TRUTH VALUES AND PROOF THEORY

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MAY 1, 2009
version 0.995

Abstract: I present an account of truth values for classical logic, intuitionistic logic, and the modal logic S5, in which truth values are not a fundamental category from which the logic is defined, but rather, an *idealisation* of more fundamental logical features in the proof theory for each system. The result is not a new set of semantic structures, but a new understanding of how the existing semantic structures may be understood in terms of a more fundamental notion of logical consequence.

My concern in this paper is threefold: (1) to examine the different structures of truth values appropriate to different accounts of logical consequence — in particular, classical logic, intuitionistic logic, and modal logic; (2) to examine the role of truth functionality — or its absence — in each of these different logical systems; and finally, (3) to examine one way we can give an account of properties of truth values from a standpoint where we do not take them as *given* but rather, in which we define them from more basic notions.

1 MOTIVATION

In this paper, the perspective is *inferentialist*. I will take the notions of logical consequence to be given in a system of *proof* — to make matters concrete, we will use sequent systems. The approach is *inferentialist* because we take the core notions of our logical theory to not be defined in terms of truth values or

*This paper is a draft, and comments from readers are very welcome. Please check the webpage `http://consequently.org/writing/tvpt` for the latest version of the paper, to post comments and to read comments left by others. ¶ Thanks to the Logic Seminar at the University of Melbourne for comments on an earlier presentation of the paper, to Shawn Standefer for helpful discussions on a draft of the paper, and to two anonymous referees of the journal for many helpful questions and comments. ¶ This research is supported by the Australian Research Council, through grant DP0343388, and the Penguin Café Orchestra.

truth preservation, but by some other means, involving inference.[1] Given this starting point, we will *define* a notion of truth appropriate to each account of logical consequence, and examine its behaviour. The result will not be a new or surprising theory of truth values: the resulting algebraic structures will be quite familiar. The novelty, such as it is, is to be found in the way these structures are obtained, the new perspective on completeness proofs that is thereby provided, and the manner in which the formal results explained here may be used at the service of inferentialist theories of *semantics*.

Everyone — whether inferentialist or not — can agree that logic and truth are intimately connected. Logical consequence and the structure of truth values go hand in hand. There are two aspects to this connection. First, there is the notion of truth as the *point* of logical consequence. Of course, one way to define logical consequence by way of preservation of truth: the valid arguments are those whose conclusions are true whenever the premises are true. This way lies truth tables, model theory, many-valued algebras, semantic structures, etc. However, not all logic is model theory, and as we have seen, not all accounts of consequence are truth-first. If we don't define validity in terms of truth but by means of proof, this does not sever the connection between validity and truth. It still may remain that one important feature of valid arguments is that they preserve truth. Any argument with true premises and an untrue conclusion is not valid. This is to be accounted for by all, not just by those who use truth preservation to define validity.

The second connection between truth and logic is given in the structure of *truth values*. In many presentations of logical consequence we determine the truth or otherwise of a statement in terms of the truth *values* of its constituent parts. To define validity in terms of truth values we need to know not only what value our premises or conclusions receive under some interpretation, and perhaps more importantly, we also need to know whether this value counts as *true*. In other words, we want some account of which of the many values we use in a compositional account of the interpretations of our statements actually counts as being *true*. For traditional truth tables, the connection is simple: there are two truth values, `true` and `false`, and to be true is to receive the value `true`. In other logics with more than two truth values (for the purposes of semantic composition), the connection between being true and taking a particular truth value may be more complicated.[2] For this second connection between the struc-

[1]There are many ways one can be an inferentialist [2, 3, 4, 7, 9, 10, 11, 15, 17]. Inferentialists propose an approach to semantics (1) where notions of proof (of valid inference) play a crucial and central role and (2) where this notion of proof and validity is can be explicated without deferring to the notion of truth, or truth preservation, but apart from this common ground, inferentialists have litte in common.

[2]For example, any three-valued logic in which two values are designated—such as Priest's 'Logic of Paradox' LP [13, 14], there are two different truth values which count as being 'true.' Coming out as true according to an interpretation does not mean taking some particular semantic value. Rather, there are two different semantic values corresponding to being true. One way to be true is to be true 'only' (to be true, but for the negation of this statement to not also be true)

ture of truth *values* and proof, the right story to tell if we start with *proof* is not so clear, for although it is clear that there is a connection between validity and truth, it is less clear, and not so obviously fundamental to the notion of truth values that validity can be understood in terms of some compositional account of truth values. Can such a story be told? Given the success of compositional theories of truth values, inferentialists would also do well to have something to say about truth values and compositionality.

2 CLASSICAL SEQUENTS

The first case study is the simplest: classical propositional logic. In the formulation I will use here, sequents take the form $X \vdash Y$ where X and Y are finite sets of formulas.[3] We write 'X, A' as a shorthand for the set union of X and the singleton of A, as usual. The sequent calculus defines the notion of a *derivation* which is a tree of sequents, each of whose leaves is an *axiom* [*Id*]

$$X, A \vdash A, Y \ [Id]$$

and each of whose transitions from leaves to root is one of the following rules. A *structural rule* of [*Cut*]

$$\frac{X \vdash A, Y \quad X, A \vdash Y}{X \vdash Y}\,[Cut]$$

or a *connective rule*, introducing the major connective of a formula as a premise or conclusion to the sequent. For example, we have:

$$\frac{X, A \vdash Y}{X, A \wedge B \vdash Y}\,[\wedge L_1] \quad \frac{X, B \vdash Y}{X, A \wedge B \vdash Y}\,[\wedge L_2] \quad \frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y}\,[\wedge R]$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y}\,[\vee L] \quad \frac{X \vdash A, Y}{X \vdash A \vee B, Y}\,[\vee R_1] \quad \frac{X \vdash A, Y}{X \vdash A \vee B, Y}\,[\vee R_2]$$

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\,[\neg L] \quad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\,[\neg R]$$

Derivations in this sequent calculus characterise classical propositional logic. A fundamental theorem concerning the sequent calculus is that any derivation of a sequent making use of the *Cut* rule can be systematically transformed into a derivation of the same conclusion in which no instance of *Cut* is used. The resulting derivation has the subformula property: the only formulas occuring in the derivation are subformulas of formulas occuring in its conclusion.

or to be both true and false (to be true and for the negation of the statement true too).

[3]For many purposes it is more useful to consider sequents to be composed of *multisets* or *sequences* of formulas. Here, explicit attention to structural rules such as conraction will be distracting, so we consider sequents as pairs of sets.

Notice that there are no truth values mentioned in the account of logical consequence supplied in the sequent calculus. This is a matter of no import if the calculus is understood merely as a way to enumerate the valid sequents of the logic. If, on the other hand, the calculus is understood as playing a *semantic* role, then perhaps this points to a semantics for the classical propositional connectives in which truth does not explain. As I mentioned in the previous section, there are many ways to be an inferentialist, and it is not my point to deliniate all of those ways here. However, it will be worth sketching just a little of my own approach, as it motivates some of the definitions and results in what follows.

