Journal of Logic, Language and Information **8:** 187–203, 1999.

Reasoning about Minimal Knowledge in Nonmonotonic Modal Logics

RICCARDO ROSATI
Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", Via Salaria 113, 00198 Roma, Italy
E-mail: rosati@dis.uniroma1.it

(Received 12 December 1997)

Abstract. We study the problem of embedding Halpern and Moses's modal logic of minimal knowledge states into two families of modal formalism for nonmonotonic reasoning, McDermott and Doyle's nonmonotonic modal logics and ground nonmonotonic modal logics. First, we prove that Halpern and Moses's logic can be embedded into all ground logics; moreover, the translation employed allows for establishing a lower bound (Π_3^p) for the problem of skeptical reasoning in all ground logics. Then, we show a translation of Halpern and Moses's logic into a significant subset of McDermott and Doyle's formalisms. Such a translation both indicates the ability of Halpern and Moses's logic of expressing minimal knowledge states in a more compact way than McDermott and Doyle's logics, and allows for a comparison of the epistemological properties of such nonmonotonic modal formalisms.

Key words: Knowledge representation, nonmonotonic reasoning, epistemic modal logics, computational complexity

1. Introduction

Research in nonmonotonic modal logics has developed from its early years along two different lines. On the one hand, several studies have proposed the definition of a nonmonotonic modal formalism by characterizing the epistemic states of the agent modeled through a fixpoint equation (McDermott and Doyle, 1980; McDermott, 1982; Moore, 1985; Konolige, 1988). The family of logics thus defined includes autoepistemic logic (Moore, 1985) and McDermott and Doyle's family of logics (MDD logics), and has been thoroughly investigated. In particular, it has been shown (Schwarz, 1990) that Moore's autoepistemic logic is an instance of MDD logics, and, more generally, that McDermott and Doyle's characterization can be considered a very powerful schema for defining nonmonotonic modal logics (Marek and Truszczyński, 1993; Schwarz and Truszczyński, 1994).

On the other hand, several researchers have proposed various notions of nonmonotonic modal logics based on a *preference semantics* (Shoham, 1987) on models of a monotonic modal logic. Among these approaches, the logic of minimal knowledge states, due to Halpern and Moses (1985), is based on a simple and "natural" preference semantics for the modal logic S5 (Shoham, 1987; Moore, 1985), which realizes the intuitive principle of minimization of knowledge of the agent modeled. Such a logic, initially proposed for modeling knowledge and ignorance of processes in a distributed computer system, constitutes the basis of several non-monotonic modal formalisms proposed in the literature (Lifschitz, 1991; Lin and Shoham, 1992, 1998; Engelfriet, 1996; Meyer and van der Hoek, 1995a, b; Donini et al., 1995; Rosati, 1997a).

Recently, there has been a considerable effort towards the reconciliation of the syntactic and the semantic approach to the definition of nonmonotonic modal logics. One of the most important steps in this direction is due to Schwarz (1992a) who provides a preference semantics, based on a partial ordering of Kripke models, for a large subset of MDD logics. Conversely, Halpern and Moses's logic of knowledge states can be given a fixpoint characterization (Tiomkin and Kaminski, 1990) which is a slight variation of McDermott and Doyle's equation, and which actually defines a whole family of logics based on the minimal knowledge paradigm, the so-called *ground* nonmonotonic modal logics (ground logics for short) (Tiomkin and Kaminski, 1990; Schwarz, 1992b; Donini et al., 1997). Furthermore, the definition of a preference semantics for ground logics (Nardi and Rosati, 1995) has shown, from the semantical viewpoint, the existence of deep analogies between this family of formalisms and MDD logics.

Finally, recent studies on the computational properties of the logic of minimal knowledge states (Donini et al., 1997) have shown that reasoning in this logic is strictly harder (unless the polynomial hierarchy collapses to NP, see Johnson, 1990) than reasoning in all the most popular propositional nonmonotonic formalisms (Cadoli and Schaerf, 1993; Gottlob, 1992), in particular with regard to many MDD logics (Marek and Truszczyński, 1993). Therefore, the question arises of how "similar" the logic of minimal knowledge states to MDD logics, and in which sense, if any, MDD logics are "less expressive" than minimal knowledge states is.

The goal of the present work is to study the possibility of representing minimal knowledge states in nonmonotonic modal logics. In particular, we study the problem of embedding Halpern and Moses's logic into ground logics and MDD logics.

The results presented in this paper can be summarized as follows: the logic of Halpern and Moses is "easily" embeddable in most of the nonmonotonic modal formalisms taken into consideration. More specifically, we first provide an embedding of Halpern and Moses's logic into ground logics which in turn allows for establishing a computational property (Π_3^p-hardness) for the deduction problem in the whole family of ground logics.

Then we show that Halpern and Moses's logic can easily be embedded into many MDD logics, in the sense that, in many cases, a simple transformation of modal theories realizes the embedding of minimal knowledge states into MDD logics.

Finally, we point out the existence of syntactic restrictions under which minimal knowledge states and preferred models in MDD logics coincide. This result allows for both establishing a computational characterization of the problem of reasoning in Halpern and Moses's logic under such syntactic restrictions, and analyzing the epistemological properties of the logic of minimal knowledge states, as compared to MDD logics. Surprisingly, it turns out that in a large subset of MDD logics modalities can be given a minimal knowledge interpretation, as far as they are not nested modalities. Thus, the higher degree of complexity of reasoning about minimal knowledge states cannot be attributed to the interpretation of the modal operator in this logic. Rather, the translation we provide indicates the ability of Halpern and Moses's logic of expressing minimal knowledge states in a more compact form than MDD logics.

The paper is structured as follows. In the next section we recall some preliminary definitions and properties of nonmonotonic modal logics. Then, in Section 3 we show the embedding of the logic of minimal knowledge states into ground logics, and in Section 4 we show that the logic of minimal knowledge states can be embedded into a large subset of MDD logics. Section 5 presents new complexity results for the problem of reasoning in ground logics, and an epistemological interpretation of such results. Finally, some conclusions are drawn in Section 6.

2. Preliminaries

In this section we briefly recall some preliminary definitions and properties of nonmonotonic modal logics. The interested reader is referred to (Marek and Truszczyński, 1993; Nardi and Rosati, 1995) for further details.

We use $\mathcal{L}$ to denote a fixed propositional language built in the usual way from an alphabet $\mathcal{A}$ of propositional symbols (containing the symbols true, false) and the propositional connectives $\vee$, $\wedge$, $\neg$, $\supset$. Formulas over $\mathcal{L}$ will often be called *objective*, because they do not contain occurrences of the modal operator.

We denote with $\mathcal{L}_K$ the modal extension of $\mathcal{L}$ with the only modality K (for knowledge), and with $\mathcal{L}_K^0$ the set $\{K\varphi \mid \varphi \in \mathcal{L}\}$. $\mathcal{L}_1$ stands for the set of formulas built from $\mathcal{L}_K^0$ and the propositional connectives $\vee, \wedge, \neg, \supset$. The symbol $\mathcal{L}_K^\Sigma$ stands for the restriction of $\mathcal{L}_K$ to the formulas in which only propositional symbols from the modal theory $\Sigma \subseteq \mathcal{L}_K$ appear. The set of *modal atoms* of a theory $\Sigma \subseteq \mathcal{L}_K$ is the set $MA(\Sigma) = \{\varphi \mid K\varphi$ is a subformula of a formula from $\Sigma\}$. A formula $\varphi \in \mathcal{L}_K$ has modal depth n if each subformula of φ lies within the scope of at most n nested modalities.

Given a modal logic $\mathcal{S}$, we denote with Cn_s the logical consequence operator in $\mathcal{S}$. Given two modal logics $\mathcal{S}_1$ and $\mathcal{S}_2$, by $\mathcal{S}_1 \subseteq \mathcal{S}_2$ we mean that all axioms of logic $\mathcal{S}_1$ are also axioms (or theorems) in logic $\mathcal{S}_2$ (e.g., K $\subseteq$ KD45 $\subseteq$ S5).

