\Gamma\Theta$ is true by definition (even if $\Sigma = \Pi$, in which case $\Gamma = \Delta = \Theta$, and we have our result immediately). To show that

$R\Sigma\Theta\Delta$, if $\alpha \to \beta \in \Sigma$ and $\alpha \in \Theta$, there is a $\gamma \in \Gamma$ where $\gamma \to \alpha \in \Sigma$, so $(\gamma \to \alpha) \wedge (\alpha \to \beta) \in \Sigma$. **C2** ensures that $\gamma \to \beta \in \Sigma$, and hence $\beta \in \Delta$ as $R\Sigma\Gamma\Delta$.

By Lemma 4, there is a prime $\Theta' \supseteq \Theta$ where $R\Sigma\Theta'\Delta$, and $R\Sigma\Gamma\Theta'$ is ensured by $\Theta' \supseteq \Theta$. If as sets $\Theta' = \Pi$, then we select it can be either Π or Π' for this case — in other cases the choice is important. Whatever we take Θ' to be, we have our result.

Before we go on to the next case, we would do well to note some features of this one. Θ, as we defined it, is the smallest set satisfying $R\Sigma\Gamma\Theta$, and fortunately for us, it is a Π-theory. We will use this construction often, and we will not rewrite the proof that the set so formed is a Π-theory.

3. Assume that **C3** holds, and consider arbitrary prime Π-theories Σ, Θ, Γ, Δ and Ξ, where $R\Sigma\Gamma\Xi$ and $R\Xi\Theta\Delta$. We wish to find a prime Π-theory Ω' where $R\Sigma\Theta\Omega'$ and $R\Gamma\Omega'\Delta$. To this end, let $\Omega = \{\beta : (\exists\alpha)(\alpha \to \beta \in \Sigma) \wedge (\alpha \in \Theta)\}$; this is a Π-theory as before. $R\Sigma\Theta\Omega$ is immediate (even when $\Sigma = \Pi$, in which case $\Gamma = \Xi$, and $\Theta = \Omega$, yielding the result). To show that $R\Gamma\Omega\Delta$, let $\alpha \to \beta \in \Gamma$ and $\alpha \in \Omega$. Then there is a $\gamma \in \Gamma$ where $\gamma \to \alpha \in \Sigma$ and $\gamma \in \Theta$, and as $\vdash_\Pi (\gamma \to \alpha) \to ((\alpha \to \beta) \to (\gamma \to \beta))$, we have $(\alpha \to \beta) \to (\gamma \to \beta) \in \Sigma$. $R\Sigma\Gamma\Xi$ then ensures that $\gamma \to \beta \in \Xi$, and $R\Xi\Theta\Delta$ then gives $\beta \in \Delta$. This means that $R\Gamma\Omega\Delta$, if $\Gamma \neq \Pi$. In this case a priming lemma then ensures the existence of a prime $\Omega' \supseteq \Omega$ where $R\Gamma\Omega'\Delta$ and $R\Sigma\Theta\Omega'$, and hence our result. Again, if as sets $\Omega' = \Pi$, then it is unimportant whether we take Ω' to be Π or Π'. (For other cases, if it is unimportant, we will fail to mention that fact.)

If, on the other hand $\Gamma = \Pi$, set $\Omega' = \Delta$. Then $R\Sigma\Theta\Delta$, as $\alpha \to \beta \in \Sigma$ and $\alpha \in \Theta$ gives $(\beta \to \beta) \to (\alpha \to \beta) \in \Sigma$, by **C3**. As $\beta \to \beta \in \Gamma$ and $R\Sigma\Gamma\Xi$, we have $\alpha \to \beta \in \Xi$. This, along with $R\Xi\Theta\Delta$ and $\alpha \in \Theta$ gives $\beta \in \Delta = \Omega'$, as we desired.

4. Assume that **C4** holds, and that Σ, Θ, Γ, Δ and Ξ are prime Π-theories such that $R\Sigma\Gamma\Xi$ and $R\Xi\Theta\Delta$. Set Ω to be $\{\beta : (\exists\alpha)(\alpha \to \beta \in \Gamma) \wedge (\alpha \in \Theta)\}$, and then it is clear that $R\Gamma\Theta\Omega$. (Even in the case where $\Gamma = \Pi$, in which case $\Omega = \Theta$, and we have our result.) To show that $R\Sigma\Omega\Delta$, consider $\alpha \to \beta \in \Sigma$ and $\alpha \in \Omega$; there must be a $\gamma \in \Theta$ where $\gamma \to \alpha \in \Gamma$. So as

$\vdash_\Pi (\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$, we have $(\gamma \to \alpha) \to (\gamma \to \beta) \in \Sigma$. $R\Sigma\Gamma\Xi$ then gives $(\gamma \to \beta) \in \Xi$, and $R\Xi\Theta\Delta$ with $\gamma \in \Theta$ gives $\beta \in \Delta$, as we wished. Ω is a Π-theory, and can be primed by a priming lemma, in the usual manner.

5. Assume that **C5** holds, and let Σ, Γ and Δ be arbitrary Π-theories where $R\Sigma\Gamma\Delta$. Let $\Omega = \{\beta : (\exists\alpha)(\alpha \to \beta \in \Sigma) \wedge (\alpha \in \Gamma)\}$, so $R\Sigma\Gamma\Omega$ is immediate. (Except if $\Sigma = \Pi$, in which case $\Delta = \Gamma$, and we simply set $\Omega = \Gamma$ as well. It follows that $R\Omega\Gamma\Delta$, as $R\Gamma\Gamma\Gamma$, because **C1** is a consequence of **C5**. This gives the result in this case.) To show that $R\Omega\Gamma\Delta$ let $\alpha \to \beta \in \Omega$ and $\alpha \in \Gamma$, so there is a $\gamma \in \Gamma$ where $\gamma \to (\alpha \to \beta) \in \Sigma$. Thus, $\alpha \wedge \gamma \in \Gamma$, and $\alpha \wedge \gamma \to \beta \in \Sigma$, as the following derivation shows.

$\vdash_\Pi \alpha \wedge \gamma \to \alpha$,	by **A3**,
$\vdash_\Pi \alpha \wedge \gamma \to \gamma$,	by **A3**,
$\vdash_\Pi (\alpha \to \beta) \to (\alpha \wedge \gamma \to \beta)$,	by suffixing,
$\vdash_\Pi (\gamma \to (\alpha \to \beta)) \to (\gamma \to (\alpha \wedge \gamma \to \beta))$,	by prefixing,
$\vdash_\Pi (\gamma \to (\alpha \wedge \gamma \to \beta)) \to (\alpha \wedge \gamma \to (\alpha \wedge \gamma \to \beta))$,	by suffixing,
$\vdash_\Pi (\gamma \to (\alpha \to \beta)) \to (\alpha \wedge \gamma \to (\alpha \wedge \gamma \to \beta))$,	by transitivity.

This ensures that $\alpha \wedge \gamma \to (\alpha \wedge \gamma \to \beta) \in \Sigma$, and **C5** gives $\alpha \wedge \gamma \to \beta \in \Sigma$. So $R\Sigma\Gamma\Delta$ gives $\beta \in \Delta$ as desired. The usual application of the priming lemma gives a prime Π-theory Ω' with the desired properties, except if $\Omega' = \Pi$ as sets, and it is taken to be Π. In that case, $R\Omega'\Gamma\Delta$ is not assured, for it is not clear that $\Gamma = \Delta$. However, if this is the case, we can take Ω' to be Π' and all is well.