In the paper 'Multiple Conclusions' [15], I argued that we can see the sequent calculus as supplying a *normative constraint* on acts of assertion and denial: a derivation of a sequent $X \vdash Y$ can be seen as making explicit the manner in which joint assertion of each member of X together with the denial of each member of Y is, in a precise sense, *out of bounds*. The manner in which a valid sequent $X \vdash Y$ tells us that the joint assertion of each member of X and denial of each member of Y is not simply a matter of saying that things are not (or cannot) be as described. Rather, the failure in such a conversational position, in which each member of X is asserted and each member of Y is denied, is of the same kind as the simplest failure of them all, the joint *assertion* and *denial* of the one statement. The fundamental normative force holds between assertion and denial. We take a denial of p to stand against an assertion of p, and *vice versa*.[4] This clash is recorded in the [*Id*] rule: $X, A \vdash A, Y$. That is, a position in which A is asserted (possibly together with other assertions, X) and denied A (possibly together with other denials, Y) is out of bounds.

The rule [*Id*] tells us that assertion and denial are incompatible, and [*Cut*] tells us the converse: it gives us conditions on when there is no such clash in assertion or denial. Reading the [*Cut*] rule from bottom to top, it tells us that if it is not out of bounds to assert X and deny Y, then at least one of adding the assertion of A or the denial of A to that position is also not out of bounds.[5] One way to understand this is to think of the aim of assertion and denial like this: the point of the denial that A is to constrain future assertions of A. To deny that A is to make any future assertion that A require a *withdrawal* of that denial. In the same way, the point of assertion is to rule out future denials. To assert that A is to make any future denial of A require a withdrawal of that assertion. Therefore, if a position in which X is asserted and Y is denied is coherent — if it involves no clash — and if the addition of the assertion that A would *add* a clash, then A's assertion is implicitly ruled out already in that very position. It would not add a clash, then, to make ruling out of A explicit. In other words, adding a

[4]There are many things one could say about the aim of assertion, and of denial. We need not commit to any one of a number of different perspectives here, except to say that we presume an account of the norms of assertion and denial for which there is such a clash, between the assertion of A and the denial of A.

[5]Note: the *order* in which assertions are made or their repetition, makes no difference to coherence. This fact is implicit in our choice of sets as the components of sequents.

denial of A to the assertions X and the denials Y would then be coherent.

The *connective* rules of the sequent calculus can then be thought of as showing how clashes among assertions or denials of complex expressions (conjunctions, disjunctions, negations, etc.) arise out of clashes concerning their constituent parts. For example, the $[\wedge R]$ rule tells us that if there is a clash in denying A (together with asserting X and denying Y) and there is a clash in denying B (together with asserting X and denying Y), then there is also a clash between denying $A \wedge B$, asserting X and denying Y. We can think of the rules for the connectives as giving instructions on how to treat assertions and denials — at least with regard to whether or not these assertions and denials are out of bounds or not.

The result then — if this story can be made out more fully — is an account of what we are to do with assertions and denials in the vocabulary of classical propositional logic. Some positions involving assertions and denials are permissible, and others are ruled out by means of these rules. The derivaton of $\neg(p \wedge q) \vdash \neg p \vee \neg q$ tells us that any position in which we assert $\neg(p \wedge q)$ and deny $\neg p \vee \neg q$ is incoherent. If I were to assert $\neg(p \wedge q)$ and I wished to take up a position on $\neg p \vee \neg q$, the only possible position on that is to assert it, as it is undeniable. This story says nothing about truth or falsity, or a structure of truth values. It is an inferentialist understanding of the meaning of this logical vocabulary.

Our target in this paper is to give an account of truth values starting from an inferentialist position like this. The goal is not to give some non-standard account of truth values, but to see if starting from this position gives us any new light on the existing theories of truth values already familiar to us.

So, consider the behaviour of truth values in traditional model theoretic semantics. There, we do not assign a truth value to a statement *tout court*. Rather, any assignment of a truth value to a statement is relative to an assignment of values to atomic expressions. A model theory does not tell us the truth value of the statement $p \wedge \neg(q \vee r)$ — it merely tells us its truth value relative to the choices of truth value for p, q and r. Does a similar phenomenon arise out of the sequent calculus? Can we find anything like truth values or items to which we can assign truth values, in the sequent calculus?

Consider a sequent $X \vdash Y$. If it is *valid*, there seems to be no reason to think of the members of X as true or as false. (Remember, we are taking a valid sequent $X \vdash Y$ to tell us that the assertion of X together with the denial of Y is ruled out.) While there may be some reason to think of a sequent $A \vdash B$ as recording an argument from the *truth* of A to the *truth* of B, this is still only of the form of a conditional. If I am already inclined to think of B as false, the argument from A to B may well lead me to consider A to be false too. A sequent $A \vdash B$ no more leads us to consider A true than it does to consider B false. The situation is symmetric between truth and falsity. On the other hand, given an invalid sequent, $X \nvdash Y$, matters are different. An invalid sequent practically begs for an interpretation in terms of truth and falsity, where X and Y are treated differently, and not symmetrically. If $X \nvdash Y$, we are naturally lead to call to mind a possibility in which each member of X is true and each member of Y is false. After all, understanding sequents inferentially, if $X \nvdash Y$, asserting each member of X and denying each member of Y is not ruled out. Here, the members of X are treated positively and the members of Y are treated negatively. This is our way in to the connection between sequents and truth.

So, instead of taking truth values or 'possibilities' as fundamental, let us start with the invalid sequents themselves, as they already play a role in our interpretation of the sequent calculus. Where $X \nvdash Y$, we will call the pair $[X : Y]$ a *position*.

DEFINITION [POSITION]: Given a collection of sentences, with a consequence relation $\vdash$ satisfying the rules of the classical sequent calculus, a pair $[X : Y]$ of sets of sentences is a *position* when $X \nvdash Y$.

A position takes its name from the interpretation in terms of assertions and denials. If $[X : Y]$ is a position, it is a 'place' that you could find yourself in a discourse, without going out of bounds. The definition of a position takes care to relativise this to the choice of consequence relation. What matters for us is that we have *at least* the rules of the classical sequent calculus. Of course, there may be more. We may care to define other vocabulary inferentially as well.[6] Nothing here takes a stand on this.

Now, given a position $[X : Y]$ some statements are explicitly asserted (those in X) and others are explicitly denied (those in Y). We will think of the members of X as *left* relative to that position, and the members of Y are *right* relative to the position. (Nothing much hangs on the words we use here: we could call them *true* and *false* respectively, if you prefer, but I would like to keep these words for later, when we have examined more closely the some of what the notions of truth and falsity *do*, and whether these properties do those things.)