We denote with S4F (also known as S4.3.2; see Segerberg, 1971) the modal logic obtained by adding to S4 the following axiom schema F:

$$(\varphi \wedge \neg K\neg K\varphi) \supset K(\neg K\neg\varphi \vee \varphi)$$

and with SW5 (also known as S4.4; see Segerberg, 1971) the modal logic obtained by adding to S4 the following axiom schema W5:

$$\varphi \supset (\neg K \neg K\varphi \supset K\varphi).$$

A *Kripke model* $\mathcal{M}$ is defined as usual by a triple $\langle W, R, V\rangle$, where W is a set of worlds, R is a binary relation on W and V is a function assigning a propositional valuation to each world $w \in W$. When R is $W \times W$ (i.e., $\mathcal{M}$ is a universal model) we simply write $\langle W, V\rangle$.

We denote with $Th(\mathcal{M})$ the set of formulas of $\mathcal{L}_K$ that are satisfied in $\mathcal{M}$, i.e., $Th(\mathcal{M}) = \{\varphi \in \mathcal{L}_K \mid \mathcal{M} \models \varphi\}$, where $\mathcal{M} \models \varphi$ for each $w \in V$, $(\mathcal{M}, w) \models \varphi$ (the satisfiability relation between a modal formula φ and a world w in a Kripke model $\mathcal{M}$ is defined in the usual way).

Given a modal logic $\mathcal{S}$, a consistent set of formulas T is an $\mathcal{S}_{\mathrm{MDD}}$ expansion for a set of initial knowledge $\Sigma \subseteq \mathcal{L}_K$ if

$$T = Cn_{\mathcal{S}}(\Sigma \cup \{\neg K\varphi \mid \varphi \in \mathcal{L}_K \backslash T\}), \tag{1}$$

where $Cn_{\mathcal{S}}$ is the consequence operator in (classical) modal logic $\mathcal{S}$.

The resulting consequence operator is defined as the intersection of all $\mathcal{S}_{\mathrm{MDD}}$ expansions for Σ: given $\Sigma \subseteq \mathcal{L}_K$, $\varphi \in \mathcal{L}_K$, we say that φ is entailed by Σ in $\mathcal{S}_{\mathrm{MDD}}$ (and write $\Sigma \models_{\mathcal{S}_{\mathrm{MDD}}} \varphi$) iff φ belongs to all $\mathcal{S}_{\mathrm{MDD}}$ expansions for Σ. Such an operator is in general nonmonotonic; thus, for every modal logic $\mathcal{S}$, the (nonmonotonic) modal logic $\mathcal{S}_{\mathrm{MDD}}$ is obtained by means of Equation (1).

Let us now briefly recall the minimal knowledge paradigm. Informally, such a principle consists of considering, among the epistemic states which are consistent with the initial knowledge Σ of the agent, only the subset composed of each state E which is minimal with respect to the objective knowledge, i.e., any other epistemic state E' either is inconsistent with Σ or contains more objective knowledge than E. In a modal logic setting, such a paradigm can be stated as follows: given a modal logic $\mathcal{S}$, a model $\mathcal{M}$ is a model of minimal knowledge for $\Sigma \subseteq \mathcal{L}_K$ in $\mathcal{S}$ if $\mathcal{M}$ is a model for Σ in $\mathcal{S}$ and for every model $\mathcal{M}'$ for Σ in $\mathcal{S}$, $Th(\mathcal{M}') \cap \mathcal{L} \not\subset Th(\mathcal{M}) \cap \mathcal{L}$.

We say that $\mathcal{S}$ is a logic of minimal knowledge if for every theory $\Sigma \subseteq \mathcal{L}_K$, every model for Σ in $\mathcal{S}$ is a model of minimal knowledge for Σ in $\mathcal{S}$.

In Halpern and Moses (1985) the notion of minimal knowledge is applied to modal logic S5: the logic of minimal knowledge states is therefore the logic obtained by considering as models for $\Sigma \subseteq \mathcal{L}_K$ only the models of minimal knowledge among the S5 models satisfying Σ (Shoham, 1987; Lifschitz, 1991).

The logic of minimal knowledge states can be given a fixpoint characterization, which is similar to the one given in the MDD case, and that gives rise to the family of *ground* nonmonotonic modal logics. Given a normal modal logic $\mathcal{S}$, a consistent set of formulas T is an $\mathcal{S}_G$ *expansion* for a set $\Sigma \subseteq \mathcal{L}_K$ if

$$T = Cn_{\mathcal{S}}(\Sigma \cup \{\neg K\varphi \mid \varphi \in \mathcal{L} \backslash T\}).$$

Note that, in this case, negative introspection is bounded to objective knowledge in the right-hand side of the equation.

Given $\Sigma \subseteq \mathcal{L}_K$, $\varphi \in \mathcal{L}_K$, we say that φ is *entailed* by Σ in $\mathcal{S}_G$ (and write $\Sigma \models_{\mathcal{S}_G} \varphi$) iff φ belongs to all $\mathcal{S}_G$ expansions for Σ.

It turns out that, if $\mathcal{S} = \mathsf{S5}$, then the logic of minimal knowledge states is obtained from the above equation (Tiomkin and Kaminski, 1990). Hence, $\mathsf{S5}_G$ corresponds to Halpern and Moses's logic.

Let us now introduce a characterization of MDD (respectively, ground) logics based on a syntactic preference criterion on Kripke models. In the next section, we will use such a characterization, thus referring to $\mathcal{S}_{\mathrm{MDD}}$ models ($\mathcal{S}_G$ models) instead of $\mathcal{S}_{\mathrm{MDD}}$ expansions ($\mathcal{S}_G$ expansions).

DEFINITION 2.1. Given a modal logic $\mathcal{S} \subseteq \mathsf{S5}$ characterized by the class $\mathcal{C}$ of Kripke models and a theory $\Sigma \subseteq \mathcal{L}_K$, a model $\mathcal{M} \in \mathcal{C}$ is an $\mathcal{S}_{\mathrm{MDD}}$ *model* for Σ iff $\mathcal{M} \models \Sigma$ and, for every model $\mathcal{N} \in \mathcal{C}$, if $\mathcal{N} \models \Sigma \cup \{\neg K\varphi \mid \varphi \in \mathcal{L}_K \backslash Th(\mathcal{M})\}$, then $Th(\mathcal{M}) = Th(\mathcal{N})$.

DEFINITION 2.2. Given a normal modal logic $\mathcal{S} \subseteq \mathsf{S5}$ characterized by the class $\mathcal{C}$ of Kripke models and a theory $\Sigma \subseteq \mathcal{L}_K$, a model $\mathcal{M} \in \mathcal{C}$ is an $\mathcal{S}_G$ *model* for Σ iff $\mathcal{M} \models \Sigma$ and, for every model $\mathcal{N} \in \mathcal{C}$, if $\mathcal{N} \models \Sigma \cup \{\neg K\varphi \mid \varphi \in \mathcal{L} \backslash Th(\mathcal{M})\}$, then $Th(\mathcal{M}) = Th(\mathcal{N})$.

It holds that an $\mathsf{S5}$ model $\mathcal{M}$ is an $\mathcal{S}_{\mathrm{MDD}}$ model for $\Sigma \subseteq \mathcal{L}_K$ iff $Th(\mathcal{M})$ is an $\mathcal{S}_{\mathrm{MDD}}$ expansion for Σ (Marek and Truszczyński, 1993), and that an $\mathsf{S5}$ model $\mathcal{M}$ is an $\mathcal{S}_G$ model for $\Sigma \subseteq \mathcal{L}_K$ iff $Th(\mathcal{M})$ is an $\mathcal{S}_G$ expansion for Σ (Nardi and Rosati, 1995). Moreover, for any modal logic $\mathcal{S}$ between K and $\mathsf{S5}$, and for any $\Sigma \subseteq \mathcal{L}_K$, each $\mathcal{S}_G$ model for Σ is an $\mathcal{S}_{\mathrm{MDD}}$ model for Σ.

