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Varieties of interlaced bilattices

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Abstract. The paper contains some algebraic results on several varieties of algebras having an (interlaced) bilattice reduct. Some of these algebras have already been studied in the literature (for instance bilattices with conflation, introduced by M. Fitting), while others arose from the algebraic study of O. Arieli and A. Avron's bilattice logics developed in the third author's PhD dissertation. We extend the representation theorem for bounded interlaced bilattices (proved, among others, by A. Avron) to unbounded bilattices and prove analogous representation theorems for the other classes of bilattices considered. We use these results to establish categorical equivalences between these structures and well-known varieties of lattices.

1. Introduction

Bilattices are algebraic structures introduced in 1988 by Matthew Ginsberg [10] as a uniform framework for inference in Artificial Intelligence. Since then they have found a variety of applications, sometimes in quite different areas from the original one. The interest in bilattices comes thus from different contexts: among others, computer science and A.I. (see especially the works of Ginsberg, Arieli and Avron), logic programming (Fitting), lattice theory and algebra [11, 12, 17] and, more recently, algebraic logic [4, 16]. An up-to-date review of the applications of this formalism and also of the motivation behind its study can be found in the dissertation [16].

This work contributes to the study of bilattices from an algebraic point of view, along the line initiated by [2, 11]. We consider various classes of algebras having a bilattice reduct, focusing on the relationship between these algebras and some well-known varieties of lattices. The novelty and the main interest of our approach lies, in our opinion, in the fact that this relationship can be exploited to obtain several results on bilattices by simply "translating" results that are known to hold for some related classes of lattices. Moreover, as we shall see, this strategy also suggests a natural way to introduce new classes of bilattices starting from their lattice counterparts.

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The paper is organized as follows. The next section contains the basic definitions on bilattices. Section 3 presents the representation theorem for interlaced bilattices, a result which plays a key role in our approach. In Section 4 we introduce some structures obtained by expanding the bilattice language, prove analogous representation theorems for these algebras, and define translations that provide a way to relate them to some known classes of lattices. These results are exploited in Section 5 to establish categorical equivalences between the categories associated with the classes of bilattices considered and those associated with the corresponding classes of lattices.

2. Bilattices

The terminology concerning bilattices is not uniform. Following [2], we reserve the name "bilattice" for the algebras that are sometimes called "bilattices with negation", while when there is no negation we use the term "pre-bilattice".

Definition 2.1. A *pre-bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus \rangle$ such that $\langle B, \wedge, \vee \rangle$ and $\langle B, \otimes, \oplus \rangle$ are both lattices.

The order associated with the lattice $\langle B, \wedge, \vee \rangle$, sometimes called the *truth lattice* or *t-lattice*, is denoted by $\leq_t$ and is called the *truth order*, while the order $\leq_k$ associated with $\langle B, \otimes, \oplus \rangle$, sometimes called the *knowledge lattice* or *k-lattice*, is the *knowledge order*.

Usually in the literature it is required that the two lattices be complete or at least bounded, but here, for the sake of generality, none of these assumptions is made. The minimum and maximum element of the lattice $\langle B, \wedge, \vee \rangle$, in case they exist, will be denoted, respectively, by f and t. Similarly, $\bot$ and $\top$ will refer to the minimum and maximum of $\langle B, \otimes, \oplus \rangle$, when they exist.

Of course the interest in pre-bilattices increases when there is some connection between the two orders. One way of establishing such a connection is to impose certain monotonicity properties on the lattice connectives, as in the following definition, due to Fitting [7].

A pre-bilattice $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus \rangle$ is *interlaced* whenever each one of the four operations $\{\wedge, \vee, \otimes, \oplus\}$ is monotonic with respect to both orders $\leq_t$ and $\leq_k$. That is, when the following quasi-equations hold:

$$x \leq_t y \Rightarrow x \otimes z \leq_t y \otimes z, \qquad x \leq_t y \Rightarrow x \oplus z \leq_t y \oplus z,$$
$$x \leq_k y \Rightarrow x \wedge z \leq_k y \wedge z, \qquad x \leq_k y \Rightarrow x \vee z \leq_k y \vee z.$$

As usual, the inequality $x \leq_t y$ is an abbreviation for the identity $x \wedge y \approx x$ and similarly $x \leq_k y$ stands for $x \otimes y \approx x$.

Pre-bilattices form a variety, axiomatized by the lattice identities for the two lattices. In [2, 17] it is proved that the class of interlaced pre-bilattices is also a variety, axiomatized by the identities for pre-bilattices plus the following ones:

$$(x \wedge y) \otimes z \leq_t y \otimes z, \qquad (x \wedge y) \oplus z \leq_t y \oplus z,$$
$$(x \otimes y) \wedge z \leq_k y \wedge z, \qquad (x \otimes y) \vee z \leq_k y \vee z.$$

From an algebraic viewpoint, the variety of interlaced pre-bilattices is perhaps the most interesting subclass of pre-bilattices. Its interest comes mainly from the fact that any interlaced pre-bilattice can be represented as a special product of two lattices. This result is well known for bounded pre-bilattices, and it has been more recently generalized to the unbounded case [12, 4].

The interlacing conditions may be strengthened through the following definition, due to Ginsberg [10]. A pre-bilattice is *distributive* when all possible distributive laws concerning the four lattice operations, i.e., all identities of the following form, hold:

$$x \circ (y \bullet z) \approx (x \circ y) \bullet (x \circ z) \quad \text{for every } \circ, \bullet \in \{\wedge, \vee, \otimes, \oplus\}.$$

The class of distributive pre-bilattices is a proper subvariety of interlaced pre-bilattices.

A second way of relating the two lattice orders of a pre-bilattice is by expanding the algebraic language with a unary operator. This is the method originally used by Ginsberg to introduce bilattices.

Definition 2.2. A *bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ such that the reduct $\langle B, \wedge, \vee, \otimes, \oplus \rangle$ is a pre-bilattice and the *negation* $\neg$ is a unary operation such that, for every $a, b \in B$,

(neg 1) if $a \leq_t b$, then $\neg b \leq_t \neg a$,
(neg 2) if $a \leq_k b$, then $\neg a \leq_k \neg b$,
(neg 3) $a = \neg\neg a$.

The interlacing and distributivity properties extend to bilattices in the obvious way. We say that a bilattice is interlaced (respectively, distributive) when its pre-bilattice reduct is interlaced (resp., distributive).

The following equations (which we call De Morgan laws) hold in any bilattice:

$$\neg(x \wedge y) \approx \neg x \vee \neg y, \qquad \neg(x \vee y) \approx \neg x \wedge \neg y,$$
$$\neg(x \otimes y) \approx \neg x \otimes \neg y, \qquad \neg(x \oplus y) \approx \neg x \oplus \neg y.$$

Moreover, if the bilattice is bounded, then $\neg\top = \top$, $\neg\bot = \bot$, $\neg\mathsf{t} = \mathsf{f}$ and $\neg\mathsf{f} = \mathsf{t}$.

So, if a bilattice $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is distributive, or at least the reduct $\langle B, \wedge, \vee \rangle$ is a distributive lattice, then $\langle B, \wedge, \vee, \neg \rangle$ is a De Morgan lattice. It is also easy to check that the four De Morgan laws imply that the negation operator satisfies **(neg 1)** and **(neg 2)**. It follows that the class of bilattices is a variety. We denote by $\mathsf{IntBiLat}$ and DBiLat the classes of *interlaced bilattices* and *distributive bilattices*, which are also varieties. Clearly $\mathsf{DBiLat} \subseteq \mathsf{IntBiLat}$, and this inclusion is strict.