However, being 'left' or 'right' in a position is only partly a matter of what is explicitly marked in that sequent. It is also a matter of consequence. Consider the position $[p \vee q, r : \neg p]$ what should we say about p? We cannot place it in the right of this position, at the cost of creating a *valid* sequent. If it is to go anywhere, it must go in the *left* of the sequent. The same goes for $p \wedge r$. So, let us expand the definition in the natural way:

DEFINITION [LEFT AND RIGHT, IN A POSITION]: The LEFT COMPONENT of the position $[X : Y]$ is X. The RIGHT COMPONENT is Y. These are the formulas explicitly on the left and in the right, respectively. We say that A is TO THE LEFT OF $[X : Y]$ if and only if $X \vdash A, Y$. A is TO THE RIGHT OF $[X : Y]$ if and only if $X, A \vdash Y$.

[6] To take it that $Ra, Ga \vdash$ is to take it that the joint assertion of Ra and Ga is out of bounds, for example. A structuralist analysis of colour terms might take it that 'is red all over' and 'is green all over' satisfy this sort of constraint.

The formulas to the left of a position are those which cannot be added to its right component at the cost of incoherence. They are *implicitly* in the left component, but not necessarily explicitly in that component. The formulas to the right of a position are those which cannot be added to its left component at the cost of incoherence. If a formula A is already in the left component of a position, then it is already to the left of that position, since $X \vdash A, Y$ is an [*Id*] sequent if $A \in X$. Similarly, if A is in the right component of a position, it is to the right of that position. Being to the left and being to the right are mutually exclusive:

FACT 1: *No statement is both to the left and to the right of the one position.*

Proof: Suppose A is both to the left of $[X : Y]$ and to its right. Then we have $X \vdash A, Y$ (by the definition of 'to the left of') and $X, A \vdash Y$ (by the definition of 'to the right of'), and hence by [*Cut*], we would $X \vdash Y$. But then, $[X : Y]$ would not be a position, contrary to our assumption. ∎

On the other hand, being to the left and to the right are not mutually exhaustive. Formulas can be neither to the left nor the right of the one position. For example, r is neither to the left nor the right of $[p : q]$, since $[p, r : q]$ and $[p : q, r]$ are both positions in their own right.

Here is how these notions interact with the logical connectives.

FACT 2: *For any position* P (i) $A \wedge B$ *is to the left of* P *iff* A *and* B *are both to the left of* P. (ii) $A \vee B$ *is to the right of* P *iff* A *and* B *are both to the right of* P. (iii) $\neg A$ *is to the left of* P *iff* A *is to the right of* P. (iv) $\neg A$ *is to the right of* P *iff* A *is to the left of* P.

Proof: For (i) $X \vdash A \wedge B, Y$ iff $X \vdash A, Y$ and $X \vdash B, Y$ (left-to-right by $A \wedge B \vdash A$, $A \wedge B \vdash B$, which come from [$\wedge L$], and [*Cut*]; right-to-left by [$\wedge R$]). The other results are similarly straightforward. ∎

These biconditionals show that being-to-the-left-of plays a lot of the role played by truth, and being-to-the-right-of plays a lot of the role played by falsity. A conjunction is to the left of P iff both conjuncts are to the left of P. A negation is to the right of P iff the thing negated is to the left of P, etc. However, not all of these roles are played by these properties. It is not the case, in general, that $A \wedge B$ is to the right of P iff either A or B are to the right of P. For example, $p \wedge q$ is to the right of $[\ : p \wedge q]$ (explicitly so) but neither p nor q are to the right of this position, since $p \not\vdash p \wedge q$ and $q \not\vdash p \wedge q$. Positions are not, in general, a faithful replacement for two-valued evaluations, since they do not determine the location (left or right) of each statement.

However, this incompleteness of positions is not an irreparable flaw. If a position is incomplete concerning a statement, it can be *extended* into a position which determines the location of that statement.

DEFINITION [EXTENSION OF POSITIONS]: $[X' : Y']$ extends $[X : Y]$ if every formula in X is in X', and every formula in Y is in Y'.

In fact, we have a slightly more general result.

FACT 3: *For any position* $[X : Y]$ *and formula* A, *if* A *is not to the left of* $[X : Y]$, *then* $[X : A, Y]$ *is a position.* (*It is an extension of* $[X : Y]$, *to which* A *is to the right.*) *Similarly if* A *is not to the right of* $[X : Y]$, *then* $[X, A : Y]$ *is a position.* (*It is an extension of* $[X : Y]$ *to which* A *is to the left.*)

Proof: Since $[X : Y]$ is a position $X \not\vdash Y$. If A is not to the left of $[X : Y]$, then by definition, $X \not\vdash A, Y$. It follows that $[X : A, Y]$ is a position. The other case is dual. ∎

So, if we have a position that does not decide on A (it is neither to the left nor the right), the position may be extended in either direction: one, which finds A to the left, and the other which finds A to the right. The *limit* of such a process of adding formulas would be something that decides *every* statement. Positions are finite, but the limit of adding formulas to the left of right is no longer a position, properly so called, but the *limit* of a process of extension.

DEFINITION [LIMIT POSITIONS]: Given a language $\mathfrak{L}$, a LIMIT POSITION is a pair $[\mathcal{X} : \mathcal{Y}]$ of sets of sentences such that (a) whenever $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ are finite sets of formulas, $[X : Y]$ is a position; and (b) $\mathcal{X} \cup \mathcal{Y} = \mathfrak{L}$.

Limit positions indicate a 'way one could go on.' No conversation determines a single limit position, for any position could be extended in more than one way, but a limit position indicates ways a single position *can* be extended.

FACT 4: *Any position* $[X : Y]$ *is extended by some limit position* $[\mathcal{X} : \mathcal{Y}]$.

Proof: Consider the tree of all positions, ordered by extension. Given the position $[X : Y]$ in that tree, take a maximal branch $[X_i : Y_i]$ (for each $i \in I$) in the tree from $[X : Y]$, and consider the pair $[\mathcal{X} : \mathcal{Y}] = [\bigcup_i X_i : \bigcup_i Y_i]$. This is an limit position, for (a) whenever $X' \subset \bigcup_i X_i$ and $Y' \subset \bigcup_i Y_i$, there is some index j where $X' \subseteq X_j$ and $Y' \subseteq Y_j$, and since $[X_j : Y_j]$ is a position, $[X' : Y']$ is too; and (b) since for every $[X_i : Y_i]$ in this branch, either $[X_i, A : Y_i]$ or $[X_i : A, Y_i]$ is also a position, any branch totally avoiding the formula A is not maximal, so every maximal branch contains every formula.[7] ∎

Given that a limit position $[\mathcal{X} : \mathcal{Y}]$, is a partition of $\mathfrak{L}$, every formula is explicitly in the left set $\mathcal{X}$ or the right set $\mathcal{Y}$. It follows that the properties of being left or right of an limit position act a great deal like truth and falsity.

[7]For this argument to work, of course, we must appeal to the well ordering of $\mathfrak{L}\mathfrak{L}$ to apply Zorn's lemma. We have assumed nothing else about $\mathfrak{L}$. In particular, we have not assumed that the language is well-founded.

