Eliminating the Ordinals from Proofs.
An Analysis of Transfinite Recursion

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Abstract: Transfinite ordinal numbers enter mathematical practice mainly via the method of definition by transfinite recursion. Outside of axiomatic set theory, there is a significant mathematical tradition in works recasting proofs by transfinite recursion in other terms, mostly with the intention of *eliminating the ordinals* from the proofs. Leaving aside the different motivations which lead each specific case, we investigate the mathematics of this action of *proof transforming* and we address the problem of formalising the philosophical notion of elimination which characterises this move.

Keywords: Ordinals, transfinite recursion, purity of method

Introduction

Transfinite ordinal numbers[1] are essential for the current development of axiomatic (Zermelo-Fraenkel) set theory (ZF). We prove theorems *about* ordinals (or other objects definable from them), and we prove theorems *by means* of ordinals. In the latter case, when the theorem to be proved does not mention the ordinals, the transfinite numbers used in the proof only appear to play an instrumental role and we can ask for alternative proofs not involving them.

As a matter of fact, in mathematics we can find a lot of examples — and a variety of reasons — for avoiding ordinals in proofs, when unnecessary. On the other hand, by carrying out a mathematical proof inside set theory, we can deal with the ordinals as sets. So, faced with a proof in which we feel the ordinals to be, in some sense, extraneous to the domain of discourse, we can try to *eliminate* the transfinite numbers either by implementing them as sets or by avoiding their use. In which cases, actually do we have the choice? Which are the salient differences between this two kinds of elimination?

In general, expressions such as "instrumental role" or "avoiding" do not convey precise mathematical concepts. Our purpose will be to investigate some cases in which it becomes possible to give a rigorous, logical content to the previous questions. For, we will focus on a particular, but ubiquitous, set-theoretical construction: the method of definition by transfinite recursion.

Informally, a general formulation of this method is the following:

Transfinite Recursion. *For every function G there exists a unique function F such that*

$$F(\alpha) = G(F \restriction \alpha^{<}), \tag{1}$$

holds for every $\alpha \in \mathsf{On}$ (where On is the class of all ordinal numbers, $\alpha^{<}$ denotes the set of the ordinal numbers less than α and $F \restriction \alpha^{<}$ denotes the restriction of F to the domain $\alpha^{<}$). We denote by $\Sigma(G)$ the function F whose existence and uniqueness are granted by transfinite recursion.

The intuitive idea, behind a proof which involves an application of transfinite recursion, is that we want to build some object by an iterative process: the function G captures the principles of construction we use at each step of the process, while the ordinals provide suitable indices for identifying the steps. So we are led to study the instances of the method of definition by transfinite recursion in which the ordinals only play an instrumental role as indices of the construction.

[1] In this paper *transfinite numbers*, *ordinals*, *ordinal numbers* all will be used as synonyms for *transfinite ordinal numbers*.

1. Kuratowski's programme from a modern perspective

Our starting point will be a well-known theorem with a lot of applications in many areas of mathematics, namely Bourbaki's Fixed Point theorem:

Theorem 1.1. *(Bourbaki, 1939, p. 37) Let $\mathbb{P} = \langle P, \preceq \rangle$ be an inductive poset and let $\Gamma : P \to P$ be expansive*[2]*. Then there is a unique $W \subseteq P$ wellordered by $\prec$ satisfying:*

(a) $x = \mathsf{lub}(\Gamma''\{y \in W \mid y \prec x\})$*, for every $x \in W$, and*

(b) $\mathsf{lub}(\Gamma''W) \in W$.

Bourbaki's theorem can be proved by using ordinal numbers. We define by transfinite recursion the following ordinal-length sequence S:

1. $S_0 = \bot$ (the bottom of $\mathbb{P}$).
2. $S_{\alpha+1} = \Gamma(S_\alpha)$.
3. $S_\gamma = \mathsf{lub}(S''\gamma)$ when γ is a limit ordinal.

Then we show that $\mathsf{ran}(S)$, the range of S, is wellordered by $\prec$ and satisfies both (a) and (b).

Alternatively, we can say that a subset X of P is Γ-*inductive* when

1. $x \in X \to \Gamma(x) \in X$, and
2. $Y \subseteq X \wedge \exists\, \mathsf{lub}(Y) \to \mathsf{lub}(Y) \in X$.

