Paradox, ZF, and the Axiom of Foundation*

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It is a great pleasure to contribute to this *Festschrift* for John Bell. No-one has done more than he has to demonstrate the fruitfulness of the interplay between technical mathematics and philosophical issues, and he is an inspiration to all of us who work somewhere in the borderland between mathematics and philosophy.

I also owe him a great personal debt. I arrived at the LSE dejected and disillusioned by my experiences of the Mathematical Tripos at Cambridge, but it is impossible to be downhearted for long in the company of John. His enthusiasm, humour and warmth were the perfect antidote to the stuffiness and inhumanity of Cambridge and helped hugely to rebuild my interest and self-confidence. John's energy levels must be seen to be believed, and an evening with him is an unforgettable experience. It generally starts about 4 p.m. and ends around 5 in the morning, when the last of his companions (never John, who always gives the impression that he could go on talking indefinitely) finally succumbs to sleep.

At John's suggestion, I wrote my M.Sc. dissertation at the L.S.E. on truth, which led on eventually to an Oxford D.Phil. which concerned both the semantic and set-theoretical paradoxes. It is the concept of set—example *par excellence* of one that straddles philosophy and mathematics—that is the subject of this essay.

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At the beginning of the twentieth century there was a crisis in the foundations of mathematics. The crisis centred around the concept of *set*, which suddenly achieved prominence in two different ways. Firstly Cantor's theory of the

*Some of the material here has been presented in Glasgow, at a St Andrews workshop on the island of Raasay, and at the Kraków meeting of the IUHPS. I thank the audiences for their comments on those occasions. In Rieger [2000] reference is made a to 'forthcoming' paper with the title *Zermelo-Fraenkel set theory: an emperor with no clothes?* In fact no such paper has appeared; much of the material intended for that paper is contained here.

transfinite showed that sets were of great intrinsic mathematical interest. And secondly through the work of Frege and Russell it emerged that sets were central to the philosophical project—logicism—of reducing mathematics to logic.

The discovery by Russell and others that, if handled carelessly, sets give rise to contradictions, threatened not only the logicist programme, but also mathematics itself; for large parts of mathematics, in particular analysis, make essential use of the completed infinite, and the paradoxes seemed to show that this was a risky practice.

A hundred years later there is no longer a foundational crisis. Why is this? By 1930, mathematicians had found a way of coping with the paradoxes. A system of axiomatic set theory, *Zermelo-Fraenkel* set theory (ZF for short) had been developed, which allowed mathematicians to do all they wanted to do with sets, whilst maintaining consistency. ZF is not the only axiomatic set theory, but at the present time it has a completely dominant position amongst such theories. For the purposes of university mathematics courses, for example, set theory just *is* ZF.

Does ZF really deserve its elevated position? Below I examine three sorts of argument which can be adduced in support of ZF:

1. Argument from the paradoxes

2. Argument from the iterative conception

3. Argument from pragmatic mathematical considerations

and conclude that none of them are convincing.

2

There is a widespread view that one needs some kind of hierarchy, and hence ZF or something like it, to avoid the paradoxes.[1] Let us go back to basics to examine the merits of this claim.

According to the *naive conception* of set, any arbitrary collection forms a set.[2] This entails the truth of the naive comprehension schema

$$\exists x \forall y (y \in x \leftrightarrow \phi(y)).$$

[1]Logical *cognoscenti* know that this is false, but the view is common amongst those who are mathematically, but not logically, well-informed.

[2]As far as I know, nobody has ever explicitly put forward the naive conception, though it is implicit in Frege's *Grundgesetze* [1893].

But this schema is in fact logically false: for as Russell noticed [1902], on letting ϕ be $y \notin y$ we obtain a contradiction.

Now whilst this shows that there is a fatal defect with the naive conception, it does not yield an illuminating explanation of what exactly is wrong with it, an explanation that might be of some use in the reform of the concept of set which must inevitably follow.

Such an explanation is, however, available.

Suppose a is a set, and consider the set

$$b = \{x \in a : x \notin x\}.$$

It is easy to see that $b \notin a$. So we have a recipe which, for any set a, gives us a set which is not a member of a. The set b "diagonalizes out" of a.

If a already has all sets as members, trouble arrives in the shape of the Russell paradox. And naively, there must be such a set a, for example the set of everything whatever. (Just take $\phi(y)$ to be $y = y$ in the naive schema above.) On this way of looking at it, contradiction arises because, under the naive conception, we have both *extensibility* (the ability to extend any given set by finding something that is not one of its members) and *universality* (the existence of a set which contains everything). Clearly these cannot co-exist consistently.