FACT 5: *For any limit position* P (i) $A \wedge B$ *is to the left of* P *iff* A *and* B *are both to the left of* P*;* (i′) $A \wedge B$ *is to the right of* P *iff either* A *or* B *is to the right of* B*.* (ii) $A \vee B$ *is to the right of* P *iff* A *and* B *are both to the right of* P*.* (ii′) $A \vee B$ *is to the left of* P *iff either* A *or* B *is to the left of* P*.* (iii) $\neg A$ *is to the left of* P *iff* A *is to the right of* P*.* (iv) $\neg A$ *is to the right of* P *iff* A *is to the left of* P*, and* (v) A *is to the left of* P *iff* A *is not to the right of* P*.*

Proof: We have already proved parts (i), (ii), (iii) and (iv) of this lemma for positions. To be to the left of an limit position P is to be to the left of some finite position extended by P, and to be to the right of P is to be the right of some finite position extended by P, so these parts of the lemma apply to the limit position P too. To prove the remaining parts, (i′) and (ii′) and (v), we start with (v).

Since P is a partition of $\mathfrak{L}$, it follows that if A is not to the left of P, it is to the right. Conversely, if A is to the right of P, it cannot be to the left of P, since $[A : A]$ is not a position.

Now, for (i′) suppose $A \wedge B$ is to the right of $P = [\mathcal{X} : \mathcal{Y}]$. If neither A nor B are to the right of P, it would follow by (v) that they would be to the left of P. If that were the case, we would have $A, B \in \mathcal{X}$ and $A \wedge B \in \mathcal{Y}$, but that cannot be, since $A, B \vdash A \wedge B$. For (ii′) the reasoning is completely dual, using $A \vee B \vdash A, B$. If $A \vee B$ is to the left of P, we cannot have both A and B to the right of P, so by (v) it follows that either A or B is to the left of P. ∎

It follows that limit positions act just like two-valued boolean evaluations. Limit positions partition the formulas of $\mathfrak{L}$ into those to the left and those to the right, and this partition satisfies the boolean evaluation conditions for the connectives. In fact, it is straightforward to see that any boolean evaluation on $\mathfrak{L}$ (a function from $\mathfrak{L}$ to `true` and `false` satisfying the boolean evaluation conditions for the connectives) determines an limit position $[\mathcal{X} : \mathcal{Y}]$ by setting $\mathcal{X}$ to be all those formulas receiving the value `true` and $\mathcal{Y}$ to be those receiving the value `false`. Limit positions are nothing more and nothing less than boolean evaluations under another guise.

Guise may matter, of course: we did not start off with a theory of `true` and `false`, but with an understanding of sequents in terms of norms governing positions in which things are asserted and denied. Given this starting point, we have defined a notion of 'truth', relative to an idealised position in a discourse: idealised to the extent of taking up every possible statement as asserted or denied. Relative to a position like *this* everything is either asserted (to the left) or denied (to the right), and its location on this divide respects the boolean evaluation conditions for the connectives. The picture is of a kind of binary truth value in which we have distinctive answers to the following three characteristic features of truth and truth value.

TRUTH AND VALIDITY Why is truth preserved in valid arguments? Here the answer is straightforward. If we have $A \vdash B$, then at any position $[X : Y]$ in which A is to the left, B must be too. Otherwise [X : Y] would be incoherent.

TRUTH VALUE AND COMPOSITIONALITY Why is the truth value of a complex sentence a function of the truth values of its components? (At least for complexes made out of propositional connectives.) The fact we need is Fact 5, and the explanation is its proof. Since limit positions are maximally inclusive—since every sentence is either on the left or the right—the sequent rules for the connectives ground their truth functionality. For example, since $A \vee B \vdash A, B$, if $A \vee B$ in the left of $[\mathcal{X} : \mathcal{Y}]$, then one of A and B must be in the left, since they cannot both be in the right. The inference rules for the connectives, together with the maximality of limit positions, ground truth functionality.

TRUTH AND ASSERTION Finally, what is the normative connection (if any) between truth and assertion? To be *true*, with respect to an limit position, is to *be* asserted in that position. However, limit positions are never the kind of thing we attain in a discourse. Given a position [X : Y] extended by a limit position $[\mathcal{X} : \mathcal{Y}]$, then we can at least conclude that if A is true in $[\mathcal{X} : \mathcal{Y}]$ then A is not false in [X : Y]. So, if A is true in $[\mathcal{X} : \mathcal{Y}]$, any position in which A is *denied* cannot be extended by $[\mathcal{X} : \mathcal{Y}]$.

Conversely, if A is asserted in position [X : Y], then A is true in any limit position extending [X : Y]. In this way, assertion the assertion of A literally does aim at the truth of A—at least, it is true in any position at the limit extending our starting point.

Given these three qualities concerning truth and truth value, it seems fair to consider the properties of being to the left and to the right of an limit position as analyses of truth and falsity, at least to the same extent that truth and falsity in a boolean interpretation deserve that name. (They are idealisations, but different idealisations than the more commonly understood ones.) Why are there two truth values? It is because there are two places a statement can be in a limit position: asserted (to the left, `true`) or denied (to the right, `false`).

Notice we have said nothing about truth *simpliciter*. This is intentional: we say no more about the simple plain truth than a theory of boolean evaluations says about truth. We have defined a notion of truth, *relative* to a limit position. We have no way, using these resources at least, of singling out a particular limit position as the one that represents all and only the truths, any more than we have a way to single out a particular boolean evaluation as representing what is really true. This is not to concede that more cannot be said on this point. However, this seems to be a point on which different inferentialists could tell the story in different ways. For now, it will be more profitable to extend our view and to look at sequent systems other than classical propositional logic.

3 INTUITIONISTIC SEQUENTS

Intuitionistic logic has a natural sequent system which differs from the system for classical logic at a purely *structural* level. Instead of working with sequents of the form $X \vdash Y$, sequents take the form $X \vdash A$ (or $X \vdash$) in which the right hand side is either a single formula or is absent entirely.[8] The structrual rules are as follows:

$$X, A \vdash A \ [Id] \qquad \frac{X \vdash A \quad X, A \vdash C}{X \vdash C} [Cut]$$

The other rules are changed only by the restriction on formulas in the right hand side.

$$\frac{X, A \vdash C}{X, A \land B \vdash C} [\land L_1] \quad \frac{X, B \vdash C}{X, A \land B \vdash C} [\land L_2] \quad \frac{X \vdash A \quad X \vdash B}{X \vdash A \land B} [\land R]$$

$$\frac{X, A \vdash C \quad X, B \vdash C}{X, A \lor B \vdash C} [\lor L] \quad \frac{X \vdash A}{X \vdash A \lor B} [\lor R_1] \quad \frac{X \vdash B}{X \vdash A \lor B} [\lor R_2]$$

$$\frac{X \vdash A \quad B, Y \vdash C}{X, A \supset B, Y \vdash C} [\supset L] \quad \frac{X, A \vdash B}{X \vdash A \supset B} [\supset R] \quad \frac{X \vdash A}{X, \neg A \vdash} [\neg L] \quad \frac{X, A \vdash}{X \vdash \neg A} [\neg R]$$

The only other difference is separate rules for disjunction and the conditional, as the they are no longer definable in terms of $\neg$ and $\land$.