PROPOSITION 2.3. *Let $\mathcal{S}^1$, $\mathcal{S}^2$ be any two normal modal logics, and let $\Sigma \subseteq \mathcal{L}_K$. If $\mathcal{S}^1 \subseteq \mathcal{S}^2$, then each $\mathcal{S}^1_{\mathrm{MDD}}$ model for Σ is an $\mathcal{S}^2_{\mathrm{MDD}}$ model for Σ and each $\mathcal{S}^1_G$ model for Σ is an $\mathcal{S}^2_G$ model for Σ.*

The following property directly follows from Marek and Truszczyński (1993: theorem 8.16).

PROPOSITION 2.4. *Let $T \subseteq \mathcal{L}_K$ be a stable theory. Let $\mathcal{M}$ be a Kripke model such that $\mathcal{M} \models (T \cap \mathcal{L}) \cup \{\neg K\varphi \mid \varphi \in \mathcal{L}\backslash T\}$. Then, for every $\varphi \in \mathcal{L}_K$, $\varphi \in T$ iff $\mathcal{M} \models \varphi$, namely $T = Th(\mathcal{M})$.*

Let us finally recall a normal form for modal theories in $\mathsf{S5}$, which is based on the fact that in $\mathsf{S5}$ every modal formula $K\varphi$ is equivalent to a formula in the set $\mathcal{L}_1$, namely a formula without nested occurrences of the modality and in which each occurrence of a propositional symbol lies within the scope of exactly one modality. The procedure for transforming the formula is conceptually simple:* informally, it

* For a detailed description of such a procedure, see Hughes and Cresswell, 1968: chapter 3).

is based on the following equivalences, which are valid in S5:

$$
\begin{aligned}
K(\varphi \wedge \psi) &\equiv K\varphi \wedge K\psi, \\
K(\varphi \vee K\psi) &\equiv K\varphi \vee K\psi, \\
K(\varphi \vee \neg K\psi) &\equiv K\varphi \vee \neg K\psi, \\
KK\varphi &\equiv K\varphi, \\
K\neg K\varphi &\equiv \neg K\varphi.
\end{aligned}
$$

We call S5 *normal form* of a modal formula $K\varphi$ the formula $\mathcal{NF}_{\mathsf{S5}}(K\varphi) \in \mathcal{L}_1$ obtained by reducing $K\varphi$ to a formula belonging to $\mathcal{L}_1$ through the above procedure.

Let Σ be a modal theory such that each formula φ from Σ is of the form $K\psi$. The S5 normal form of Σ, denoted as $\mathcal{NF}_{\mathsf{S5}}(\Sigma)$, is defined as

$$\mathcal{NF}_{\mathsf{S5}}(\Sigma) = \{\mathcal{NF}_{\mathsf{S5}}(\varphi) \mid \varphi \in \Sigma\}.$$

Note that, in general the size of $\mathcal{NF}_{\mathsf{S5}}(\Sigma)$ is exponential in the size of Σ, which is informally due to the fact that, in order to transform a modal formula of the form $K\varphi$, it is necessary to put φ in a "modal conjunctive normal form" (see Hughes and Cresswell, 1968, for more details). Moreover, every propositional symbol in $\mathcal{NF}_{\mathsf{S5}}(\Sigma)$ lies within the scope of exactly one modal operator.

3. Minimal Knowledge in Ground Logics

In this section we show how to embed the logic of minimal knowledge states into ground logics. In particular, we first provide a very simple translation for theories without nested occurrences of the modality. Then, we extend such a translation in order to deal with general modal theories; notably, this last translation allows for a computational characterization of the entailment problem in all ground logics, which generalizes a previous result shown in Donini et al. (1997).

In the following, we use the term *embedding* (or translation) to indicate a transformation function for modal theories, i.e., $\mathcal{E} : \mathcal{L} \rightarrow \mathcal{L}_K$. Sometimes we will abuse terminology, using this term also to indicate the application of a transformation to a modal theory.

We are interested in finding *faithful* embeddings, in the following sense: given two modal logics $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{E}$ is a faithful embedding from $\mathcal{S}_1$ into $\mathcal{S}_2$ if, for each $\Sigma \subseteq \mathcal{L}_K$ and for each Kripke model $\mathcal{M}$, $\mathcal{M}$ is an $\mathcal{S}_1$ model for Σ iff $\mathcal{M}$ is an $\mathcal{S}_2$ model for $\mathcal{E}(\Sigma)$.

We now define the following transformation functions for modal theories:

$$
\begin{aligned}
\mathcal{K}(\Sigma) &= \{K\varphi \mid \varphi \in \Sigma\}, \\
\mathcal{T}(\Sigma) &= \Sigma \cup \{(K\varphi \supset \varphi) \mid \varphi \in MA(\Sigma)\}, \\
\mathcal{T}_N(\Sigma) &= \mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma)) \cup \{(K\varphi \supset \varphi) \mid \varphi \in MA(\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma)))\}.
\end{aligned}
$$

It is easy to see that any S5 model $\mathcal{M}$ is a model for Σ iff it is a model for $\mathcal{K}(\Sigma)$. Moreover, since each formula $K\varphi \supset \varphi$ is valid in S5, the following lemma holds:

LEMMA 3.1. *Let $\Sigma \subseteq \mathcal{L}_K$. Then, $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ iff $\mathcal{M}$ is an $\mathsf{S5}_G$ model for $\mathcal{T}(\Sigma)$. Moreover, $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ iff $\mathcal{M}$ is an $\mathsf{S5}_G$ model for $\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma))$.*

We now show that minimal knowledge states are easily embeddable into *any* ground logic $\mathcal{S}_G$. We start by analyzing the case of theories $\Sigma \subseteq \mathcal{L}_1$.

LEMMA 3.2. *Let $\Sigma \subseteq \mathcal{L}_1$. Let $\mathcal{M}$ be an $\mathsf{S5}_G$ model for Σ and let $\mathcal{N}$ be a* K *model. If $\mathcal{N} \models \mathcal{T}(\Sigma) \cup \{\neg K\varphi \mid \varphi \in \mathcal{L}\backslash Th(\mathcal{M})\}$, then $Th(\mathcal{M}) \cap \mathcal{L} = Th(\mathcal{N}) \cap \mathcal{L}$.*

Proof. Let $\varphi \in \mathcal{L}_1$ and let $D(\varphi)$ be the DNF of φ, obtained considering each modal atom as a propositional symbol. Let $D(\Sigma) = \{D(\varphi) \mid \varphi \in \Sigma\}$. By hypothesis $\mathcal{N} \models \Sigma \cup \{\neg K\varphi \mid \varphi \in \mathcal{L}\backslash Th(\mathcal{M})\}$, and since $\varphi \equiv D(\varphi)$ is valid in K, $\mathcal{N} \models D(\Sigma)$. Now consider each theory $H_i(\Sigma)$ obtained from $\{\neg K\varphi \mid \varphi \in \mathcal{L}\backslash Th(\mathcal{M})\}$ by adding one conjunct of φ, for each $\varphi \in D(\Sigma)$. Since $\mathcal{N} \models D(\Sigma) \cup \{\neg K\varphi \mid \varphi \in \mathcal{L}\backslash Th(\mathcal{M})\}$, $\mathcal{N}$ must satisfy at least one of such theories $H_i(\Sigma)$. On the other hand, the fact that $\mathcal{M}$ is an $\mathsf{S5}_G$ model implies that each $H_i(\Sigma)$ either is S5-inconsistent or is minimal, in the sense that for each pair of S5-consistent theories $H_i(\Sigma)$, $H_j(\Sigma)$, $\{\varphi \mid K\varphi \in H_i(\Sigma)\}$ is propositionally equivalent to $\{\varphi \mid K\varphi \in H_j(\Sigma)\}$. Hence, for each S5-consistent $H_i(\Sigma)$, the theory $\{K\varphi \mid K\varphi \in H_i(\Sigma)\}$ is equivalent in S5 to the theory $\{K\varphi \mid \varphi \in MA(\Sigma)$ and $\mathcal{M} \models \varphi\}$. Now, due to the form of the $H_i(\Sigma)$'s, each $H_i(\Sigma)$ is S5-consistent iff it is K-consistent; consequently, the above equivalence also holds in the logic K. Therefore, $\mathcal{N} \models \{K\varphi \mid \varphi \in MA(\Sigma)$ and $\varphi \in Th(\mathcal{M}) \cap \mathcal{L}\}$, and since $\mathcal{N} \models \mathcal{T}(\Sigma)$, it follows that $\mathcal{N} \models \{\varphi \mid \varphi \in MA(\Sigma)$ and $\mathcal{M} \models \varphi\}$. Now, since $\Sigma \subseteq \mathcal{L}_1$, it follows that $Th(\mathcal{M}) \cap \mathcal{L} = Cn(\{\varphi \mid \varphi \in MA(\Sigma)$ and $\mathcal{M} \models \varphi\}$, consequently $\mathcal{N} \models Th(\mathcal{M}) \cap \mathcal{L}$. On the other hand, $\mathcal{N} \models \{\neg K\varphi \mid \varphi \in \mathcal{L}\backslash Th(\mathcal{M})\}$ implies $Th(\mathcal{N}) \cap \mathcal{L} \subseteq Th(\mathcal{M}) \cap \mathcal{L}$, therefore $Th(\mathcal{N}) \cap \mathcal{L} = Th(\mathcal{M}) \cap \mathcal{L}$. □