Essentially the same diagnosis can be given for the other standard set-theoretic paradoxes. In the so-called Cantor paradox, extensibility arises from the power set operation—since the power set $P(a)$ of a is always strictly larger than a, there must be elements of $P(a)$ which are not in a. So again the existence of a universal set leads to contradiction.

In some of the paradoxes the extensibility and universality are relativised to particular sub-collections of the universe. For example, the Burali-Forti paradox hinges on the tension between (i) the principle that, for any initial segment of ordinals, we can, by considering the ordinal number of the segment itself, find an ordinal not in that segment, and (ii) the principle that there is a well-ordered set of *all* ordinals. The slightly less well-known Mirimanoff paradox concerns the set W of all well-founded sets.[3] Since all the members of W are well-founded, so is W, but then it should be a member of itself, and so, after all, not well-founded. Here extensibility is obtained by considering, for any set a of well-founded sets, the set $b = a \cup \{a\}$; this is another set of well-founded sets (since a itself must be well-founded), yet it has a member (namely a itself) which is not a member of a (else we have $a \in a$, so a is not well-founded).[4]

[3] For the definition of well-founded, see below.

[4] For more details, including attempts to apply the same idea to the semantic para-

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A consequence of this diagnosis is that there is a neat and rather natural way to solve the paradoxes: ban universality by not allowing very large collections (e.g. the universe and the collection of all ordinals) to be sets. Remarkably, this solution was hit upon by the originator of set theory, Georg Cantor, before the "paradox industry" had even got under way. In a letter he wrote to Dedekind [1899] we find the following passage:

> ... it is necessary, as I discovered, to distinguish two kinds of multiplicity... For a multiplicity can be such that the assumption that *all* of its elements "are together" leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as "one finished thing". Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities* ... If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as "being together"... I call it a *consistent multiplicity* or *set*.[5]

Cantor has discovered what in modern parlance is called the set/class distinction, usually attributed to von Neumann [1925]. The key idea is that some infinite collections are all right, and they form the sets; others are just too big, and are either abolished altogether, or allowed in as some other kind of entity (proper classes).[6] The idea is, in the light of our diagnosis, thoroughly motivated and not at all *ad hoc*.

And, indeed, this is exactly what happens with ZF. This is sometimes expressed, somewhat misleadingly, by saying that ZF incorporates the doctrine of *limitation of size*—misleading because this phrase suggests that there is some cardinal magnitude below which collections are safe but above which

doxes, see Chapter 1 of Rieger [1996] or the article Priest [1994] (Priest uses the terms *transcendence* and *closure*). Dummett's idea of an indefinitely extensible concept ([1991] p. 316), and Grim's book *The Incomplete Universe* [1991] are also relevant. The basic idea can be found in Russell's paper [1906c] which I discuss below.

[5] Though it is not quite as explicit, the distinction between the transfinite and the absolute infinite can be found much earlier in Cantor's writings (e.g. Cantor [1883] p. 205). It seems likely that, having realized that any set can be enlarged by the power set operation, Cantor drew immediately the conclusion that there can be no universal set. Cantor is sometimes accused of believing in naive set theory (e.g. Körner [1960] p. 44: "Cantor's theory of classes, by admitting as a class any collection, however formed, leads to contradictions"). This is quite unjustified: rather "his conception of set... was one in which the paradoxes cannot arise" (Menzel [1984] p. 92). See also Hallett [1984] p. 38 and *passim*.

[6] More precisely, the principle that a collection is too big to form a set iff it can be put into 1-1 correspondence with the universe can be taken as the basis for an axiomatization of set theory, as is done in von Neumann [1925].

they are paradoxical, whereas the point is not that sets must be below some particular size but that they must not be as big as the universe.

Nothing we have said so far, however, requires sets to be arranged in a *hierarchy.* But the ZF axioms embody such a requirement. In particular, the *axiom of foundation* states that every (non-empty) set x has a member y which is minimal, in the sense that no member of x belongs to y.[7] Another way of putting this, equivalent in the presence of the axiom of choice, is that there is no infinite descending membership chain $x \ni x_1 \ni x_2 \ni \ldots$. So there cannot be, for example, a set which is a member of itself, or a member of a member of itself.

A suspicion therefore arises that ZF restricts the notion of set more than is necessary to avoid the paradoxes, and therefore offends against the following methodological principle: *when forced by paradox to reform a naive concept, preserve as much of it as possible.* The naive concept of set does not obey the axiom of foundation: it allows such self-membered sets as the set of absolutely everything, and the set of all things discussed in this paper. ZF rules out both of these, but the principle of restricting universality seems to deny sethood only to the first.[8] Can there be a consistent theory which allows the second?