We can define positions just as we did before, taking care to attend to the new structure of sequents. So, instead of taking a position to be simply an invalid sequent (in which either only one or no formulas at all take the right hand side), we may consider a generalisation where we allow for multiple formulas on the right hand side, for there is no problem with denying more than one sentence, even if we make do with sequents with no more than one consequent formula.

DEFINITION [INTUITIONIST POSITIONS]: Given a language and a consequence relation satisfying the rules of the intuitionistic sequent calculus, an *intuitionist position* is a pair $[X : Y]$ of finite sets X and Y where $X \not\vdash A$ for any $A \in Y$ (where if Y is empty, we demand $X \not\vdash$ too). A formula A is said to be *to the left of* $[X : Y]$ if $X \vdash A$; and A is *to the right* of $[X : Y]$ iff $X, A \vdash B$ for some $B \in Y$ (or $X, A \vdash$, if Y is empty).

A position, therefore, is a pair of sets formulas (those asserted and those denided), where by the lights of the logic we can never validly deduce any formula denied from the assertions. This much is unchanged from the classical case, but the difference in the underlying logic is significant.

For example, now $[\neg\neg p : p]$ is a position, since by the lights of the intuitionistic sequent calculus, $\neg\neg p \not\vdash p$. It follows from this that the properties of being

[8] In these rules the schematic formula C can also stand in for an empty consequent.

to-the-left-of and being to-the-right-of these positions interact differently with negation. In this position, $\neg\neg p$ is to the left and p is to the right, so somewhere, the general *classical* condition for negation ($\neg A$ is to the left of P iff A is to the right of P; and $\neg A$ is to the right of P iff A is to the left of P) must break down. It is not too difficult to see where. We also have $[\ : p, \neg p]$ as a perfectly acceptable position. Here both p and $\neg p$ are to the right of the position. On the other hand, $[p, \neg p :\ ]$ is *not* a position; we cannot in general have a formula and its negation both to the left of the one position.

Distinctive behaviour is not restricted to negation: $[p \vee q : p, q]$ is a position too, since $p \vee q \nvdash p$ and $p \vee q \nvdash q$. It follows that we can have positions where $p \vee q$ is to the left and *both* p and q are to the right. The evaluation conditions satisfied by positions are reduced in number:

FACT 6: *For any position* P (i) $A \wedge B$ *is to the left of* P *iff* A *and* B *are both to the left of* P. (ii) *If* $A \vee B$ *is to the right of* P *then* A *and* B *are both to the right of* P. (iii) *If* $\neg A$ *is to the left of* P *then* A *is to the right of* P. (iv) *If* A *is to the left of* P *then* $\neg A$ *is to the right of* P. (v) *If* $A \supset B$ *is to the left of* P *and* A *is to the left of* P *then* B *is to the right of* P*; and if* $A \supset B$ *is to the right of* P *then* B *is to the right of* P.

Proof: As before, for (i) $X \vdash A \wedge B, Y$ iff $X \vdash A, Y$ and $X \vdash B, Y$ (left-to-right by $A \wedge B \vdash A$, $A \wedge B \vdash B$, which come from [$\wedge L$], and [*Cut*]; right-to-left by [$\wedge R$]). For (ii) if $X, A \vee B \vdash C$ then $X, A \vdash C$ (using $A \vdash A \vee B$ and [*Cut*]) and similarly, $X, B \vdash C$. For (iii) if $X \vdash \neg A$ then we have $X, A \vdash$ (using $A, \neg A \vdash$ and [*Cut*]). For (iv) if $X \vdash A$ then by [$\neg L$] have $X, \neg A \vdash$. For (v) If $X \vdash A \supset B$ and $X \vdash A$ then by [*Cut*] on $A \supset B, A \vdash B$ we have $X \vdash B$. Similarly, if $X, A \supset B \vdash C$ then by [*Cut*] on $B \vdash A \supset B$ we have $X, B \vdash C$. ∎

So, we have some of the same evaluation conditions for left and right as in the classical case, but not all of them. However, despite the restriction on structural rules in sequents, positions are *completeable* in just the same way they are in the classical sequent calculus.

FACT 7: *If* $[X : Y]$ *is a position, then so is either* $[X : A, Y]$ *or* $[X, A : Y]$.

Proof: Consider what would be the case were neither $[X : A, Y]$ nor $[X, A : Y]$ positions. Were $[X : A, Y]$ not to be a position, we would have either $X \vdash A$ or $X \vdash B$, for some $B \in Y$. In the second case, this would mean that $[X : Y]$ were not a position, but we have assumed that it is. So the only case remaining is the first case, $X \vdash A$. Were $[X, A : Y]$ is not to be a position, we would have $X, A \vdash C$ for some $C \in Y$, and by the following instance of [*Cut*]

$$\frac{X \vdash A \quad X, A \vdash C}{X \vdash C}\,[Cut]$$

we would conclude that $[X : Y]$ is not a position, which we have ruled out by assumption. So, if $[X : Y]$ is a position, then so is either $[X : A, Y]$ or $[X, A : Y]$. ∎

Therefore, it makes sense to consider LIMIT POSITIONS, as before.

DEFINITION [LIMIT POSITIONS]: As before, $[\mathcal{X} : \mathcal{Y}]$ is a LIMIT POSITION in a language $\mathfrak{L}$ iff (a) for any finite $X \subseteq \mathcal{X}$ and $Y \subseteq \mathcal{Y}$, $[X : Y]$ is a position; and (b) $\mathcal{X} \cup \mathcal{Y} = \mathfrak{L}$.

And just as in the classical case, we have the following fact:

FACT 8: *Any position* $[X : Y]$ *is extended by some limit position* $[\mathcal{X} : \mathcal{Y}]$.

Proof: As before, consider the tree of all positions ordered by extension. Any maximal branch $[X_i : Y_i]$, $(i \in I)$ through the tree defines a limit position. In this case, we use Fact 7 to show that any truly maximal branch does not leave out any formula in $\mathfrak{L}$. ∎

When is $\neg A$ in the left of some limit position $[\mathcal{X} : \mathcal{Y}]$? For this, we would need $X \vdash \neg A$ for some finite $X \subset \mathcal{X}$. It follows that $X, A \vdash$, and hence, that A is to the right not only of $[\mathcal{X} : \mathcal{Y}]$, but also of any *other* $[\mathcal{X}' : \mathcal{Y}']$ where $\mathcal{X} \subseteq \mathcal{X}'$, no matter what we find in $\mathcal{Y}'$. Call such positions $[\mathcal{X}' : \mathcal{Y}']$ *strengthenings* of $[\mathcal{X} : \mathcal{Y}]$.

DEFINITION: A position $[X' : Y']$ is a STRENGTHENING of $[X : Y]$ if $X \subseteq X'$. Similarly, a limit position $[\mathcal{X}' : \mathcal{Y}']$ is a STRENGTHENING of $[\mathcal{X} : \mathcal{Y}]$ if $\mathcal{X} \subseteq \mathcal{X}'$

So, extensions are strengthenings, but strengthenings need not be extensions. In strengthenings of positions, more may be asserted but less may be denied.