Then we can show that, when Γ is expansive, the least Γ-inductive set $\mathcal{I}(\Gamma)$ always exists and that if, further, $\mathbb{P}$ is inductive, then $\mathcal{I}(\Gamma)$ satisfies all the requirements of Bourbaki's theorem. This latter was the route followed by Bourbaki in his (1949/50) proof of the theorem.

Both Bourbaki's theorem and proof are straightforward abstract versions of an older theorem proved by Kuratowski (1922) for the special case of $\mathbb{P} = \mathcal{P}(E)$ being the powerset of a fixed set E partially ordered by inclusion[3].

Actually, Kuratowski was mostly concerned with the method, provided by the proof of his theorem, of eliminating the use of the transfinite numbers from a wide range of mathematical proofs. His purpose was explicitely stated in the title (*A method of eliminating the transfinite numbers from the mathematical reasoning*) and illustrated in details in the first part of his paper.

Kuratowski first fixes the scope and the goals of his work by focusing on a particular kind of construction by transfinite recursion which occurs in the original proofs of several theorems; then establishes a general result and shows its equivalence with the given construction; finally, he illustrates how his theorem can be applied for recasting old proofs in ordinals-free terms, by showing that similar achievements previously obtained by *ad hoc* procedures all can be addressed in a uniform way by his general method. We will refer to this complex of motivations and results as to *Kuratowski's programme*.

By sticking, for convenience, to Bourbaki's abstract setting, we can identify the schema of transfinite recursion taken under consideration by Kuratowski with the instances $\Sigma(G)$ in which G is defined by cases from an element $a \in P$, an expansive function $\Gamma : P \to P$ and the least upper bound of the range taken at limits, as in the above definition of the sequence S

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[2] A partially ordered set ("poset" for short) is *inductive* if exists the least upper bound (lub) of any nonempty chain. A map $\Gamma : P \to P$ is *expansive* if $x \preceq \Gamma(x)$ for every $x \in P$. For X a subset of P, $\Gamma''X$ denotes the *image of X under* Γ, i.e., the set $\{\Gamma(x) \mid x \in X\}$.

[3] By the way, Bourbaki's version is not really more general than Kuratowski's, since every partial order can be represented by inclusion (see, for instance, Moore, 1982, p. 226) and Kuratowski's proof only uses the inductiveness of $\mathcal{P}(E)$.

(we will call this schema *Kuratowski recursion*). Kuratowski gives the definition of the set $\mathcal{I}(\Gamma, a)$ (as above, with a in the role of $\perp$) then prove that $\mathcal{I}(\Gamma, a)$ and $\mathsf{ran}(\Sigma(\Gamma, a))$ coincide. This identity grounds his method of the elimination of the ordinals which consists, he says, in "replacing in each process represented by $\Sigma(\Gamma, a)$ the definition of the set $\mathsf{ran}(\Sigma(\Gamma, a))$ by the definition of the set $\mathcal{I}(\Gamma, a)$" (Kuratowski, 1922, p. 79). Kuratowski stresses the fact that his definition of the set $\mathcal{I}(\Gamma, a)$ does not appeal to the transfinite numbers but that, obviously, the proof of its equivalence with the definition of $\mathsf{ran}(\Sigma(\Gamma, a))$ does. Hence, this equivalence acts as a *promise* of an effective elimination of the ordinals, as resulting from an application of the general method. In Kuratowski's words "in each singular case, where we will have the purpose of eliminating the transfinite numbers, we will not make any use of this equivalence; it will never appear as a premise" (Kuratowski, 1922, p. 79). In particular, Kuratowski directly proves the properties of the set $\mathcal{I}(\Gamma, a)$ which we have summarised in the statement of Bourbaki's theorem.

Kuratowski was writing before von Neumann implementation of the ordinal numbers in set theory, hence his programme was well rooted in contemporary concerns about using these objects. The reception of the transfinite numbers by the mathematical community was affected by the general difficulties in accepting the new Cantorian theory of sets, but also by the fact that, at this stage, the theory of ordinal numbers was not established as a sound mathematical theory yet. Nevertheless, transfinite induction and transfinite recursion as methods in establishing new theorems, entered mathematics through the works given by Cantor himself and others. And these theorems, despite some scepticism about the techniques used in proving them, generally *was* accepted by the mathematical community.