Indeed there can. One sort of theory with this property is discussed by Aczel [1988]. Briefly, the idea is to take the axioms of ZF *except* foundation, and add to them some version of an *anti-foundation* axiom. This is best understood in terms of membership graphs. There is a natural sense in which (directed) graphs can be regarded as pictures of sets: for example, Figure 1 is a picture of the von Neumann ordinal $2 = \{\emptyset, \{\emptyset\}\}$.

Only well-founded graphs (graphs without infinite paths) can be pictures of sets in the ZF universe; to obtain the richer non-well-founded universes we allow *any* graph to be a picture of a set. Thus Figure 2 is the picture of a set $a = \{a, \emptyset\}$.

By constructing a graph model from a model of ZF, Aczel proves that these systems are consistent if ZF is.[9]

Summary: it can be rigorously proved that ZF restricts the notion of set more than the paradoxes demand.

[7] A set x satisfying this condition is said to be *well-founded.*

[8] To make the example work, interpret "discussed in this paper" so that it applies to only a small (e.g. finite) number of things.

[9] For more details see Aczel [1988]. I discuss the merits of the various anti-foundation axioms in Rieger [2000].

Figure 1: An exact picture of 2

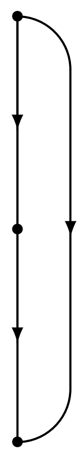
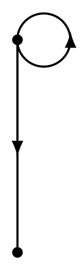


Figure 2: A non-well-founded graph



4

How did the idea take hold that a hierarchy is necessary to solve the paradoxes? To answer this it will be necessary to take a short historical detour.

In December 1905 Russell read a remarkable paper to the London Mathematical Society, later published as Russell [1906c]. In it he states clearly that the lesson of the paradoxes is that naive comprehension must be rejected:

> What is demonstrated by the contradictions we have considered is broadly this: 'A propositional function of one variable does not always determine a class.' (Russell [1906c] pp. 144–5)

And he gives essentially the diagnosis above:

> ...there are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any class of terms[10] all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property. (Russell [1906c] p. 144)

This insight would seem to lead naturally to the conclusion outlined above, that a solution to the paradoxes may be obtained by ensuring that there is no universal class, no class of all ordinals, etc. However, Russell does not simply draw this inference; rather he considers three different responses, all of which would indirectly ban the offending classes: the "zigzag" theory, the "limitation of size" theory, and the "no-classes" theory, tentatively suggesting that the last of these offers the most promising route for a solution.[11]

Less than a year later, Russell had changed his mind about the paradoxes. In a paper published in September 1906, he wrote this:

> I recognise, however, that the clue to the paradoxes is to be found in the vicious-circle suggestion. (Russell [1906b] p. 198)

The "vicious-circle suggestion" is

[10] "Term" here just means "object".

[11] It might be thought that "limitation of size" embodies exactly the idea of restricting universality, but it is clear that Russell does not think of it in this way: rather he sees the theory as posing the question "how far up the series of ordinals it is legitimate to go" (p. 53), a question which he cannot see any prospect of answering.

> ...that whatever in any way concerns *all* o[r][12] *any* or *some* (undetermined) of the members of a class must not itself be one of the members of a class.[13] (Russell [1906b] p. 198)

A new concept, *circularity*, has now entered the discussion. Russell believes that all the paradoxes result from the (allegedly) circular practice of allowing totalities containing members which, in some appropriate sense to be discussed below, "concern" that very totality.

It might perhaps seem at first sight that this is just another way of stating the original diagnosis. For is not this circularity the key feature of the "self-reproductive" classes identified in the earlier paper? But in fact the difference is dramatic. According to the previous diagnosis, there cannot be a class of *all* ordinals or *all* things (for this would lead to the contradictory consequence that there is a new ordinal (thing) which both must and must not be in the class). But in the later paper, Russell advocates the very much stronger principle that there can be *no class whatever*, large or small, which has members concerning that very class. This inevitably imposes a hierarchical structure on the universe of classes. *At this point an alien constructivism was imported into classical mathematics, vestiges of which are still visible today.* For the vicious circle principle is inextricably linked with a constructivist view of the metaphysics of mathematics.[14]

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What had happened to Russell in the few months between the two papers? He had been reading Poincaré. The second paper was, in fact, written in reply to Poincaré [1905-6]. In that work Poincaré blames the paradoxes on circularity. His treatment is sketchy, and he discusses only Richard's paradox[15] in any detail, claiming that "the same explanation serves for the

[12]The original has "of", which seems to be a misprint.