FACT 9: $\neg A$ *is to the left of an limit position* P *iff* A *is to the right of every strengthening of* P.

Proof: If $\neg A$ is to the left of $[\mathcal{X} : \mathcal{Y}]$, it follows that $X \vdash \neg A$ for some $X \subset \mathcal{X}$. Therefore, for any extension $[\mathcal{X}' : \mathcal{Y}']$ of $[\mathcal{X} : \mathcal{Y}]$, since $X \vdash \neg A$ and $X \subset \mathcal{X}'$ too, we have $X, A \vdash$ and A is to the right of $[\mathcal{X}' : \mathcal{Y}']$.

Conversely, if $\neg A$ is *not* to the left of $[\mathcal{X} : \mathcal{Y}]$, it follows that $\neg A$ is to the right of $[\mathcal{X} : \mathcal{Y}]$, i.e. $\neg A \in Y$. For each $X \subset \mathcal{X}$ we must have $X, A \not\vdash$, lest $X \vdash \neg A$ and $[\mathcal{X} : \mathcal{Y}]$ not be a position. Therefore, $[\mathcal{X}, A : \ ]$ is a position and can be extended to an limit position $[\mathcal{X}' : \mathcal{Y}']$, a strengthening of $[\mathcal{X} : \mathcal{Y}]$ to which A is *not* to the right. ∎

Extensions not only feature in the behaviour of negation at limit positions, but also with disjunction and conditionals.

FACT 10: $A \vee B$ *is to the left of* $[\mathcal{X} : \mathcal{Y}]$ *iff each strengthening of* $[\mathcal{X} : \mathcal{Y}]$ *has some further strengthening at which either* A *or* B *is to the left.*

Proof: If $A \vee B$ is to the left of $[\mathcal{X} : \mathcal{Y}]$, since this is a limit position, we have $A \vee B \in \mathcal{X}$, and hence, $A \vee B \in \mathcal{X}'$ for any strengthening $[\mathcal{X}' : \mathcal{Y}']$. Take some such strengthening $[\mathcal{X}' : \mathcal{Y}']$. We must have either $\mathcal{X}', A \not\vdash$ or $\mathcal{X}', B \not\vdash$ for otherwise, we would have $\mathcal{X}', A \vee B \vdash$ and hence $\mathcal{X}' \vdash$ by [*Cut*]. So, either there is an extension $[\mathcal{X}'' : \mathcal{Y}'']$ where A is to the left, or B is to the left.

Conversely, if $A \vee B$ is *not* to the left of $[\mathcal{X} : \mathcal{Y}]$, it is a member of $\mathcal{Y}$, since $[\mathcal{X} : \mathcal{Y}]$ is a limit position. We want some strengthening of $[\mathcal{X} : \mathcal{Y}]$ with *no* extension where to which either A or B is to the left. One way to do this is to find a strengthening of $[\mathcal{X} : \mathcal{Y}]$ where both A and B are to the right. Consider $[\mathcal{X} : A, B]$. For no $X \subset \mathcal{X}$ do we have $X \vdash A$ nor do we have $X \vdash B$. If we had either $X \vdash A$ or $X \vdash B$ for any $X \subset \mathcal{X}$, we would have $X \vdash A \vee B$, and thus, $A \vee B$ would be to the left of $[\mathcal{X} : \mathcal{Y}]$. So, $[\mathcal{X} : A, B]$ is a position, and it has some extension $[\mathcal{X}' : \mathcal{Y}']$ where $A, B \in \mathcal{Y}'$, and which is a strengthening of $[\mathcal{X} : \mathcal{Y}]$ and which itself has *no* strengthening with either A or B to the left, since A and B are already in the right. ▪

FACT 11: $A \supset B$ *is to the left of a limit position* $[\mathcal{X} : \mathcal{Y}]$ *iff for each strengthening of* $[\mathcal{X} : \mathcal{Y}]$ *to which* A *is to the left, so is* B.

Proof: If $A \supset B$ is to the left of $[\mathcal{X} : \mathcal{Y}]$, then $A \supset B \in \mathcal{X}$, and hence, for any strengthening $[\mathcal{X}' : \mathcal{Y}']$ where $A \in \mathcal{X}'$ we have $A \supset B \in \mathcal{X}'$ too, and by $A \supset B, A \vdash B$, $B \in \mathcal{X}'$ too (since $[\mathcal{X}' : \mathcal{Y}']$ is itself a limit position).

Conversely, if $A \supset B$ is to the right of $[\mathcal{X} : \mathcal{Y}]$ we have no $X \subset \mathcal{X}$ where $X, A \vdash B$, lest we have $X \vdash A \supset B$ and $[\mathcal{X} : \mathcal{Y}]$ is not a limit position. So, $[\mathcal{X}, A : B]$ is a position and it is extended by a limit position $[\mathcal{X}' : \mathcal{Y}']$ to which A is to the left and B is not. So, this is an extension of $[\mathcal{X} : \mathcal{Y}]$ to which A is to the left and B isn't. ▪

However, conjunction behaves just as it does in the classical case.

FACT 12: $A \wedge B$ *is to the left of a limit position* P *iff* A *and* B *are to the left of* P.

Proof: The proof is exactly the same as the classical case. ▪

Let's summarise the conditions for left and right, dropping the ruse of calling these 'left' and 'right' for 'true' and 'untrue.' We have the following conditions on truth at limit sequents.

- $A \wedge B$ is true at P iff A and B are both true at P.
- $A \vee B$ is true at P iff each strengthening of P has a further strengthening at which either A or B is true.
- $A \supset B$ is true at P iff at every strengthening of P, if A is true, so is P.
- $\neg A$ is true at P iff at every strengthening of P, A is not true.

The result, of course, is that limit positions, partially ordered by strengthening, form a model of intuitionistic logic. This is a *Beth* semantics for intuitionistic logic [5, 6, 8]. The natural application of our technique to the standard sequent system of intuitionistic logic gives us a known model theory for the logic.

However, there is another model theory for intuitionist logic, somewhat simpler than Beth semantics. The Kripke semantics for intuitionist logic also evaluates formulas for truth at a family of points, partially ordered by extension. However, the truth condition for disjunction is simpler: In a Kripke model, a disjunction is true at a point iff a disjunct is true at that point.