As a result, proofs involving ordinals were mostly seen as a kind of informal arguments or as heuristic methods in finding solutions and became rather a common attitude to look for alternative proofs avoiding the ordinals[4]. As representative of this kind of attitude towards the transfinite numbers we can quote Borel's position, generally concerning "Cantor's arguments", expressed in a letter to Hadamard in 1905: "One may wonder what is the real value of these arguments that I do not regard as absolutely valid but that still lead ultimately to effective results. In fact, it seems that if they were completely devoid of value, they could not lead to anything, since they would be meaningless collections of words"[5].

Kuratowski's 1922 work, as a strictly intended programme of elimination of the ordinals, was superseded one year later by Von Neumann's implementation of ordinal numbers in set theory. Von Neumann (1923) established transfinite ordinals as legitimate mathematical objects, in fact *sets*. Even though Von Neumann ordinals provided a sound set-theoretical basis for doing transfinite recursion, Kuratowski's method of proof was rediscovered and survived in mathematical practice at least in two different forms, one requiring the axiom of Choice and one not requiring the axiom.

The Choice horn of the story is represented by a maximality principle already proved in Kuratowski's 1922 paper as an application of his method, and later popularised as a fruitful algebraic tool by Zorn (1935) (nowadays known from this latter as *Zorn's lemma*), and successfully used in other mathematical branches too, alongside with other equivalent maximality principles[6].

The Choice-free horn is represented by Kuratowski's 1922 main result emphasised as a fixed-point theorem by Bourbaki and, in this form and in later more specific variants, now acting as a central tool in order-theoretic applications, in particular to computer science issues[7].

Hence, even in current mathematical practice, we have evidence that Kuratowski's (and re-

[4]See Moore (1982, p. 222), Potter (2004, p. 292-293), Ferreirós (2007, p. 373).

[5]In (Baire et al., 1905), English translation from Moore (1982, p. 320).

[6]See Rosser (1953, pp. 481-482), Kuratowski and Mostowski (1976, p. 256, n. 1), Moore (1982, pp. 220-235).

[7]See Dugundji and Granas (1982), Davey and Priestley (1990, pp. 94ff), Carl and Heikkilä (2011).

lated) method of definition can be perceived as alternative to transfinite recursion. But what can we say in general? When does a method of definition have to count as an elimination of the ordinals from the proofs? Which is the mathematical content of Kuratowski's programme that survives von Neumann implementation of the ordinals in set theory?

Besides the empirical motivation represented by pre-existing examples of proofs recast in ordinals-free terms, Kuratowski gave two other reasons for pursuing his programme:

> *Even though, sometimes, transfinite numbers can be shown to be fruitful in making the exposition shorter or easier, the existence of a process that allows to avoid ordinals, in proving theorems that do not deal with the transfinite, is important for the following two reasons: in reasoning about ordinals we implicitly appeal to axioms which ensure their existence; but weakening the axioms system we use in proving something is desirable both from a logical and from a mathematical point of view. Moreover, this strategy expunges from the arguments the unnecessary elements, increasing their æsthetic value*[8].

In modern terms, the willing expressed by Kuratowsi could be formulated in precise, logical terms as follows [9].

Let $\mathcal{L}$ be a two-sorted first order language, with lower case Latin letters as variables for *sets* and lower case Greek letters for *ordinal numbers*. We assume Zermelo's set theory be axiomatised in this language and we add the Axiom of Ordinals:

Axiom of Ordinals. *For every wellordered set* $\langle A, R\rangle$ *there exists exactly one ordinal number* α*, the* order type *of* $\langle A, R\rangle$*, denoted by* $\mathsf{ot}(A, R)$*, such that*

$$\mathsf{ot}(A, R) = \mathsf{ot}(B, S) \iff \langle A, R\rangle \sim \langle B, S\rangle,$$

where "$\sim$" denotes the notion of isomorphism (or similarity*) between wellordered sets.*

In this formal setting, the purpose of "eliminating the ordinals from the proofs" can be stated as follows:

> *Given a theorem whose statement is formulated in the* pure *language of sets — i.e., not mentioning ordinal numbers — we look for a proof of this theorem in the same language, from Zermelo's axioms, not using the Axiom of Ordinals.*

We can recognise in Kuratowski's programme the following two motivations for eliminating the ordinals from a proof: (1) an unnecessary use of the Axiom of Ordinals; and (2) an instance of the "purity of method" concern: namely, if the statement of a theorem does not mention the ordinal numbers we want the proof neither does.

