

Paradoxes and failures of cut*

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Abstract

This paper presents and motivates a new philosophical and logical approach to truth and semantic paradox. It begins from an inferentialist, and particularly bilateralist, theory of meaning—one which takes meaning to be constituted by assertibility and deniability conditions—and shows how the usual multiple-conclusion sequent calculus for classical logic can be given an inferentialist motivation, leaving classical model theory as of only derivative importance. The paper then uses this theory of meaning to present and motivate a logical system—ST—that conservatively extends classical logic with a fully transparent truth predicate. This system is shown to allow for classical reasoning over the full (truth-involving) vocabulary, but to be nontransitive. Some special cases where transitivity does hold are outlined. ST is also shown to give rise to a familiar sort of model for nonclassical logics: Kripke fixed points on the Strong Kleene valuation scheme. Finally, to give a theory of paradoxical sentences, a distinction is drawn between two varieties of assertion and two varieties of denial. On one variety, paradoxical sentences cannot be either asserted or denied; on the other, they must be both asserted and denied. The target theory is compared favorably to more familiar related systems, and some objections are considered and responded to.

1 Introduction

This paper presents and motivates a new logical approach to truth and semantic paradox. The approach is that of [Ripley, 2012], which lays the technical groundwork, showing how to conservatively extend classical logic with a fully transparent truth predicate.¹ This results in a *nontransitive* logical theory. Here the approach is integrated into an *inferentialist*, and particularly *bilateralist*, theory of meaning, where it has a natural home.

In §2, I introduce the theoretical framework for the paper—a bilateralist theory of meaning—and show how it applies to secure the meanings of the usual classical logical connectives. I also consider the role of the sequent calculus rule of cut from a bilateralist perspective, and show how

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¹This is, in turn, inspired by [Cobreros et al., 2012], which shows how to conservatively extend classical logic so as to handle vagueness, by validating the so-called principle of tolerance without falling into the problems associated with it. [Cobreros et al., 2014] discusses related issues.

to introduce a transparent truth predicate into our classical language. This changes things slightly, resulting in a *nontransitive* logic, ST. §3 presents and discusses the logic ST, both proof- and model-theoretically. As emerges there, ST—despite its classical-logic-based proof theory—enjoys a model theory based on familiar nonclassical models for truth: those used by the logics K3 and LP. §4 applies ST to the familiar truth-based paradoxes, introducing a crucial distinction between *strict* and *tolerant* assertion and denial. §5 points to a number of advantages of the ST-based approach to truth, both over classical approaches with nontransparent truth and over K3- and LP-based nonclassical approaches; I also consider and reply to some possible objections.

2 Bilateralism, cut, and truth

This section sketches the bilateralist form of inferentialism that will be the guiding philosophical perspective in this paper. Inferentialism, as I'll use the term here, is the view that the meanings of linguistic items are to be explained in terms of which inferences in the language are valid. §2.1 introduces the position, and uses it to explain and justify a classical sequent calculus. §2.2 passes to a discussion of the rule of cut, and shows the role this rule plays in constructing models from invalid sequents. These two sections draw heavily on the work of Restall, particularly [Restall, 2005] (in §2.1) and [Restall, 2009] (in §2.2); expanded discussion of these issues can be found there. Finally, §2.3 introduces the notion of *transparent truth* from a bilateralist perspective.

2.1 Bilateralism and the classical sequent calculus

Inferentialism is the view that meanings are to be explained in terms of which inferences are valid. *Bilateralism*, a particular form of inferentialism defended in [Rumfitt, 2000, Restall, 2005], is the view that which inferences are valid is itself to be explained in terms of conditions on *assertion* and *denial*.

Thus, meaning, for a bilateralist, is a matter of conditions on assertion and denial.² Importantly, a bilateralist should not understand denial as [Geach, 1965] does, as assertion of a negation. This is because bilateralism commits one to going the other way, as in [Price, 1990], explaining the meaning of *negation* in terms of the prior notions of assertion and denial.

Bilateralism, as both Rumfitt and Restall argue, can be used to put classical logic on secure inferentialist footing. (This is in contrast to more familiar forms of inferentialism, like those defended in [Dummett, 1976, Dummett, 1991], which take inferentialism to give aid and comfort to intuitionist logic, and to provide a basis for criticism of classical logic.) This paper will show that the secure footing provided by bilateralism extends to the logic described in [Ripley, 2012]: a conservative extension of classical logic featuring a transparent truth predicate.

I'll work throughout with a Gentzen-style sequent calculus; this section gives a brief tour of such a calculus for classical logic from a bilateralist

²It will emerge in §4 that there is need for two distinct notions of assertion, and similarly two distinct notions of denial: a *strict* and a *tolerant* version of each. What I'll call 'assertion' and 'denial' until the distinction becomes important are in fact strict assertion and strict denial.

perspective. The particular perspective taken on the sequent calculus here is that of [Restall, 2005].

Let a *position* be some pair $\langle \Gamma, \Delta \rangle$ of assertions and denials. These might be, for example, the assertions and denials actually made by one party to a particular debate. The position $\langle \Gamma, \Delta \rangle$ asserts each thing in Γ and denies each thing in Δ . Some positions are *in bounds*, and some are *out of bounds*. For example, the position that consists of asserting ‘Melbourne is bigger than Brisbane’ and ‘Brisbane is bigger than Darwin’ while denying ‘Melbourne is bigger than Darwin’ is out of bounds. It’s part of what gives ‘bigger’ its meaning, on this view, that it is transitive. On the other hand, the position that consists of asserting ‘Melbourne is bigger than Brisbane’ and ‘Brisbane is bigger than Darwin’, without any denials, is in bounds. It hasn’t violated any of the inferential properties of the expressions contained within it.³

$\Gamma \vdash \Delta$ can now be read as the claim that the position $\langle \Gamma, \Delta \rangle$ is out of bounds. Thus, this account gives an understanding of consequence in terms of the prior notions of constraints on assertions and denials.

For the moment, let’s be fully classical in our predilections. What structure can we find in \vdash ? This, of course, depends on which assertions and denials are out of bounds. We should notice first that asserting and denying the very same thing seems clearly to be out of bounds. Thus, we have $A \vdash A$, for every A .⁴ It is no coincidence that this is the usual axiom of identity for sequent-calculus based proof systems; Restall’s insight is that we can understand sequent calculi in these terms. (In fact, identity need only be assumed for atomic A ; in the system to be considered here, it can be derived for all other A ; this is important, and will be discussed presently.)

We also can acknowledge the rule of weakening: if a position is out of bounds already, then adding more assertions and denials cannot bring it back into bounds. In symbols:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Much of the structure of \vdash , as we saw in the example involving ‘bigger’, will depend on the particular vocabulary involved in the cases under consideration. Besides the above principles—identity and weakening—there’s not much left that’s totally vocabulary-independent. (Restall [Restall, 2005, Restall, 2013] argues for one more content-independent constraint, corresponding to the sequent rule of cut. That’s for §2.2.) However, we can use what we have so far to give an account of the usual logical vocabulary.

For example, we can give an account of classical negation as follows: a negation $\neg A$ is assertible just when its negatum A is deniable, and $\neg A$ is deniable just when A is assertible. These conditions justify, respectively, the usual pair of sequent rules for classical negation:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$

³It may have failed to draw a possible inference, but any position actually encountered will do this; there are just too many possible inferences.

⁴Capital Greek letters are for sets of formulas, capital Roman letters for individual formulas. I’ll write things like $\Gamma, A \vdash \Delta, \Delta'$ to mean $\Gamma \cup \{A\} \vdash \Delta \cup \Delta'$. Similarly, I’ll talk of ‘asserting Γ, A ’ or ‘denying Δ, Δ' ’ to mean asserting each thing in $\Gamma \cup \{A\}$ or denying each thing in $\Delta \cup \Delta'$.

$$\begin{array}{c}
\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \supset B \vdash \Delta} \quad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \supset B, \Delta} \\
\\
\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \quad \frac{\Gamma \vdash A \text{ (or } B), \Delta}{\Gamma \vdash A \vee B, \Delta} \\
\\
\frac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall x A(x) \vdash \Delta} \quad \frac{\Gamma \vdash A(a), \Delta}{\Gamma \vdash \forall x A(x), \Delta}
\end{array}$$

Figure 1: Classical sequent rules

Thus, we can see how conditions on assertion and denial can be used to give an account of the meaning of classical negation.