6. Assume that **C6** holds. We will work in the almost-canonical model of Π-theories as usual, but it will have a different relation R' defined as follows: $R'\Sigma\Gamma\Delta$ if and only if for each $\alpha \to \beta \in \Sigma$, if $\alpha \in \Gamma$ then $\beta \in \Delta$ for Σ, $\Gamma \neq \Pi$. Otherwise, $R'\Pi\Gamma\Delta$ and $R'\Gamma\Pi\Delta$ if and only if $\Gamma = \Delta$. We need to show that in this model of Π-theories

that $1 = I(\Sigma, \alpha)$ if and only if $\alpha \in \Sigma$, and that it satisfies $R'\Sigma\Gamma\Delta \Rightarrow R'\Gamma\Sigma\Delta$. The latter part is simpler.

If either of Σ or Γ is Π, then the condition is satisfied by fiat. If Σ and Γ are both not Π, then let $\alpha \to \beta \in \Gamma$, $\alpha \in \Sigma$ and $R'\Sigma\Gamma\Delta$. **C6** gives $(\alpha \to \beta) \to \beta \in \Sigma$, and $R'\Sigma\Gamma\Delta$ then gives $\beta \in \Delta$, and so we have $R'\Gamma\Sigma\Delta$, as desired.

To show that the model structure satisfies $1 = I(\Sigma, \alpha)$ if and only if $\alpha \in \Sigma$, we need only consider the case where α is $\gamma \to \delta$, and where Σ is not Π. The rest of the proof is unaltered from the almost-canonical structure. We need to show that $1 \in I(\Sigma, \gamma \to \delta)$ if and only if $\gamma \to \delta \in \Sigma$. From right to left, it is enough to note that $\gamma \to \delta \in \Sigma$ ensures that for all prime Π-theories Γ and Δ where $R\Sigma\Gamma\Delta$, if $\gamma \in \Gamma$, then $\delta \in \Delta$, by the definition of R. As $R\Sigma\Gamma\Delta \Rightarrow R'\Sigma\Gamma\Delta$, we have that all prime Π-theories Γ and Δ where $R'\Sigma\Gamma\Delta$, if $\gamma \in \Gamma$, then $\delta \in \Delta$.

And in the other direction, if $1 \in I(\Sigma, \gamma \to \delta)$, then $\forall\Gamma, \Delta$ where $R'\Sigma\Gamma\Delta$, if $\gamma \in \Gamma$ then $\delta \in \Delta$. If $\gamma \to \delta \notin \Sigma$, then by Lemma 3 there are prime Π-theories Γ and Δ where $R\Sigma\Gamma\Delta$, $\gamma \in \Gamma$ and $\delta \notin \Delta$. In this case, $R'\Sigma\Gamma\Delta$ unless $\Gamma = \Pi$ (the case where we've been mucking about with R'). In this case, if $R\Sigma\Pi\Delta$, and $\alpha \in \Sigma$, then $\vdash_{\Pi} \alpha \to ((\alpha \to \alpha) \to \alpha)$ gives $(\alpha \to \alpha) \to \alpha \in \Sigma$, which with $R\Sigma\Pi\Delta$ and $\alpha \to \alpha \in \Pi$ gives $\alpha \in \Delta$. So $\Sigma \subseteq \Delta$, which means that $\delta \notin \Sigma$, and as $R'\Sigma\Pi\Sigma$, we have our result, that not all Π-theories Γ and Δ where $R'\Sigma\Gamma\Delta$ satisfy $\gamma \in \Gamma \Rightarrow \delta \in \Delta$. Contraposing gives us the desired result.

7. **C7** is a stronger version of **C6**, so we need R' in this case too. Assume that **C7** holds, and let Σ, Γ, Δ, Θ and Ξ be Π-theories such that $R'\Sigma\Gamma\Delta$ and $R'\Delta\Theta\Xi$. Define $\Omega = \{\beta : (\exists\alpha)(\alpha \to \beta \in \Sigma) \wedge (\alpha \in \Theta)\}$; this satisfies $R\Sigma\Theta\Omega$ by definition.

If $\Sigma = \Pi$, then $\Theta = \Omega$ and all is well. If $\Theta = \Pi$, then $\Omega = \Sigma$, as is $\beta \in \Sigma$, then $(\beta \to \beta) \to \beta \in \Sigma$ too, by **C7** (Derive **C6** from **C7**, and this is enough.), so as $\beta \to \beta \in \Pi$, $\Sigma \subseteq \Omega$. Conversely, if $\beta \in \Omega$, then $\beta \in \Sigma$, as Σ is a Π-theory. So, we have $R'\Sigma\Theta\Omega$ in any case.

Let $\alpha \to \beta \in \Omega$ and $\alpha \in \Gamma$, so there is a $\gamma \in \Theta$ where $\gamma \to (\alpha \to \beta) \in \Sigma$. This gives $\alpha \to (\gamma \to \beta) \in \Sigma$ by **C7**, and hence $\gamma \to \beta \in \Delta$ as $R\Sigma\Gamma\Delta$. This, with $R\Delta\Theta\Xi$ and $\gamma \in \Theta$ gives $\beta \in \Xi$, and hence

$R\Omega\Gamma\Xi$, as desired. We want $R'\Omega\Gamma\Xi$. If $\Omega = \Pi$ as sets, then take $\Omega = \Pi'$, and so $R'\Omega\Gamma\Xi$. If $\Gamma = \Pi$, then $\Sigma = \Delta$, and as $R'\Delta\Phi\Xi$, we have $R'\Sigma\Phi\Xi$, and we are safe to take Ξ for Ω. In this case $R'\Sigma\Phi\Xi$, as $\Xi \supseteq \Omega$, and $R'\Xi\Gamma\Xi$ as $\Gamma = \Pi$.

A priming lemma gives a prime Π-theory Ω', with the desired properties.

8. Let Σ, Θ, Γ, Δ and Ξ be arbitrary prime Π-theories such that $R\Sigma\Theta\Gamma$ and $R\Gamma\Delta\Xi$. Let $\Psi = \{\beta : (\exists\alpha)(\alpha \to \beta \in \Theta) \wedge (\alpha \in \Delta)\}$ and $\Phi = \{\beta : (\exists\alpha)(\alpha \to \beta \in \Sigma) \wedge (\alpha \in \Delta)\}$. Then it is immediate that $R\Theta\Delta\Psi$ and $R\Sigma\Delta\Phi$. (Even when $\Theta = \Pi$, for in that case $\Psi = \Delta$, and if $\Sigma = \Pi$, $\Theta = \Delta$).

   It remains for us to see that $R\Phi\Psi\Xi$. To see this, let $\alpha \to \beta \in \Phi$ (so there is a $\gamma \in \Delta$ where $\gamma \to (\alpha \to \beta) \in \Sigma$) and $\alpha \in \Psi$ (so there is a $\delta \in \Delta$ where $\delta \to \alpha \in \Theta$). We then see that $\gamma \wedge \delta \in \Delta$, $\gamma \wedge \delta \to (\alpha \to \beta) \in \Sigma$ (by prefixing), and $\gamma \wedge \delta \to \alpha \in \Theta$ (also by prefixing). But **C8** gives $(\delta \wedge \gamma \to \alpha) \to (\delta \wedge \gamma \to \beta) \in \Sigma$, and so $R\Sigma\Theta\Gamma$ ensures that $\delta \wedge \gamma \to \beta \in \Gamma$. This, in turn gives $\beta \in \Xi$, as $R\Gamma\Delta\Xi$. The result follows from an application Lemma 6 to Ψ and Φ.