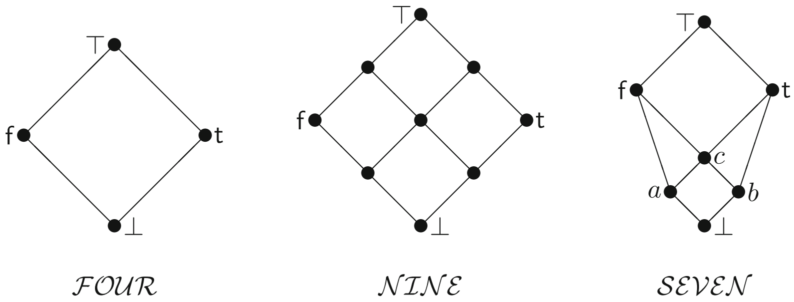


FIGURE 1. Some examples of (pre-)bilattices

Figure 1 shows the double Hasse diagrams of some of the best-known (pre-) bilattices. They should be read as follows: $a \leq_t b$ if there is a path from a to b which goes uniformly from left to right, while $a \leq_k b$ if there is a path from a to b which goes uniformly from the bottom to the top. The four lattice operations are thus uniquely determined by the diagram, while negation, if there is one, corresponds to reflection along the vertical axis connecting $\bot$ and $\top$. It is then clear that all the pre-bilattices shown in Figure 1 can be endowed with a negation in a unique way and turned in this way into bilattices. When no confusion is likely to arise, we will use the same name to denote a concrete pre-bilattice and its associated bilattice; the names used in the diagrams are by now more or less standard in the literature.

The smallest non-trivial bilattice, $\mathcal{FOUR}$, has a fundamental role among bilattices, both from an algebraic and a logical point of view. $\mathcal{FOUR}$ is distributive and, as a bilattice, it is a simple algebra. It is in fact, up to isomorphism, the only subdirectly irreducible distributive bilattice (this was proved for the bounded case in [11], then generalized in [4] to the unbounded).

3. Representation theorems

The representation theorem for interlaced (pre-)bilattices has a key role in our approach to the study of bilattices. In order to state this result, we introduce the following construction, due to Fitting [7].

Let $\mathbf{L_1} = \langle L_1, \sqcap_1, \sqcup_1 \rangle$ and $\mathbf{L_2} = \langle L_2, \sqcap_2, \sqcup_2 \rangle$ be lattices with associated orders $\leq_1$ and $\leq_2$. The *product pre-bilattice* $\mathbf{L_1} \odot \mathbf{L_2} = \langle L_1 \times L_2, \wedge, \vee, \otimes, \oplus \rangle$ is defined as follows. For all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L_1 \times L_2$,

$$\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle := \langle a_1 \sqcap_1 b_1,\ a_2 \sqcup_2 b_2 \rangle,$$
$$\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle := \langle a_1 \sqcup_1 b_1,\ a_2 \sqcap_2 b_2 \rangle,$$
$$\langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle := \langle a_1 \sqcap_1 b_1,\ a_2 \sqcap_2 b_2 \rangle,$$
$$\langle a_1, a_2 \rangle \oplus \langle b_1, b_2 \rangle := \langle a_1 \sqcup_1 b_1,\ a_2 \sqcup_2 b_2 \rangle.$$

$\mathbf{L_1} \odot \mathbf{L_2}$ is always an interlaced pre-bilattice, and it is distributive if and only if both $\mathbf{L_1}$ and $\mathbf{L_2}$ are distributive lattices. From the definition it follows immediately that

$$\langle a_1, a_2 \rangle \leq_k \langle b_1, b_2 \rangle \quad \text{iff} \quad a_1 \leq_1 b_1 \quad \text{and} \quad a_2 \leq_2 b_2,$$
$$\langle a_1, a_2 \rangle \leq_t \langle b_1, b_2 \rangle \quad \text{iff} \quad a_1 \leq_1 b_1 \quad \text{and} \quad a_2 \geq_2 b_2.$$

If $\mathbf{L_1}$ and $\mathbf{L_2}$ are isomorphic, then it is possible to define a negation in $\mathbf{L_1} \odot \mathbf{L_2}$, and we speak of the *product bilattice* instead of the product pre-bilattice. If $h\colon \mathbf{L_1} \cong \mathbf{L_2}$ is an isomorphism, then the negation is defined as

$$\neg\langle a_1, a_2 \rangle := \langle h^{-1}(a_2), h(a_1) \rangle.$$

In particular, if $\mathbf{L_1} = \mathbf{L_2}$, the definition gives $\neg\langle a_1, a_2 \rangle := \langle a_2, a_1 \rangle$.

The product pre-bilattice construction can be regarded as a special case of direct product. In fact, any lattice $\mathbf{L} = \langle L, \sqcap, \sqcup \rangle$ can be seen as a degenerate pre-bilattice in at least four different ways. We can consider the following algebras:

$$\mathbf{L}^{++} = \langle L, \sqcap, \sqcup, \sqcap, \sqcup \rangle, \qquad \mathbf{L}^{+-} = \langle L, \sqcap, \sqcup, \sqcup, \sqcap \rangle,$$
$$\mathbf{L}^{-+} = \langle L, \sqcup, \sqcap, \sqcap, \sqcup \rangle, \qquad \mathbf{L}^{--} = \langle L, \sqcup, \sqcap, \sqcup, \sqcap \rangle.$$

The first superscript, $+$ or $-$, says whether we are taking as t-order the same order as in the original lattice or the dual one; the second superscript refers to the same for the k-order. Using this notation, it is easy to see that the product pre-bilattice $\mathbf{L_1} \odot \mathbf{L_2}$ coincides with the direct product $\mathbf{L_1}^{++} \times \mathbf{L_2}^{-+}$. On the other hand, the product bilattice cannot be regarded as a direct product, and in general the factor lattice need not have a negation.

The following results were proved by Avron [2] for bounded (pre-)bilattices, then generalized in [12, 4] to the unbounded case:

Theorem 3.1 (Representation of pre-bilattices)**.** *A pre-bilattice* $\mathbf{B}$ *is interlaced if and only if there exist two lattices* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *such that* $\mathbf{B} \cong \mathbf{L_1} \odot \mathbf{L_2}$*. Moreover,* $\mathbf{B}$ *is distributive if and only if both* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are distributive lattices.*

The idea of the proof in [4], that differs essentially from all the proofs that can be found in the literature on bounded bilattices (and also from that in [12]), is the following. Given an interlaced pre-bilattice $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus \rangle$, define two relations $\sim_1$ and $\sim_2$ as follows:

$$\sim_1 = \{\langle a, b \rangle \in B \times B : a \vee b = a \otimes b\},$$
$$\sim_2 = \{\langle a, b \rangle \in B \times B : a \wedge b = a \otimes b\}.$$

Verify that the above relations are congruences of $\mathbf{B}$ (thus, in particular, they are compatible with $\otimes$ and $\oplus$) and prove that, setting $\mathbf{L_1} = \langle B, \otimes, \oplus \rangle / {\sim_1}$ and $\mathbf{L_2} = \langle B, \otimes, \oplus \rangle / {\sim_2}$, we have $\iota_{\mathbf{B}}\colon \mathbf{B} \cong \mathbf{L_1} \odot \mathbf{L_2}$, with the isomorphism $\iota_{\mathbf{B}}$ defined, for all $a \in B$, as $\iota_{\mathbf{B}}(a) = \langle [a]_1, [a]_2 \rangle$, where $[a]_1$ and $[a]_2$ denote the equivalence classes of a modulo, respectively, $\sim_1$ and $\sim_2$.

As a corollary of the representation theorem, we obtain a characterization of the congruences of any interlaced pre-bilattice $\mathbf{B} \cong \mathbf{L_1} \odot \mathbf{L_2}$: we have

(i) $\langle \mathrm{Con}(\mathbf{B}), \subseteq \rangle \cong \langle \mathrm{Con}(\mathbf{L_1}), \subseteq \rangle \times \langle \mathrm{Con}(\mathbf{L_2}), \subseteq \rangle$,
(ii) $\mathrm{Con}(\mathbf{B}) = \mathrm{Con}(\langle B, \wedge, \vee \rangle) = \mathrm{Con}(\langle B, \otimes, \oplus \rangle)$.