We can now prove that, for theories $\Sigma \subseteq \mathcal{L}_1$, $\mathcal{T}(\Sigma)$ is a faithful embedding of $\mathsf{S5}_G$ into any ground logic $\mathcal{S}_G$.

THEOREM 3.3. *Let $\Sigma \subseteq \mathcal{L}_1$, and let $\mathcal{S}$ be any modal logic such that* $\mathsf{K} \subseteq \mathcal{S} \subseteq \mathsf{S5}$. *Then, a Kriple model $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ iff $\mathcal{M}$ is an $\mathcal{S}_G$ model for $\mathcal{T}(\Sigma)$.*

Proof. If part. Follows straightforward from the fact that every $\mathcal{S}_G$ model for $\mathcal{T}(\Sigma)$ is also an $\mathsf{S5}_G$ model for $\mathcal{T}(\Sigma)$ and from Lemma 3.1.

Only-if part. Suppose $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ. Then, by Lemma 3.1, it follows that $\mathcal{M}$ is an $\mathsf{S5}_G$ model for $\mathcal{T}(\Sigma)$. Now suppose that $\mathcal{M}$ is not an $\mathcal{S}_G$ model for $\mathcal{T}(\Sigma)$. Therefore, there exists an $\mathcal{S}$ model $\mathcal{N}$ such that $\mathcal{N} \models \mathcal{T}(\Sigma) \cup \{\neg K\varphi \mid \varphi \in \mathcal{L}\backslash Th(\mathcal{M})\}$ and $Th(\mathcal{N}) \neq Th(\mathcal{M})$. Now, Lemma 3.2

implies that $Th(\mathcal{M}) \cap \mathcal{L} = Th(\mathcal{N}) \cap \mathcal{L}$. Consequently, $\mathcal{N} \models Th(\mathcal{M}) \cap \mathcal{L}$, and by Proposition 2.4 it follows that $Th(\mathcal{M}) = Th(\mathcal{N})$. Contradiction. Therefore, $\mathcal{M}$ is an $\mathcal{S}_G$ model for $\mathcal{T}(\Sigma)$. □

We now prove that $\mathsf{S5}_G$ can be embedded into any ground logic $\mathcal{S}_G$ through a polynomial-time transformation of modal theories. First, we prove the following lemma.

LEMMA 3.4. *Let* $\Sigma \subseteq \mathcal{L}_K$ *and let*

$$\begin{aligned}\Sigma_1 = \Sigma \ &\cup \left(\bigcup_{K\varphi \in MA(\Sigma)} \{K\varphi \supset \varphi\}\right)\\ &\cup \left(\bigcup_{K\varphi \in MA(\Sigma)} \{K\varphi \supset KK\varphi\}\right)\\ &\cup \left(\bigcup_{K\varphi \in MA(\Sigma)} \{\neg K\varphi \supset K\neg K\varphi\}\right).\end{aligned}$$

Then, an $\mathsf{S5}$ *model* $\mathcal{M}$ *is a* K_G *model for* Σ_1 *iff* $\mathcal{M}$ *is an* $\mathsf{S5}_G$ *model for* Σ.

Proof. Only-if part. Follows from the fact that Σ is $\mathsf{S5}$-equivalent to Σ_1 and from Proposition 2.3.

If part. Let $\mathcal{M}$ be an $\mathsf{S5}_G$ model for Σ. Then, since Σ is $\mathsf{S5}$-equivalent to Σ_1, $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ_1. Suppose $\mathcal{M}$ is not a K_G model for Σ_1. Then there exists a model $\mathcal{M}' = \langle W', R', V'\rangle$ such that $\mathcal{M}' \models \Sigma_1 \cup \{\neg K\varphi : \varphi \in \mathcal{L}\backslash Th(\mathcal{M})\}$. Now, let $K\varphi$ be a modal atom of Σ such that $\varphi \in \mathcal{L}$. Suppose there exists a world w in $\mathcal{M}'$ such that $\langle \mathcal{M}', w\rangle \models \neg\varphi$. Then, since $K\varphi \supset \varphi \in \Sigma_1$ and $\mathcal{M}' \models \Sigma_1$, it follows that $\langle \mathcal{M}', w\rangle \models \neg K\varphi$. And since $\neg K\varphi \supset K\neg K\varphi \in \Sigma_1$, $\langle \mathcal{M}', w\rangle \models K\neg K\varphi$. It follows that $\langle \mathcal{M}', w'\rangle \models \neg K\varphi$ holds for every w' such that $(w, w') \in R'$. Now let w'' be a world in $\mathcal{M}'$ such that $(w'', w) \in R'$. Clearly, $\langle \mathcal{M}', w''\rangle \models \neg K\varphi$. Let w''' be a world such that $(w''', w'') \in R'$. Then $\langle \mathcal{M}', w'''\rangle \models \neg KK\varphi$, and since $K\varphi \supset KK\varphi \in \Sigma_1$, it follows that $\langle \mathcal{M}', w'''\rangle \models \neg K\varphi$. By iteration we arrive at the conclusion that $\langle \mathcal{M}', w'\rangle \models \neg K\varphi$ holds for each w' in the connected component to which w belongs. That is, for each connected component $\mathcal{M}'_i$ of $\mathcal{M}'$, and for each $K\varphi \in MA(\Sigma)$ such that $\varphi \in \mathcal{L}$, either $\mathcal{M}'_i \models K\varphi$ or $\mathcal{M}'_i \models \neg K\varphi$. Now let $P_1 = \{K\varphi : \varphi \in \mathcal{L}$ and $K\varphi \in MA(\Sigma)$ and $\mathcal{M}'_i \models K\varphi\}$, let $N_1 = \{K\varphi : \varphi \in \mathcal{L}$ and $K\varphi \in MA(\Sigma)\}\backslash P_1$, and let $\Sigma(P_1, N_1)$ be the theory obtained by substituting each occurrence of $K\varphi$ in Σ with true if $K\varphi \in P_1$ and each occurrence of $K\varphi$ in Σ with false if $K\varphi \in N_1$. Clearly, $\mathcal{M}'_i \models \Sigma(P_1, N_1)$. Again, we can conclude that, for each $K\varphi \in MA(\Sigma)$ such that $\varphi P_1, N_1) \in \mathcal{L}$, either $\mathcal{M}'_i \models K\varphi$ or $\mathcal{M}'_i \models \neg K\varphi$. Therefore, either $\mathcal{M}'_i \models K\varphi$ or $\mathcal{M}'_i \models \neg K\varphi$ for each modal atom in $MA(\Sigma)$ of modal depth 2. By iteration we arrive at the conclusion that, for each