   If $\Phi' = \Pi$ as sets, then take Φ' to be Π', and the result that $R\Phi'\Psi'\Xi$ is then preserved.

9. Assume **C9**, and let Σ, Γ, Δ, Θ and Ξ be arbitrary Π-theories such that $R\Sigma\Gamma\Xi$ and $R\Xi\Delta\Theta$. Let $\Phi = \{\beta : (\exists\alpha)(\alpha \to \beta \in \Gamma) \wedge (\alpha \in \Delta)\}$ and $\Psi = \{\beta : (\exists\alpha)(\alpha \to \beta \in \Sigma) \wedge (\alpha \in \Delta)\}$, so $R\Sigma\Delta\Psi$ and $R\Gamma\Delta\Phi$ are immediate. (If $\Sigma = \Pi$, $\Psi = \Delta$, and if $\Gamma = \Pi$, $\Phi = \Delta$.)

   We have only to demonstrate that $R\Phi\Psi\Theta$ (as priming lemmas give us the rest of the result). To show this, let $\alpha \to \beta \in \Phi$ (so there is a $\gamma \in \Delta$ where $\gamma \to (\alpha \to \beta) \in \Gamma$) and $\alpha \in \Psi$ (so there is a $\delta \in \Delta$ where $\delta \to \alpha \in \Sigma$). This ensures that $\gamma \wedge \delta \in \Delta$, and that, by prefixing, $\gamma \wedge \delta \to (\alpha \to \beta) \in \Gamma$ and $\gamma \wedge \delta \to \alpha \in \Sigma$. **C9** then gives $(\gamma \wedge \delta \to (\alpha \to \beta)) \to (\gamma \wedge \delta \to \beta) \in \Sigma$, and $R\Sigma\Gamma\Xi$ gives $\gamma \wedge \delta \to \beta \in \Xi$, and $R\Xi\Delta\Theta$ gives $\beta \in \Theta$, as we set out to show. Lemma 6 then completes the proof.

   Again, if $\Phi' = \Pi$ as sets, then set $\Phi' = \Pi'$, and the result that $R\Phi'\Psi'\Theta$ is preserved.

10. Assume that **C10** holds, and let Σ be a prime Π-theory. We wish to show that $R\Sigma\Pi\Sigma$, so let $\alpha \to \beta \in \Sigma$ and $\alpha \in \Pi$. By **C10**, $(\alpha \to \beta) \to \beta \in \Pi$, and $R\Pi\Sigma\Sigma$ gives $\beta \in \Sigma$ as we desired.

This completes the list, and our equivalences have been shown. ■

5. AN ORDERING ON WORLDS

In RLR, more axioms are listed, along with their corresponding restriction on the relation R. As an example, $\beta \to (\alpha \to \beta)$ is shown to 'correspond to' the condition $Rabc \Rightarrow a \leqslant c$. The relation $\leqslant$ on worlds needs some explanation, as we have not introduced it in this paper. Simply put, $a \leqslant b$ if and only if $Rgab$, where g is the base world. (Or in the case of more than one base world, $a \leqslant b \Leftrightarrow Rxab$ for *some* base world x.) In the original semantics, this has the pleasing property of ensuring that if α is true in a, then α is true in b. Its corresponding condition in the canonical model structure is represented by the relation of containment, that is, $\Sigma \leqslant \Delta \Leftrightarrow \Sigma \subseteq \Delta$. Unfortunately, in the simplified semantics such a connection does not exist, for we have $Rgab$ if and only if $a = b$, so the definition of $\leqslant$ collapses into equality. It might be thought that the occurrences of $\leqslant$ in modelling conditions could be replaced by $=$, but this fails in general. For example, the class of simplified interpretations satisfying $Rabc \Rightarrow a = c$ is certainly sound with respect to the axiom $\beta \to (\alpha \to \beta)$, but completeness fails. We would need to show that in the almost-canonical model, $R\Sigma\Gamma\Delta \Rightarrow \Sigma = \Delta$, which, when $\Sigma = \Pi$, ensures that $\Pi = \Delta$ for each Π-theory Δ, and thus there is only one world. The condition on R is too strict, and we need to find another way to model the relation $\leqslant$.

The way to proceed seems to be as follows. We can define a $\leqslant$ as a primitive binary relation on worlds, with conditions that are relatively simple to check practically. Then we can show that this relation has the desired properties (namely that $a \leqslant b \Rightarrow (I(a, \alpha) = 1 \Rightarrow I(b, \alpha) = 1)$ for each formula α), and define an *extended interpretation* to be an interpretation with such an additional binary relation. Then the extra modelling results hold for extended interpretations. This is what we shall do.

Given an interpretation $\langle g, W, R, I\rangle$, a binary relation $\leqslant$ on W satisfying

$$a \leqslant b \Rightarrow \begin{cases} (I(a, p) = 1 \Rightarrow I(b, p) = 1) & \text{for every propositional variable } p \\ Rbcd \Rightarrow Racd & \text{if } a \neq g \\ Rbcd \Rightarrow c \leqslant d & \text{if } a = g \end{cases}$$

is said to be a *containment* relation on $\langle g, W, R, I\rangle$. We can then prove the following result.

THEOREM 10. *Given a containment relation* $\leqslant$ *on* $\langle g, W, R, I\rangle$, $a \leqslant b \Rightarrow (I(a, \alpha) = 1 \Rightarrow I(b, \alpha) = 1)$ *for every formula* α.

Proof. We will prove this by induction on the complexity of formulae. The result holds (for all worlds a and b where $a \leqslant b$) for propositional variables, and the inductive cases for $\wedge$ and $\vee$ are immediate. Now assume that the result holds for α and β, that $a \leqslant b$, and $I(a, \alpha \to \beta) = 1$.

If $a \neq g$ then we have that for all c and d where $Racd$, $I(c, \alpha) = 1 \Rightarrow I(d, \beta) = 1$, and as $Rbcd \Rightarrow Racd$, we have that for all c and d where $Rbcd$, $I(c, \alpha) = 1 \Rightarrow I(d, \beta) = 1$, and hence $I(b, \alpha \to \beta) = 1$.

If $a = g$, then for each c, $I(c, \alpha) = 1 \Rightarrow I(c, \beta) = 1$. We wish to show that $I(b, \alpha \to \beta) = 1$. We have by the condition on $\leqslant$ that $Rbcd \Rightarrow c \leqslant d$, so for each c and d where $Rbcd$, if $I(c, \alpha) = 1$ then $I(c, \beta) = 1$ (as $I(g, \alpha \to \beta) = 1$), which gives $I(d, \beta) = 1$ (as $Rbcd$ gives $c \leqslant d$). Hence $I(b, \alpha \to \beta) = 1$. This completes the proof. ■

We can now use this relation to prove soundness of further extensions of $\mathbf{B}^+$. These are catalogued in the following theorem.

THEOREM 11. *For each row n in the list below, the logic* $\mathbf{B}^+$ *with the axiom (or rule)* **C***n added is sound with respect to the class of extended* $\mathbf{B}^+$ *interpretations* $\langle g, W, R, I, \leqslant\rangle$ *where R satisfies* **D***n*.