The representation theorem for bilattices can be regarded as a special case of the former one:

Theorem 3.2 (Representation of bilattices)**.** *A bilattice* $\mathbf{B}$ *is interlaced if and only if there is a lattice* $\mathbf{L}$ *such that* $\mathbf{B}$ *is isomorphic to* $\mathbf{L} \odot \mathbf{L}$*. Moreover,* $\mathbf{B}$ *is distributive if and only if* $\mathbf{L}$ *is a distributive lattice.*

As to the congruences, we have $\mathrm{Con}(\mathbf{L} \odot \mathbf{L}) \cong \mathrm{Con}(\mathbf{L})$. This result is very important, as it allows us to characterize the subdirectly irreducible members of IntBiLat as those that can be obtained as a product bilattice of a subdirectly irreducible lattice. This implies that the only subdirectly irreducible distributive bilattice is $\mathcal{FOUR}$, being isomorphic to the bilattice product $\mathbf{2} \odot \mathbf{2}$, where $\mathbf{2}$ denotes the two-element lattice. Therefore, we conclude that *the variety* DBiLat *is generated by* $\mathcal{FOUR}$.

Among the corollaries of Theorem 3.2, let us cite the fact that an interlaced pre-bilattice is distributive if and only if its t-lattice (or, equivalently, its k-lattice) reduct is distributive.

Let us note that in the case of bilattices we can exploit the negation operator to obtain an alternative and straightforward proof of the representation theorem [4, 16]. We describe the constructions involved, as they will be used in the next sections.

Given a bilattice $\mathbf{B}$, we consider the set $\mathrm{Reg}(\mathbf{B}) = \{a \in B : a = \neg a\}$ of *regular* elements, i.e., the fixed points of the negation operator. It is easy to see that $\mathrm{Reg}(\mathbf{B})$ is closed under $\otimes$ and $\oplus$, hence is the universe of a sublattice of the k-lattice of $\mathbf{B}$. Now, to every $a \in B$ we associate a regular element according to the following definition:

$$\mathrm{reg}(a) := (a \vee (a \otimes \neg a)) \oplus \neg(a \vee (a \otimes \neg a)).$$

It can be easily checked that, for any interlaced bilattice $\mathbf{B}$, the following properties hold [4]:

(i) $\mathrm{reg}(a) = (a \wedge (a \oplus \neg a)) \otimes \neg(a \wedge (a \oplus \neg a))$,
(ii) $a \in \mathrm{Reg}(\mathbf{B})$ iff $a = \mathrm{reg}(a)$ iff $a = \mathrm{reg}(b)$ for some $b \in B$,
(iii) $\mathrm{reg}(a \otimes b) = \mathrm{reg}(\mathrm{reg}(a) \otimes \mathrm{reg}(b)) = \mathrm{reg}(a) \otimes \mathrm{reg}(b) = \mathrm{reg}(a \wedge b)$,
(iv) $\mathrm{reg}(a \oplus b) = \mathrm{reg}(\mathrm{reg}(a) \oplus \mathrm{reg}(b)) = \mathrm{reg}(a) \oplus \mathrm{reg}(b) = \mathrm{reg}(a \vee b)$.

Using the previous properties, it is not difficult to prove that

$$\mathbf{B} \cong \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus \rangle \odot \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus \rangle.$$

The isomorphism is $\iota_{\mathbf{B}} \colon B \to \mathrm{Reg}(\mathbf{B}) \times \mathrm{Reg}(\mathbf{B})$ defined, for all $a \in B$, as $\iota_{\mathbf{B}}(a) := \langle \mathrm{reg}(a), \mathrm{reg}(\neg a) \rangle$. The inverse map $\iota_{\mathbf{B}}^{-1} \colon \mathrm{Reg}(\mathbf{B}) \times \mathrm{Reg}(\mathbf{B}) \to B$ is defined, for all $a, b \in \mathrm{Reg}(\mathbf{B})$, as

$$\iota_{\mathbf{B}}^{-1}(\langle a, b\rangle) := (a \otimes (a \vee b)) \oplus (b \otimes (a \wedge b)).$$

Note that this implies that $\mathbf{B}$ is generated by the set $\mathrm{Reg}(\mathbf{B})$.

In any product bilattice $\mathbf{L} \odot \mathbf{L}$, the regular elements are those of the form $\langle a, a\rangle$ for some $a \in L$. Given $a_1, a_2 \in L$, we have that $\mathrm{reg}(\langle a_1, a_2\rangle) = \langle a_1, a_1\rangle$, so $\iota_{\mathbf{B}}(\langle a_1, a_2\rangle) = \langle\langle a_1, a_1\rangle, \langle a_2, a_2\rangle\rangle$. Conversely, for any $a_1, a_2 \in L$, we have $\iota_{\mathbf{B}}^{-1}(\langle\langle a_1, a_1\rangle, \langle a_2, a_2\rangle\rangle) = \langle a_1, a_2\rangle$.

Let us state a property concerning congruences that will be useful in the next section:

Proposition 3.3. *Let* $\mathbf{B}$ *be any interlaced bilattice and* $\theta \in \mathrm{Con}(\mathbf{B})$. *Then, for all* $a, b \in B$*:*

$$\langle a, b\rangle \in \theta \quad \text{iff} \quad \langle \mathrm{reg}(a), \mathrm{reg}(b)\rangle, \langle \mathrm{reg}(\neg a), \mathrm{reg}(\neg b)\rangle \in \theta.$$

Proof. Recalling the definition of a regular element, it is easy to see that $\langle a, b\rangle \in \theta$ implies $\langle \mathrm{reg}(a), \mathrm{reg}(b)\rangle \in \theta$ and $\langle \mathrm{reg}(\neg a), \mathrm{reg}(\neg b)\rangle \in \theta$. Taking the definition of $\iota_{\mathbf{B}}^{-1}$ into account, the converse is also easy. We have that, for all $a \in B$:

$$a = (\mathrm{reg}(a) \otimes (\mathrm{reg}(a) \vee \mathrm{reg}(\neg a))) \oplus (\mathrm{reg}(\neg a) \otimes (\mathrm{reg}(a) \wedge \mathrm{reg}(\neg a))). \qquad \square$$

Note that this last property is independent of the language considered, as long as the algebra has an interlaced bilattice reduct: hence it will also hold for the classes of algebras obtained through language expansions that we are going to introduce in the next section.

4. Language expansions

The algebraic signature considered in the previous sections has been expanded in various ways and for different purposes in the literature on bilattices. In this section we are interested in proving that for some of the algebras thus obtained one can obtain representation theorems analogous to the one described in the previous section.

4.1. Bilattices with conflation. The first expansion we shall consider, due to Fitting [8], consists in adding an operator that behaves as a dual of the bilattice negation, called *conflation*.

Definition 4.1. An algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg, -\rangle$ is called a *bilattice with conflation* if the reduct $\langle B, \wedge, \vee, \otimes, \oplus, \neg\rangle$ is a bilattice and the *conflation* $-\colon B \to B$ is an operation satisfying, for all $a, b \in B$,

(conf 1) if $a \leq_k b$, then $-b \leq_k -a$,
(conf 2) if $a \leq_t b$, then $-a \leq_t -b$,
(conf 3) $a = --a$.

We say that $\mathbf{B}$ is *commutative* if it also satisfies the equation: $\neg{-x} \approx -\neg x$.

If a bilattice with conflation is distributive, or at least the k-lattice of $\mathbf{B}$ is distributive, then the reduct $\langle B, \otimes, \oplus, - \rangle$ is a De Morgan lattice. The class of bilattices with conflation is a variety, axiomatized by the equations defining bilattices together with **(conf 3)** and the following ones:

$$-(x \otimes y) \approx -x \oplus -y, \qquad -(x \oplus y) \approx -x \otimes -y,$$
$$-(x \wedge y) \approx -x \wedge -y, \qquad -(x \vee y) \approx -x \vee -y.$$

Adding the appropriate equations to a presentation of this class, we may define the varieties of interlaced (distributive) bilattices with conflation and commutative (interlaced, distributive) bilattices with conflation.