$K\varphi \in MA(\Sigma)$ and for each connected component $\mathcal{M}'_i$ of $\mathcal{M}'$, either $\mathcal{M}'_i \models K\varphi$ or $\mathcal{M}'_i \models \neg K\varphi$. Now, there are two possible cases:

1. for all $\mathcal{M}'_i$'s, $\{K\varphi : K\varphi \in MA(\Sigma)$ and $\mathcal{M}'_i \models K\varphi\} = P$. Therefore, $\mathcal{M}' \models \Sigma(P, N)$, and since $\mathcal{M}' \models \Sigma_1$, it follows that $\mathcal{M}' \models \varphi(P, N)$ for each $K\varphi \in P$. Thus, $Th(\mathcal{M}') \cap \mathcal{L} = Th(\mathcal{M}) \cap \mathcal{L}$, and since $\mathcal{M}' \models \{\neg K\varphi : \varphi \in \mathcal{L} \backslash Th(\mathcal{M})\}$, from Proposition 2.4 it follows that $Th(\mathcal{M}') = Th(\mathcal{M})$, thus contradicting the hypothesis.
2. there exists an $\mathcal{M}'_i$ such that $\{K\varphi : K\varphi \in MA(\Sigma)$ and $\mathcal{M}'_i \models K\varphi\} \neq P$. Therefore, there exists a modal atom $K\varphi \in MA(\Sigma)$ such that φ is of depth j, all $K\psi$'s in $MA(\Sigma)$ of modal depth less than j are such that $\mathcal{M}'_i \models K\psi$ iff $K\psi \in P$, $\mathcal{M}'_i \not\models K\varphi$, and $\mathcal{M} \models K\varphi$. Let $P' = \{K\varphi : K\varphi \in MA(\Sigma)$ and $\mathcal{M}'_i \models K\varphi\}$, let $N' = MA(\Sigma) \backslash P_1$, and let $\mathcal{M}'' = \langle V'', W'' \rangle$ be the S5 model such that $V''(w) = w$ and $W'' = \{w : w \models \Sigma(P', N') \cup \{\psi(P', N') : K\psi \in P'\}\}$. It is easy to see that $\mathcal{M}'' \models \Sigma$ and $Th(\mathcal{M}'') \cap \mathcal{L} \subset Th(\mathcal{M}) \cap \mathcal{L}$, due to the existence of the above mentioned modal atom $K\varphi$. Therefore, $\mathcal{M}$ is not an $\mathsf{S5}_G$ model for Σ, thus contradicting the hypothesis.

Hence, $\mathcal{M}$ is a K_G model for Σ_1. □

THEOREM 3.5. *Let $\Sigma \subseteq \mathcal{L}_1$, and let $\mathcal{S}$ be any modal logic such that* $\mathsf{K} \subseteq \mathcal{S} \subseteq \mathsf{S5}$. *Then a Kripke model $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ iff $\mathcal{M}$ is an $\mathcal{S}_G$ model for the theory*

$$\begin{aligned}\Sigma_1 = \Sigma \; &\cup \left(\bigcup_{K\varphi \in MA(\Sigma)} \{K\varphi \supset \varphi\} \right) \\ &\cup \left(\bigcup_{K\varphi \in MA(\Sigma)} \{K\varphi \supset KK\varphi\} \right) \\ &\cup \left(\bigcup_{K\varphi \in MA(\Sigma)} \{\neg K\varphi \supset K\neg K\varphi\} \right).\end{aligned}$$

Proof. Follows straightforward from Lemma 3.4 and Proposition 2.3. □

Hence, it is possible to embed Halpern and Moses's logic into any ground logic by means of a very simple transformation, which consists of adding the instances of modal axiom schemas **T**, **4**, and **5**, relative to the modal subformulas of the form $K\varphi$ appearing in the theory. In Section 6, we investigate the computational implications of this result.

4. Minimal Knowledge in MDD Logics

In this section we show how to embed the logic of minimal knowledge states into MDD logics. We start by taking into consideration the embedding of theories $\Sigma \subseteq \mathcal{L}_1$ into the logic $\mathsf{K}_{\mathrm{MDD}}$. To this aim we use the following lemma, which is a direct consequence of a property shown by Schwarz (1992: proposition 5.2):

LEMMA 4.1. *Let $\Sigma \subseteq \mathcal{L}_1$. Then $\mathcal{M}$ is a $\mathsf{K}_{\mathrm{MDD}}$ model for Σ iff $\mathcal{M}$ is a K_G model for Σ.*

The above lemma allows us to prove the following property.

THEOREM 4.2. *Let $\Sigma \subseteq \mathcal{L}_1$. Then a Kripke model $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ iff $\mathcal{M}$ is a $\mathsf{K}_{\mathrm{MDD}}$ model for $\mathcal{T}(\Sigma)$.*

Proof. If part. If $\mathcal{M}$ is a $\mathsf{K}_{\mathrm{MDD}}$ model for $\mathcal{T}(\Sigma)$, then, since $\mathcal{T}(\Sigma) \subseteq \mathcal{L}_1$, from Lemma 4.1 it follows that $\mathcal{M}$ is a K_G model for $\mathcal{T}(\Sigma)$, which in turn implies that $\mathcal{M}$ is an $\mathsf{S5}_G$ model for $\mathcal{T}(\Sigma)$. Therefore, by Lemma 3.1, $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ.

Only-if part. If $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ, then, by Lemma 3.1, $\mathcal{M}$ is an $\mathsf{S5}_G$ model for $\mathcal{T}(\Sigma)$. Now, since $\mathcal{T}(\Sigma) \subseteq \mathcal{L}_1$, it follows from Theorem 3.3 that $\mathcal{M}$ is a K_G model for $\mathcal{T}(\Sigma)$, which in turn implies that $\mathcal{M}$ is a $\mathsf{K}_{\mathrm{MDD}}$ model for Σ. □

Now we show that, for theories contained in $\mathcal{L}_1$, $\mathsf{S4F}_{\mathrm{MDD}}$ models and $\mathsf{S5}_G$ models coincide. To this aim, we make use of the two following lemmas, which derive directly from the possible-world semantic characterizations of MDD and ground logics given, respectively, in Schwarz (1992a) and Nardi and Rosati (1995).

LEMMA 4.3. *Let $\Sigma \subseteq \mathcal{L}_K$ and let $\mathcal{M}$ be an $\mathsf{S5}$ model. Then $\mathcal{M}$ is an $\mathsf{S4F}_{\mathrm{MDD}}$ model for Σ iff $\mathcal{M} \models \Sigma$ and, for every $\mathsf{S4F}$ model $\mathcal{N}$, if $\mathcal{N} = \mathcal{M}' \odot \mathcal{M}$ and $\mathcal{N} \models \Sigma$, then for each world $w \in W_N$ and for each finite set of propositional symbols $P \subset \mathcal{A}$, $V_n(w)|_P = V_M(w')|_P$, for some $w' \in W_M$.*

Informally, Lemma 4.3 states that an $\mathsf{S5}$ model $\mathcal{M}$ is preferred for Σ in $\mathsf{S4F}_{\mathrm{MDD}}$ iff there is no $\mathsf{S4F}$ model $\mathcal{N}$ satisfying Σ such that $\mathcal{M}$ is the lower cluster of $\mathcal{N}$ and $\mathcal{N}$ contains at least one interpretation different from those in $\mathcal{M}$.

LEMMA 4.4. *Let $\Sigma \subseteq \mathcal{L}_K$ and let $\mathcal{M}$ be an $\mathsf{S5}$ model. Then $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ iff $\mathcal{M} \models \Sigma$ and, for every $\mathsf{S5}$ model $\mathcal{N}$, if $\mathcal{N} \models \Sigma$ and for each $w \in W_M$ and for each finite set of propositional symbols $P \subseteq \mathcal{A}$, $V_M(w)|_P = V_N(w')|_P$, for some $w' \in W_N$, then for each world $w \in W_N$ and for each finite set of propositional symbols $P \subseteq \mathcal{A}$, $V_N(w)|_P = V_M(w')|_P$, for some $w' \in W_M$.*

Roughly speaking, Lemma 4.4 establishes that an S5 model $\mathcal{M}$ is preferred for Σ in $S5_G$ iff there is no S5 model $\mathcal{N}$ satisfying Σ and containing all the interpretations in $\mathcal{M}$ such that $\mathcal{N}$ contains at least one interpretation different from those in $\mathcal{M}$.