A similar account can be given of all the usual classical vocabulary, as is discussed in [Restall, 2005]. ([Rumfitt, 2000] gives a related account.) This leaves us with the usual classical sequent rules, given in Figure 1 for use in later discussion. (I omit the rules for \wedge and \exists ; they are definable in the usual ways, and will obey the usual sequent rules as well. In the rules for \forall , t is any term, and a is any term not occurring in $\Gamma \cup \Delta$.)

This is a sound and complete presentation of first-order classical logic.⁵ A sequent is provable in this system iff it is valid on the usual sort of classical models. From an inferentialist perspective, soundness and completeness serve to legitimate talk of *reference*, *denotation*, *semantic value*, and the like; these model-theoretic terms derive their sense from the connection between models and valid inference. §2.2 will show how this derivation works.

Crucially, the above presentation already shows how the meanings of compound sentences formed with \neg , \supset , \forall , etc., are *compositionally* determined from the meanings of their components. There is no need for a detour through model theory to do this. When an assertion or denial of $\neg A$ is in or out of bounds is determined entirely by when assertions and denials of A are out of bounds, via the rules governing the introduction of \neg , and so for the other connectives and quantifiers.

This is one of the reasons it's important that the only identity axioms needed for this sequent calculus are for *atomic* sentences. It's out of bounds to assert $\neg A$ and deny $\neg A$ at the same time, but this should not be merely *assumed*; as part of the meaning of a complex sentence, it should be derivable from the meanings of A and \neg . In the sequent calculus given here, identity for all compound sentences can be derived by induction from identity for atomics only; if this were not so, the sequent calculus would be, from a bilateralist point of view, problematically noncompositional. (This account of compositionality provides the start of an answer to the so-called Frege-Geach problem for inferentialism, but I won't explore that here. For more, see eg [Smiley, 1996].)

2.2 Cut and models

Many discussions of sequent calculi focus on the rule of cut:

⁵For proof, see eg [Smullyan, 1995]. This presentation does not include $=$; I won't worry about that here, but it can be added to the eventual target system without difficulty, as in [Ripley, 2012].

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}$$

Given our present understanding, this is an *extensibility* constraint on assertion and denial, as [Restall, 2009] makes clear. It tells us that if, given assertion of Γ and denial of Δ , it is out of bounds to assert A and out of bounds to deny A , then asserting Γ and denying Δ is *already* out of bounds. Any position that is in bounds must be extensible, either to an in-bounds position that asserts A or to one that denies A .

One might think that this is a vital part of the structure of assertion and denial, on par with the axioms of identity (as in [Restall, 2013]). But that is not the only option. After all, the rule of cut is *eliminable* in the sequent calculus presented above. In other words, the set of provable sequents is already closed under this rule. We don't need to make any special provision for it; it might just *happen* to be part of the structure of assertion and denial when attention is restricted to classical logical vocabulary. For all we've said so far, it might play no crucial role. (Of course, cut plays crucial technical roles in proof theory; there is no need to deny that. The concern here is for a philosophical theory of meaning.) In fact, this is the perspective taken in §3.

However, [Restall, 2005] has argued for the centrality of cut. If this argument is successful, then my treatment of cut as at best epiphenomenal is mistaken. He writes (referring to in-bounds positions as 'coherent states', notation changed):

[Cut] follows from the intuitive picture of the connection between assertion and denial. To deny A is to place it out of further consideration. To go on and to accept A is to change one's mind. Dually, to accept A is to place its denial out of further consideration. To go on to deny A is to change one's mind. If one cannot coherently assert A , in the context of a coherent state $\langle \Gamma, \Delta \rangle$, then it must at least be coherent to place it out of further consideration, for the inference relation itself has already, in effect, done so.

But this is too quick. If, given one's other commitments, an assertion of A is out of bounds, it simply does not follow that a denial of A is in bounds. Restall seems to suppose that a denial of A places an assertion of A out of bounds *and does nothing else*; then if an assertion of A is already out of bounds its denial must be safe. But denials can have effects besides placing their corresponding assertions out of bounds. For example, as we've seen, a denial of 'Melbourne is bigger than Darwin' can place the joint assertion of 'Melbourne is bigger than Brisbane' and 'Brisbane is bigger than Darwin' out of bounds; this is an effect above and beyond its placing an assertion of 'Melbourne is bigger than Darwin' out of bounds. So knowing that an assertion of A is out of bounds is not enough to allow us to conclude that its denial is in bounds. Thus, pending some further argument, we don't need to see cut as crucial. Extensibility might just happen to hold in the language we've so far considered, without being a fundamental feature of assertion and denial.

Whatever view one takes on cut, though, its presence allows for an easy way to understand how proof-theoretic properties can explain model-theoretic properties, providing a demonstration of the ways in which inferential properties can be used to explain representational properties. This section proceeds to build a model theory out of the proof theory so far defended, with an eye to being able to demonstrate soundness and completeness. (What follows here is substantially the story of [Restall, 2009].)

We eventually want, then, that a sequent $\Gamma \vdash \Delta$ is unprovable iff there is a countermodel to it.

This suggests a very direct approach: build models (and so countermodels) out of unprovable sequents. There are a number of ways to do this, some corresponding to familiar model theories and some to unfamiliar model theories; here it'll stay familiar. An unprovable sequent, recall, is an in-bounds position. Now, models of the familiar sort are crucially *maximally opinionated*: they assign a value to every sentence. It would be nice, then, to be able to find a maximally opinionated in-bounds position that extends any given in-bounds position; these maximally opinionated in-bounds positions could serve as our models.

It is the rule of cut that guarantees that this will go smoothly. So long as the position we start with is in bounds, Cut tells us that there is an in-bounds position that extends that position by taking a stand on A , for any formula A . And since that position is in bounds, cut tells us that there is an in-bounds position that extends it by taking a stand on B as well, for any formula B . By following this chain of reasoning, we can guarantee that for any in-bounds position, there is a maximally opinionated position that extends it while remaining in bounds. This maximally opinionated in-bounds position can be understood as a *valuation* of the usual sort: say it assigns value 1 to anything it asserts, and value 0 to anything it denies.⁶

Given the above understandings of the logical connectives, the values assigned to compound formulas by maximally opinionated in-bounds positions will always be determined (according to the sequent rules) by the values assigned to the components from which they are assembled. (This is not so for out-of-bounds positions, and it is not so for non-maximally opinionated positions.) Thus, models defined in this way will have their usual (representational) compositional structure as well, deriving from the nonrepresentational compositionality remarked on in §2.1. This thus provides a way to understand our usual models as deriving from proofs, rather than vice versa.

Note that the generation of models by this method crucially relied on cut. If we did not have a guarantee that we can extend any in-bounds position to one that takes a stand on A , for any A , then we might have found no way to extend an in-bounds position to a maximally opinionated in-bounds position; we would have failed to find a countermodel. Cut is vital in this construction. Nevertheless, we will see in §3 that a similar construction can work for the same purposes, even in the absence of cut. There, the absence of cut will be forced by the addition of a transparent truth predicate to the classical sequent calculus given above.

2.3 Transparent truth

By a *transparent* truth predicate I mean one that is “see-through” for purposes of meaning. That is, where $\langle \rangle$ is a name-forming operation, so $\langle A \rangle$ is the name of a formula A , T is a transparent truth predicate iff $T\langle A \rangle$ always means the same thing as A itself. Given a bilateralist understanding of meaning, this gives us two new sequent rules:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, T\langle A \rangle \vdash \Delta} \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash T\langle A \rangle, \Delta}$$

⁶As should be clear, there is nothing special about 1 and 0 in this role; they are mere bookkeeping devices.

Wherever A is out of bounds, there too $T\langle A \rangle$ is out of bounds. Here, I won't be concerned to evaluate whether we need a transparent truth predicate; I will simply assume that we do. (For defense of this position, see [Field, 2008, Beall, 2009]; for opposition, see [Priest, 2006b].) These rules yield the full 'intersubstitutable' behavior of truth, as can be shown by induction on proof length.