C11 $\alpha \to (\beta \to \beta)$

C12 $\beta \to (\alpha \to \beta)$

C13 $\alpha \to (\beta \to (\gamma \to \alpha))$

C14 $\alpha \to (\beta \to \alpha \wedge \beta)$

C15 $(\alpha \wedge \beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$

C16 $(\alpha \to \beta) \vee (\beta \to \alpha)$

C17 $\alpha \to (\alpha \to \alpha)$

C18 $(\alpha \wedge \beta \to \gamma) \to ((\alpha \to \gamma) \vee (\beta \to \gamma))$

D11 $Rabc \Rightarrow b \leqslant c$

D12 $Rabc \Rightarrow a \leqslant c$

D13 $R^2abcd \Rightarrow a \leqslant d$

D14 $Rabc \Rightarrow a \leqslant c$ *and* $b \leqslant c$

D15 $R^2abcd \Rightarrow$ *for some* x
$b \leqslant x$, $c \leqslant x$ *and* $Raxd$

D16 $a \leqslant b$ *or* $b \leqslant a$

D17 $Rabc \Rightarrow a \leqslant c$ *or* $b \leqslant c$

D18 $Rabc$ *and* $Rade \Rightarrow$ *for some* x
$b \leqslant x$, $d \leqslant x$ *and (*$Raxc$ *or* $Raxe$*)*

Proof. We proceed exactly as in the previous collection of soundness results, except for using the fact that $a \leqslant b$ gives $I(a, \alpha) = 1 \Rightarrow I(b, \alpha) = 1$ for any formula α.

11. Assume that R satisfies $Rabc \Rightarrow b \leqslant c$ for each $a, b, c \in W$. Then,

$$I(g, \alpha \to (\beta \to \beta)) = 0$$

if and only if there is a w (perhaps g itself) where $I(w, \alpha) = 1$ and $I(w, \beta \to \beta) = 0$. So, there are x, y where $Rwxy$, $I(x, \beta) = 1$ and $I(y, \beta) = 0$. However by assumption, $x \leqslant y$, which is impossible. So **C11** holds at g.

12. Assume that R satisfies $Rabc \Rightarrow a \leqslant c$. Then,

$$I(g, \beta \to (\alpha \to \beta)) = 0$$

if and only if there is a w (perhaps g — we will take this as read from now on) where $I(w, \beta) = 1$ and $I(w, \alpha \to \beta) = 0$. So, there

are x, y (again, perhaps g) where $Rwxy$ (so, if $w = g$, this means that $x = y$), $I(x, \alpha) = 1$ and $I(y, \beta) = 0$. However by assumption, $w \leqslant y$, which is impossible. So **C12** holds at g.

13. Assume that $R^2abcd \Rightarrow a \leqslant d$. Then,

$$I(g, \alpha \to (\beta \to (\gamma \to \alpha))) = 0$$

if and only if there is a w where $I(w, \alpha) = 1$ and $I(w, \beta \to (\gamma \to \alpha)) = 0$, which ensures that there are x, y where $Rwxy$, $I(x, \beta) = 1$ and $I(y, \gamma \to \alpha) = 0$. This then gives us z and t where $Ryzt$, $I(z, \gamma) = 1$ and $I(t, \alpha) = 0$. However, R^2wxzt gives $w \leqslant t$, which means that $I(t, \alpha) = 1$, contradicting this result. Hence **C13** holds.

14. **C14** is an immediate corollary of **C11** and **C12**, so soundness follows from the results for those axioms.

15. Assume that $R^2abcd \Rightarrow$ for some x, $b \leqslant x$, $c \leqslant x$ and $Raxd$. Then,

$$I(g, (\alpha \wedge \beta \to \gamma) \to (\alpha \to (\beta \to \gamma))) = 0$$

ensures that there is a w where $I(w, \alpha \wedge \beta \to \gamma) = 1$ and $I(w, \alpha \to (\beta \to \gamma)) = 0$. This gives us x, y where $Rwxy$, $I(x, \alpha) = 1$ and $I(y, \beta \to \gamma) = 0$, which in turn means that there are z, t where $Ryzt$, $I(z, \beta) = 1$ and $I(t, \gamma) = 0$. Because of this we have that R^2wxzt and hence there is a v where $x \leqslant v$, $z \leqslant v$ and $Rwvt$. Hence, $I(v, \alpha) = I(v, \beta) = 1$, and as $Rwvt$, $I(t, \gamma) = 1$, contradicting what we have seen. So, **C15** holds.

16. Assume that for each a and b either $a \leqslant b$ or $b \leqslant a$. Then

$$I(g, (\alpha \to \beta) \vee (\beta \to \alpha)) = 0$$

ensures that $I(g, \alpha \to \beta) = 0$ and $I(g, \beta \to \alpha) = 0$, giving us w and w' where $I(w, \alpha) = I(w', \beta) = 1$ and $I(w', \alpha) = I(w, \beta) = 0$. But this contradicts $w \leqslant w'$ or $w' \leqslant w$, hence our result.

17. Assume that $Rabc \Rightarrow a \leqslant c$ or $b \leqslant c$. Then

$$I(g, \alpha \to (\alpha \to \alpha)) = 0$$

ensures that for some w, $I(w, \alpha) = 1$ and $I(w, \alpha \to \alpha) = 0$. Hence there are x, y where $Rwxy$, $I(x, \alpha) = 1$ and $I(y, \alpha) = 0$. But by hypothesis, either $w \leqslant y$ or $x \leqslant y$, contradicting what we have just seen and giving us our result.

18. Assume that $Rabc$ and $Rab'c' \Rightarrow$ for some x, $b \leqslant x$, $b' \leqslant x$ and $Raxc$ or $Raxc'$. Then

$$I(g, (\alpha \wedge \beta \to \gamma) \to ((\alpha \to \gamma) \vee (\beta \to \gamma))) = 0$$

means that for some w, $I(w, \alpha \wedge \beta \to \gamma) = 1$ and $I(w, (\alpha \to \gamma) \vee (\beta \to \gamma)) = 0$. So, there are x, y and x', y' where $Rwxy$, $I(x, \alpha) = 1$, $I(y, \gamma) = 0$ and $Rwx'y'$, $I(x', \beta) = 1$ and $I(y', \gamma) = 0$. By hypothesis, there is a z where x, $x' \leqslant z$, so $I(z, \alpha \wedge \beta) = 1$, and either $Rwzy$ or $Rwzy'$. But these ensure that either $I(y, \gamma) = 1$ or $I(y', \gamma) = 1$, both contradicting what we have seen. This ensures that **C18** holds. ■

For the completeness proof we need a containment relation in the canonical models. Thankfully the obvious candidate works.