It turns out that intuitionist logic has more than one sequent system. It has a sequent system in which sequents have the form $X \vdash Y$ where we allow more than one formula in consequent position. The structural rules of [*Id*] and [*Cut*] are kept from classical logic

$$X, A \vdash A, Y \ [Id] \qquad \frac{X \vdash A, Y \quad X, A \vdash Y}{X \vdash Y}\,[Cut]$$

and the rules for conjunction and disjunction are the same:

$$\frac{X, A \vdash Y}{X, A \land B \vdash Y}\,[\land L_1] \quad \frac{X, B \vdash Y}{X, A \land B \vdash Y}\,[\land L_2] \quad \frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \land B, Y}\,[\land R]$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \lor B \vdash Y}\,[\lor L] \quad \frac{X \vdash A, Y}{X \vdash A \lor B, Y}\,[\lor R_1] \quad \frac{X \vdash A, Y}{X \vdash A \lor B, Y}\,[\lor R_2]$$

The *right* rules for the conditional and negation, however, are resricted.

$$\frac{X \vdash A, Y \quad B, X \vdash Y}{X, A \supset B \vdash Y}\,[\supset L] \qquad \frac{X, A \vdash B}{X \vdash A \supset B}\,[\supset R]$$

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\,[\neg L] \qquad \frac{X, A \vdash}{X \vdash \neg A}\,[\neg R]$$

Here, sequents allow for multiple statements in the consequent, but the right rules for negation and the conditional are restricted to cases where the consequent is a singleton. In this case, we can define positions and limit positions in exactly the same manner as in classical logic or the standard semantics for intuitionist logic, except the change in sequents makes a significant difference.

For example, now we can prove $p \lor q \vdash p, q$, there is no limit position at which $p \lor q$ is true and p and q are not. It follows that a disjunction is now true at a limit position if and only if either disjunct is true at a limit sequence. The other proofs remain unchanged from before, and we are able to show that limit positions in *this* sequent system have the following properties:

- $A \land B$ is true at P iff A and B are both true at P.

- $A \vee B$ is true at P iff either A or B is true at P.
- $A \supset B$ is true at P iff at every strengthening of P, if A is true, so is P.
- $\neg A$ is true at P iff at every strengthening of P, A is not true.

The result is a Kripke model for intuitionist logic. In fact, the limit positions from *this* sequent system are all limit positions from the first Gentzen sequent system for intuitionist logic, so this model is a submodel of the Beth model for intuitionist logic. We have two different classes of positions at which statements are either true or false, depending on how liberal we are with the treatment of denials. Is $[p \vee q : p, q]$ a position? On the liberal line, with Gentzen sequents and Beth models, it is, as neither p nor q follow from $p \vee q$. On the more strict line, with multiple conclusion sequents and Kripke models, it is not, as p together with q cover all of the options, given the assumption of $p \vee q$. To argue that one or other analysis is more suited to some purpose would take us too far afield here. Instead, I will consider how either of these approaches give a perspective on the role of truth and truth value.

TRUTH AND VALIDITY Why is truth preserved in valid arguments? Here the answer is the same as in the classical case. If we have $A \vdash B$, then at any position in which A is to the left, B must be too. Otherwise the position would be incoherent. So truth at limit position (of either kind) is preserved in intuitionistically valid arguments. Furthermore, if we have any argument from premises X to a conclusion A which is *invalid*, then we have some position, with respect to which X is true and A is false (that is $[X : A]$). We may extend this position to an ideal position in the usual way.

TRUTH VALUE AND COMPOSITIONALITY In what way do the truth values of `true` and `false` play a role in a compositional 'semantics' for formulas considered intuitionistically? Does the truth value of a formula depend just on the truth value of its components? This happens less in our new context than was the case with classical logic. Both $[p : \neg p]$ and $[\ : p, \neg p]$ are positions, so we may have p `true` and $\neg p$ `false` (relative to a limit position extending $[p : \neg p]$) and we may also have p and $\neg p$ both `false` (relative to a limit position extending $[\ : p, \neg p]$).

In other words, the location of a complex formula in a limit position does not determine its subformulas' locations in *that* sequent, at least not in many cases. (It does, in the case of conjunction, in the same way as in classical logic). However, the position *is* determined in a more attenuated sense: the location of a statement in a limit position is a function of the location of its subformulas in *other* positions. A statement and its negation can both fail to be true (be false, to the *right* of a position). However, if $\neg p$ is to the right of a position, it follows that the assumption of p is not inconsistent with what is *asserted* in that position, and hence, that some strengthening of that position has p to the left. Here, the more limited rules for the negation (or dually, the more liberal account of what counts as a coherent position) are why we alow for this shift in positions.

TRUTH AND ASSERTION What is the normative connection (if any) between truth and assertion? Here, the connection is as straightforward as in the case of classical logic. To be true, with respect to an limit position, is to *be* asserted in that position. Now, truth is more fine-grained because there are more limit positions from which truth may be evaluated, but as before, if A is asserted in a position $[X : Y]$, then relative to any limit position $[\mathcal{X}' : \mathcal{Y}']$ extending $[X : Y]$, the formula A is true.

4 MODAL HYPERSEQUENTS

I will end this exploration of the connection betwen truth, truth values and sequent systems with a speculative look at a possible extension of these results to a hypersequent calculus for the modal logic S5 [16].[9] The sequent system governs *hypersequents*: multisets of sequents, here written as sequents separated by a vertical bar. (The schematic variable Δ is used to range over hypersequents. In other words, $X \vdash Y \mid \Delta$ is a hypersequent, one sequent of which is $X \vdash Y$, and the others of which are in Δ.)

$$X, A \vdash A, Y \mid \Delta \; [Id] \qquad \frac{X \vdash A, Y \mid \Delta \quad X, A \vdash Y \mid \Delta}{X \vdash Y \mid \Delta} [Cut]$$

$$\frac{X \vdash A, Y \mid \Delta}{X, \neg A \vdash Y \mid \Delta} [\neg L] \qquad \frac{X, A \vdash Y \mid \Delta}{X \vdash \neg A, Y \mid \Delta} [\neg R]$$

$$\frac{X, A, B \vdash Y \mid \Delta}{X, A \wedge B \vdash Y \mid \Delta} [\wedge L] \qquad \frac{X \vdash A, Y \mid \Delta \quad X \vdash B, Y \mid \Delta}{X, A \wedge B \vdash Y \mid \Delta \mid \Delta} [\wedge R]$$

$$\frac{X \vdash Y \mid X', A \vdash Y' \mid \Delta}{X, \Box A \vdash Y \mid X' \vdash Y' \mid \Delta} [\Box L] \qquad \frac{\vdash A \mid X \vdash Y \mid \Delta}{X \vdash \Box A, Y \mid \Delta} [\Box R]$$

We can interpret a sequent as constraining a varigated family of assertions and denials: assertions and denials separated into different contexts or *zones* of a discourse. For example, suppose I assert that p, deny that q and then consider a hypothetical possibility in which q and in which r fails. This is a stratified position $[p : q] \mid [q : r]$. The denial of q at one point does not conflict with the assertion of q elsewhere, because they are separated into different 'zones'.

[9] The idea of a hypersequent presentation of modal logic is not new. It dates back at least to Avron's work in the 1990s [1]. This presentation, however, is relatively recent. As far as I know, onlymy work and Poggiolesi's tree sequents [12, 16] admit of an interpretation in terms of assertion and denial stratified into *zones* as I am considering here.