Given some straightforward (and also not-to-be-questioned-here) principles of sentence formation, the presence of a predicate like T is going to lead us into the familiar paradoxes. Of these, the simplest, and the one that will occupy us to the greatest extent in this paper, is the liar: a sentence λ that is $\neg T\langle \lambda \rangle$. λ is thus the claim that λ is not true.

The liar sentence, and other paradoxes like it, are supposed to cause difficulty for theories based on classical logic. To see how, return to our classical setup. Given what's gone so far, we can quickly derive both $\vdash \lambda$ and $\lambda \vdash$:

$$\frac{\frac{\frac{T\langle \lambda \rangle \vdash T\langle \lambda \rangle}{\neg T\langle \lambda \rangle \vdash \neg T\langle \lambda \rangle}}{\lambda \vdash \lambda}}{T\langle \lambda \rangle \vdash \lambda}}{\vdash \neg T\langle \lambda \rangle, \lambda}}{\vdash \lambda}$$

$$\frac{\frac{\frac{T\langle \lambda \rangle \vdash T\langle \lambda \rangle}{\neg T\langle \lambda \rangle \vdash \neg T\langle \lambda \rangle}}{\lambda \vdash \lambda}}{\lambda \vdash T\langle \lambda \rangle}}{\lambda, \neg T\langle \lambda \rangle \vdash}}{\lambda \vdash}$$

Now, a single application of cut allows us to prove the empty sequent:

$$\frac{\vdash \lambda \quad \lambda \vdash}{\vdash}$$

If this is right, the empty position is out of bounds. And this, plus the rule of weakening, tells us that *every* position is out of bounds, since every position extends the empty position. Worse, since the bilateralist account of meaning derives meaning from which positions are in and out of bounds, this would force the conclusion that everything means the same thing, trivializing the account of meaning altogether. This is completely unacceptable; something must have gone wrong. The position to be developed in §3 locates the trouble in the rule of cut.

3 Living without cut

This section presents and discusses the logic ST, the target logic for this paper. We already have a full Gentzenization of ST on the table: it is simply the Gentzenization of classical logic given in §2.1, together with the rules for T given in §2.3. Crucially, ST does *not* include the rule of cut, nor can cut be added to ST without triviality, as we've seen. However, ST alone is safe: there is no derivation of the empty sequent in ST. (Proof of this is straightforward: no axiom of ST is empty, and no rule of ST can go from non-empty premises to an empty conclusion.) That is, the rules for T are perfectly compatible with classical logic, even in the presence of paradoxical sentences, so long as we can live without cut.

Of course, just because there's no derivation of the empty sequent doesn't on its own show that ST provides a sensible logical framework. We need to look at the logic in more detail. This section does just that. §3.1 limns the structure of ST, and §3.2 shows how ST can connect to a familiar model theory even in the absence of cut.

3.1 The logic ST

In this subsection, I'll present and discuss some features of ST. Throughout, I'll refer to the classical sequent calculus presented in 2.1 as CL.⁷

Fact 1. *Suppose that neither Γ nor Δ contains T . Then CL derives $\Gamma \vdash \Delta$ iff ST derives $\Gamma \vdash \Delta$.*

Proof. Every CL derivation is an ST derivation, so LTR is immediate. To RTL: Suppose that ST derives $\Gamma \vdash \Delta$ but CL does not. Then every ST derivation of this sequent must use a T -rule somewhere, since this is the only difference between CL and ST. But every rule in ST has the property that if there is a T -involving formula in one of its premises, then there is a T -involving formula in one of its conclusions. So there must be a T somewhere in Γ or Δ . Contradiction. \square

Fact 1 tells us that ST extends CL only with regard to T -involving formulas; sequents not involving T are equiderivable in CL and ST. Adding the T rules thus does nothing to disrupt the classical sequent calculus already in place.

Fact 2. *Suppose that CL derives $\Gamma \vdash \Delta$. Then ST derives $\Gamma^* \vdash \Delta^*$, where $*$ is any uniform substitution on the language.*

Proof. We know (by induction) that $B \vdash B$ is derivable in ST, for any sentence B . Take a CL derivation of $\Gamma \vdash \Delta$. It must begin from some number of axioms of the form $A \vdash A$, where A is atomic. For each axiom $A \vdash A$ used in the derivation, replace it with the ST derivation of $A^* \vdash A^*$. This will result in an ST derivation of $\Gamma^* \vdash \Delta^*$. \square

(Note that this will not work to show that ST-derivable sequents are closed under uniform substitution; they are not, as uniform substitution does not respect the ST-enforced connection between A and $T\langle A \rangle$.)

Fact 2 shows just how classical ST is: *every* substitution instance of *every* CL-derivable sequent is ST-derivable. This includes, for example, excluded middle ($\vdash A \vee \neg A$), explosion ($A \wedge \neg A \vdash$), material modus ponens ($A, A \supset B \vdash B$), contraction ($A \supset (A \supset B) \vdash A \supset B$), and every other classical validity that has been given up in the face of paradox. Fact 1 ensures that these continue to hold where T is not involved, and Fact 2 ensures that their validity extends to cases where T is involved.⁸

Although every CL-derivable sequent is ST-derivable, the set of CL-derivable sequents is closed under certain rules that the set of ST-derivable sequents is not closed under. In other words, although all CL-derivable *inferences* are ST-derivable, some CL-derivability-preserving *meta-inferences* fail to preserve ST-derivability. ST is thus a *weakly classical* logic, in the sense of [Field, 2008]. We've already seen an example of such a meta-inference: the rule of cut. Other philosophically and

⁷Note that CL, like ST, does not contain the rule of cut, although cut is admissible in CL. Note as well that formulas of the form $T\langle A \rangle$ fit into CL without difficulty; they are simply treated no differently from any other atomic formulas. The difference between CL and ST is in the T rules, not in the vocabulary involved.

⁸Unlike supervaluationist variations on classical logic, these results extend to the multiple-conclusion presentation of classical logic that's been assumed throughout this paper. The relation between ST and classical logic is thus much tighter than the relation between supervaluationist approaches and classical logic.

logically important meta-inferences, however, continue to hold in ST. As examples, all of the following hold:⁹

Reflexivity: If $\Gamma \cap \Delta$ is nonempty, then $\Gamma \vdash \Delta$.

Monotonicity: If $\Gamma \vdash \Delta$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$.

Classical reductio: If $\Gamma, \neg A \vdash A, \Delta$, then $\Gamma \vdash A, \Delta$.

Proof by cases: If $\Gamma, A \vdash \Delta$ and $\Gamma', B \vdash \Delta'$, then $\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'$.

Deduction theorem: $\Gamma, A \vdash B, \Delta$ iff $\Gamma \vdash A \supset B, \Delta$.

Further, although cut does not always preserve ST-derivability, a large number of instances of it do, and we can find a number of areas in which it's safe to apply cut. Most directly, Fact 2, plus the fact that cut is eliminable in CL, guarantees:

Fact 3 (Limited cut (I)). *If CL derives both $\Gamma, A \vdash \Delta$ and $\Gamma \vdash A, \Delta$, then ST derives $\Gamma^* \vdash \Delta^*$, for any uniform substitution $*$.*

This allows for some use of cut in constructing ST derivations; so long as cut is applied to sequents whose derivations do not themselves involve the T rules distinctive of ST, it will preserve ST-derivability. (This is not a vocabulary restriction: it is perfectly safe to cut on sequents containing T , so long as those sequents are not derived by application of the T rules.) In the next section, after showing how to provide models for ST, we will use them to extend the ground covered by cut in ST. However, Fact 3 alone already covers a huge amount of ground. Cut is thus usable in ST much of the time; it just can't be used indiscriminately. Some care is required.

3.2 Models without cut

As we saw in §2.2, the rule of cut is intimately involved in generating models of the usual sort for classical logic. Without cut in play, we will have to generate a different sort of model. Eventually, it will turn out that we can use models of a familiar (albeit nonclassical) sort for ST. (These will be the three-valued models used in [Kripke, 1975] for transparent truth, on the Strong Kleene valuation scheme; they will be presented below.) This connection will prove useful for exploring the relations between ST and the familiar nonclassical logics K3T and LPT (the extensions of K3 and LP, respectively, with a transparent truth predicate). However, we will not reach these models in a single step.