In order to obtain a representation theorem for bilattices with conflation, we introduce the following construction, also due to Fitting. Let $\mathbf{L} = \langle L, \sqcap, \sqcup, ' \rangle$ be an *involutive lattice*, i.e., an algebra such that the reduct $\langle L, \sqcap, \sqcup \rangle$ is a lattice and the operation $'\colon A \to A$ satisfies that, for all $a, b \in A$,

(inv 1) if $a \leq b$, then $b' \leq a'$,
(inv 2) $a = a''$.

Given an involutive lattice $\mathbf{L} = \langle L, \sqcap, \sqcup, ' \rangle$, we denote by $\mathbf{L} \odot \mathbf{L}$ the bilattice with conflation whose bilattice reduct is the product bilattice $\langle L, \sqcap, \sqcup \rangle \odot \langle L, \sqcap, \sqcup \rangle$ defined above and where the conflation is defined, for all $a, b \in L$, as $-\langle a, b \rangle = \langle b', a' \rangle$. It can be easily checked that $\mathbf{L} \odot \mathbf{L}$ is always an interlaced bilattice with conflation; in addition, it is commutative.

Conversely, given a commutative interlaced bilattice with conflation $\mathbf{B}$, we have that the set $\mathrm{Reg}(\mathbf{B})$ is closed under conflation; this implies that the algebra $\langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle$ is an involutive lattice. The following result was proved by Fitting [8] for the case of bounded distributive bilattices, then generalized in [16] to unbounded interlaced bilattices.

Theorem 4.2 (Representation of bilattices with conflation)**.** *Let* $\mathbf{B}$ *be a commutative interlaced bilattice with conflation. Then:*

(i) $\mathrm{Reg}(\mathbf{B})$ *is closed under conflation, so* $\langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle$ *is an involutive lattice,*
(ii) $\mathrm{reg}(-a) = -\,\mathrm{reg}(\neg a)$ *for all* $a \in B$,
(iii) $\mathbf{B} \cong \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle \odot \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle$.

Proof. (i): Using commutativity, we have that, for every $a \in \mathrm{Reg}(\mathbf{B})$, $\neg{-a} = -\neg a = -a$, i.e., $-a \in \mathrm{Reg}(\mathbf{B})$.

(ii): Recall that, for any $a \in B$, we have $\mathrm{reg}(a) = (a \wedge (a \oplus \neg a)) \otimes \neg(a \wedge (a \oplus \neg a))$. Applying De Morgan laws and commutativity, we have

$$\begin{aligned}
-\operatorname{reg}(\neg a) &= -((\neg a \wedge (a \oplus \neg a)) \otimes \neg(\neg a \wedge (a \oplus \neg a)))\\
&= -(\neg(a \vee (a \oplus \neg a)) \otimes (a \vee (a \oplus \neg a)))\\
&= -\neg(a \vee (a \oplus \neg a)) \oplus -(a \vee (a \oplus \neg a))\\
&= \neg-(a \vee (a \oplus \neg a)) \oplus -(a \vee (a \oplus \neg a))\\
&= \neg(-a \vee -(a \oplus \neg a)) \oplus (-a \vee -(a \oplus \neg a))\\
&= \neg(-a \vee (-a \otimes -\neg a)) \oplus (-a \vee (-a \otimes -\neg a))\\
&= \neg(-a \vee (-a \otimes \neg -a)) \oplus (-a \vee (-a \otimes \neg -a))\\
&= \operatorname{reg}(-a).
\end{aligned}$$

(iii) Since $\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle$ is a lattice with involution, we can construct the bilattice with conflation $\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle \odot \langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle$ as defined above. We know that $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle \cong \langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus \rangle \odot \langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus \rangle$ via the map $\iota_{\mathbf{B}} \colon B \to \operatorname{Reg}(\mathbf{B}) \times \operatorname{Reg}(\mathbf{B})$ defined as $\iota_{\mathbf{B}}(a) = \langle \operatorname{reg}(a), \operatorname{reg}(\neg a) \rangle$. We shall prove that $\iota_{\mathbf{B}}$ also preserves conflation, i.e., that $\iota_{\mathbf{B}}(-a) = -\iota_{\mathbf{B}}(a)$. This is easy, for using (ii) and commutativity, we have

$$\begin{aligned}
\iota_{\mathbf{B}}(-a) &= \langle \operatorname{reg}(-a), \operatorname{reg}(\neg -a) \rangle = \langle -\operatorname{reg}(\neg - -a), -\operatorname{reg}(\neg - \neg -a) \rangle\\
&= \langle -\operatorname{reg}(\neg a), -\operatorname{reg}(\neg\neg - -a) \rangle = \langle -\operatorname{reg}(\neg a), -\operatorname{reg}(a) \rangle\\
&= -(\langle \operatorname{reg}(a), \operatorname{reg}(\neg a) \rangle) = -\iota_{\mathbf{B}}(a).
\end{aligned}$$ □

As in the case of bilattices, the previous result allows us to obtain some information on the congruences:

Theorem 4.3. *For any commutative interlaced bilattice with conflation* $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg, - \rangle$*, it holds that* $\langle \operatorname{Con}(\mathbf{B}), \subseteq \rangle \cong \langle \operatorname{Con}(\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle), \subseteq \rangle$.

Proof. The isomorphism is given by the map

$$H \colon \operatorname{Con}(\mathbf{B}) \to \operatorname{Con}(\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle)$$

defined, for all $\theta \in \operatorname{Con}(\mathbf{B})$, as $H(\theta) = \theta \cap (\operatorname{Reg}(\mathbf{B}) \times \operatorname{Reg}(\mathbf{B}))$.

Clearly $H(\theta) \in \operatorname{Con}(\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle)$. It is also easy to see that H is order-preserving, for $\theta_1 \subseteq \theta_2$ implies

$$\theta_1 \cap (\operatorname{Reg}(\mathbf{B}) \times \operatorname{Reg}(\mathbf{B})) \subseteq \theta_2 \cap (\operatorname{Reg}(\mathbf{B}) \times \operatorname{Reg}(\mathbf{B})).$$

To see that it is order-reflecting, assume $H(\theta_1) \subseteq H(\theta_2)$ and $\langle a, b \rangle \in \theta_1$. By Proposition 3.3 we have $\langle \operatorname{reg}(a), \operatorname{reg}(b) \rangle \in \theta_1$ and $\langle \operatorname{reg}(\neg a), \operatorname{reg}(\neg b) \rangle \in \theta_1$. So the assumptions imply $\langle \operatorname{reg}(a), \operatorname{reg}(b) \rangle \in \theta_2$ and $\langle \operatorname{reg}(\neg a), \operatorname{reg}(\neg b) \rangle \in \theta_2$, so, applying again Proposition 3.3, we obtain $\langle a, b \rangle \in \theta_2$. This proves that H is an order embedding. To see that it is onto, we will show that its inverse is

$$H^{-1} \colon \operatorname{Con}(\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle) \to \operatorname{Con}(\mathbf{B})$$

defined, for all $\theta \in \operatorname{Con}(\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle)$, as follows:

$$\langle a, b \rangle \in H^{-1}(\theta) \text{ iff } \langle \operatorname{reg}(a), \operatorname{reg}(b) \rangle \in \theta \text{ and } \langle \operatorname{reg}(\neg a), \operatorname{reg}(\neg b) \rangle \in \theta.$$

Clearly $H^{-1}(\theta)$ is an equivalence relation and, as noted in the previous section,

$$\mathrm{reg}(a \wedge b) = \mathrm{reg}(a \otimes b) = \mathrm{reg}(a) \otimes \mathrm{reg}(b),$$
$$\mathrm{reg}(a \vee b) = \mathrm{reg}(a \oplus b) = \mathrm{reg}(a) \oplus \mathrm{reg}(b).$$