[13]The occurrence of "any" (and "some") may seem puzzling here: since anything presumably concerns itself, the principle seems to rule out anything ever being a member of a class. But Russell should be read as forbidding any member of a class concerning quantification over the class.

[14]To avoid confusion, I should perhaps make it clear that here and throughout the paper I use "constructivism" as a name for a metaphysical view about mathematics, roughly that mathematical objects are brought into existence by some activity of human minds. The term is sometimes now used for mathematics without the law of excluded middle, but I shall use it in its earlier sense.

[15]This is a semantic paradox, introduced in Richard [1905], which concerns the collection E of all reals definable in a finite number of words; by a diagonal argument we can obtain a new real, not in E yet definable in a finite number of words.

other antinomies, as may be easily verified" (p. 190). Thus applied to classes,

> ...the definitions that must be regarded as non-predicative are those which contain a vicious circle.[16] (p. 190)

Though he does not explicitly formulate the VCP, it is clear from a later passage in the paper that he has the same conception of it as Russell:

> ...if the definition of a notion N depends on *all* the objects A, it may be tainted with the vicious circle, if among the objects A there is one that cannot be defined without bringing in the notion N itself. (p. 194)

Now Poincaré's views on the VCP arise completely naturally from his wider views on the philosophy of mathematics, in particular, his view on mathematical existence. He is explicit in his constructivism. In a paper written in 1912, he declares that a mathematical object "exists only when it is conceived by the mind" (Poincaré [1963] p. 72). He considers a "genus" (set) G with a member X, and writes of the members of G

> ...they will exist only after they have been constructed; that is, after they have been defined; *X exists only by virtue of its definition*,[17] which has meaning only if all the members of G are known beforehand, and X in particular.

This conception of existence of course provides a motivation for the VCP. If mathematical objects are brought into existence by their definitions, then it seems that no totality can possibly contain members defined in terms of that very totality. However, Russell adopted Poincaré's views on impredicativity without accepting the constructivist outlook. By doing so he landed the classical mathematical community in a philosophical confusion from which it has yet to emerge.[18]

[16] The original italicises this sentence. At this point "non-predicative" means simply "not defining a class"; confusingly, Russell, having accepted the diagnosis, started using "impredicative" to *mean* "violating the vicious circle principle".

[17] My italics.

[18] Goldfarb [1989] attempts to reconcile Russell's predicativism with his lack of constructivism, arguing that his views on variables and their ranges of significance can lead to ramification of intensional entities (in particular propositions and propositional functions) even on a realist conception. But even if this is right—and Goldfarb says he is only making a "first step" (p. 27) towards a full treatment of the issue—the fact remains that Russell advocates the VCP in full generality. As Goldfarb admits (pp. 30–1), it is hard to see how the ramification of *sets* can be justified except on a constructivist view.

6

Armed with his diagnosis of the paradoxes and aided by Whitehead, Russell embarked on reworking mathematics whilst obeying the VCP: the result was the ramified type theory of *Principia Mathematica* [1910–3]. This is not the place for a detailed discussion of that work, but for present purposes it is enough to note that the system obtained is a (rather complex) hierarchy of propositional functions; the position of a function in the hierarchy depends not only on (i) its arguments but also on (ii) the ranges of any quantifications in its definition (this latter refinement making the hierarchy "ramified"), both (i) and (ii) being required to be lower down in the hierarchy.

But despite Whitehead and Russell's efforts, their system has never been accepted as a foundation for mathematics. Instead, the system of axiomatic set theory developed in continental Europe, mostly by Zermelo, proved much easier to work with. Most of the axioms appear first in Zermelo's paper [1908], which contains versions of: extensionality, empty set, pairs, separation, power set, union, choice and infinity, that is, all the axioms in the now-standard theory except for replacement and foundation. Interestingly Zermelo's motivation at this point seems only partly to have been the paradoxes; primarily he was concerned to analyse exactly which principles concerning sets he had used in his proof that every set can be well-ordered (Zermelo [1904]). The central idea is to replace naive comprehension by separation: that is, we cannot in general form the set of absolutely all y such that $\phi(y)$, but only the set of all members of some set a such that $\phi(y)$. Paradox is avoided because there is no way to prove that the universe is a set; indeed the Russell paradox becomes a proof that it is *not* a set.[19]

The replacement axiom was added later by Fraenkel [1922] and Skolem [1922]. As for the axiom of foundation: the issue first seems to have been considered by Mirimanoff ([1917a] and [1917b]), who distinguishes "ordinary" sets which do not have infinite descending membership chains from "extraordinary" ones which do. He does not assert, however, that there is anything wrong with the extraordinary sets. Von Neumann [1925] describes non-well-founded sets as "superfluous" (p. 404) and gives an axiom (p. 412) which excludes some, but not all, of them. Three years later [1928] he formulates the axiom of foundation in the form $\forall x(x \neq \emptyset \rightarrow \exists y \in x(y \cap x = \emptyset))$. However, it is not until the paper Zermelo [1930] that the axiom of foundation is explicitly adopted as a postulate. With this paper all the axioms of standard modern set theory are in place.