This section will first follow the plan of §2.2 (and so of [Restall, 2009]) as far as is possible without cut, to generate a wide class of models directly from unprovable sequents. The familiar models that are the target are special cases of these resulting models. It will then be shown that whenever a sequent has a counterexample of the general sort, it has a counterexample of the special sort; thus, we lose no counterexamples by restricting our attention to the special cases.¹⁰

So we will generate a model from each in-bounds position, by extending it to a more opinionated position. However, since cut fails in ST, we won't be able to extend to maximally opinionated positions (ones that either assert or deny every formula). Rather, we extend only as far as we know to be safe.

⁹[Ripley, 2012] proves these claims model-theoretically, but most of them can also be read directly off of the sequent calculus presented here.

¹⁰This two-stage plan is used in [Restall, 2012], for purposes that are similar philosophically but quite different in logical detail. (That paper explores a two-dimensional modal logic for which cut remains admissible.)

For example, if $\Gamma, \neg A \not\vdash \Delta$, then it must be that $\Gamma, \neg A \not\vdash A, \Delta$, by the rule for \neg . Thus, we can safely extend this particular position with a denial of A . Each connective rule gives us further cases of safe extension. Sometimes—in the case of two-premise rules—these are disjunctive. For example, if $\Gamma, A \supset B \not\vdash \Delta$, then either $\Gamma, A \supset B \not\vdash A, \Delta$ or $\Gamma, A \supset B, B \not\vdash \Delta$, by the rule for \supset . Either we can extend with a denial of A or we can extend with an assertion of B . Rules for T are involved here as well; for example, if $\Gamma, T\langle A \rangle \not\vdash \Delta$, then $\Gamma, T\langle A \rangle, A \not\vdash \Delta$.

This, then, is the strategy for extending any in-bounds position: add to it those assertions and denials that are *safe*; this safety is guaranteed, when it is, by the sequent rules, applied in reverse to the particular formulas occurring in the in-bounds position. When we extend this as far as we can safely, we stop: the in-bounds positions reached in this way, despite not being maximally opinionated, can serve as models.

These models divide formulas into three classes: those asserted, those denied, and those that are neither asserted nor denied (because it wouldn't be safe to do so). We can thus understand them as three-valued models: say a position assigns value 1 to what it asserts, value 0 to what it denies, and value $\frac{1}{2}$ to everything else. Call anything that assigns values 1, $\frac{1}{2}$, and 0 in a pattern determinable from an in-bounds position in this way a *general model*.¹¹

Completeness is immediate: every unprovable sequent extends to some general model, since general models are just whatever unprovable sequents extend to. However, general models are quite unconstrained. For example, there will be a general model that assigns value 1 to the atomic sentence p and value $\frac{1}{2}$ to everything else, including $p \vee q$. It would be nice to have models in which the values assigned to compound formulas are in some way determined by the values assigned to their components.¹²

We can provide a model theory constrained in this way by turning to the three-valued models of [Kripke, 1975], on the Strong Kleene valuation scheme. I'll call these *KK models*, for 'Kleene-Kripke'.

Definition 1. A *KK model* for a language \mathcal{L} is a structure $\langle D, I \rangle$, such that:

- D is a nonempty domain such that $\mathcal{L} \subseteq D$, and
- I is an interpretation function:
 - For a name or variable a , $I(a) \in D$,
 - for a distinguished name $\langle A \rangle$ of a formula A , $I(\langle A \rangle) = A$,
 - for an n -ary predicate P , $I(P)$ is a function from n -tuples of members of D to $\{1, \frac{1}{2}, 0\}$,
 - for an atomic sentence $A = P(a_1, a_2, \dots, a_n)$,
 $I(A) = I(P)(\langle I(a_1), I(a_2), \dots, I(a_n) \rangle)$,
 - $I(\neg A) = 1 - I(A)$,
 - $I(A \supset B) = \max(1 - I(A), I(B))$,

¹¹This would rule out, for example, something that assigns value 1 to $\neg p$ but value $\frac{1}{2}$ to p , since no in-bounds position can extend to produce such a pattern. Extending from an assertion of $\neg p$ will force a denial of p . General models are very closely related to the Schütte models discussed in [Girard, 1987]; the only difference is due to the T rules.

¹²There is no particular *philosophical* need for such a thing. If there were such a need, it would presumably come from a compositionality constraint on meanings. But meanings are already guaranteed to be compositional in ST, because the sequent calculus encoding these meanings is. Models play no role in securing meanings, so it's not important if they don't assign values compositionally. Nonetheless, it would certainly be *convenient* to have a model theory for ST that is a little more constrained and well-behaved.

- $I(\forall x A(x)) = \min(\{I'(A(x))\})$, for all x -variants I' of I , and
- $I(T\langle A \rangle) = I(A)$.

As usual, \wedge , \vee , \exists , and \equiv can be treated as defined. The final condition on I is Kripke's fixed-point condition, ensuring that T behaves transparently; the remainder of the conditions simply set up the usual Strong Kleene-style three-valued models, with distinguished names for the formulas in our language.

From these models, we can define a model-theoretic notion of validity as follows:

Definition 2. *An ST-counterexample to a sequent $\Gamma \vdash \Delta$ is a KK model $\langle D, I \rangle$ such that: $I(\gamma) = 1$ for every $\gamma \in \Gamma$ and $I(\delta) = 0$ for every $\delta \in \Delta$. A sequent is ST-valid iff no KK model is a counterexample to it.*

Now, I'll sketch proofs of soundness and completeness for ST. From soundness it will follow that KK models are a special case of the general models we have considered. From completeness it will follow that only these special cases are necessary, that whenever there is a general-model counterexample to an argument, there is an ST-counterexample of the special (KK) sort.

Fact 4 (Soundness). *If a sequent is ST-derivable, then it is ST-valid.*

Proof. The axioms of ST are ST-valid, and all the rules of ST can be seen to preserve ST-validity. \square

From this, we can conclude that KK models are special cases of general models. To see this, take a KK model $\langle D, I \rangle$, and consider the sequent $\{A : I(A) = 1\} \vdash \{A : I(A) = 0\}$. Since the model is a counterexample to the sequent, the sequent is not ST-valid, so it is not ST-derivable. Thus, $\langle \{A : I(A) = 1\}, \{A : I(A) = 0\} \rangle$ is an in-bounds position. What's more, as can be checked by comparing Definition 1 to the ST rules, it is already extended as far as is safe. Thus, $\langle D, I \rangle$ assigns values in a way that can be generated by extending an underivable sequent: it is a general model.

Fact 5 (Completeness). *If a sequent is ST-valid, then it is ST-derivable.*

This can be proved by the method of *reduction trees*, as explored in eg [Takeuti, 1987], adding a twist due to [Kremer, 1988]. The general plan is this: one takes an underivable sequent and extends it to a general model. Then one takes just the valuations of atomic sentences from the general model and builds a Strong Kleene model using those atomic valuations. Finally, one applies the fixed-point construction of [Kripke, 1975] to convert the Strong Kleene model into a KK model. This KK model is an ST-counterexample to the original underivable sequent. The proof is given in more detail in the Appendix.

KK models are not just of use in exploring ST. They can be used to explore more familiar nonclassical logics involving transparent truth:

Definition 3. *A KK model $\langle D, I \rangle$ is a K3T-counterexample to a sequent $\Gamma \vdash \Delta$ iff $I(\gamma) = 1$ for every $\gamma \in \Gamma$ and $I(\delta) \neq 1$ for every $\delta \in \Delta$. A sequent is K3T-valid iff no KK model is a K3T-counterexample to it.*

Definition 4. *A KK model $\langle D, I \rangle$ is an LPT-counterexample to a sequent $\Gamma \vdash \Delta$ iff $I(\gamma) \neq 0$ for every $\gamma \in \Gamma$ and $I(\delta) = 0$ for every $\delta \in \Delta$. A sequent is LPT-valid iff no KK model is an LPT-counterexample to it.*

The logical system of [Field, 2008] is an extension of K3T with an additional conditional connective, and that of [Beall, 2009] is an extension of LPT with an additional conditional connective. These systems will be discussed and compared to ST in §§4 and 5. The connection between ST and these systems is not just useful for comparing ST to the literature; it can also be used to further extend the range of cases in which the rule of cut preserves ST-validity (and thus derivability, by Facts 4 and 5).