It is then clear that $H^{-1}(\theta)$ is compatible with all the lattice operations of both orders, as well as with negation. As to conflation, assume $\langle a, b\rangle \in H^{-1}(\theta)$, i.e., $\langle \mathrm{reg}(a), \mathrm{reg}(b)\rangle \in \theta$ and $\langle \mathrm{reg}(\neg a), \mathrm{reg}(\neg b)\rangle \in \theta$. From the latter, applying Theorem 4.2 (ii), we obtain $\langle - \mathrm{reg}(\neg - \neg a), - \mathrm{reg}(\neg - \neg b)\rangle \in \theta$. Now, using commutativity and the fact that θ is compatible with conflation, we conclude that $\langle \mathrm{reg}(-a), \mathrm{reg}(-b)\rangle \in \theta$. A similar reasoning shows that $\langle \mathrm{reg}(a), \mathrm{reg}(b)\rangle \in \theta$ implies $\langle \mathrm{reg}(\neg -a), \mathrm{reg}(\neg -b)\rangle \in \theta$. Hence $\langle -a, -b\rangle \in H^{-1}(\theta)$. Finally, it follows immediately from the definitions that $H(H^{-1}(\theta)) = \theta$. □

From the previous result we may obtain more information on commutative distributive bilattices with conflation. In fact, for any algebra $\mathbf{L} \odot \mathbf{L}$ in this variety, we have that $\mathbf{L}$ is a De Morgan lattice. It is known [9] that the only subdirectly irreducible De Morgan lattices are the four-element non-linear one $\mathbf{M_4}$, which is the $\{\wedge, \vee, \neg\}$-reduct of the bilattice $\mathcal{FOUR}$, the two-element chain $\mathbf{B_2}$ and the three-element chain $\mathbf{K_3}$. Thus we have the following:

Theorem 4.4. *The only subdirectly irreducible commutative distributive bilattices with conflation are* $\mathbf{B_2} \odot \mathbf{B_2}$*,* $\mathbf{K_3} \odot \mathbf{K_3}$ *and* $\mathbf{M_4} \odot \mathbf{M_4}$*. Moreover, the variety of commutative distributive bilattices with conflation is generated by* $\mathbf{M_4} \odot \mathbf{M_4}$*.*

Proof. It follows from Theorem 4.3 that the only subdirectly irreducible commutative distributive bilattices with conflation are $\mathbf{M_4} \odot \mathbf{M_4}$, $\mathbf{K_3} \odot \mathbf{K_3}$ and $\mathbf{B_2} \odot \mathbf{B_2}$. Therefore, these algebras generate the variety, and indeed $\mathbf{M_4} \odot \mathbf{M_4}$ alone generates it, for it is easy to see that $\mathbf{K_3} \odot \mathbf{K_3}$ and $\mathbf{B_2} \odot \mathbf{B_2}$ are isomorphic to subalgebras of $\mathbf{M_4} \odot \mathbf{M_4}$. □

An easy consequence of the previous results is that, as happens with De Morgan lattices, the variety of commutative distributive bilattices with conflation has exactly two proper subvarieties, namely the variety generated by $\mathbf{K_3} \odot \mathbf{K_3}$, which we call *Kleene bilattices with conflation* (KBiLatCon) and the one generated by $\mathbf{B_2} \odot \mathbf{B_2}$, that we call, following [1], *classical bilattices with conflation* (CBiLatCon). It is easy to provide an equational presentation for these varieties:

Theorem 4.5. *The variety of Kleene bilattices with conflation is axiomatized by the identities defining commutative distributive bilattices with conflation plus either of the following two:*

$$(x \wedge \neg - x) \wedge (y \vee \neg - y) \approx (x \wedge \neg - x),$$
$$(x \otimes \neg - x) \otimes (y \oplus \neg - y) \approx (x \otimes \neg - x).$$

Proof. It is known [9] that the variety of Kleene lattices $\mathbf{L} = \langle L, \sqcap, \sqcup, ' \rangle$ is generated by $\mathbf{K_3}$ and axiomatized by the identities for De Morgan lattices plus the following: $(x \sqcap x') \sqcap (y \sqcup y') \approx (x \sqcap x')$. It is easy to check that if a commutative distributive bilattice with conflation $\mathbf{L} \odot \mathbf{L}$ satisfies either one of the two above equations, then $\mathbf{L} = \langle L, \sqcap, \sqcup, ' \rangle$ satisfies $(x \sqcap x') \sqcap (y \sqcup y') \approx (x \sqcap x')$. For instance, using the first equation, we have that for any $a_1, a_2, b_1, b_2 \in L$:

$$\begin{aligned}\langle a_1, a_2 \rangle \wedge \neg - \langle a_1, a_2 \rangle &= \langle a_1, a_2 \rangle \wedge \langle a_1', a_2' \rangle = \langle a_1 \sqcap a_1', a_2 \sqcup a_2' \rangle \\ &\leq_t \langle b_1 \sqcup b_1', b_2 \sqcap b_2' \rangle = \langle b_1, b_2 \rangle \vee \langle b_1', b_2' \rangle \\ &= \langle b_1, b_2 \rangle \vee \neg - \langle b_1, b_2 \rangle.\end{aligned}$$

That is, $a_1 \sqcap a_1' \leq b_1 \sqcup b_1'$ and $a_2 \sqcup a_2' \geq b_2 \sqcap b_2'$. Hence $\mathbf{L}$ is a Kleene lattice. Conversely, for any Kleene lattice $\mathbf{L}$, the bilattice with conflation $\mathbf{L} \odot \mathbf{L}$ will satisfy both of the above equations. □

Theorem 4.6. *The variety of classical bilattices with conflation is axiomatized by the identities defining commutative distributive bilattices with conflation plus any of the following ones:*

$$x \wedge (y \vee \neg - y) \approx x, \qquad x \otimes (y \oplus \neg - y) \approx x,$$
$$x \vee (y \wedge \neg - y) \approx x, \qquad x \oplus (y \otimes \neg - y) \approx x.$$

Proof. Similar to the proof of the previous theorem. The variety of Boolean algebras $\mathbf{L} = \langle L, \sqcap, \sqcup, ' \rangle$ is generated by $\mathbf{B_2}$ and axiomatized by the identities for De Morgan lattices plus either of the following: $x \sqcap (y \sqcup y') \approx x$ or $x \sqcup (y \sqcap y') \approx x$. Again, if a commutative distributive bilattice with conflation $\mathbf{L} \odot \mathbf{L}$ satisfies any of the above equations, then $\mathbf{L} = \langle L, \sqcap, \sqcup, ' \rangle$ satisfies $x \sqcap (y \sqcup y') \approx x$ and $x \sqcup (y \sqcap y') \approx x$. For instance, using the first equation, we have that for any $a_1, a_2, b_1, b_2 \in L$:

$$\langle a_1, a_2 \rangle \leq_t \langle b_1 \sqcup b_1', b_2 \sqcap b_2' \rangle = \langle b_1, b_2 \rangle \vee \langle b_1', b_2' \rangle = \langle b_1, b_2 \rangle \vee \neg - \langle b_1, b_2 \rangle.$$

That is, $a_1 \leq b_1 \sqcup b_1'$ and $a_2 \geq b_2 \sqcap b_2'$. So $\mathbf{L}$ is a Boolean algebra. Conversely, for any Boolean algebra $\mathbf{L}$, the bilattice with conflation $\mathbf{L} \odot \mathbf{L}$ will satisfy the above equations. □

4.2. Brouwerian bilattices. The second way of expanding the bilattice language that we are going to consider consists in adding a binary connective that plays (on a logical level) the role of an implication. These enriched algebras arose from the study developed in [16] of the algebraic models of the "logic of logical bilattices" introduced by Arieli and Avron [1].