THEOREM 12. *In the canonical model, and in the almost canonical model, $\subseteq$ is a containment relation.*

Proof. That $\Sigma \subseteq \Gamma \Rightarrow (p \in \Sigma \Rightarrow p \in \Gamma)$ is immediate. If $\Sigma \subseteq \Gamma$ and $\Sigma \neq \Pi$ then $R\Gamma\Delta\Phi \Rightarrow R\Sigma\Delta\Phi$ by the definition of R, and if $R\Gamma\Delta\Phi$, and $\Pi \subseteq \Gamma$, then each formula $\alpha \to \alpha \in \Gamma$, and hence $\Delta \subseteq \Phi$. ■

One further result we need is that for certain extensions of $\mathbf{B}^+$, we can do without the empty and full Π-theories, and still have an interpretation. (The collection of *all* formulae is the full Π-theory.) These two rather excessive theories are appropriately called degenerate theories, and this result is called a *non-degeneracy theorem.*

THEOREM 13. *Provided that $\alpha \to \alpha \in \Sigma$ for each formula α and each non-empty prime Π-theory Σ, then the canonical (or almost-canonical) interpretation, which is limited to non-degenerate prime Π-theories is an interpretation of $\mathbf{B}^+$.*

Proof. To show that this structure *is* an interpretation, it is sufficient to show that the assignment $I(\Sigma, \alpha) = 1$ iff $\alpha \in \Sigma$ satisfies the inductive characterisation of an interpretation. Because the structure is a *reduction* of the earlier structure, inductive cases are exactly the same, except for showing that when $\alpha \to \beta \notin \Sigma$ (for non-degenerate Σ), there are non-degenerate prime Γ and Δ where $R\Sigma\Gamma\Delta$, $\alpha \in \Gamma$ and

$\beta \notin \Delta$. To this end, define $\Gamma' = \{\gamma : \vdash_\Pi \alpha \to \gamma\}$ and $\Delta' = \{\delta : (\exists \gamma)(\gamma \in \Gamma' \ \& \ \gamma \to \delta \in \Sigma)\}$. We will show that $\alpha \in \Gamma' \cap \Delta'$ and $\beta \notin \Gamma' \cup \Delta'$, so that these theories are non-degenerate.

First note that $\nvdash_\Pi \alpha \to \beta$. For otherwise we have $\vdash_\Pi (\alpha \to \alpha) \to (\alpha \to \beta)$ by prefixing, and $\alpha \to \alpha \in \Sigma$ gives $\alpha \to \beta \in \Sigma$, which we know does not obtain. So it follows that $\beta \notin \Gamma'$. That $\alpha \to \alpha \in \Sigma$ and $\alpha \in \Gamma'$ gives $\alpha \in \Delta'$, as we desired. Noting that $\beta \notin \Delta'$ completes the first part of the result — Γ' and Δ' are non-degenerate.

We only need to find non-degenerate prime Γ and Δ to complete the theorem. This is done by appling Lemma 7 — we need just show that the Γ and Δ so obtained are non-degenerate. As $\Gamma' \subseteq \Gamma$, Γ is non-empty. To see that $\alpha \notin \Gamma$, note that in the proof Γ is disjoint with Θ_2, and as $\alpha \to \alpha \in \Sigma$, $\alpha \in \Theta_2$, giving $\alpha \notin \Gamma$. The result of the lemma ensures that $\beta \notin \Delta$ and that $\Delta' \subseteq \Delta$, so Δ is also non-degenerate.

If $\Sigma = \Pi$, then $\Delta' = \Gamma'$, and as noting that $R\Sigma\Delta\Delta$ (where Δ was constructed by Lemma 7), $\alpha \in \Delta$ and $\beta \notin \Delta$ is sufficient to complete the proof.

We now give some example conditions which enable us to use non-degenerate models.

THEOREM 14. *Conditions* **C11**, **C12** *and* **C13** *ensure that* $\alpha \to \alpha \in \Sigma$ *for each non-empty prime* Π*-theory* Σ.

Proof. **C11** is obvious. For **C12**, note that $\vdash_\Pi (\alpha \to \alpha) \to (\beta \to (\alpha \to \alpha))$ is an instance of **C12**, and hence $\vdash_\Pi \beta \to (\alpha \to \alpha)$. For **C13**, note that $\vdash_\Pi (\alpha \to \alpha) \to ((\alpha \to \alpha) \to (\beta \to (\alpha \to \alpha)))$ is an instance of **C13**, and hence $\vdash_\Pi \beta \to (\alpha \to \alpha)$.

This gives us enough machinery to prove completeness for the rest of the positive extensions of $\mathbf{B}^+$. They are of the same form as the other completeness proofs, except that they use the fact that $\subseteq$ is a containment relation in the canonical model and in the almost canonical model.

THEOREM 15. *For each row n in the list given in Theorem 11, the logic* $\mathbf{B}^+$ *with the axiom (or rule)* **C***n added is complete with respect to the class of extended* $\mathbf{B}^+$ *interpretations* $\langle g, W, R, I, \leqslant \rangle$ *where R satisfies condition* **D***n*.

Proof. We take these individually as before, using the almost canonical model:

11. Assume that **C11** holds. We can use the non-degenerate model, by Theorem 13. Assume also that Σ, Γ, Δ are non-degenerate Π-theories satisfying $R\Sigma\Gamma\Delta$. We wish to show that $\Gamma \subseteq \Delta$. This is immediate for the case $\Sigma = \Pi$, and otherwise, note that for some α, $\alpha \in \Sigma$, and hence $\beta \to \beta \in \Sigma$ for each β, $\beta \in \Gamma$ gives $\beta \in \Delta$, as we desired.
12. Assume that **C12** holds. To see that we can use the non-degenerate model, note that $\vdash_\Pi (\gamma \to \gamma) \to (\delta \to (\gamma \to \gamma))$, as this is an instance of **C12**, and so $\vdash_\Pi \delta \to (\gamma \to \gamma)$. Assume also that $R\Sigma\Gamma\Delta$. Take some $\alpha \in \Gamma$ and some $\beta \in \Sigma$, then **C12** gives $\alpha \to \beta \in \Sigma$ and hence $\beta \in \Delta$. This means that $\Sigma \subseteq \Delta$ as desired. The result holds, even if $\Sigma = \Pi$.
13. Assume **C13**. We can use the non-degenerate model, as $\vdash_\Pi (\gamma \to \gamma) \to ((\theta \to \theta) \to (\delta \to (\gamma \to \gamma)))$ is an instance of **C13**, so we have $\vdash_\Pi \delta \to (\gamma \to \gamma)$. Then take $R\Sigma\Gamma\Delta$ and $R\Delta\Theta\Xi$. Take $\alpha \in \Sigma$, $\beta \in \Gamma$ and $\gamma \in \Theta$. **C13** ensures that $\beta \to (\gamma \to \alpha) \in \Sigma$, and hence $\alpha \in \Xi$, as desired (even if $\Sigma = \Pi$).
14. This is a combination of **C11** and **C12**.
15. Assume **C15**, and that $R\Sigma\Gamma\Delta$ and $R\Delta\Theta\Xi$. We wish to find a prime Π-theory Φ' where $R\Sigma\Phi'\Xi$ and both $\Gamma, \Theta \subseteq \Phi'$. To this end, set $\Phi = \{\alpha: (\exists \gamma \in \Gamma, \delta \in \Theta) \vdash_\Pi \gamma \wedge \delta \to \alpha\} \cup \Gamma \cup \Theta$. It is clear that $\Gamma, \Theta \subseteq \Phi$, and to show that Φ is a Π-theory, note that if Γ and Θ are both non-empty, then $\Phi = \{\alpha: (\exists \gamma \in \Gamma, \delta \in \Theta) \vdash_\Pi \gamma \wedge \delta \to \alpha\}$, and this is clearly a Π-theory, as Γ, Θ are both Π-theories, and is transitive (in that if $\vdash_\Pi \alpha \to \beta$ and $\vdash_\Pi \beta \to \gamma$ then $\vdash_\Pi \alpha \to \gamma$). So if both Θ and Γ are non-empty, Φ is a Π-theory. Otherwise (if one of Θ and Γ are empty), Φ is the union of Θ and Γ, which is then also a Π-theory.
    To show that $R\Sigma\Phi\Xi$, let $\alpha \to \beta \in \Sigma$ and $\alpha \in \Phi$. By definition, there are $\gamma \in \Gamma$ and $\delta \in \Theta$ where $\vdash_\Pi \gamma \wedge \delta \to \alpha$. Hence, $\vdash_\Pi (\alpha \to \beta) \to (\gamma \wedge \delta \to \beta)$ and so $\gamma \wedge \delta \to \beta \in \Sigma$, which by **C15** gives $\gamma \to (\delta \to \beta) \in \Sigma$. $R\Sigma\Gamma\Delta$ and $R\Delta\Theta\Xi$ then give us $\beta \in \Xi$ as desired. The case for $\Sigma = \Pi$ is given below.
    A priming lemma then completes the proof except for the case where $\Sigma = \Pi$. In that case, instead of using Φ, we simply need to