Fact 6 (Limited cut (II)). *If $\Gamma, A \vdash \Delta$ is ST-valid and $\Gamma \vdash A, \Delta$ is K3T-valid, then $\Gamma \vdash \Delta$ is ST-valid. If $\Gamma \vdash A, \Delta$ is ST-valid and $\Gamma, A \vdash \Delta$ is LPT-valid, then $\Gamma \vdash \Delta$ is ST-valid.*

Proof of this fact can be read directly off the definitions of the various sorts of validity involved. Since K3T and LPT are both sublogics of ST (which also can be read off the definitions of validity), Fact 6 gives another range of cases in which cut preserves ST provability; these cases extend beyond those provided by Fact 3 (and vice versa).

4 What to say about paradoxes

A detailed picture of ST has emerged in §3. We've seen that ST includes a fully transparent truth predicate, and preserves all classical inferences in the face of the paradoxes. It does not allow for cut. However, I've so far said relatively little about the paradoxes themselves. We know that they will not trivialize ST; the empty sequent is not derivable. This section will show what picture of the paradoxes this leaves us with.

§4.1 makes clear what the philosophical upshot is of the failure of cut in ST. Given the reading of the sequent calculus endorsed in §2.1, it will have the upshot that we should neither assert nor deny paradoxical sentences. §4.2 expands our theoretical vocabulary, introducing the notions of *tolerant* assertion and denial and showing how to use them to treat the paradoxes.

4.1 Neither assert nor deny

Recall that cut is an *extensibility* constraint on in-bounds positions. If indeed extensibility is a *sine qua non*, if it must be built into the inferential structure of assertion and denial, then we're in for trouble. But we saw in §2.2 that there is another option: we might think that extensibility only *happens* to hold over the usual classical rules. If this is so, there's no particular reason to expect it to continue to hold when we consider a richer set of rules, such as one including appropriate constraints on *T*. As we have seen, it does not: there are things (like the empty sequent) provable in ST plus cut that are not provable in ST alone.

This suggests that in the presence of paradoxes, assuming extensibility is a mistake. Some in-bounds positions, and some formulas *A*, are such that the position cannot be extended to include *either* asserting *A* or denying it. In the above proof of the empty sequent, the crucial formula was λ . It is simply not safe to assume that a liar sentence can be either asserted or denied. In fact, this can give us an understanding of what it is to be paradoxical: a paradoxical sentence is one such that extensibility does not hold for it, such that it's not in bounds either to assert or deny it. One should thus do neither.

Whether extensibility holds for a given sentence is relative to a position. For example, the sentence "The first mentioned sentence in the third

paragraph of §4.1 is false' is not on its own paradoxical. One needs the further information that that sentence *is* the first mentioned sentences in the third paragraph of §4.1; it is only in the context of a position that asserts that further information that the sentence is paradoxical. So-called “contingent paradoxes” fit this mold: they are only paradoxical given certain extra information, and, given that information, extensibility fails for such a sentence.

This applies to the liar sentence, as we’ve seen, but it also applies to Curry paradoxes. A Curry sentence κ is a sentence such that $\kappa = T\langle\kappa\rangle \supset p$, where p is anything you like. Given just this, we can derive $T\langle\kappa\rangle \vdash p$:

$$\frac{\frac{\frac{T\langle\kappa\rangle \vdash T\langle\kappa\rangle \quad p \vdash p}{T\langle\kappa\rangle, T\langle\kappa\rangle \supset p \vdash p}}{T\langle\kappa\rangle, \kappa \vdash p}}{T\langle\kappa\rangle \vdash p}$$

From there, it is a short step to derive $\vdash T\langle\kappa\rangle$:

$$\frac{\frac{\frac{T\langle\kappa\rangle \vdash p}{\vdash T\langle\kappa\rangle \supset p}}{\vdash \kappa}}{\vdash T\langle\kappa\rangle}$$

If we could cut these two sequents together, we’d have a derivation of $\vdash p$, and so a demonstration that denying p is out of bounds:

$$\frac{\vdash T\langle\kappa\rangle \quad T\langle\kappa\rangle \vdash p}{\vdash p}$$

Since p was arbitrary, this would be bad: we ought to be able to deny *something!* But we cannot cut these two sequents together in ST, as there is no rule of cut. Moreover, we know that there is no proof of $\vdash p$ for arbitrary p in ST. Let p be a T -free atomic sentence. Since p contains no T , there is only a derivation of $\vdash p$ in ST if there is one in CL. But there is no classical derivation of $\vdash p$; so there is no such derivation in ST either. Cut must fail for $T\langle\kappa\rangle$; it too is not in bounds either to assert or to deny.¹³

Although reminiscent, this is importantly different from a more familiar sort of response to paradox: *paracomplete* responses, like those of [Field, 2008, Tappenden, 1993]. According to these paracomplete responses, one should assert neither a paradoxical sentence nor its negation. Instead, one should deny both it and its negation. Crucially, this sort of response breaks the tight connection between negation and denial used in §2.1 and [Price, 1990] to give a bilateralist account of the meaning of negation. A paracompletist of this stripe must find some other way to account for the meaning of negation.

What’s more, a paracompletist of this sort loses the account of compositionality outlined in §2.1. She must reject that assertion and denial are governed by the rule introducing \neg on the right:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$

¹³The same goes for κ itself, as variations on the above derivation will reveal.

Without this rule or some replacement, though, one cannot derive the meaning of a negation from the meaning of its negatum, so long as meaning is understood to depend only on constraints governing assertion and denial. However, the usual paracompletists cannot offer a replacement: for them, when $\neg A$ can and cannot be asserted and denied depends on more than when A can be asserted and denied and the meaning of \neg . Paracompletists must thus reject either the bilateralist account of meaning, or else reject compositionality of meaning. Of course either of these is possible; this merely makes plain how different paracomplete accounts are from the ST-based account advanced here, which holds to both bilateralism and compositionality.

The paracompletist has something to say about the paradoxes: she denies them. The ST partisan, however, does not have this option. Nonetheless, there is no need for an STer to remain mute in the face of the paradoxes, struck dumb by an inability either to assert or to deny them. We must recognize two types of assertion, and two corresponding types of denial. To distinguish them, what I've so far called assertion and denial will from here on be called *strict* assertion and *strict* denial; the types presently coming onto stage will be called *tolerant* assertion and *tolerant* denial.¹⁴

4.2 Strict and tolerant

Tolerant assertion and denial can be understood in terms of their relations to strict assertion and denial, the kind we've been considering so far. A tolerant assertion of something is in bounds exactly when a strict denial of that thing is out of bounds, and a tolerant denial of something is in bounds exactly when a strict assertion of that thing is out of bounds.

Since there are things that fall into a gap between strict assertibility and strict deniability, tolerant assertibility is *weaker* than strict assertibility, and tolerant deniability is *weaker* than strict deniability. If there were no gap between strict assertibility and strict deniability, if cut held, then strict and tolerant would coincide. As this is not the case, they do not. Some things (the paradoxes) can be neither strictly asserted nor strictly denied, and so they can be *both* tolerantly asserted and tolerantly denied.

There is a deep symmetry between strict and tolerant. Just as we can understand tolerant assertion and denial in terms of strict assertion and denial, we could as well have begun from tolerant assertion and denial and used these notions to give a reading of the sequent calculus. On this reading, $\Gamma \vdash \Delta$ records that any in-bounds position taking a stand on everything in $\Gamma \cup \Delta$ must either tolerantly deny something in Γ or else tolerantly assert something in Δ . This reading then motivates all the same sequent rules as have already been considered. However, the rules of identity and cut swap interpretations. Where identity tells us that strict assertion may not overlap strict denial, it also tells us that one or the other of tolerant assertion and tolerant denial must always be possible. Where the failure of cut in ST tells us that some formulas may be neither strictly asserted nor strictly denied, it also tells us that tolerant assertion may overlap tolerant denial.