What is the significance of these two motivations after Von Neumann's implementation of the ordinals in set theory?

Since Von Neumann ordinals are sets, hardly we can attribute a precise, logical content to the expression "a theorem whose statement is formulated in the *pure* language of sets". Since Von Neumann ordinals are a proper subclass of the class of all sets we can, of course, positively say that "the ordinals occur in the statement of a theorem (or in a step of a proof)", but negating this statement raises some puzzles: on one hand, the universally quantified variables of the statement of the theorem, if any, range over *all* sets, ordinals included; besides, how we can

[8] Kuratowski, 1922, p. 77 (my translation). Of course, "æstethic value" is a matter of taste: for an opposite evaluation see Hausdorff (1935, p. 188, 1957 English translation)

[9] See Kuratowski and Mostowski (1976) for details.

exclude that the statement speaks about the ordinals in some indirect way, perhaps via Scott-Tarski's implementation[10] instead of Von Neumann's, or even by speaking about other sets which can play the role of ordinals, even though we are not aware of this possibility?

Even the Axiom of Ordinals hardly can help in distinguishing between proofs involving or not involving the ordinals. Even though the Axiom of Ordinals for Von Neumann ordinals is not provable in Zermelo set theory, the instances of transfinite recursion which Kuratowski's method aims to eliminate *are* provable in Zermelo set theory. Hence, after Kuratowski's "elimination" of the ordinals, we obtain a "new" proof of the theorem in the same language and from the same axioms.

Puzzles of this sort about the logical content of the expression "eliminating the ordinals from the proofs" in a Zermelo context lead to the unpleasant conclusion that the "purity of method" concern might make sense only from a historical or epistemological point of view: the possibility of distinguishing ordinals from sets would only depend on our knowledge about the existence of a set-theoretical implementation of the ordinals, not on what sets and ordinals actually are.

Yet, in the informal mathematical language, we still speak about "theorems and proofs involving (or not involving) the ordinals" and we understand that "the ordinals play an instrumental role in a proof of a theorem" when, first, "the theorem does not talk about the ordinals" and, secondly, we can find another proof of the same theorem in which "the ordinals do not appear"[11].

Hence we are looking for other ways of recasting Kuratowski's programme in formal terms which can capture the sense of the expression "eliminating the ordinals from the proofs".

Referring to the analysis given by Quine (1960, Chapter VII) of the elimination of the abstract objects from the mathematical discourse, we can say that the implementation of the ordinal numbers in set theory counts as an *elimination* in the sense of Quine's example of the implementation of the ordered pairs[12]. Yet, it remains the possibility of an elimination by paraphrasing, as in Quine's example about the infinitesimals.

We will try to give an operative sense to the word "elimination", the sense implicitly given by Kuratowski when, speaking about his programme, says:

> *After we have defined this schema* [*transfinite recursion by an expansive function*] I will establish a general method which allows us to transform every construction represented by this schema into another which no longer makes appeal to any notion of transfinite number.[13]

Our strategy will be the following. We look at instances $\Sigma(G)$ of transfinite recursion where G takes values from a fixed set P, and try to identify properties of G which allow to transform the definition of $\mathsf{ran}(\Sigma(G))$ in a way recognisable, in some sense, as an "elimination of the ordinals". We will consider a few such properties which lead to corresponding progressively stronger notion of "eliminability". Finally we observe that Kuratowski's schema and method fit the stronger of such notions.

Our criteria for eliminating the ordinals will be extracted from an analysis of the intuitive process which is formalised by transfinite recursion, namely, the iterated application of an operator G, continued into the transfinite. The outcome of an application of transfinite recursion is a definable transfinite sequence $\Sigma(G)$ provably recursive on the full class of the ordinals. As a set-theoretical definition, $\Sigma(G)$ is a correspondence between sets implementing the ordinals

[10]Scott (1955) and Tarski (1955).