[19] The axiom of foundation immediately rules out a universal set, for such a set would be a member of itself. But the point is that such a set is ruled out anyway by the other axioms. Foundation plays no role in solving the paradoxes.

7

In comparing type theory with ZF, it is useful to try to get clearer about what the VCP is saying. Russell never provided a single clear statement of it. Here are two attempts:

> I. No totality may contain members defined in terms of itself. (Russell [1908] p. 75)
>
> II. Whatever involves *all* of a collection must not be one of the collection. (Russell [1908] p. 63)

Now it seems that Russell took these, and other, statements to be different formulations of the same principle.[20] However, Gödel [1944] pointed out that it looks like there is more than one principle here, in particular one to do with *definitions*, the other to do with the notion of *involving*.[21] Let us call these VCP I and VCP II, and try to see more precisely what implications they have for sets.

VCP I seems closest to the constructivist spirit of Poincaré. It rules out *impredicative definitions*: for example definitions of x which quantify over a collection of which x is a member.[22]

Since sets presumably "involve" their members, VCP II rules out sets which are members of themselves. It also seems reasonable that "involving" is transitive, so that a set also involves the members of its members, and so on. Hence a reasonable explication of "x involves y" in the case of sets is "y is a member of the transitive closure of x",[23] in which case VCP II will be obeyed if sets satisfy the axiom of foundation.[24]

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[20] Some other statements from Russell's writings are to be found at the pages cited above, and also Russell [1906b] p. 204, Whitehead and Russell [1910] p. 37.

[21] Gödel claims to discern a third principle, concerning "pre-supposing", which I shall not discuss here.

[22] This will not do as it stands as a *characterization* of impredicative definitions. For example it will be equally objectionable if, instead of x itself being a member of the totality, some second object y, defined using x, is a member. Presumably to make this rigorous we would require some notion of well-foundedness for definitions; I shall not attempt to supply details here.

[23] The transitive closure of x is the set whose members are the members of x, the members of the members of x, and so on.

[24] Though he does not state it explicitly, this seems to be what Gödel has in mind in his paper (see p. 131 with its footnote reference to Mirimanoff). It is a little too strong to say that VCP II *entails* the axiom of foundation, for an infinite descending membership chain $x_1 \ni x_2 \ni \ldots$ in which all the x_i are *different* violates foundation without circularity. Such a chain seems equally offensive to the constructivist intuitions underpinning the VCPs, and suggests that they do not fully capture those intuitions.

Now the system of *Principia Mathematica* obeys both VCP I and VCP II. Impredicative definitions are rigorously avoided, and the universe has a hierarchical structure. That ZF obeys VCP II is, as I have said, guaranteed by the axiom of foundation. But ZF violates VCP I. The axiom of separation allows us, for any set a, to define a new set b by admitting only those members of a satisfying some formula $\phi(x)$. But there is no restriction on the quantifiers that may occur in $\phi(x)$: they may range over the whole universe. Impredicative definitions are perfectly allowable in ZF.

8

As I recounted above, most of the axioms of ZF resulted from Zermelo's attempt to defend his proof that every set could be well-ordered. This does not apply, however, to the axiom of foundation. I conjecture that it was inspired by type theory, but I do not know of anything explicit in the early literature which supports this. In any case, somewhat later a new way justifying the axioms developed. This is the second of the three ways I mentioned of arguing in favour of ZF.

The idea is as follows. We all have an intuitive grasp of the concept of natural number, that is, we grasp a structure which we refer to as "the natural numbers". If someone wanted to justify the Peano axioms for number theory, they would appeal to the evident truth of the axioms in this intuitively understood structure. The claim is that something analogous can be done for set theory. There is an intuitive conception of set, the *iterative* conception, which gives rise to an intuitively understood model, the *cumulative hierarchy.* The axioms (or at least, a number of them)[25] are then justified by appealing to the fact that they are true in this model.