show that $\Gamma, \Theta \subseteq \Xi$. Note that $\Sigma = \Pi$ gives $\Gamma = \Delta$, and hence we have $R\Gamma\Theta\Xi$. For this it is sufficient to note that as $\vdash_\Pi (\alpha \wedge \beta \to \alpha) \to (\alpha \to (\beta \to \alpha))$ and $\vdash_\Pi (\alpha \wedge \beta \to \beta) \to (\alpha \to (\beta \to \beta))$, **C15** gives both **C11** and **C12**, which in turn ensures that $R\Gamma\Theta\Xi$ gives $\Gamma, \Theta \subseteq \Xi$ by our results from before.

16. Assume **C16** and take Π-theories Σ and Γ where $\Sigma \not\subseteq \Gamma$. Hence there is some β where $\beta \in \Sigma$ and $\beta \notin \Gamma$. Given $\alpha \in \Gamma$, it is sufficient to show that $\alpha \in \Sigma$. As $\alpha \to \beta \notin \Pi$ (since $\alpha \in \Gamma$ and $\beta \notin \Gamma$), and $(\alpha \to \beta) \vee (\beta \to \alpha) \in \Pi$ we must have $\beta \to \alpha \in \Pi$, which with $\beta \in \Sigma$ gives us our result.
17. Assume **C17**, and that $R\Sigma\Gamma\Delta$. We wish to show that either $\Sigma \subseteq \Delta$ or $\Gamma \subseteq \Delta$. To show that this obtains, take $\alpha \in \Sigma$ where $\alpha \notin \Delta$ and $\beta \in \Gamma$ where $\beta \notin \Delta$. then $\alpha \vee \beta \notin \Delta$, but $\alpha \vee \beta \in \Sigma, \Gamma$. **C17** gives $\alpha \vee \beta \to \alpha \vee \beta \in \Sigma$, and $R\Sigma\Gamma\Delta$ gives us $\alpha \vee \beta \in \Delta$. Hence our result.
18, Assume **C18**, and that $R\Sigma\Gamma\Delta$ and $R\Sigma\Gamma'\Delta'$. We wish to find a prime Π-theory Φ' where $\Gamma, \Gamma' \subseteq \Phi'$, and either $R\Sigma\Phi'\Delta$ or $R\Sigma\Phi'\Delta'$. The proof in RLR recommends that to this end we define a set $\Phi = \{\alpha : (\exists \gamma \in \Gamma, \gamma' \in \Gamma') \vdash_\Pi \gamma \wedge \gamma' \to \alpha\}$ and show that either $R\Sigma\Phi\Delta$ or $R\Sigma\Phi\Delta'$. To do this you take $\alpha \to \beta \in \Sigma$ and $\alpha \in \Phi$. Then $\vdash_\Pi \gamma \wedge \gamma' \to \alpha$ for some γ, γ' in Γ, Γ' respectively. So as $\vdash_\Pi (\alpha \to \beta) \to (\gamma \wedge \gamma' \to \beta)$ we see that $\gamma \wedge \gamma' \to \beta \in \Sigma$, and hence $(\gamma \to \beta) \vee (\gamma' \to \beta) \in \Sigma$, giving either $\gamma \to \beta$ or $\gamma' \to \beta$ in Σ. $R\Sigma\Gamma\Delta$ and $R\Sigma\Gamma'\Delta'$ then gives either $\beta \in \Delta$ or $\beta \in \Delta'$. And the text leaves us there. The astute will note that this is *not* enough to give us the result, as for a range of values β, there is nothing to ensure that they land in the same place. Some might end up in Δ, and some in Δ'. All we have shown is that $R\Sigma\Theta'(\Delta \cup \Delta')$. Fortunately, all is not lost, as **C18** gives **C16**, as $\vdash_\Pi (\alpha \wedge \beta \to \alpha \wedge \beta) \to (\alpha \to \alpha \wedge \beta) \vee (\beta \to \alpha \wedge \beta)$, so $\vdash_\Pi (\alpha \to \alpha \wedge \beta) \vee (\beta \to \alpha \wedge \beta)$ which easily yields $\vdash_\Pi (\alpha \to \beta) \vee (\beta \to \alpha)$, as we wished.) So, by the proof for **C16**, it follows that either $\Delta' \subseteq \Delta$, or $\Delta \subseteq \Delta'$, so $\Delta \cup \Delta'$ is one of them, giving the result. For those who prefer a smoother proof, abandon all thoughts of Φ, and take the larger of Γ and Γ' as our required prime Π-theory. The result follows immediately. ■

In RLR, there are a few more extensions that are considered – such as $\alpha \vee (\alpha \to \beta)$ – these seem to require the non-degenerate model structure to push through the completeness proofs, but it seems that Theorem 14 cannot be proved for these extensions, despite what is said in RLR. On p. 314 non-degeneracy is assumed for this axiom, but on p. 317, it is only shown to work for axioms like our **C11**.[3] Despite this setback, it is possible to extend the structure of an interpretation yet again, by adding an explicit empty world e, satisfying certain obvious conditions. Then a phrase like $a \neq e$ is used in a modelling condition whenever it is needed that a be non-empty. The details of this approach can be found on p. 380 of RLR, and the interested reader is referred there. We will extend the semantics to deal with a more pressing need, and that is to add negation.

6. ADDING NEGATION

The addition of negation to the story complicates things somewhat. In SS it is shown that there are (at least) two different ways of expanding the simplified semantics to deal with negation. In this section we will show that the semantics using the Routley '*' operation can model common negation extensions of **B**. In SS, a four-valued interpretation was also used to model negation, and that can also be done for extensions of **B**. However, that opens up a whole range of other issues, which will be covered in a forthcoming paper. Here we will deal solely with the '*' modelling of negation.

6.1. *The Systems* **BM** *and* **B**, *with* '*'

One logic extending $\mathbf{B}^+$ by adding negation is **BM**, which is obtained from $\mathbf{B}^+$ by adding the rule:

$$\mathbf{R4}\ \frac{\alpha \to \beta}{\neg\beta \to \neg\alpha},$$

along with the De Morgan laws

$$\mathbf{A7}\ \neg(\alpha \vee \beta) \leftrightarrow \neg\alpha \wedge \neg\beta,$$

$$\mathbf{A8}\ \neg(\alpha \wedge \beta) \leftrightarrow \neg\alpha \vee \neg\beta.$$

In SS it is shown that if we extend interpretations to contain a function *: $W \to W$, and define the truth conditions for negation as:

$$1 = I(w, \neg\alpha) \Leftrightarrow 1 \neq I(w^*, \alpha)$$

the logic **BM** is sound with respect to these conditions. To show completeness, define * on the set of prime Π-theories by:

$$\Sigma^* = \{\alpha : \neg\alpha \notin \Sigma\}.$$

This is shown to send prime Π-theories to prime Π-theories, and to give the desired results. The details of the completeness proof are not difficult, and the interested reader is referred to SS for the details.