[11]See Fitting (1986), Moschovakis (1994), Back and Von Wright (1998), for some examples of "post-Von Neumann" works in which similar attitudes, towards the role played by the ordinals in mathematics, are implicitly adopted.

[12]See Kuratowski and Mostowski (1976, Chapter VII, §9).

[13]Kuratowski, 1922, p. 77 (the translation is mine, italics are in the original).

and other sets. As in Cantor's first formulation of this idea, in the iterative process the ordinals just play the role of *indices*, no matter which is their internal structure. So, our first criterion is:

First Criterion. *The range* $\mathsf{ran}(\Sigma(G))$ *has to be independent from the specific set-theoretical implementation of the ordinals used to formalise the definition of the transfinite sequence* $\Sigma(G)$.

The second criterion concerns the length of the defined sequence. Transfinite recursion produces an ordinal-length sequence, i.e., a sequence defined on the proper class of all ordinals. But hardly this is needed when the purpose of the iterative process is just to extract the relevant information from some *set*. Cantor's original need was to be able to perform the iteration as long as necessary, as long as some closure condition — identifying the goal of the iteration — was met. After the stage where this condition holds is reached, the iteration produces information redundancies and resuming the process adds nothing to the relevant information carried by the already constructed sequence. This remark leads to the idea of an indefinitely prolongeable process, up to a variable length depending on some side condition, rather than to an actual ordinal-length sequence, as the output provided by transfinite recursion is.

The process of iteration abstracts from the idea of looking for an object which satisfies certain properties by examining step by step a list of candidates. When the candidates are bounded to belong to some set, we want to look at a searching inside this set, not at an endless process. This idea also has impact on the possibility of avoiding any reference to proper classes, since any sequence defined by transfinite recursion is a proper class, while, under the usual implementation of the ordinals, for any fixed length the process stopped at this length produces a set.

Therefore, we can state our second criterion as

Second Criterion. *We say that* G *is* local *if there exists an ordinal* δ *such that* $\mathsf{ran}(\Sigma(G)) = \mathsf{ran}(\Sigma(G) \restriction \delta)$.

If there exists an ordinal such as δ, there exists also a least one. We call this latter the *characteristic length* of G and denote it by $\delta(G)$.

The naïve idea is that, if G is local, then we can just define S from G up to $\delta = \delta(G)$, by using the local version of transfinite recursion, which we denote by $\Sigma(G, \delta)$. Besides, if G is also invariant, we can choose any wellordered set similar to δ as the base for recursion, so even dispensing with a particular implementation of the ordinal numbers.

However, to realise this project, we need to find a uniform way for estimating the characteristic length of G *before* running the recursion, i.e., we need a criterion on G to say that G is local, and we need a function $\mathcal{H}$ which computes a wellordered set $\mathcal{H}(G)$ which turns out to be similar to the characteristic length $\delta(G)$.

Finally, the last criterion will deal with the possibility of recovering the relevant information without appealing to the intuitive idea of an iterative process: we need to positively formulate an alternative method, as Kuratowski did for the special case of the transfinite iteration of a progressive operator. This means to find a definition of the range of $\Sigma(G)$ based on resources already present in the set of values taken by G, despensing at all with extraneous sets acting as "indices" in a recursive process.

Since G is defined on sequences from P, in particular G is defined on wellordering from P. Since arbitrary wellordering from P can be identified with families of subsets of P, we can see G as an operator from $\mathcal{P}(\mathcal{P}(P))$ into P. Hence our third criterion will be stated as follows:

Third Criterion. *Let* $\mathbb{P}$ *be some fixed structure with domain* P. *We say that* G *is* internal to $\mathbb{P}$ *if* $\mathsf{ran}(\Sigma(G))$ *is definable in some finite-order structure over* $\langle \mathbb{P}, G \rangle$.

2. Hartogs recursion

In this section we characterise a schema of transfinite recursion which meets all three criteria for eliminating the ordinals and generalises Kuratowski's method.