Definition 4.7. A *Brouwerian bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$ such that $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is a bilattice and the following equations are satisfied:

(B1) $(x \supset x) \supset y \approx y$,
(B2) $x \supset (y \supset z) \approx (x \wedge y) \supset z \approx (x \otimes y) \supset z$,

(B3) $(x \vee y) \supset z \approx (x \supset z) \wedge (y \supset z) \approx (x \oplus y) \supset z$,
(B4) $x \wedge ((x \supset y) \supset (x \otimes y)) \approx x$,
(B5) $\neg(x \supset y) \supset z \approx (x \wedge \neg y) \supset z$.

Brouwerian bilattices obviously form a variety, denoted BrBiLat. An interesting subclass of Brouwerian bilattices is the variety of *implicative bilattices*, defined as the Brouwerian bilattices that additionally satisfy the following equation:

$$((x \supset y) \supset x) \supset x \approx x \supset x.$$

This class was introduced and studied in [16], where it is proved that implicative bilattices are the equivalent algebraic semantics (in the sense of [3]) of Arieli and Avron's logic of logical bilattices with implication. Brouwerian bilattices can be considered a natural generalization of the implicative ones, and the relation between the two classes is analogous, as we will see below, to the relation between (generalized) Heyting algebras and (generalized) Boolean algebras.

Our next aim is to prove a representation theorem for Brouwerian bilattices analogous to the ones stated above for bilattices and bilattices with conflation. In the following propositions we prove some facts about Brouwerian bilattices that will be needed to obtain this result. In order to simplify the notation, we use the following abbreviations: for any element a of a Brouwerian bilattice, we use $\top(a)$ to denote the element $(a \supset a) \oplus \neg(a \supset a)$ and we write $E(a)$ instead of the expression $a = a \supset a$.

Proposition 4.8. *Let* **B** *be a Brouwerian bilattice. For all $a, b, c \in B$:*

(i) *if $a = b \supset b$, then $a \supset c = c$ and $E(a)$,*
(ii) $a \leq_t b \supset a$,
(iii) $E(a \supset (b \supset a))$,
(iv) $\top(a) = \neg\top(a)$,
(v) $\top(a) \supset b = b$,
(vi) $E(a \supset \top(b))$,
(vii) *$E(a \supset b)$ if and only if $a \leq_t a \otimes b$,*
(viii) *if $E(a \supset b)$ and $E(\neg b \supset \neg a)$, then $a \leq_t b$,*
(ix) *if $E(a \supset b)$ and $E(\neg a \supset \neg b)$, then $a \leq_k b$,*
(x) *$\top(a) \leq_t b$ if and only if $E(b)$,*
(xi) $\top(a) = \top(b)$,
(xii) $a \leq_k \top(a)$,
(xiii) *$a \leq_t b$ if and only if $\top(c) \leq_t a \supset b$ and $\top(c) \leq_t \neg b \supset \neg a$,*
(xiv) *$a \leq_k b$ if and only if $\top(c) \leq_t a \supset b$ and $\top(c) \leq_t \neg a \supset \neg b$.*

Proof. (i) By (B1) we have $(b \supset b) \supset c = c$ and $E(b \supset b)$, so the result immediately follows.

(ii) Using (B1) and (B3) we have that, for all $a, b, c \in B$,

$$\begin{aligned} a \wedge (b \supset a) &= ((c \supset c) \supset a) \wedge (b \supset a) = ((c \supset c) \vee b) \supset a \\ &= (c \supset c) \supset (((c \supset c) \vee b) \supset a) = ((c \supset c) \wedge ((c \supset c) \vee b)) \supset a \\ &= (c \supset c) \supset a = a. \end{aligned}$$

(iii) Notice that (ii) implies $a \leq_t b \supset a \leq_t a \supset (b \supset a)$. Then, using (B2), we have

$$\begin{aligned} (a \supset (b \supset a)) \supset (a \supset (b \supset a)) &= ((a \supset (b \supset a)) \wedge a) \supset (b \supset a) \\ &= a \supset (b \supset a). \end{aligned}$$

(iv) Immediate, using the properties of the bilattice negation.

(v) By (B3) we have $\top(a) \supset b = ((a \supset a) \supset b) \wedge (\neg(a \supset a) \supset b)$. By (B1) and (ii) we have

$$((a \supset a) \supset b) \wedge (\neg(a \supset a) \supset b) = b \wedge (\neg(a \supset a) \supset b) = b.$$

(vi) By (v) we have $a \supset \top(b) = \top(b) \supset (a \supset \top(b))$. Then the result follows from (iii).

(vii) Assume $E(a \supset b)$. Then, by (B4) and (i), we have

$$a \leq_t (a \supset b) \supset (a \otimes b) = a \otimes b.$$

Conversely, assume $a \leq_t a \otimes b$, i.e., $a = a \wedge (a \otimes b)$. Applying (B2) several times, we have

$$\begin{aligned} (a \wedge (a \otimes b)) \supset b &= (a \otimes b) \supset (a \supset b) = (a \wedge b) \supset (a \supset b) \\ &= (a \wedge b) \supset b = b \supset (a \supset b). \end{aligned}$$

Then the result follows by (iii).

(viii) Assume $E(a \supset b)$ and $E(\neg b \supset \neg a)$. Then, using (vi), we obtain $a \leq_t a \otimes b$ and $\neg b \leq_t \neg b \otimes \neg a$. By the properties of the bilattice negation, this implies $b = \neg\neg b \geq_t \neg(\neg b \otimes \neg a) = a \otimes b$. Hence $a \leq_t a \otimes b \leq_t b$, so the result immediately follows.

(ix) Assume $E(a \supset b)$ and $E(\neg a \supset \neg b)$. Reasoning as in (vii), we obtain $a \leq_t a \otimes b$ and $a \geq_t a \otimes b$. Hence $a = a \otimes b$, i.e., $a \leq_k b$.

(x) Assume $\top(a) \leq_t b$. Then we have

$$\begin{aligned} b \supset b &= (b \vee \top(a)) \supset b && \text{from the assumption} \\ &= (b \supset b) \wedge (\top(a) \supset b) && \text{by (B3)} \\ &= (b \supset b) \wedge b && \text{by (v)} \\ &= b && \text{by (ii).} \end{aligned}$$

Conversely, assume $E(b)$. Then by (v) we have $E(\top(a) \supset b)$. On the other hand, by (iv) we have $\neg b \supset \neg\top(a) = \neg b \supset \top(a)$, so by (vi) we obtain $E(\neg b \supset \top(a))$. Hence, applying (viii), we have $\top(a) \leq_t b$.

(xi) By symmetry it is sufficient to show that $\top(a) \leq_t \top(b)$, i.e., using (viii), that $E(\top(a) \supset \top(b))$ and $E(\neg\top(b) \supset \neg\top(a))$. By (iv) we have $\top(a) = \neg\top(a)$

for any $a \in B$, so it will be enough to check that $E(\top(a) \supset \top(b))$. By (v) we have $\top(a) \supset \top(b) = \top(b)$, so the result immediately follows.

(xii) By (ix), it is enough to prove that $E(a \supset \top(a))$ and $E(\neg a \supset \neg\top(a))$. The first one has been proved in (vi), while the second one follows from (iv), since $\neg\top(a) = \top(a)$.

(xiii) The leftwards implication follows from (viii). Conversely, assume $a \leq_t b$, which implies $\neg b \leq_t \neg a$. Then $a \wedge b = a$, so by (B2) we have $a \supset b = (a \wedge b) \supset b = a \supset (b \supset a)$. Then, applying (iii) and (x), we obtain the desired result. The same reasoning shows that $\neg b \supset \neg a \geq_t \top(c)$, hence $(a \supset b) \wedge (\neg b \supset \neg a) \geq_t \top(c)$.

(xiv) Similar to the previous case, as we can use (B2) to show that $a \leq_k b$ implies $a \supset b = (a \otimes b) \supset b = a \supset (b \supset a)$ and so on. □

From Proposition 4.8 (xi) it follows that $\top(a) = (a \supset a) \oplus \neg(a \supset a)$ defines an algebraic constant in every $\mathbf{B} \in \mathsf{BrBiLat}$. Moreover, by (xii), this constant is the top element of the k-order. So we denote it just by $\top$. Using this notation, let us state some more arithmetical properties of Brouwerian bilattices.