We now prove the following key property.

LEMMA 4.5. *Let $\Sigma \subset \mathcal{L}_1$. Then a Kripke model $\mathcal{M}$ is an $S5_G$ model for Σ iff $\mathcal{M}$ is an $S4F_{MDD}$ model for Σ.*

Proof. If part. Suppose $\mathcal{M}$ is an $S5_G$ model for Σ, and suppose $\mathcal{M}$ is not an $S4F_{MDD}$ model for Σ. From Lemma 4.3, it follows that there exists an S4F model $\mathcal{N} = \mathcal{M}' \odot \mathcal{M}$ such that there exists a world $w \in W'_M$ and a finite set of propositional symbols $P \subseteq \mathcal{A}$ such that $V_N(w)|_P \neq V_M(w')|_P$, for each $w' \in W_M$. Now, let $\mathcal{N}'$ be the model $\langle W_N, W_N \times W_N, V_N \rangle$, i.e., the S5 model obtained from $\mathcal{N}$ by connecting each world of the lower cluster $\mathcal{M}$ with each world of the upper cluster $\mathcal{M}'$. Let $\varphi \in \mathcal{L}_1$, and let w be a world in $\mathcal{N}'$. It is easy to see that the set of worlds accessible from w in $\mathcal{N}'$ is the same set of worlds accessible from the worlds of the upper cluster in $\mathcal{N}$, i.e., $\{w' \mid (w, w') \in R'_N\} = \{w' \mid w' \in W'_M \text{ and } (w, w') \in R_N\}$. Therefore, for each $\varphi \in \mathcal{L}_1$, $(\mathcal{N}', w) \models \varphi$ iff $(\mathcal{N}, w) \models \varphi$. And since $\Sigma \subseteq \mathcal{L}_1$ and $(\mathcal{N}, w) \models \Sigma$, it follows that for each world $w \in \mathcal{N}'$, $(\mathcal{N}', w) \models \Sigma$, therefore $\mathcal{N}' \models \Sigma$. In addition, by hypothesis there exists a world $w \in W'_M$ and a finite set of propositional symbols $P \subseteq \mathcal{A}$ such that $V_N(w)|_P \neq V_M(w')|_P$, for each $w' \in W_M$, consequently, by Lemma 4.4, $\mathcal{M}$ is not an $S5_G$ model for Σ. Contradiction. Therefore, $\mathcal{M}$ is an $S4F_{MDD}$ model for Σ.

Only-if part. Suppose $\mathcal{M}$ is not $S5_G$ model for Σ. Then there exists an S5 model $\mathcal{M}'$ such that $\mathcal{M}' \models \Sigma$ and there exists a world $w \in W'_M$ and a finite set of propositional symbols $P \subseteq \mathcal{A}$ such that $V'_M(w)|_P \neq V_M(w')|_P$, for each $w' \in W_M$. Now, the S4F model $\mathcal{N} = \mathcal{M}' \odot \mathcal{M}$ is such that the set of worlds accessible from each world in $\mathcal{N}$ is the same set of worlds accessible from each world in $\mathcal{M}'$, i.e., $\{w' \mid (w, w') \in R_N\} = \{w' \mid w' \in W'_M \text{ and } (w, w') \in R'_M\}$. Therefore, for each $\varphi \in \mathcal{L}_1$, $(\mathcal{N}, w) \models \varphi$ iff $(\mathcal{M}, w) \models \varphi$. And since $\Sigma \subseteq \mathcal{L}_1$ and $\mathcal{M}' \models \Sigma$, $\mathcal{N} \models \Sigma$. Moreover, by construction of $\mathcal{N}$ it follows that there exists a world $w \in W_N$ and a finite set of propositional symbols $P \subseteq \mathcal{A}$ such that $V_N(w)|_P \neq V_M(w')|_P$, for each $w' \in W_M$. Therefore, by Lemma 4.3, $\mathcal{M}$ is not an $S4F_{MDD}$ model for Σ. □

The above lemma allows us to prove that the logic $S5_G$ is easily embeddable into a large subset of MDD logics.

THEOREM 4.6. *Let $\Sigma \subseteq \mathcal{L}_K$, and let $\mathcal{S}$ be any modal logic such that $K \subseteq \mathcal{S} \subseteq S4F$. Then a Kripke model $\mathcal{M}$ is an $S5_G$ model for Σ iff $\mathcal{M}$ is an $\mathcal{S}_{MDD}$ model for $\mathcal{T}_N(\Sigma)$.*

Proof. If part. If $\mathcal{M}$ is an $\mathcal{S}_{MDD}$ model for $\mathcal{T}_N(\Sigma)$ ($K \subseteq \mathcal{S} \subseteq S4F$), then, by Proposition 2.3, $\mathcal{M}$ is an $S4F_{MDD}$ model for $\mathcal{T}_N(\Sigma)$. Now, since axiom schema **T** is valid in S4F, it follows that for each Kripke model $\mathcal{M}$, $\mathcal{M}$ is an S4F model for

$\mathcal{T}_N(\Sigma)$ iff $\mathcal{M}$ is an S4F model for $\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma))$, hence $\mathcal{M}$ is an $\mathsf{S4F}_{\mathrm{MDD}}$ model for $\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma))$. And since $\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma)) \subseteq \mathcal{L}_1$, it follows from Lemma 4.5 that $\mathcal{M}$ is an $\mathsf{S5}_G$ model for $\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma))$. Consequently, by Lemma 3.1, $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ.

Only-if part. If $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ, then, by Lemma 3.1, $\mathcal{M}$ is an $\mathsf{S5}_G$ model for $\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma))$. Consequently, since $\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma)) \subseteq \mathcal{L}_1$, by Theorem 4.2 it follows that $\mathcal{M}$ is a $\mathsf{K}_{\mathrm{MDD}}$ model for $\mathcal{T}_N(\Sigma)$. Therefore, by Proposition 2.3, for any $\mathcal{S}$ such that ($\mathsf{K} \subseteq \mathcal{S} \subseteq \mathsf{S4F}$), $\mathcal{M}$ is an $\mathcal{S}_{\mathrm{MDD}}$ model for $\mathcal{T}_N(\Sigma)$. □

COROLLARY 4.7. *Let $\Sigma \subseteq \mathcal{L}_K$, and let $\mathcal{S}$ be any modal logic such that* $\mathsf{T} \subseteq \mathcal{S} \subseteq \mathsf{S4F}$. *Then a Kripke model $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ iff $\mathcal{M}$ is an $\mathcal{S}_{\mathrm{MDD}}$ model for $\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma))$.*

COROLLARY 4.8. *Let $\Sigma \subseteq \mathcal{L}_1$, and let $\mathcal{S}$ be any modal logic such that* $\mathsf{T} \subseteq \mathcal{S} \subseteq \mathsf{S4F}$. *Then a Kripke model $\mathcal{M}$ is an $\mathsf{S5}_G$ model for Σ iff $\mathcal{M}$ is an $\mathcal{S}_{\mathrm{MDD}}$ model for Σ.*

The previous theorem shows a translation of logic $\mathsf{S5}_G$ in the subset of MDD logics between K and S4F.

5. On the Complexity of Reasoning about Minimal Knowledge States

In this section, we present some new complexity results for the problem of reasoning in ground logics, which follow from the properties shown in previous sections.