We can also understand sequents as reporting connections between strict and tolerant assertion or denial. For example, $\Gamma \vdash \Delta$ reports that any position that strictly asserts everything in Γ must, if it takes a stand

¹⁴This terminology is adapted from [Cobreros et al., 2012]; the name ST is derived there from a combination of 'strict' and 'tolerant'.

on everything in Δ , tolerantly assert at least something in Δ ;¹⁵ and it reports that any position that strictly denies everything in Δ must, if it takes a stand on everything in Γ , tolerantly deny at least something in Γ . All four of these understandings are equivalent, given the connections between strict and tolerant outlined above.

Although the overlap between tolerant assertion and tolerant denial may be reminiscent of the usual *dialetheic* responses to paradox, like those of [Priest, 2006b, Beall, 2009], this approach is importantly different. According to these dialetheic responses, one should assert *both* a paradoxical sentence and its negation (at least for some paradoxes; I'll consider Curry paradox in a moment).¹⁶

These dialetheic accounts, like the above-considered paracomplete accounts, break the tight connection between negation and denial considered here. They reject that assertion and denial are governed by the rule introducing \neg on the left:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

Again, this disrupts the derivation of the meaning of a negation from the meaning of its negatum, so long as meanings are understood bilaterally. These dialetheists, like the usual paracompletists, must not accept both bilateralism and compositionality. This marks a striking difference from the ST-based account advanced here.

Despite these differences, the present account can be understood, and I think should be understood, as both paracomplete and dialetheist, just not in the usual ways. What is characteristic of a paracomplete account is its willingness to, for some A , deny both A and $\neg A$; and indeed I (tolerantly) deny both the liar sentence and its negation. What is characteristic of a dialetheic account is its willingness to, for some A , assert both A and $\neg A$; and indeed I (tolerantly) assert both the liar sentence and its negation. (It is this kinship with both paracomplete and dialetheic accounts that explains why KK models are connected so tightly to ST.)

5 Upshots and discussion

This paper has worked from a bilateralist conception of meaning, and shown that transparent truth is compatible with classical logic, so long as one is willing to let go of cut, and with its transitivity of consequence. This is to be understood as a failure of *extensibility*; there are some sentences—paradoxical ones—that it is out of bounds either to assert or to deny, when assertion and denial are taken strictly. We've also seen, however, that there are more tolerant notions of assertion and denial that leave us able to discuss these paradoxes and our situation regarding them. The purpose of this section is to give a brief overview of advantages to this way of thinking about the paradoxes, and to present and answer some objections.

Bilateralism has been argued for in [Rumfitt, 2000, Restall, 2005], and a number of other places; I will not point to its advantages here. Instead I'll point to the advantages that accrue to an ST-based theory of truth

¹⁵It is a relative of this reading used to understand a relative of ST in [Cobreros et al., 2012].

¹⁶[Priest, 2006a] allows that one should both assert and deny certain paradoxical sentences, but he does not take seriously the possibility of *doing so*; instead, he treats the obligation as a rational dilemma. This complicates the relation between ST and Priest's picture, in ways I won't consider here.

and paradox. Most of these advantages do not depend on the bilateralism that has been its motivation here; even if motivated differently, ST would retain these features.

5.1 Advantages

This way of thinking about paradox has a number of advantages over its closest relatives.

Transparency This is the key advantage of the ST approach over a strongly classical approach. The rejection of cut allows for fully transparent truth. This is not the place to argue for a transparent conception of truth; it can be motivated from a number of angles. (For examples, see [Beall, 2009, Field, 2008].) Although there are well-developed non-transparent theories of truth that fit with strongly classical logic (as in [Maudlin, 2004, Halbach, 2011]), it is hard to believe that these ever would have been developed if transparent truth *were* compatible with strongly classical logic. Although ST does not achieve this (impossible) integration, it comes very close.

Classicality This is the core of the advantages of the ST approach over K3T- and LPT-based approaches like those of [Field, 2008, Beall, 2009],¹⁷ along with other nonclassical approaches such as that of Zardini [Zardini, 2011]. There is no need, from an ST-based perspective, ever to criticize (on logical grounds) any classically-valid inference. As such, there is no need for the ‘classical recapture’ that so exercises many nonclassical theorists. If the classical bird is never let out of the cage in the first place, there is no need to recapture it.¹⁸

Suitable conditional Most K3T- and LPT-based theories do not content themselves with just the vocabulary considered in this paper. For example, [Beall, 2009] and [Field, 2008] extend LPT and K3T, respectively, with extra conditional connectives. This is because the material conditional exhibits odd behavior in these logics: in LPT it does not validate modus ponens ($A, A \supset B \vdash B$), and in K3T it does not validate identity ($\vdash A \supset A$). Beall’s and Field’s extra conditionals, however, validate both of these principles.

As is well-known [Restall, 1993], any transitive logic with transparent truth is in trouble (is in fact trivial, due to Curry paradox) if it features a conditional connective \rightarrow validating identity, modus ponens, and contraction ($A \rightarrow (A \rightarrow B) \vdash A \rightarrow B$). As a result, neither Beall’s nor Field’s conditional validates contraction. The trouble is that contraction is an intuitively attractive principle.

If there were really no sense to be made of a conditional satisfying all of identity, modus ponens, and contraction, then that would be that. Intuitions have to give sometimes. But ST (via Fact 2) shows us that we are not in so sorry a state: we can have our good old classical material

¹⁷Although [Priest, 2006b] provides an LP-based approach to truth and paradox, it does not feature a transparent truth predicate, and so is not based on what I’ve here called LPT.

¹⁸There are a number of motivations that might push one away from classical logic other than the desire for transparent truth (as in eg [Anderson and Belnap, 1975, Routley et al., 1982, Priest, 2006b, Brady, 2006]). For those moved by these motivations, ST’s classicality is not an unmixed blessing.

conditional, complete with identity, modus ponens, and contraction, all in the presence of transparent truth.¹⁹

Restricted quantification This advantage is intimately related to the presence of a fully classical conditional in ST. Many logical theories use their conditional not just to represent conditional sentences, but for purposes of restricted quantification. For example, ‘Every ball is being tripped’ can be regimented as $\forall x(Bx \rightarrow Tx)$, where \rightarrow is some conditional connective. In ST, \supset works fine for this purpose, as it does classically.

For a number of nonclassical theories, though, including those considered above, things don’t work out so smoothly. The main difficulty is the same as for the conditional: Curry paradox. In particular, in a transitive logic with transparent truth, no restricted universal quantifier $[Ax/Bx]$ (for ‘All As are Bs’) can validate all of the following, on pain of Curry-flavored triviality:

- $\vdash [Ax/Ax]$
- $At, [Ax/Bx] \vdash Bt$
- $[Ax/[Ax/Bx]] \vdash [Ax/Bx]$

As should be clear, these are the restricted quantification analogues of identity, modus ponens, and contraction, respectively. Again, contraction is the usual casualty. But while there are some conceptions of conditionality on which it might be plausible to reject contraction, it is very hard to make sense of rejecting its quantified analogue. From the premise that all As are such that all As are Bs, surely it is appropriate to conclude that all As are Bs. After all, all As are such that the conclusion holds, and what besides all the As could matter for whether all As are Bs?

This is not the only problem afflicting nonclassical theories of restricted quantification (for more, see eg [Beall et al., 2006]), but it is a problem common to a number of these theories. ST, because of its classical conditional, provides a theory of restricted quantification free from all these difficulties.

Validity Curry One form of the Curry paradox discussed in the recent literature [Field, 2008, Beall and Murzi, 2013] uses a validity predicate as its ‘conditional’. The paradox can be set up with a simple binary validity predicate V ; we suppose, as principles governing V , that if $A \vdash B$ then $\vdash V(\langle A \rangle, \langle B \rangle)$, and that $A, V(\langle A \rangle, \langle B \rangle) \vdash B$.

Consider a sentence ν that is $V(\langle \nu \rangle, \langle p \rangle)$. Given the above principles, we can quickly derive $\nu \vdash p$:

$$\frac{\nu, V(\langle \nu \rangle, \langle p \rangle) \vdash p}{\nu \vdash p}$$

From there, it is two short steps to $\vdash \nu$:

$$\frac{\nu \vdash p}{\vdash V(\langle \nu \rangle, \langle p \rangle)}$$

$$\frac{\vdash V(\langle \nu \rangle, \langle p \rangle)}{\vdash \nu}$$

If we could cut these together, we’d be in trouble: we’d have derived $\vdash p$. Beall and Murzi point out that that most nonclassical truth theories must

¹⁹It is not only Beall and Field that give up contraction in the face of the paradoxes; so too do [Priest, 2006b, Zardini, 2011]. [Brady, 2006] gives up contraction as well, and treats the paradoxes, but has quite different motivations for giving up contraction.

treat this paradox differently from the ordinary Curry. ST, of course, has no such obligation; ν is simply not the sort of sentence it's safe to assume extensibility for. Because this paradox is already accounted for, there is no difficulty in introducing a validity predicate of this sort into a system governed by ST.