The system **B** can be obtained from **BM** by adding the axiom:

A9 $\alpha \leftrightarrow \neg\neg\alpha$.

(Or alternatively, add to $\mathbf{B}^+$ $\neg\neg\alpha \to \alpha$ and the rule from $\alpha \to \neg\beta$ to $\beta \to \neg\alpha$.) To obtain semantics for **B** we simply require that * satisfy $w^{**} = w$ in each interpretation. Soundness and completeness is simple to show. The only other construction we need to consider is the containment relation $\leqslant$ on worlds. It no longer follows that containment relations as they stand satisfy the condition $a \leqslant b \Rightarrow (I(a, \alpha) = 1 \Rightarrow I(b, \alpha) = 1)$, for another condition must be added to deal with negation. This is dealt with in the following theorem.

THEOREM 16. *Let $\langle g, W, R, I, * \rangle$ be an interpretation, and $\leqslant$ a binary relation on W satisfying:*

$$a \leqslant b \Rightarrow \begin{cases} (I(a, p) = 1 \Rightarrow I(b, p) = 1) & \text{for every propositional variable } p \\ Rbcd \Rightarrow Racd & \text{if } a \neq g \\ Rbcd \Rightarrow c \leqslant d & \text{if } a = g \\ b^* \leqslant a^* & \end{cases}$$

Then $a \leqslant b \Rightarrow (I(a, \alpha) = 1 \Rightarrow I(b, \alpha) = 1)$ for every formula a. Any relation satisfying these conditions is said to be a containment relation.

Proof. We add a clause for $\neg$ to the induction on the complexity of formulae. If $a \leqslant b$ and the result holds for α, then if $I(a, \neg\alpha) = 1$ it follows that $I(a^*, \alpha) = 0$, and as $b^* \leqslant a^*$ it must be that $I(b^*, \alpha) = 0$ and hence that $I(b, \neg\alpha) = 1$ as desired. ∎

6.2. *Extensions of* **B**

The extension results in the previus sections carry over to the logic **B** with no modification. What we are interested in is the possibility of extending **B** with axioms or rules that use negations. This can be done, as the following theorem shows.

THEOREM 17. *For each row n in the list below, the logic* **B** *with the axiom* **C***n added is sound and complete with respect to the class of* **B** *interpretations* $\langle g, W, R, I, {}^* \rangle$ *where R satisfies condition* **D***n, and for the last axiom, the interpretations are assumed to be extended with a containment relation* $\leqslant$.

C19 $(\alpha \to \neg\alpha) \to \neg\alpha$

C20 $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$

C21 $\alpha \vee \neg\alpha$

D19 Raa^*a *for* $a \neq g$, *and* $g^* \leqslant g$

D20 $Rabc \Rightarrow Rac^*b^*$

D21 $g^* \leqslant g$

Proof. These are proved in exactly the same way as the other extensions.

19. Assume that Raa^*a and that

$$I(g, (\alpha \to \neg\alpha) \to \neg\alpha) = 0.$$

Hence there must be a w where $I(w, \alpha \to \neg\alpha) = 1$ and that $I(w, \neg\alpha) = 0$, so $I(w^*, \alpha) = 1$. If $w \neq g$, Rww^*w then gives $I(w, \neg\alpha) = 1$, contradicting what we have seen. If $w = g$, $g^* \leqslant g$ gives $I(g, \alpha) = 1$, and hence $I(g, \neg\alpha) = 1$, contradicting what we have seen. So **C19** must hold.

Now assume that **C19** holds and that Σ is a prime Π-theory, distinct from Π. We wish to show that $R\Sigma\Sigma^*\Sigma$, so let $\alpha \to \beta \in \Sigma$ and $\alpha \in \Sigma^*$. The thing to note is that if **C19** holds, so must $(\alpha \to \beta) \to (\neg\alpha \vee \beta)$. To see this, consider the following derivation:

$\vdash_\Pi \alpha \wedge \neg\beta \to \alpha$, by **A3**,

$\vdash_\Pi \beta \to \neg\alpha \vee \beta$, by **A2**,

$\vdash_\Pi \neg\alpha \vee \beta \to \neg(\alpha \wedge \neg\beta)$, by **A8** and **A9**,

$\vdash_\Pi \beta \to \neg(\alpha \wedge \neg\beta)$, by transitivity,

$\vdash_\Pi (\alpha \to \beta) \to ((\alpha \wedge \neg\beta) \to$

$\to \neg(\alpha \wedge \neg\beta))$, by **R3**,

$\vdash_\Pi ((\alpha \wedge \neg\beta) \to \neg(\alpha \wedge \neg\beta)) \to$

$\to \neg(\alpha \wedge \neg\beta)$, by **C19**,

$\vdash_\Pi (\alpha \to \beta) \to \neg(\alpha \wedge \neg\beta)$, by transitivity,

$\vdash_\Pi \neg(\alpha \vee \neg\beta) \to (\neg\alpha \vee \beta)$, by **A8**, **A9**,

$\vdash_\Pi (\alpha \to \beta) \to (\neg\alpha \vee \beta)$, by transitivity.

So we have $\neg\alpha \vee \beta \in \Sigma$, and $\neg\alpha \notin \Sigma$, giving $\beta \in \Sigma$, as we wanted.

To show that $\Pi^* \subseteq \Pi$, note that $\alpha \vee \neg\alpha$ is a theorem. Take $\alpha \in \Pi^*$, so $\neg\alpha \notin \Pi$. But $\alpha \vee \neg\alpha \in \Pi$, so $\alpha \in \Pi$ as desired.

20. Assume that $Rabc \Rightarrow Rac^*b^*$ and that

$$I(g, (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)) = 0.$$

There must be a w where $I(w, \alpha \to \beta) = 1$ and $I(w, \neg\beta \to \neg\alpha) = 0$, which in turn gives x, y where $Rwxy$, $I(x, \neg\beta) = 1$ and $I(y, \neg\alpha) = 0$. We must have Rwy^*x^*, by assumption, and hence, as $I(y^*, \alpha) = 1$ we must have $I(x^*, \beta) = 1$. This gives $I(x, \neg\beta) = 0$, which contradicts what we have seen. Thus **C20** must hold.

Now assume that **C20** holds and that Σ, Γ and Δ are prime Π-theories such that $R\Sigma\Gamma\Delta$. Let $\alpha \to \beta \in \Sigma$ and $\alpha \in \Delta^*$, i.e., $\neg\alpha \notin \Delta$. By **C20** we must have $\neg\beta \to \neg\alpha \in \Sigma$, so we must have $\neg\beta \notin \Gamma$, lest $\neg\alpha \in \Delta$. This then gives $\beta \in \Gamma^*$, which ensures that $R\Sigma\Delta^*\Gamma^*$. (If $\Sigma = \Pi$, the result is even easier to prove.)

21. Assume that $g^* \leqslant g$ and that

$$I(g, \alpha \vee \neg\alpha) = 0.$$

It follows that $I(g, \alpha) = I(g, \neg\alpha) = 0$, and hence that $I(g^*, \alpha) = 1$. This contradicts $g^* \leqslant g$, so **C21** holds.