Our formalisation of the First Criterion will be provided by the notion of indexed mapping. An *indexed mapping* S will be a pair $(F, <)$, where F is a mapping and $<$ is a wellordering of $\mathsf{dom}(F)$, the domain of F. This notion intends to be a generalisation of both notions of *sequence* and *wellordering*, since every sequence is associated to its domain wellordered by the less than relation between ordinals and every wellordering $\langle A, R\rangle$ can be seen as the indexed mapping (id_A, R), where id_A is the identity on A.

Definition 2.1. Let S, T be two indexed mappings whose domains are, respectively, A and B. We say that S and T are *similar* if and only if

- A and B are similar wellorderings, and
- $S(x) = T(\chi(x))$ for every $x \in A$, where χ is the unique isomorphism between A and B.

Definition 2.2. We say that a mapping G is *invariant* if and only if $G(S) = G(T)$ whenever S and T are similar indexed mappings.

For instance, it is straightforward to see that the map $S \mapsto \mathsf{ran}(S)$ is an invariant mapping.

The formal notion of invariant captures the intuitive idea that, when we are only interested in the information given by $H(S)$, where H is invariant, and not in the full S, the wellordering on the domain of S just plays the role of a system of indices, freely interchangeable with any other similar wellordering.

Definition 2.3. Let G be any mapping. We say that an indexed mapping S is *G-recursive* if and only if

$$S(x) = G(S \restriction x^<),$$

for every $x \in \mathsf{dom}(S)$, where $x^<$ denotes the initial segment[14] of the domain of S determined by x.

The method of definition by transfinite recursion naturally extends to indexed functions in the form: for every mapping G and for every wellordering A there exists a unique G-recursive indexed mapping F such that $\mathsf{dom}(F) = A$. The indexed mapping F uniquely determined by G and A will be denoted by $\Sigma(G, A)$.

It is straightforward to prove by induction the following

Lemma 2.4. *Let G be an invariant mapping. Let S and T be two G-recursive indexed mappings. If the domains of S and T are similar wellorderings then S and T are similar.*

By Lemma 2.4, when G is invariant all indexed mappings defined from G by transfinite recursion are similar, no matter which wellordering (up to similarity) we choose to carry out the recursive definition. This leads to the following:

Remark 2.5. If G is invariant then the definition by transfinite recursion of $\mathsf{ran}(\Sigma(G))$ is independent from the implementation of the ordinal numbers, i.e., whatever wellordered proper class of sets we choose as system of indices for the recursion we obtain the same result.

Definition 2.6. We say that G is *Hartogs* if and only if G is invariant and

$$G(f) \notin \mathsf{ran}(f) \to f \quad \text{is injective},$$

for every G-recursive indexed function f.

[14] An *initial segment* of an ordered set $\langle A, <\rangle$ is a subset $X \subseteq A$ such that $y < x \to y \in X$ for every $x \in X$ and $y \in A$. X is a *proper* initial segment of A if $X \neq A$. For $x \in A$, $x^< = \{y \in A \mid y < x\}$ is a proper initial segment of A.

Theorem 2.7. *Let G be an invariant Hartogs mapping. Then there exists a wellordering $\mathcal{H}(G)$ such that*
$$\mathsf{ran}(\Sigma(G)) = \mathsf{ran}(\Sigma(G, \mathcal{H}(G))).$$

Proof. Let G be a Hartogs mapping.

Claim: For every wellordering X, if $\Sigma(G, X)$ is not injective then $\mathsf{ran}(\Sigma(G, X)) = \mathsf{ran}(\Sigma(G, A))$ for all A end-extending[15] X.

Proof of the claim. Assume that $\Sigma(G, X)$ is non-injective. We will show, by induction, that $\mathsf{ran}(\Sigma(G, A)) \subseteq \mathsf{ran}(\Sigma(G, X))$, for every wellordering A end-extending X.

Assume, by the inductive hypothesis, $\mathsf{ran}(\Sigma(G, Y)) = \mathsf{ran}(\Sigma(G, X))$, for every initial segment Y such that $X \subseteq Y \subset A$. Let $y \in \mathsf{ran}(\Sigma(G, A))$.