The following result is of particular interest to our approach:

Proposition 4.9. *Let $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg\rangle$ be a Brouwerian bilattice. Then the reduct $\langle B, \wedge, \vee, \otimes, \oplus, \neg\rangle$ is an interlaced bilattice.*

Proof. Let $a, b \in B$ be such that $a \leq_t b$. To see that $a \otimes c \leq_t b \otimes c$ and $a \oplus c \leq_t b \oplus c$ for all $c \in B$, using Proposition 4.8 (viii) and (x), we will prove that $(a \otimes c) \supset (b \otimes c) \geq_t \top$, $\neg(b \otimes c) \supset \neg(a \otimes c) \geq_t \top$, $(a \oplus c) \supset (b \oplus c) \geq_t \top$ and $\neg(b \oplus c) \supset \neg(a \oplus c) \geq_t \top$.

As to the first, using (B2) and Proposition 4.8 (xiv), we have

$$(a \otimes c) \supset (b \otimes c) = ((a \wedge b) \otimes c) \supset (b \otimes c) = (a \otimes b \otimes c) \supset (b \otimes c) \geq_t \top$$

and, applying De Morgan laws and Proposition 4.8 (ii),

$$\begin{aligned}\neg(b \otimes c) \supset \neg(a \otimes c) &= (\neg b \otimes \neg c) \supset (\neg a \otimes \neg c) = (\neg(a \vee b) \otimes \neg c) \supset (\neg a \otimes \neg c)\\ &= ((\neg a \wedge \neg b) \otimes \neg c) \supset (\neg a \otimes \neg c)\\ &= (\neg a \otimes \neg b \otimes \neg c) \supset (\neg a \otimes \neg c) \geq_t \top.\end{aligned}$$

As to the second, using (B3), (B2) and Proposition 4.8 (ii) and (xiv), we have

$$\begin{aligned}(a \oplus c) \supset (b \oplus c) &= (a \supset (b \oplus c)) \wedge (c \supset (b \oplus c))\\ &= ((a \wedge b) \supset (b \oplus c)) \wedge (c \supset (b \oplus c))\\ &= (a \supset (b \supset (b \oplus c))) \wedge (c \supset (b \oplus c)) \geq_t \top,\end{aligned}$$

where the last inequality holds because, by Proposition 4.8 (xiv), $b \leq_k b \oplus c$ implies $b \supset (b \oplus c) \geq_t \top$ and similarly $c \leq_k b \oplus c$ implies $c \supset (b \oplus c) \geq_t \top$.

Applying also De Morgan laws, we have

$$\begin{aligned}\neg(b \oplus c) \supset \neg(a \oplus c) &= (\neg b \oplus \neg c) \supset (\neg a \oplus \neg c)\\ &= (\neg b \supset (\neg a \oplus \neg c)) \wedge (\neg c \supset (\neg a \oplus \neg c))\\ &= (\neg(a \vee b) \supset (\neg a \oplus \neg c)) \wedge (\neg c \supset (\neg a \oplus \neg c))\\ &= ((\neg a \wedge \neg b) \supset (\neg a \oplus \neg c)) \wedge (\neg c \supset (\neg a \oplus \neg c))\\ &= (\neg b \supset (\neg a \supset (\neg a \oplus \neg c))) \wedge (\neg c \supset (\neg a \oplus \neg c))\\ &= ((\neg b \otimes \neg a) \supset (\neg a \oplus \neg c)) \wedge (\neg c \supset (\neg a \oplus \neg c)) \geq_t \top,\end{aligned}$$

because $\neg b \otimes \neg a \leq_k \neg a \oplus \neg c$ and $\neg c \leq_k \neg a \oplus \neg c$, which imply

$$\top \leq_t (\neg b \otimes \neg a) \supset (\neg a \oplus \neg c) \text{ and } \top \leq_t \neg c \supset (\neg a \oplus \neg c).$$

Now assume $a \leq_k b$. To see that $a \wedge c \leq_k b \wedge c$, we will prove that $(a \wedge c) \supset (b \wedge c) \geq_t \top$ and $\neg(a \wedge c) \supset \neg(b \wedge c) \geq_t \top$. Then using Proposition 4.8 (xiv) we will obtain the desired conclusion.

As to the former, using (B2) and Proposition 4.8 (xiii), we have

$$(a \wedge c) \supset (b \wedge c) = ((a \otimes b) \wedge c) \supset (b \wedge c) = (a \wedge b \wedge c) \supset (b \wedge c) \geq_t \top.$$

As to the latter, using De Morgan laws and (B3), we have

$$\begin{aligned}\neg(a \wedge c) \supset \neg(b \wedge c) &= (\neg a \vee \neg c) \supset (\neg b \vee \neg c)\\ &= (\neg a \supset (\neg b \vee \neg c)) \wedge (\neg c \supset (\neg b \vee \neg c)).\end{aligned}$$

Since $\neg c \leq_t \neg b \vee \neg c$, by Proposition 4.8 (xiii) we have that $\top \leq_t \neg c \supset (\neg b \vee \neg c)$. It will then suffice to show that $\top \leq_t \neg a \supset (\neg b \vee \neg c)$ and the result will follow by the monotonicity of $\wedge$ with respect to $\leq_t$. Using the assumption that $a \leq_k b$, De Morgan laws and (B3), we have

$$\begin{aligned}\neg a \supset (\neg b \vee \neg c) &= \neg(a \otimes b) \supset (\neg b \vee \neg c) = (\neg a \otimes \neg b) \supset (\neg b \vee \neg c)\\ &= \neg a \supset (\neg b \supset (\neg b \vee \neg c)) = (\neg a \wedge \neg b) \supset (\neg b \vee \neg c) \geq_t \top,\end{aligned}$$

where the last inequality follows from Proposition 4.8 (xiii) and the fact that $\neg a \wedge \neg b \leq_t \neg b \vee \neg c$.

To see that $a \vee c \leq_k b \vee c$, note that $a \leq_k b$ if and only if $\neg a \leq_k \neg b$. Applying what we have just proved, we have $\neg a \wedge \neg c \leq_k \neg b \wedge \neg c$ and, therefore, $\neg(\neg a \wedge \neg c) \leq_k \neg(\neg b \wedge \neg c)$. Now, using De Morgan laws, we have $a \vee c = \neg(\neg a \wedge \neg c) \leq_k \neg(\neg b \wedge \neg c) = b \vee c$. □

Proposition 4.9 implies that the bilattice reduct of any Brouwerian bilattice **B** can be represented as a product bilattice $\mathbf{L} \odot \mathbf{L}$. Moreover, since **B** has a top element with respect to $\leq_k$, **L** will have a maximum element. Now, in order to obtain a full representation, we introduce the product Brouwerian bilattice construction, which is a slight modification of the product bilattice.

Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1 \rangle$ be a Brouwerian lattice, i.e., an algebra such that $\langle L, \sqcap, \sqcup, 1 \rangle$ is a lattice with maximum element 1 and the following residuation condition is satisfied: for all $a, b, c \in L$, $a \sqcap b \leq c$ if and only if $b \leq a \backslash c$. These algebras are also called *generalized Heyting algebras* [5], *Brouwerian* *algebras* [6], implicative lattices [13] or *relatively pseudo-complemented lattices* [15]. Note also that some authors call "Brouwerian lattices" structures that are dual to those defined above.

It is known that the lattice reduct of any Brouwerian lattice is distributive. We recall some arithmetical properties of these algebras that we will need (see [15] for all proofs).