How does the model work? Start with the empty set.[26] Call this V_0. V_1 is the power set of V_0, and in general, we obtain the next level after V_n by taking the power set. V_ω is just the union of all the V_n for finite n, and $V_{\omega+1}$ is the power set of V_ω. Continue through the ordinals, forming power sets at each stage and taking unions at limit ordinals. The result is a hierarchy in which sets only have members from lower down the hierarchy. As Lavine

[25] There is disagreement, for example, on whether the axiom of replacement is derivable from the iterative conception.

[26] A variation is possible in which instead we start with some *atoms* or *urelements*, that is, some non-sets. Though this is probably more natural from a naive point of view, mathematicians standardly work with a universe of *pure* sets, where everything is a set, since this is technically smoother (for example the quantifiers can simply be taken to range over all sets) and does not result in any limitation in structure. For present purposes the difference in the two approaches is not important.

[1994] points out, "the iterative conception gives the Axiom of Foundation center stage" (p. 144).

The cumulative hierarchy was hinted at by Mirimanoff [1917a] and introduced explicitly by von Neumann [1929] for the purposes of a consistency proof, but the idea of using it as an intuitive model justifying the axioms only came later. It is suggested by Gödel [1947] (pp. 474–5) but only becomes explicit around 1970, when a number of papers appeared roughly simultaneously. Shoenfield [1967] and [1977], Boolos [1971] and Wang [1974] are representative. Let us examine some of the passages in which they justify that part of their conception of set which gives rise to the axiom of foundation.[27] First Shoenfield:

> Sets are formed in *stages*. For each stage S, there are certain stages which are before S. At each stage S, each collection consisting of sets formed at stages before S is formed into a set. ...When we are forming a set z by choosing its members, we do not yet have the object z, and hence cannot use it as a member of z. The same reasoning shows that certain other sets cannot be members of z. For example, suppose that $z \in y$. Then we cannot form y until we have formed z. Hence y is not available as an object when z is formed, and therefore cannot be a member of z. (Shoenfield [1977] p. 323)

Boolos actually claims that the iterative conception of set has an intuitive plausibility independent of the paradoxes, and that one might have come to see it as superior to naive set theory (as embodied in the naive comprehension axiom) even if the paradoxes had never been discovered. That is (though this is not the way Boolos expresses it), there are really two versions of naive set theory, one captured by naive comprehension, the other by the iterative

[27] More detailed marshallings of evidence against the iterative conception may be found in Lavine [1994], Chapter V, and Hallett [1984], Chapters 5–6. The overall conclusion of Hallett's book, however, that "we have no satisfactory simple heuristic explanation of *why* it [ZF] works", seems to me to be too strong. It is not mysterious that ZF avoids the paradoxes, since it is apparent from the axioms that the paradoxical collections are denied sethood. Hallett also makes much (in Chapter 5) of the technical result that we have very little idea of the size of the power set of ω, arguing that this refutes ZF's claim to embody a "limitation of size" conception. This, however, seems to depend on thinking of "limitation of size" in the style of Russell, as "no sets allowed that are bigger than such-and-such a cardinal"; rather, as I have been trying to convey, the point is that however big it is, $P(\omega)$ is still a set, and therefore not as large as the universe. There is, however, another sense of "why ZF works" considered by Hallett: why it (or indeed any set theory) is adequate as a foundation for mathematics. I agree that this is genuinely mysterious, and I shall not try to solve the mystery here.

conception, and the latter has at least as great an intuitive appeal as the former:

> ZF . . . is not only a consistent (apparently) but also an independently motivated theory of sets: there is, so to speak, a "thought behind it" about the nature of sets which might have been put forth even if, impossibly, naive set theory had been consistent. (Boolos [1971] p. 490)

Boolos observes that naive comprehension implies that there is a set of all sets, and that this set is then a member of itself. He continues

> It is important to realise how odd the idea of something's containing itself is. Of course a set can and must *include* itself (as a subset). But *contain* itself? Whatever tenuous hold on the concepts of *set* and *member* were given one by Cantor's definitions of "set" and one's ordinary understanding of "element", "set", "collection", etc. is altogether lost if one is to suppose that some sets are members of themselves. The idea is paradoxical not in the sense that it is contradictory to suppose that some set is a member of itself, for, after all, "$(\exists x)(Sx \,\&\, x \in x)$"[28] is obviously consistent, but that if one understands "$\in$" as meaning "is a member of", it is very, very peculiar to suppose it true. For when one is told that a set is a collection into a whole of definite elements of our thought, one thinks: Here are some things. Now we bind them up into a whole (footnote: We put a "lasso" around them, in a figure of Kripke's.). *Now* we have a set. We don't suppose that what we come up with after combining some elements into a whole could have been one of the very things we combined (not, at least, if we are combining two or more elements). (pp. 490–1)