First, we briefly recall some basic notions from complexity theory (see, e.g., Johnson, 1990, for further details). We denote as P the class of problems solvable in polynomial time by a deterministic Turing machine. The class NP contains all problems that can be solved by a nondeterministic Turing machine in polynomial time. The class coNP comprises all problems that are the complement of a problem in NP. A problem P_1 is said to be NP-complete if it is in NP and for every problem P_2 in NP, there is a polynomial-time reduction from P_2 to P_1. If there is a polynomial-time reduction from an NP-complete problem P_2 to a problem P_1, then P_1 is said to be NP-hard. With a slight abuse of terminology, we call NP-algorithm a nondeterministic algorithm that runs in polynomial time. P^A (NP^A) is the class of problems that are solved in polynomial time by deterministic (nondeterministic) Turing machines using an oracle for A (i.e., that solves in constant time any problem in A). The classes Σ_k^p, Π_k^p and Δ_k^p of the polynomial hierarchy are defined by $\Sigma_0^p = \Pi_0^p = \Delta_0^p = \mathrm{P}$, and for $k \geq 0$, $\Sigma_{k+1}^p = \mathrm{NP}^{\Sigma_k^p}$, $\Pi_{k+1}^p = \mathrm{co}\Sigma_{k+1}^p$ and $\Delta_{k+1}^p = \mathrm{P}^{\Sigma_k^p}$.

We also recall the following computational characterization for the problem of reasoning in the logic $\mathsf{S5}_G$, which has been proved in Donini et al. (1997).

PROPOSITION 5.1. *Entailment in* $\mathsf{S5}_G$ *is a* Π_3^p*-complete problem.*

We now present two new results concerning the complexity of reasoning in ground logics. First, Lemma 3.4 allows for establishing a lower bound for the problem of reasoning in all ground logics.

THEOREM 5.2. *Given a modal logic* $\mathcal{S}$ *such that* $\mathsf{K} \subseteq \mathcal{S} \subseteq \mathsf{S5}$, *entailment in* $\mathcal{S}_G$ *is* Π_3^p*-hard.*

Proof. Follows from Lemma 3.4, from Proposition 5.1, and from the fact that

$$\begin{aligned}\Sigma_1 = \Sigma\ &\cup \left(\bigcup_{K\varphi \in MA(\Sigma)} \{K\varphi \supset \varphi\} \right) \\ &\cup \left(\bigcup_{K\varphi \in MA(\Sigma)} \{K\varphi \supset KK\varphi\} \right) \\ &\cup \left(\bigcup_{K\varphi \in MA(\Sigma)} \{\neg K\varphi \supset K\neg K\varphi\} \right).\end{aligned}$$

can be computed in time polynomial with regard to the size of Σ. □

The above theorem generalizes a previous result (Donini et al., 1997: theorem 4.2) which established Π_3^p as a lower bound for reasoning in a subset of ground logics. Theorem 5.2 shows that reasoning in *all* ground logics is harder than reasoning in the most famous propositional formalisms for nonmonotonic reasoning, such as default logic, circumscription, autoepistemic logic (Cadoli and Schaerf, 1993), several MDD logics (Marek and Truszczyński, 1993), and Levesque's logic of only knowing (Rosati, 1997b): reasoning in all such logics lies at the second level of the polynomial hierarchy.

Next we show that, in the case of modal theories contained in $\mathcal{L}_1$, reasoning in Halpern and Moses's logic also lies at the second level of the polynomial hierarchy. Indeed, Theorem 4.6 shows that there exists a polynomial translation of $\mathsf{S5}_G$ into all MDD logics between $\mathsf{K}_{\mathrm{MDD}}$ and $\mathsf{S4F}_{\mathrm{MDD}}$ for theories in $\mathcal{L}_1$. Now, since entailment in $\mathsf{S4F}_{\mathrm{MDD}}$ is Π_2^p-complete (Marek and Truszczyński, 1993), it follows that entailment in $\mathsf{S5}_G$ for theories in $\mathcal{L}_1$ is in Π_2^p.

We now prove that Π_2^p is also a lower bound for entailment in $\mathsf{S5}_G$ for theories in $\mathcal{L}_1$.

LEMMA 5.3. *Let* $\Sigma \subseteq \mathcal{L}_1$, *and let* $\varphi \in \mathcal{L}_K$. *The problem of establishing whether* $\Sigma \models_{\mathsf{S5}_G} \varphi$ *is* Π_2^p*-hard.*

Proof. We reduce the problem of query answering in positive disjunctive logic programs under the stable model semantics to an entailment problem in $\mathsf{S5}_G$ for theories in $\mathcal{L}_1$. Query answering in positive disjunctive logic programs under the stable model semantics is a Π_2^p-complete problem (Eiter and Gottlob, 1995: theorem 3.2). We consider the following translation τ of a positive disjunctive logic program into $\mathsf{S5}_G$ (which is a restriction of Lifschitz's translation of logic programs into the logic *MKNF*, see Lifschitz, 1991). Each program rule r is of the form

$$r = p_1 \mid \ldots \mid p_n \leftarrow q_1, \ldots, q_m,$$

where $m \geq 0$, $n \geq 0$, and each p_i, q_i is an atom (propositional symbol). Such a rule is translated into the formula

$$\tau(r) \neq Kq_1 \vee \ldots \vee \neg Kq_m \vee Kp_1 \vee \ldots \vee Kp_n.$$

Given a positive disjunctive logic program P, $\tau(P) = \{\tau(r) \mid r \in P\}$. Notice that for each program P the theory $\tau(P)$ is contained in $\mathcal{L}_1$.

Correctness of the above translation follows straightforwardly from the results presented in Lifschitz (1991); in particular, it follows that a literal l is entailed by a positive disjunctive logic program P under the stable model semantics iff:

1. $\tau(P) \models_{\mathsf{S5}_G} Kl$, if 1 is an atom; and
2. $\tau(P) \models_{\mathsf{S5}_G} \neg Kl$, if 1 is a negated atom.

And since τ is a transformation which can be performed in time linear with regard to the size of P, it follows that entailment in $\mathsf{S5}_G$ is Π_2^p-hard for theories in $\mathcal{L}_1$. □

Hence, the following property holds.

THEOREM 5.4. *Let* $\Sigma \subseteq \mathcal{L}_1$, *and let* $\varphi \in \mathcal{L}_K$. *The problem of establishing whether* $\Sigma \models_{\mathsf{S5}_G} \varphi$ *is* Π_2^p*-complete.*

We now give an epistemological reading of the results presented above, based on the following considerations:

- Corollary 4.8 implies that, under some syntactical restrictions (i.e., for theories $\Sigma \subseteq \mathcal{L}_1$), minimal knowledge states and a significant subset of MDD logics coincide;
- from Theorem 4.6 it follows that minimal knowledge states are embeddable *in polynomial time* into a significant subset of MDD logics, if the initial theory Σ is such that $\mathcal{NF}_{\mathsf{S5}}(\mathcal{K}(\Sigma))$ can be computed in polynomial time; and
- for general theories $\Sigma \subseteq \mathcal{L}_K$, the additional degree of complexity of reasoning about minimal knowledge states is due to the fact that the translation

of Σ in S5 normal form causes a growth of the size of the theory which is exponential with regard to the size of Σ. This indicates the ability of Halpern and Moses's logic of expressing minimal knowledge states in a more compact way than MDD logics.

Summarizing, from an epistemological point of view it emerges that the two different approaches to nonmonotonicity in modal logics produce, for a subclass of theories, the same epistemic interpretation of the modality. In particular, it turns out that, for the subset of theories $\Sigma \subseteq \mathcal{L}_1$, the logic of minimal knowledge states and reflexive MDD logics contained in $\mathsf{S4F}_{\mathrm{MDD}}$ coincide. In the case of general theories (with nested modalities) the embedding is realized through a conceptually very simple translation, which basically consists of putting the initial theory in normal form. This implies that, for such a large subset of reflexive MDD logics, the modality K can be interpreted as a minimal knowledge operator, as long as it does not occur within the scope of another modality. And for other, nonreflexive, MDD logics, such a minimal knowledge interpretation can be obtained by simply adding *few* instances of the modal axiom schema **T**.