Validity and truth-preservation This same fact allows the ST proponent to treat validity as guaranteeing truth-preservation, in a sense that Field [Field, 2008] despairs of. In fact a simple argument (appealing to rules governing V , T , and \supset) yields as a theorem that if an argument is valid, then if its premises are true, so is its conclusion:

$$\frac{\frac{\frac{A, V(\langle A \rangle, \langle B \rangle) \vdash B}{T\langle A \rangle, V(\langle A \rangle, \langle B \rangle) \vdash T\langle B \rangle}}{V(\langle A \rangle, \langle B \rangle) \vdash T\langle A \rangle \supset T\langle B \rangle}}{\vdash V(\langle A \rangle, \langle B \rangle) \supset (T\langle A \rangle \supset T\langle B \rangle)}$$

See [Field, 2008, p. 43]: ‘[O]n every possible theory of truth the argument [just given] is problematic’ (emphasis in original). The argument is not problematic at all in ST. What would be problematic would be to assume cut as well; then the validity Curry would rear its head. As a result, Field’s own theory cannot sanction this eminently plausible result. (For similar reasons, neither can most other well-known theories of transparent truth.)

Uniformity This advantage, too, is related to the classicality of ST’s conditional. By tracing all the truth-based paradoxes back to a single source—the failure of extensibility, and so of cut—ST reveals what is common to them. This is in sharp contrast to other dialethic theories, which give strikingly different accounts of the liar and liar-like paradoxes on the one hand and Curry and Curry-like paradoxes on the other.

The accounts in [Priest, 2006b, Beall, 2009] take liar sentences to be both true and false; they assert liar sentences and their negations. This works well as a way to dissolve the liar paradox; however, it cannot for them extend to Curry paradox. The systems they advocate are such that any assertion of a Curry sentence, even if coupled with assertion of its negation, will lead inexorably to triviality: the assertion of every sentence whatever. As such, they are blocked from applying the same approach to Curry paradox that they apply to the liar. They must refrain completely from asserting Curry sentences. This is done by giving an account of the conditional—which, recall, cannot be material—on which contraction fails. Contraction, however, has nothing to do with the liar, on their view.²⁰

ST, on the other hand, adopts a uniform approach to liar paradox and Curry paradox. What’s more, ST extends naturally to provide a natural and psychologically plausible account of vagueness as well, as is argued in [Cobreros et al., 2012]. This account, too, is based on failures of extensibility. ST thus provides a way to see the commonality in the range of truth-based paradoxes, as well as between the truth-based paradoxes and the paradoxes due to vagueness.

²⁰For a unified approach to Curry and the liar based on restricting contraction only, see [Zardini, 2011].

5.2 Objections and replies

But it's not transitive! One might worry that ST's nontransitivity is itself a significant disadvantage. All the other systems I've compared ST to above are transitive.²¹ If transitivity is itself a major advantage, this might be thought to outweigh the above-cited advantages of ST. Why, then, might we think that transitivity is important? The main reason, I think, is twofold: we want to use chains of reasoning to extend our knowledge, and we have experience of that process going well.

However, ST can accommodate this without difficulty. First, there are Facts 3 and 6, which give a wide range of cases in which cut, and so transitivity, is safe. So long as we work within these cases, there is no trouble in chaining our reasoning together to extend our knowledge. Further, our experience in having chains of reasoning work well is entirely *within* this safe ground. The instances of cut given up by ST are in areas where we are used to things not working out well at all: in the realm of paradox, outside areas covered by LPT and K3T. ([Ripley, 2012] shows that the *only* failures of transitivity in ST are due to the paradoxes.) ST thus does nothing to hamper our ordinary reasoning.

Second, there is a sense in which ST allows unrestricted extension of chains of reasoning even about paradoxical cases. The structure of the sequent calculus itself allows for proving sequents by chaining together as many rule applications as one likes, of any sort, to sequents that are themselves already proved. Although cut is not an admissible rule to be chained together in this way, the sequent rules we do have can be applied repeatedly, without restriction. ST allows us to extend our knowledge *about consequence* as much as we like with chains of reasoning. It's just that cut is not an acceptable link in these chains.

But it's not strongly classical! ST, as discussed above, is weakly classical, but one might worry that its non-strong-classicality, while limited, is nonetheless a significant disadvantage. This worry is, I think, either ill-founded or else reduces to the worry about nontransitivity. After all, transitivity *is* the difference between ST and strong classical logic, as can be seen by examining the sequent calculus presented here for ST. If ST's deviation from strong classicality is a disadvantage, this must be because nontransitivity is a disadvantage; that's the extent of the deviation. So this worry reduces to the previous one.

Uniform substitution is a necessary condition for a logic ST is not closed under uniform substitution. For example, $T\langle p \rangle \vdash p$ is ST-derivable, but $q \vdash r$ is not, despite the fact that the latter comes from a uniform substitution on the former. (Recall that $T\langle p \rangle$ is just another atomic sentence, as far as any uniform substitution can tell.) If one thinks that uniform substitution is a necessary condition for a logic, then ST is not a logic.

This is not, I think, a major worry. I'm happy to respond in either of two ways. It might be that uniform substitution is not a necessary condition for a logic, or it might be that ST is not a logic. Either of these

²¹This is not because there are no nontransitive systems available. For example, [Tennant, 2004], [Weir, 2005], and [Zardini, 2008] all advocate nontransitive weakenings of classical logic, for a variety of reasons more and less similar to my own. The key advantage of ST over these systems is in its classicality; there is no need to weaken classical logic, as these systems all do, in the face of the paradoxes.

is fine by me; the issue strikes me as purely terminological. The question is about whether T is or is not a logical constant, and I have nothing in particular to say about that. There would be a substantive issue if the objector were claiming that *constraints on assertions and denials* must be closed under uniform substitution, but that's not remotely plausible.

The distinction between strict and tolerant is costly The ST-based approach outlined here has appealed to a distinction between strict and tolerant, both for assertion and denial. It might be thought that solutions that don't need this distinction have an advantage over the ST-based solution. There's something right about this; there is a cost to having to introduce new distinctions, especially distinctions not needed by one's competitor solutions. However, I think the price of this particular distinction is not high.

Firstly, it is not a primitive distinction; we can understand tolerant assertion and denial in terms of their strict cousins, as I've presented them here, or we can equally well understand strict in terms of tolerant. So long as we have a grip on one, there is no difficulty in coming to understand the other.

Second, it allows us to understand senses in which paracompletists and dialetheists of the usual sorts have advanced our understandings of the paradoxes: the paracompletists assert strictly and deny tolerantly, and the dialetheists assert tolerantly and deny strictly. Neither has the whole story, but they are not badly misled, only incomplete. By getting clear on the ways in which strict and tolerant fit together, ST allows us to see that classical logic is safe for use even when denying tautologies, and even when asserting contradictions.

6 Conclusion

This paper has presented a theory of truth and paradoxes based on the logic ST and a bilateralist theory of meaning. It's been shown that ST allows us to maintain all classical inferences in the presence of a fully transparent truth predicate; we need only to give up the rule of cut. Without cut, one way to understand models inferentially is blocked, but a related idea still works. By distinguishing strict from tolerant assertion, we can give a theory on which paradoxes fall into a gap between strict assertion and denial, and into the overlap of tolerant assertion and denial.