Wang says simply:

> A set is a collection of previously given objects. (Wang [1974] p. 530)

What I want to emphasize here is the constant appeal, in these passages, to constructivist images and terminology. All three authors use temporal words: "before", "yet", "until", "when", "now", "previously". The question is, in what sense are we to take this? Clearly all agree that it is not to be taken literally: there is not actually a time t_0 at which only the empty

[28] Boolos is using "Sx" for "x is a set".

set exists, another (later) time by which the singleton of the empty set has been formed, and so on. The constructivist language is supposed only to be metaphorical. Boolos for example, having presented the intuitive idea in constructivist language, then back-pedals: "From the rough description it sounds as if sets were continually being created, which is not the case" (p. 491).

It is clear, then, that the conception of set advanced is not supposed to be literally constructivist, but apparently only constructivist "in principle", under some liberal interpretation. The trouble is, I shall argue, that the sense has to be so liberal that it is no longer entitled to be called constructivist at all.

Wang admits (p. 531) that there is an element of "idealization" in supposing that we can "run through" an infinite number of objects in the way required in his description of the cumulative hierarchy. But all the authors are silent on what *exactly* this means. If the talk of "formation", "collection" and so on are to have any force, there must surely be envisaged an *agent* who is doing the forming and collecting. What properties do we take this agent to have? Parsons ([1977] p. 507) raises some problems concerning this:

> It is hard to see what the conception of an idealized mind is that would fit here: it would differ not only from finite minds but also from the divine mind as conceived in philosophical theology, for the latter is thought of either as in time, and therefore as doing things in an order with the same structure as that in which finite beings operate, or its eternity is interpreted as complete liberation from succession.

To elaborate: if the agent is conceived of as working in ordinary time, there is just not enough of it to generate the whole hierarchy (at least if time consists of continuum-many instants). The agent needs to occupy a "super-time" with perhaps a class of instants isomorphic to the ordinals. On the other hand, we must not let the agent be too powerful; if he could move backwards and forwards in time at will then it is mysterious why the sets need to be constructed in order at all.

Even if the notion of the ideal agent could be satisfactorily clarified there remains the problem of the status of the ordinals. The cumulative hierarchy is obtained by iterating the power set operation up the entire collection of ordinals. If these are assumed as given from the start this seems a platonistic rather than constructive foundation for the whole enterprise. Wang (p. 532) suggests that the conception of what ordinals there are can develop as the hierarchy is generated. But only countably many ordinals can ever be defined, so it seems that some kind of platonistic conception is inevitable.

Worse than this, however, is the issue of impredicativity. A *sine qua non* of constructivism is that objects are conceived of as occurring in an order, such that at any point in the construction process, only those objects occurring earlier in the order are available. It seems therefore that no theory allowing impredicative definitions can rightly claim to be constructive: one simply cannot quantify over objects which, if the constructivism is taken seriously, do not exist ("at the time"). But the ZF axioms of which the hierarchy is an intuitive model involve impredicative quantifications. Most striking is the axiom of power set in tandem with the axiom of separation. From the power set axiom we know that for any set x the power set $P(x)$ is also a set; the axiom of separation can then be used to pick out individual subsets by means of a formula ϕ. But this formula can contain quantifications over anywhere in the universe. To put it informally, what subsets there are of a particular set depends not only on what happens at the level of the set, and the next higher level, but also on what happens in the whole hierarchy—as Bell and Machover [1977] put it, "the size of a power set Pu of a given set u is proportional not only to the size of u but also to the 'richness' of the entire universe" (p. 509).[29] This seems incompatible with any constructive interpretation.

It is not that the authors are at all unaware of this; it is just that they are silent on the conflict between it and the constructivist heuristic which they give for the iterative conception of set. Wang for example (p. 532) says explicitly "we do not concern ourselves over how a set is defined, e.g. whether by an impredicative definition" and admits (p. 560) that "if we adopt a constructive approach, then we do have a problem in allowing unlimited quantifiers to define other sets", but he seems to see no conflict between his own use of constructivist terminology and his advocacy of impredicativity.