On the other hand, the existence of the normal form $\mathcal{NF}_{\mathsf{S5}}$ for $\mathsf{S5}_G$ implies the impossibility of expressing other notions in these logics through iterated modalities, whereas in many MDD logics iterated modalities can be given interesting interpretations (e.g., $K\neg K\neg$ can be interpreted as a "default assumption operator" in $\mathsf{S4F}_{\mathrm{MDD}}$, see Schwarz and Truszczyński, 1994). In this sense, the logic of minimal knowledge states can be considered "less expressive" than the most studied logics in MDD family, if (as, e.g., in Gottlob, 1993) we consider the expressive power of a nonmonotonic formalism as its ability to express sets of epistemic states through sets of premises.

6. Conclusions

In this work we have shown that Halpern and Moses's logic of minimal knowledge states $\mathsf{S5}_G$ is easily embeddable into a large subset of the modal logics for nonmonotonic reasoning. In particular, we have proved that Halpern and Moses's logic can be embedded into all ground logics. This result is not surprising, since ground logics can be considered as a generalization of the minimal knowledge paradigm on which Halpern and Moses's logic is based. However, the translation presented allows for establishing a lower bound for the problem of reasoning in any ground logic, which proves that deduction in all ground logics is harder than in all the best known propositional formalisms for nonmonotonic reasoning.

We have also shown that the logic of minimal knowledge states can be embedded into a significant subset of McDermott and Doyle's family of nonmonotonic modal formalisms. This result provides a first explanation of the higher degree of complexity of deduction in minimal knowledge states with regard to the major propositional nonmonotonic logics, since the embedding presented allows for identifying the additional source of complexity which makes deduction in minimal knowledge states harder than in MDD logics. Moreover, the translation can be given an epistemological interpretation, thus showing that minimal knowledge states can be easily expressed in MDD logics, and it is therefore possible to give an interpretation of the modality in terms of minimal knowledge for MDD logics.

As a byproduct, the study of the relationships between "classical" (MDD) nonmonotonic modal logics and minimal knowledge states contributes to explain the connections between MDD logics and a conspicuous number of modal formalisms based upon the notion of minimal knowledge (Lifschitz, 1991; Lin and Shoham, 1992, 1998; Engelfriet, 1996; Meyer and van der Hoek, 1995a, b; Donini et al., 1995), thus allowing for a better understanding of the epistemological and computational properties of such logics.

References

Cadoli, M. and Schaerf, M., 1993, "A survey of complexity results for nonmonotonic logics," *Journal of Logic Programming* **17**, 127–160.

Donini, F.M., Nardi, D., and Rosati, R., 1995, "Non-first-order features in concept languages," pp. 91–102 in *Proceedings of AI*IA-95*, G. Soda, ed., Lecture Notes in Artificial Intelligence, Vol. 992, Berlin: Springer-Verlag.

Donini, F.M., Nardi, D., and Rosati, R., 1997, "Ground nonmonotonic modal logics," *Journal of Logic and Computation* **7**(4), 523–548.

Eiter, T. and Gottlob, G., 1995, "On the computational cost of disjunctive logic programming: Propositional case," *Annals of Mathematics and Artificial Intelligence* **15**(3, 4), 289–323.

Engelfriet, J., 1996, "Minimal temporal epistemic logic," *Notre Dame Journal of Formal Logic* **37**(2), 233–259.

Gottlob, G., 1992, "Complexity results for nonmonotonic logics," *Journal of Logic and Computation* **2**, 297–425.

Gottlob, G., 1993, "Translating default logic into standard autoepistemic logic," *Journal of ACM* **42**(4), 711–740.

Halpern, J.Y. and Moses, Y., 1985, "Towards a theory of knowledge and ignorance: Preliminary report," Technical Report CD-TR 92/34, IBM.

Hughes, G.E. and Cresswell, M.J., 1968, *An Introduction to Modal Logic*, London: Methuen.

Johnson, D.S., 1990, "A catalog of complexity classes," pp. 67–161 in *Handbook of Theoretical Computer Science*, Vol. A, J. van Leeuwen, ed., Amsterdam: Elsevier Science Publishers/North Holland.

Konolige, K., 1988, "On the relationship between default and autoepistemic logic," *Artificial Intelligence Journal* **35**, 343–382.

Lifschitz, V., 1991, "Nonmonotonic databases and epistemic queries," pp. 381–386 in *Proceedings of IJCAI-91*, Sydney, D. Driankow, P.W. Eklund, and A.L. Ralescu, eds., Berlin: Springer-Verlag.

Lin, F. and Shoham, Y., 1992, "Epistemic semantics for fixed-point non-monotonic logics," *Artificial Intelligence Journal* **57**, 271–289.

Lin, F. and Shoham, Y., 1998, "On non-forgetting and minimal learning," in *Proceedings of the 1993. International Colloquium on Cognitive Science*, N. Asher, K. Korta, and J. Ezquerro, eds., Dordrecht: Kluwer Academic Publishers, to appear.

Marek, W. and Truszczyński, M., 1993, *Nonmonotonic Logics – Context-Dependent Reasoning*, Berlin: Springer-Verlag.

McDermott, D., 1982, “Non-monotonic logic II: Non-monotonic modal theories,” *Journal of the ACM* **29**, 33–57.

McDermott, D. and Doyle, J., 1980, “Non-monotonic logic I,” *Artificial Intelligence Journal* **13**, 41–72.

Meyer, J.-J.Ch. and van der Hoek, W., 1995a, “A default logic based on epistemic states,” *Fundamenta Informaticae* **23**(1) 33–65.

Meyer, J.-J.Ch. and van der Hoek, W., 1995b, *Epistemic Logic for AI and Computer Science*, Cambridge: Cambridge University Press.

Moore, R.C., 1985, “Semantical considerations on nonmonotonic logic,” *Artificial Intelligence Journal* **25**, 75–94.

Nardi, D. and Rosati, R., 1995, “A preference semantics for ground nonmonotonic modal logics,” pp. 225–236 in *Proceedings of EPIA-95*, C. Pinto-Ferreira and N. Mamede, eds., Lecture Notes in Artificial Intelligence, Vol. 990, Berlin: Springer-Verlag.

Rosati, R., 1997a, “Reasoning with minimal belief and negation as failure: Algorithms and complexity,” pp. 430–435 in *Proceedings of the Fourteenth National Conference on Artificial Intelligence (AAAI-97)*, Cambridge, MA: MIT Press.

Rosati, R., 1997b, “Complexity of only knowing: The propositional case,” pp. 76–91 in *Proceedings of the Fourth International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR-97)*, J. Dix, U. Furbach, and A. Nerode, eds., Lecture Notes in Artificial Intelligence, Vol. 1265, Berlin: Springer-Verlag.

Schwarz, G., 1990, “Autoepistemic modal logics,” pp. 97–109 in *Proceedings of TARK-90*, R. Parikh, ed., San Mateo, CA: Morgan Kaufmann.

Schwarz, G., 1991, “Autoepistemic logic of knowledge,” pp 260–274 in *Proceedings of LPNMR-91*, A. Nerode, V. Marek, and V.S. Subrahmanien, eds., Cambridge, MA: MIT Press.

Schwarz, G., 1992a, “Minimal model semantics for nonmonotonic modal logics,” pp. 34–43 in *Proceedings of LICS-92*, New York: IEEE Computer Society Press.

Schwarz, G., 1992b, “Bounding introspection in nonmonotonic logics,” pp. 581–590 in *Proceedings of KR-92*, B. Nebel, C. Rich, and W. Swertout, eds., Los Altos, CA: Morgan Kaufmann.

Schwarz, G. and Truszczyński, M., 1994, “Minimal knowledge problem: A new approach,” *Artificial Intelligence Journal* **67**, 113–114.

Segerberg, K., 1971, *An Essay in Classical Modal Logic*, Filosofiska Studier, Vol. 13, Uppsala, Sweden: Uppsala University.

Shoham, Y., 1987, “Nonmonotonic logics: Meaning and utility,” pp. 388–392 in *Proceedings of IJCAI-87*, J. McDermott, ed., San Mateo, CA: Morgan Kaufmann.

Tiomkin, M. and Kaminski, M., 1990, “Nonmonotonic default modal logics,” pp. 73–84 in *Proceedings of TARK-90*, R. Parikh, ed., San Mateo, CA: Morgan Kaufmann.