If A has no maximum, then $\mathsf{ran}(\Sigma(G, A)) = \bigcup\{\mathsf{ran}(\Sigma(G, Y)) \mid Y \subset A\}$ so, trivially, $y \in \mathsf{ran}(\Sigma(G, X))$. If not, let $S = \Sigma(G, A)$ and let x be the maximum of A, and suppose, without loss of generality, $y = S(x) = G(S \restriction x^<) = G(\Sigma(G, x^<))$. By hypothesis, if $G(\Sigma(G, x^<)) \notin \mathsf{ran}(\Sigma(G, x^<))$ then $\Sigma(G, x^<)$ is injective, contradicting the non-injectivity of $\Sigma(G, X)$. Hence $y = G(\Sigma(G, x^<)) \in \mathsf{ran}(\Sigma(G, x^<)) = \mathsf{ran}(\Sigma(G, X))$. $\dashv$

From the claim immediately follows that if $\Sigma(G, X)$ is not injective then $\mathsf{ran}(\Sigma(G, X)) = \mathsf{ran}(\Sigma(G))$ so it only remains to show that such an X exists.

By Hartogs' theorem (1915) there exists a wellorderable set $\mathcal{H}(P)$ not injectable in P. Hence $\Sigma(G, \mathcal{H}(P))$ is not injective. $\square$

Theorem 2.7 shows that any Hartogs mapping G generates a transfinite sequence $\Sigma(G)$ which satisfies the first and the second criterion for eliminating the ordinals.

Theorem 2.8. *Let G be an invariant Hartogs mapping. Then* $\mathsf{ran}(\Sigma(G))$ *is definable in the third-order structure over* $\langle \mathbb{P}, G \rangle$.

Proof. We say that a wellordering $X \in \mathcal{P}(\mathcal{P}(P))$ is *G-recursive* if and only if $x = G(x^<)$ for every $x \in X$. By a straightforward generalisation of the second proof of Zermelo's wellordering theorem[16], we can prove in the third-order structure over $\langle \mathbb{P}, G \rangle$ that there exists a unique G-recursive wellordering $\mathcal{K}(G) = K$ such that $G(K) \in K$. We want to show that $K = \mathsf{ran}(\Sigma(G))$. By Hartogs recursion there exists a unique injective G-recursive function f whose domain X is a proper initial segment of $\mathcal{H}(P)$ and such that $G(f) \in \mathsf{ran}(f)$. Since f is injective and G is invariant, f induces on its range a G-recursive wellordering Y such that $G(Y) \in Y$. Hence, by uniqueness, $X = \mathsf{ran}(f) = K$. Since X is a proper initial segment of $\mathcal{H}(P)$, $X = a^<$ for a unique $a \in \mathcal{H}(P)$. Let $X' = X \cup \{a\}$. Since $\Sigma(G, X')(a) = G(f) \in \mathsf{ran}(f)$, $\Sigma(G, X')$ is not injective, hence by Thm 2.7, $K = \mathsf{ran}(f) = \mathsf{ran}(\Sigma(G, X')) = \mathsf{ran}(\Sigma(G))$. $\square$

Definition 2.9. We say that G is *Kuratowski* if and only if
$$\mathsf{ran}(f) = \mathsf{ran}(g) \to G(f) = G(g),$$
for every G-recursive indexed functions f and g.

By definition, G is Kuratowski if G is a function of the map $f \mapsto \mathsf{ran}(f)$ restricted to G-recursive indexed functions. Since the map $f \mapsto \mathsf{ran}(f)$ is invariant it follows that every Kuratowski mapping is an invariant mapping.

Lemma 2.10. *Every Kuratowski mapping is a Hartogs mapping.*

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[15] An ordered set $\langle A, < \rangle$ *end-extends* an ordered set $\langle B, \prec \rangle$ if and only if B is an initial segment of A and $\prec$ is the restriction of $<$ to B.