There is a lot that has not been done here. Because of the similarities between transparent truth and naive comprehension, there is some hope for ST to provide a fitting logical basis for naive set theory. Because of the similarities between ST, LPT, and K3T, there is some hope for an ST-based approach to paradox to prove as resistant to "revenge" worries as approaches based on those logics. Because of the similarities between ST and the logical system defended for vague language in [Cobreros et al., 2012], there is some hope for a unified ST-based approach to the semantic and soritical paradoxes. But all these are for future work. The goal here has simply been to make plain a philosophical perspective from which ST seems an appropriate guide to reasoning about paradoxes.²²

²²This paper has been helped tremendously by discussions with, and comments from, Conrad Asmus, Jc Beall, Pablo Cobreros, Paul Égré, Lloyd Humberstone, Toby Meadows, Graham Priest, Greg Restall, Robert van Rooij, Zach Weber, and Stefan Wintein, as well as comments from two anonymous referees.

A Completeness

This appendix shows that ST-provability guarantees ST-validity. The proof of completeness follows the outline in §3.2: first, we take an unprovable sequent and extend it as far as is safe, with safety determined by applying our sequent rules in reverse. We use this extended sequent to form the atomic basis of a Strong Kleene model. Finally, the Strong Kleene model is extended to a KK model by the method of [Kripke, 1975].²³

Before we begin the proof proper, a quick reformulation of the ST sequent calculus itself will help. Remove the rule of weakening, and replace the identity axioms ($A \vdash A$ for each atomic A) with weakened identity axioms ($\Gamma, A \vdash A, \Delta$ for each atomic A and sets Γ, Δ of formulas). This results in an equivalent sequent system, as can be proven by induction on proof length, and it makes things slightly easier below. We will also need an enumeration of the language's terms. Now, on to the proof.

First, extending the sequent. We start with an unprovable sequent $\Gamma_0 \vdash \Delta_0$, and build a tree above it, with each node consisting of a sequent that extends the sequent below it. (A sequent $\Gamma' \vdash \Delta'$ *extends* a sequent $\Gamma \vdash \Delta$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.) These extensions are determined in stages. At stage 0, we simply write $\Gamma_0 \vdash \Delta_0$, to start the tree off as its root.

As we extend the tree, we will sometimes find that the tip of a branch is an axiom. (Recall that axioms now have the form $\Gamma, A \vdash A, \Delta$, with A atomic.) When this happens, call the branch *closed*, and don't bother extending it any more. Call a branch that hasn't closed yet *open*.

At each successive stage, we apply the following steps to reduce each formula in the tip of each open branch, except for formulas that have themselves been generated during the stage itself. (Those get reduced starting in the next stage.)

- To reduce a negation on the left in $\Gamma, \neg A \vdash \Delta$, extend the branch with $\Gamma, \neg A \vdash A, \Delta$.
- To reduce a negation on the right in $\Gamma \vdash \neg A, \Delta$, extend the branch with $\Gamma, A \vdash \neg A, \Delta$.
- To reduce a conditional on the left in $\Gamma, A \supset B \vdash \Delta$, extend the branch by splitting it in two. To one new branch, add $\Gamma, A \supset B, B \vdash \Delta$; to the other, add $\Gamma, A \supset B \vdash A, \Delta$.
- To reduce a conditional on the right in $\Gamma \vdash A \supset B, \Delta$, extend the branch with $\Gamma, A \vdash B, A \supset B, \Delta$.
- To reduce a universal quantification on the left in $\Gamma, \forall x A(x) \vdash \Delta$, extend the branch with $\Gamma, \forall x A(x), A(t) \vdash \Delta$, where t is the first term that hasn't yet been used in a reduction of this formula in this position.
- To reduce a universal quantification on the right in $\Gamma \vdash \forall x A(x), \Delta$, extend the branch with $\Gamma \vdash A(a), \forall x A(x), \Delta$, where a is the first term not occurring in $\Gamma, A(x)$, or Δ .
- To reduce a truth predication of a distinguished name on the left in $\Gamma, T(A) \vdash \Delta$, extend the branch with $\Gamma, T(A), A \vdash \Delta$.

²³The extension of an unprovable sequent follows the method of *reduction trees* [Takeuti, 1987], and the addition of the Kripke construction to the end of the proof is a technique taken from [Kremer, 1988], where it is used for a different logic over the same models.

- To reduce a truth predication of a distinguished name on the right in $\Gamma \vdash T\langle A \rangle, \Delta$, extend the branch with $\Gamma \vdash A, T\langle A \rangle, \Delta$.

If \wedge , \vee , and \exists are not taken to be defined, similar reduction steps can be set up for them in the appropriate way; I'll skip that here, taking these to be defined.

If this process is followed ω times, one of two results will occur: either every branch will close, or some branch will remain open. If every branch has closed, then the tree is itself an ST proof of $\Gamma_0 \vdash \Delta_0$, as can be verified by comparing the reduction steps to the sequent rules, so $\Gamma_0 \vdash \Delta_0$ is provable after all. Contradiction.

Thus, there is a branch that remains open even after the above construction is completed. Each sequent on this branch extends the sequent below it. Take the union of all the sequents on the branch, and call it $\Gamma_\omega \vdash \Delta_\omega$.²⁴

Now, we can use $\Gamma_\omega \vdash \Delta_\omega$ to define a Strong Kleene model.²⁵ The model $\langle D, I \rangle$ is defined as follows.

Let D' be the set of nondistinguished terms occurring in $\Gamma_\omega \vdash \Delta_\omega$. Then $D = D' \cup \mathcal{L}$, where \mathcal{L} is the set of formulas in the language. Now, to I :

- For each nondistinguished term t , $I(t) = t$.
- For each distinguished term $\langle A \rangle$, $I(\langle A \rangle) = A$.
- For each n -ary predicate P (including T), $I(P)$ is the function that takes $\langle I(t_1), I(t_2), \dots, I(t_n) \rangle$ to 1 iff $P(t_1, t_2, \dots, t_n) \in \Gamma_\omega$, to 0 iff $P(t_1, t_2, \dots, t_n) \in \Delta_\omega$, and to $\frac{1}{2}$ otherwise.
- I for compound sentences is determined recursively as in §3.2.

Call this model M_0 . M_0 assigns value 1 to every formula in Γ_ω and 0 to every formula in Δ_ω , as can be shown by induction on formula length, and so assigns 1 to everything in Γ_0 and 0 to everything in Δ_0 . It is almost an ST-counterexample. The only problem is that it's not guaranteed to be a KK model; we might have A such that $I(T\langle A \rangle) \neq I(A)$.

This can only happen in certain circumstances. If $I(T\langle A \rangle) = 1$, then $T\langle A \rangle \in \Gamma_\omega$, and so $A \in \Gamma_\omega$; since M_0 assigns 1 to everything in Γ_ω , we know that $I(T\langle A \rangle) = I(A)$. Similarly, if $I(T\langle A \rangle) = 0$, we can show that $I(T\langle A \rangle) = I(A)$. The only risk is when $I(T\langle A \rangle) = \frac{1}{2}$. (This happens when $T\langle A \rangle$ does not occur in $\Gamma_\omega \vdash \Delta_\omega$.) In this case, we have no guarantee that $I(A) = \frac{1}{2}$ as well.

However, when a Strong Kleene model has this property (that whenever $I(T\langle A \rangle) \neq \frac{1}{2}$, $I(T\langle A \rangle) = I(A)$), it is *fixable*. That is, it is a suitable ground model for the fixed-point construction of [Kripke, 1975], discussed as well in [Kremer, 1988, Field, 2008], and a number of other places, and generalized in [Leitgeb, 1999]. Since M_0 is fixable, run this construction. The result will be a KK model $M = \langle D, I' \rangle$ with the property that $I(A) = I'(A)$ for every formula except possibly formulas of the form $T\langle A \rangle$ such that $I(T\langle A \rangle) = \frac{1}{2}$.

M is an ST-counterexample to $\Gamma_0 \vdash \Delta_0$. It differs from M_0 only on sentences that do not occur in $\Gamma_\omega \vdash \Delta_\omega$, and so do not occur in $\Gamma_0 \vdash \Delta_0$. But we've already seen that M_0 assigns 1 to everything in Γ_0 and 0 to everything in Δ_0 . Since M is a KK model, it is an ST-counterexample.

²⁴That is, take the set of everything occurring on the left of any sequent in the branch, and call it Γ_ω , and similarly for the right and Δ_ω .

²⁵A Strong Kleene model is a KK model without the restriction on the extension of T ; T is treated like any other unary predicate.

Thus, any sequent that is not ST-provable has an ST-counterexample; KK models are all the counterexamples we need for ST.

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