Now assume that **C21** holds. We want to show that $\Pi^* \subseteq \Pi$ – to this end, note that if $\alpha \in \Pi^*$, $\neg\alpha \notin \Pi$ and so $\alpha \vee \neg\alpha \in \Pi$ ensures that $\alpha \in \Pi$ as desired. ■

7. THE LOGICS WE HAVE COVERED

It is time to take stock and consider what logics have a semantics as the result of these investigations. It is clear that we have covered $\mathbf{B}^+$, **BM** and **B**, but with the addition of any logic that can be obtained by adding various axioms amongst **C1–21**. We enumerate some of those covered; the details are taken from RLR and other sources. (It must be understood that these logics are all disjunctive systems.)

DW = **B** + **C20**; this is the basic logic covered in Slaney's 'A General Logic'. **DJ** = **DW** + **C2**, **G** = **B** + **C21**; **G** is the weakest of the affixing systems that includes each of the classical tautologies as theorems. **DK** = **G** + {**C2**, **C20**}, **DL** = **DK** + **C19**, **TW** = **B** + {**C3** + **C4** + **C20**}, **C**(or **RW**) = **TW** + **C6**, **CK**(or **RWK**) = **C** + **C12**, **T** = **TW** + **C5** + **19**, **R** = **T** + **C6** = **C** + **C5**, $\mathbf{EW}^d$ = **TW** + **C10**, and $\mathbf{E}^d$ = **T** + **C10**.

8. BOOLEAN NEGATION

As a formal construction, it is possible to add to these logics a 'negation' commonly called "Boolean Negation", which we will write as '$-$'. It is characterised by the following axioms. (See Giambrone and Meyer's 'Completeness and Conservative Extension Results for some Boolean Relevant Logics' for this characterisation.)

BA1 $\alpha \to (\beta \to \gamma \vee -\gamma)$

BA2 $-(\alpha \to \beta) \vee (-\alpha \vee \beta)$

BA3 $\alpha \wedge -\alpha \to \beta$

If a logic **L** is without Boolean negation, the logic resulting from adding such a negation is called '**CL**'. It is well-known that Boolean negation satisfies $\vdash - -\alpha \leftrightarrow \alpha$, $\vdash \alpha \wedge -\alpha \to \beta$ and $\vdash \alpha \to \beta \vee -\beta$, and I will not prove that here. To model Boolean negation in the simplified semantics, we add the obvious condition that:

$$I(w, -\alpha) = 1 \text{ if and only if } I(w, \alpha) = 0$$

It is trivial to show that the semantics for **L** with this extension is sound and complete for **CL**, using well-known properties of Boolean negation. However, this gives us a conservative extension result, which is a corollary of the following lemma.

LEMMA 18. *Given any* **BM** *or* $\mathbf{B}^+$ *interpretation, not using a containment relation, the structure given by adding the rule for Boolean negation' has exactly the same evaluation as the original on formulae that do not contain '$-$'.*

Proof. By inspection. The reason a containment relation is not permitted is that the hereditariness condition on the relation fails in general, given the presence of '$-$'. (See Note 1.) ■

THEOREM 19. *If* **L** *is a logic which has a sound and complete simplified semantics, not using a containment relation, then* **CL** *is a conservative extension of* **L**.

Proof. This is a simple corollary of the lemma. ■

It follows that **CR**, **CC**, **CTW**, **CDJ** and **CDW** are conservative extensions of **R**, **C**, **TW**, **DJ** and **DW** respectively – and other less known logics are also conservatively extended. The results for **R**, **C** and **TW** were known, but those for **DW** and **DJ** are new.

Other logics such as **CDL**, **CDK** and **CCK** are not proved to conservatively extend **DL**, **DK** and **CD** – as their semantics use the inclusion relation. For **CK**, there is a good reason why the extension result cannot be proved.

THEOREM 20. **CCK** *is not a conservative extension of* **CK**.

Proof. Firstly, $\alpha \vee \neg\alpha$ is not a theorem of **CK**. We will show that it is a theorem of **CCK**. To do this, note that $\alpha \vee -\alpha$ holds in **CCK**, so it is enough to show that $-\alpha \to \neg\alpha$.

In **CK**, it is simple to show that $\neg\alpha \leftrightarrow (\alpha \to \bot)$ for some contradiction '$\bot$'. For example, $\neg\alpha \leftrightarrow (\alpha \to \neg(\alpha \to \alpha))$ holds in **CK**, and hence in **CCK**. Now we have $\alpha \to (-\alpha \to \alpha \wedge -\alpha)$ and $\alpha \wedge -\alpha \to \neg(-\alpha \to -\alpha)$, so we have $\alpha \to (-\alpha \to \neg(-\alpha \to -\alpha))$. But this gives $\alpha \to \neg -\alpha$, which contraposed is $-\alpha \to \neg\alpha$, as desired.

Whether or not **DL** and **DK** *fail* to be conservatively extended by Boolean negation is another story, and it is one that will not be answered here, but rather in a subsequent paper that deals with four-valued semantics for these systems.[4]

NOTES

[1] It should be noted that the provability relation '$\vdash$' used here is distinct from the '$\vdash$' that appears in other sections of the relevant logic literature. In our case, $\Theta \vdash \alpha$ iff there is a proof of α that uses premises from among the elements of Θ. In 'A General Logic' by Slaney, for example, $\Theta \vdash \alpha$ iff there is a proof of $\beta_1 \wedge \ldots \wedge \beta_n \rightarrow \alpha$ for some $\beta_i \in \Theta$. These notions are distinct. In the notion Slaney uses, it turns out that $\Theta \vdash \alpha$ iff for every theory in which the elements of Θ are true, so is α. In our notion, the theories in question are restricted to those that are *regular* (or *detached* – meaning that if $\alpha \rightarrow \beta$ and α are in the theory, so is β) and *normal* (containing all the theorems).

[2] If Boolean negation is present, it can be used to *show* that $R\Pi\Gamma\Delta$ if and only if $\Gamma = \Delta$ (given that Γ is non-empty and Δ is not full). It is quite simple to do: Boolean negation (written as '$-$') satisfies $\vdash \alpha \rightarrow \beta \vee -\beta$ and $\vdash \alpha \wedge -\alpha \rightarrow \beta$. It follows that for any non-empty, non-full Π-theories Γ, $\alpha \vee -\alpha \in \Gamma$, and $\alpha \wedge -\alpha \notin \Gamma$, so exactly *one* of α and $-\alpha$ are in Γ. If $R\Pi\Gamma\Delta$, then it is clear that $\Gamma \subseteq \Delta$ (as Π contains all identities), and so, as Γ is non-empty, it contains exactly one element of each $\{\alpha, -\alpha\}$ pair, for each α. So, Δ contains at least this element for each pair. However, it cannot contain *both*, for any pair (being non-full), so Δ contains exactly the same elements of each pair as does Γ. Hence, $\Gamma = \Delta$. The other direction of the biconditional is obvious, given that Γ and Δ are Π-theories.

The behaviour of Boolean negation is important, when we come to the last section, where we show that Boolean negation conservatively extends a large class of logics.

[3] For completeness' sake, the candidates given in RLR that seem to need non-degeneracy, but for which the current results will not hold (and the proofs in RLR do not seem to work) are $\alpha \vee (\alpha \rightarrow \beta)$, $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta \wedge \gamma))$ and $\alpha \vee \beta \rightarrow ((\alpha \rightarrow \beta) \rightarrow \beta)$.

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