[16] See Kanamori (1997, Theorem 2.1, p. 292)

Proof. Let G be a Kuratowski mapping and f a G-recursive non-injective indexed function. Let y be first in $\mathsf{dom}(f)$ such that $f(y) = f(x)$ for some $x < y$. We will show, by induction, that $f(z) = f(x)$ for every $z \geq y$. Suppose, by the inductive hypotesis, $f(u) = f(x)$ for every u such that $y \leq u < z$. Hence $\mathsf{ran}(f \restriction y^{<}) = \mathsf{ran}(f \restriction z^{<})$. Since G is Kuratowski, it follows $f(z) = G(f \restriction z^{<}) = G(f \restriction y^{<}) = f(y) = f(x)$. Hence $\mathsf{ran}(f) = \mathsf{ran}(f \restriction y)$, so $G(f) = G(f \restriction y) = f(y) \in \mathsf{ran}(f)$, i.e., G is Hartogs. □

The mapping G defined by cases from an expansive operator $\Gamma : P \to P$ can also be defined as $G(f) = \mathsf{lub}(\Gamma''\mathsf{ran}(f))$, so it is immediate to see that G, accordingly to our definition, is Kuratowski. Hence Lemma 2.10 and Theorem 2.8 together give a direct account of why Kuratowski's theorem is successful in eliminating the ordinals (according to our three Criteria) from arguments involving ordinal-length sequences which are generated by an expansive operator on an inductive poset.

Actually, Hartogs recursion is more general than Kuratowski. A Hartogs mapping G generates an ordinal-length sequence $\Sigma(G)$ pictured as follows: an injective sequence $f = \Sigma(G) \restriction \delta$, followed by an ordinal-length sequence S such that $S(\alpha) \in \mathsf{ran}(f)$ for every $\alpha \geq \delta$. For contrast, an ordinal-length sequence generated by a mapping G which satisfies the hypotheses of Bourbaki-Kuratowski's theorem, looks as an injective sequence $f = \Sigma(G) \restriction \delta$, followed by a *constant* ordinal-length sequence S such that $S(\alpha) = f(\delta^{-})$ for every $\alpha \geq \delta$, where δ^{-} is the immediate predecessor of δ.

On the other hand, under the hypothesis of Bourbaki's theorem, $\mathsf{ran}(\Sigma(G))$ admits a *second-order* definition over the structure $\langle \mathbb{P}, G \rangle$. Indeed we can prove that, in this case, is not necessary to consider arbitrary G-recursive wellordering from P but only subset of P wellordered by the partial order $\prec$ of $\mathbb{P}$. These latter live in $\mathcal{P}(P)$ and $K = \mathsf{ran}(\Sigma(G)))$ can be characterized as the greatest G-recursive subset of P.

Moreover, for every poset $\mathbb{P}$, the following holds:

Lemma 2.11. *If Γ is expansive, then the least Γ-inductive set $\mathcal{I}(\Gamma)$ exists and $\mathcal{I}(\Gamma) = K$.*

Proof. For the existence and G-recursiveness of $\mathcal{I}(\Gamma) = I$ see Bourbaki, 1949/50. Hence $I \subseteq K$. We will show by induction on the wellordering of K that $K = I$. Suppose, towards a contradiction, that there exists an element in $K - I$, and let x be the first one. Since $y \in K \wedge y \prec x$ implies $y \in I$, it follows $\{y \in I \mid y \prec x\} = \{y \in K \mid y \prec x\}$. Let $Y = \{\Gamma(y) \in P \mid y \in U \wedge y \prec x\}$. Since K is G-recursive, $x = \mathsf{lub}(Y)$. Since $Y \subseteq I$, $\mathsf{lub}(Y)$ exists and I is Γ-inductive, $x = \mathsf{lub}(Y) \in I$: contradiction. Hence $K - I = \emptyset$, so $I = K$. □

Lemma 2.11 is just an abstract and concise formulation of the well known possibility of giving definitions "from below" (K) or "from above" ($\mathcal{I}(\Gamma)$) of the same set $\mathsf{ran}(\Sigma(G))$ when G is defined from an expansive function; a possibility first shown by the two Zermelo's proofs (1904 and 1908) of the wellordering theorem or, for the special case of the natural numbers, by Dedekind (1888, §131)[17]. Hence, the possibility of giving a definition "from above" of the set $\mathsf{ran}(\Sigma(G))$ can be seen as a stronger condition satisfied by Kuratowski recursion but not shared with (the more general) Hartogs recursion.

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[17]See Kanamori, 2004, p. 502.

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