The justification of ZF as constructivist in principle is an attempt to have the best of two incompatible worlds, and results in a hybrid position which is philosophically bankrupt and ought to satisfy nobody. A symptom of the philosophical confusion upon which ZF rests is the status of the axiom of choice. This is accepted by most mathematicians, but is not usually regarded as just another of the axioms of set theory—it has a more dubious status. It is customary to state carefully whether or not any theorem requires it, and to do without it if possible. It is almost as though people feel a little guilty in using it. Why is this? I suggest that the explanation is that the strongly non-constructive feel of the axiom conflicts with the (false) idea that the rest of ZF is constructive. But in fact the axiom of choice is fully in the spirit of

[29] In technical terms, the power set operation is not *absolute*. The issue is discussed by Hallett [1984] pp. 206–7, 221, and [1994] pp. 83–92.

the rest of set theory—the damage its absence does to the theory of cardinal arithmetic is one demonstration of this. If it were clearly realised that ZF is not constructive at all, the axiom of choice would cease to be regarded as a second-class citizen and take up its rightful position as just another of the axioms of set theory.

The conclusion of this section, then, is that ZF does not embody a philosophically coherent notion of set. There is a coherent constructivist position, which entails repudiating impredicative definitions, obeying VCP I, and ending up perhaps with something like ramified type theory. It seems, however, that such a position will not lead to a foundation for classical mathematics. (Whitehead and Russell famously had to postulate the axiom of reducibility to make possible the derivation of mathematics in their system, but this axiom is unmotivated in the light of the VCP. And alternative versions of constructivism, for example intuitionism, are more damaging yet to classical mathematics.) There is also a coherent anti-constructivist position, which rejects the metaphysics of constructivism and its resultant inability to justify classical mathematics. This position rejects the VCP in all its forms. But ZF is an uneasy compromise between these two: it pays lip-service to constructivism without really meaning it, and in doing so forfeits its claim to philosophical justification.

9

Suppose it is admitted that ZF cannot be given a coherent philosophical justification. It seems there is still a third and final argument a defender of it might use: we might call it the argument from mathematical pragmatics.

ZF has proved adequate as a foundation for mathematics, in the sense that all known mathematics can be carried out in ZF. It is convenient to work with: for example, the well-foundedness of the sets allows inductive definitions to be handled smoothly. So whether or not it can be thought of as the axiomatization of a coherent notion of set, it is reasonable—so the argument goes—for it to occupy the position it does as the dominant theory of sets.

One reply to this is that it is no longer clear that ignoring non-well-founded sets gives a theory which is optimal for applications. In recent years uses have started to be found for non-well-founded theories—indeed the current revival of interest started with Aczel's realization that the modelling work he was doing in computer science (on parallel processing) was much simpler if one abandoned foundation (Aczel [1988] Chapter 8). Rather than attempt to describe this application in detail, I will try to give the general flavour with some simpler examples.

It is very common in mathematical modelling to use (ordered) n-tuples $\langle x_1, \ldots, x_n \rangle$. There is a standard way of handling n-tuples in set theory: for example for the pair $\langle a, b \rangle$ we use the set $\{\{a\}, \{a, b\}\}$. It can happen that we want an entire n-tuple to be equal to one of its elements, and this will be forbidden by the axiom of foundation.

Thus in Barwise and Etchemendy's treatment of the liar [1987] (so far the best-known application of non-well-founded sets) the aim is to model a proposition which asserts its own falsehood. Propositions are modelled by pairs,[30] so what we need is a proposition p which satisfies $p = \langle \mathbf{F}, p \rangle$ (where $\mathbf{F}$ is an atom representing falsehood). This is possible only if we abandon foundation.

A similar example, this time from computer science: a *stream* is a sequence of data items, and can neatly be defined as an ordered pair where the first element is an item of data and the second element is a stream. Then an infinite sequence of zeroes is a stream s satisfying $s = \langle 0, s \rangle$. Once again the axiom of foundation prevents this from being modelled in a natural way. This example is from Barwise and Moss [1996], p. 34. The book explores applications in other areas, for example the theory of games and the model theory of modal logic.

It is true that a hard-line supporter of ZF cannot be *forced* to repudiate foundation. We can always carry out these modellings by choosing appropriate objects in the well-founded universe. But such an approach is analogous to a hard-line disbeliever in complex numbers insisting on them as mere pairs of reals. As more and more applications are discovered, it becomes clearer that there is no good reason for not accepting non-well-founded sets as genuine sets.

10

There is a second and deeper reply to the "pragmatic" argument for ZF. A theory of sets should, I think, be answerable to our informal concept of set as completely arbitrary collection, as well as to the needs of mathematicians. Thus, even if mathematicians can get by using only some special class of sets, it does not follow that we should rest content with a theory which says that these are all the sets there are. Only a non-well-founded theory can convincingly be shown to modify the naive conception as much as, but no more than, is required by the paradoxes; and only in adopting such a theory can we obtain a truly satisfactory solution.

[30] I am simplifying the details of the theory to bring out the essential point.

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