Here is a derivation in the hypersequent system. The starting hypersequent is an instance of [*Id*], a multiset of sequents, one of which contains a formula in antecedent and in conseqent position.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \ \mid\ p \vdash p}{\Box p \vdash \ \mid\ \vdash p}[\Box R]}{\Box p \vdash \ \mid\ \vdash \Box p}[\neg R]}{\vdash \neg\Box p \ \mid\ \vdash \Box p}[\neg L]}{\neg\Box p \vdash \Box\neg\Box p}[\Box R]$$

To develop a theory of truth values in the same vein as that for classical and intuitionist logic, we need to choose what we take for positions. A natural choice, generalising what we have seen before, is to take a position to be an invalid hypersequent, and a limit position is a limit of filling extending a position. The [*Cut*] rule allows for the filling in of *each* sequent in a position with formulas in the left or in the right.

$$\dfrac{X \vdash A, Y \mid \Delta \qquad X, A \vdash Y \mid \Delta}{X \vdash Y \mid \Delta}[Cut]$$

So, we could define a limit position as a *set* of pairs of sets $\{[\mathcal{X}_i : \mathcal{Y}_i] \mid i \in I\}$ where not only can we add no more formulas in any position in that set, but we can also not add any more sequents. Why do this? It is a natural analogue of the classical and intuitionist where we take the position to be the entire hypersequent, and a limit is something to which *nothing* can be added. In other words, we have the following definitions:

DEFINITION: A POSITION is a set of pairs of sets $\{[X_i : Y_i] \mid i \in I\}$ such that there is no valid hypersequent Δ where every component sequent $X \vdash Y$ in Δ there is some $i \in I$ where $X \subseteq X_i$ and $Y \subseteq Y_i$. In other words, no hypersequent covered by that position is valid.

A position $\{[X_j : Y_j] \mid j \in J\}$ EXTENDS another position $\{[X_i : Y_i] \mid i \in I\}$ when for every $i \in I$ there is some $j \in J$ where $[X_j : Y_j]$ extends $[X_i : Y_i]$. A LIMIT POSITION (in language $\mathfrak{L}$) is a position which is extended by no position (in $\mathfrak{L}$).

Can any finite position may be extended to a limit position? Yes, for exactly the same reason as before.

FACT 13: *Any position is extended by some limit position.*

Proof: A limit position (in $\mathfrak{L}$) is the limit of a maximal branch in the set of all positions (in $\mathfrak{L}$) ordered by extension, just as before. But now, not only do we extend positions by adding in formulas to the left or the right, but also by adding in whole extra sequents. ■

Now for limit positions, each component is itself a limit sequent-position $[\mathcal{X} : \mathcal{Y}]$, and hence (since the classical rules for $\wedge$ and $\neg$ apply), being in the left or right of an individual component satisfies the classical two-valued conditions. Given that limit positions can have more than one component[10] there is no sense that a limit position defines a *categorical* notion of truth or falsity. A limit position defines truth or falsity only relative to each different component. In a position extending $\{[p : q], [q : r]\}$ for example, q is true at one component (a component extending $[q : r]$) and false at another (a component extending $[p : q]$). A limit position then defines a *collection* of locations at which statements in the modal language are true and false. A single position, then, corresponds not to a point in a model structure (as it did in the case of sequents for intuitionist logic and Beth or Kripke models), but a position corresponds to a model structure. What kind of structure? The following fact shows its properties.

FACT 14: *For any limit position* $\{[X_i : Y_i] \mid i \in I\}$, *truth at a component satisfies the following three conditions:*

* $A \wedge B$ *is true at* $[X_i : Y_i]$ *iff* A *and* B *are both true at* $[X_i : Y_i]$.
* $\neg A$ *is true at* $[X_i : Y_i]$ *iff* A *is not true at* $[X_i : Y_i]$.
* $\Box A$ *is true at* $[X_i : Y_i]$ *iff* A *is true at each* $[X_j : Y_j]$ *where* $j \in I$.

Proof: The first two conditions follow from the locally classical behaviour of the sequent rules. The work occurs in the proof of the condition for $\Box$.

If $\Box A$ is true at $[X_i : Y_i]$ and A were *not* true at some component $[X_j : Y_j]$, then since $[X_j : Y_j]$ is a limit sequent, we would have $A \in Y_j$. But $\Box A \vdash \ \mid\ \vdash A$ is a valid sequent (by $[\Box L]$, from the axiom $\vdash \ \mid\ A \vdash A$), so $\{[X_i : Y_i], [X_j : Y_j]\}$ would not be a position (since $\Box A \in X_i$ and $A \in Y_j$), and *a fortiori*, $\{[X_i : Y_i] \mid i \in I\}$ would not be a position. But we have assumed it is, so whenever $\Box A$ is true at $[X_i : Y_i]$, A is true at every component $[X_j : Y_j]$.

Conversely, we use $[\Box R]$. If $\Box A$ is not true at $[X_i : Y_i]$, we have $\Box A \in Y_i$. We need to show that there is some component $[X_j : Y_j]$ where $A \in Y_j$. For this, we will show that $\{[X_i : Y_i] \mid i \in I\} \cup \{[\ : A]\}$ is a position. It must be a position, for if it weren't, there would be a provable hypersequent $\vdash A \mid X \vdash Y \mid \Delta$ (where $X \subseteq X_i$ and $Y \subseteq Y_i$, and Δ is extended by the other components $[X_j : Y_j]$) extended by $\{[X_i : Y_i] \mid i \in I\} \cup \{[\ : A]\}$. If *that* were the case, then we could conclude by $[\Box R]$ that $X \vdash \Box A, Y \mid \Delta$, but this sequent is extended by our original position $\{[X_i : Y_i] \mid i \in I\}$, so it is *not* valid. As a result, $\{[X_i : Y_i] \mid i \in I\} \cup \{[\ : A]\}$ is itself a position. It is extended by a maximal position, which itself must be $\{[X_i : Y_i] \mid i \in I\}$, since $\{[X_i : Y_i] \mid i \in I\}$ is extended by no other position. In other words, we must have $A \in Y_j$ for some $j \in I$. ∎

[10] But it *need not*. See the later result characterising limit positions. Each one-world Kripke model for S5 determines a limit position with exactly one component.

Each limit position defines a universal Kripke model for the modal logic S5. Our techniques, applied to a hypersequent system, instead of a traditional sequent system, has produced limit positions which are themselves entire models of the modal logic, instead of points in a model structure.

In each logic under study, we have seen a *standard* model theory defined out of the raw materials given by the proof theory, not in some *ad hoc*, case by case manner, but in the same way in each logic. Truth can be defined relative to a limit position, for positions place formulas in the left (asserted) or the right (denied) and in the limit, every formula is so placed. So the family of limit positions provides a fitting structure, not only for a model theory of each of our logics, but also, perhaps, for further discussion of the significance of proof theory for semantics.

It is to be hoped that these techniques shed light on the relationship between proof theory and models for non-classical logics, and more importantly, the different *interpretation* of both sorts of theories. If these results go some way to clarify these issues and to provoke further research into the interface between proof theory and model theory—and in particular, structures inovling truth and falsity—this paper will have done its job.

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