Proposition 4.10. *Let* $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1\rangle$ *be a Brouwerian lattice. Then, for all* $a, b, c \in L$*:*

(i) $a \leq b$ *if and only if* $a \backslash b = 1$,
(ii) $1 \backslash a = a$,
(iii) $(a \sqcap b)\backslash c = a\backslash(b\backslash c)$,
(iv) $a\backslash(b \sqcap c) = (a\backslash b) \sqcap (a\backslash c)$,
(v) $a \leq (b\backslash a)$,
(vi) $(a\backslash b)\backslash a \leq (a\backslash b)\backslash b$,
(vii) $(a \sqcup b)\backslash c = (a\backslash c) \sqcap (b\backslash c)$,
(viii) $a\backslash(b\backslash c) = (a\backslash b)\backslash(a\backslash c) = (a \sqcap b)\backslash c$,
(ix) $a \sqcup (b\backslash c) = (a \sqcup b)\backslash(a \sqcup c)$.

Given a Brouwerian lattice $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1\rangle$, we denote by $\mathbf{L} \odot \mathbf{L}$ the algebra $\langle L \times L, \wedge, \vee, \otimes, \oplus, \supset, \neg\rangle$ whose bilattice reduct is the usual product bilattice $\langle L, \sqcap, \sqcup\rangle \odot \langle L, \sqcap, \sqcup\rangle$ and where the operation $\supset$ is defined, for all $a_1, a_2, b_1, b_2 \in L$, as

$$\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle = \langle a_1\backslash b_1, a_1 \sqcap b_2\rangle.$$

The next result shows that $\mathbf{L}\odot\mathbf{L}$ is indeed a Brouwerian bilattice. The product Brouwerian bilattice construction is very similar to the "twist-structure" used in [13, 14] to represent the algebras there called *N4-lattices*, which provide an algebraic semantics for paraconsistent Nelson's logic, as special products of two copies of a Brouwerian lattice. In fact, it can be proved that N4-lattices coincide with the $\{\wedge, \vee, \supset, \neg\}$-subreducts of Brouwerian bilattices.

Proposition 4.11. *Let* $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1\rangle$ *be a Brouwerian lattice. Then the product* $\mathbf{L} \odot \mathbf{L}$ *is a Brouwerian bilattice.*

Proof. Using the properties stated in Proposition 4.10, we will show that $\mathbf{L}\odot\mathbf{L}$ satisfies equations (B1) to (B5) of Definition 4.7. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in L$. Then:

(B1) $$(\langle a_1, a_2\rangle \supset \langle a_1, a_2\rangle) \supset \langle b_1, b_2\rangle = \langle 1, a_1 \sqcap a_2\rangle \supset \langle b_1, b_2\rangle = \langle 1\backslash b_1, 1 \sqcap b_2\rangle = \langle b_1, b_2\rangle.$$

(B2) $$\begin{aligned}\langle a_1, a_2\rangle \supset (\langle b_1, b_2\rangle \supset \langle c_1, c_2\rangle) &= \langle a_1\backslash(b_1\backslash c_1), a_1 \sqcap b_1 \sqcap c_2\rangle \\ &= \langle (a_1 \sqcap b_1)\backslash c_1, a_1 \sqcap b_1 \sqcap c_2\rangle = (\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle) \supset \langle c_1, c_2\rangle \\ &= (\langle a_1, a_2\rangle \otimes \langle b_1, b_2\rangle) \supset \langle c_1, c_2\rangle.\end{aligned}$$

(B3) $(\langle a_1,a_2\rangle \vee \langle b_1,b_2\rangle) \supset \langle c_1,c_2\rangle = \langle (a_1 \sqcup b_1)\backslash c_1, (a_1 \sqcup b_1) \sqcap c_2\rangle$
$$= (\langle a_1,a_2\rangle \oplus \langle b_1,b_2\rangle) \supset \langle c_1,c_2\rangle = \langle (a_1\backslash c_1) \sqcap (b_1\backslash c_1), (a_1 \sqcup b_1) \sqcap c_2\rangle$$
$$= (\langle a_1,a_2\rangle \supset \langle c_1,c_2\rangle) \wedge (\langle b_1,b_2\rangle \supset \langle c_1,c_2\rangle).$$

(B4) $(\langle a_1,a_2\rangle \supset \langle b_1,b_2\rangle) \supset (\langle a_1,a_2\rangle \otimes \langle b_1,b_2\rangle)$
$$= \langle (a_1\backslash b_1)\backslash(a_1 \sqcap b_1), (a_1\backslash b_1) \sqcap a_2 \sqcap b_2\rangle$$
$$= \langle ((a_1\backslash b_1)\backslash a_1) \sqcap ((a_1\backslash b_1)\backslash b_1), (a_1\backslash b_1) \sqcap a_2 \sqcap b_2\rangle$$
$$= \langle (a_1\backslash b_1)\backslash a_1, (a_1\backslash b_1) \sqcap a_2 \sqcap b_2\rangle \geq_t \langle a_1,a_2\rangle.$$

(B5) $\neg(\langle a_1,a_2\rangle \supset \langle b_1,b_2\rangle) \supset \langle c_1,c_2\rangle = \langle a_1 \sqcap b_2, a_1\backslash b_1\rangle \supset \langle c_1,c_2\rangle$
$$= \langle (a_1 \sqcap b_2)\backslash c_1, a_1 \sqcap b_2 \sqcap c_2\rangle = (\langle a_1,a_2\rangle \wedge \neg\langle b_1,b_2\rangle) \supset \langle c_1,c_2\rangle. \quad \square$$

Given a Brouwerian bilattice $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg\rangle$, consider the algebra $\langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, \backslash\rangle$, where $\langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus\rangle$ is defined as before and the operation $\backslash\colon \mathrm{Reg}(\mathbf{B}) \times \mathrm{Reg}(\mathbf{B}) \to \mathrm{Reg}(\mathbf{B})$ is defined, for all $a, b \in \mathrm{Reg}(\mathbf{B})$, as

$$a\backslash b = \mathrm{reg}(a \supset b).$$

As shown in [4], in any interlaced bilattice the relation

$$\{\langle a, b\rangle \in B \times B : \mathrm{reg}(a) = \mathrm{reg}(b)\}$$

is a congruence of the reduct $\langle B, \wedge, \vee, \otimes, \oplus\rangle$, and it is not difficult to prove that it is also compatible with the operation $\supset$. Since $\mathrm{reg}(a) = \mathrm{reg}(\mathrm{reg}(a))$ for all $a \in B$, it is also easy to conclude that, for all $a, b \in B$,

$$a\backslash b = \mathrm{reg}(a \supset b) = \mathrm{reg}(\mathrm{reg}(a) \supset \mathrm{reg}(b)) = \mathrm{reg}(\mathrm{reg}(a) \supset b) = \mathrm{reg}(a \supset \mathrm{reg}(b)).$$

In the following proofs we will sometimes use this fact without notice. Our next aim is to show that $\langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, \backslash\rangle$ is indeed a Brouwerian lattice.

Proposition 4.12. *Let $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg\rangle$ be a Brouwerian bilattice and $a, b, c \in \mathrm{Reg}(\mathbf{B})$. Then:*

(i) *$a \leq_k b$ if and only if $a\backslash b = \top$,*
(ii) *$a \otimes b \leq_k c$ if and only if $a \leq_k b\backslash c$.*

Proof. (i) Recalling that the reduct $\langle B, \wedge, \vee, \otimes, \oplus, \neg\rangle$ is interlaced, it is easy to see that, for all $a \in B$, it holds that $a \geq_t \top$ if and only if $\mathrm{reg}(a) = \top$. The rightwards implication is easy: applying the definition of reg, one can check that $\mathrm{reg}(a \vee \top) = \mathrm{reg}(\top) = \top$. As to the converse, we may use the fact that, as we have seen in the proof of Proposition 3.3, for all $a \in B$,

$$a = (\mathrm{reg}(a) \otimes (\mathrm{reg}(a) \vee \mathrm{reg}(\neg a))) \oplus (\mathrm{reg}(\neg a) \otimes (\mathrm{reg}(a) \wedge \mathrm{reg}(\neg a))).$$

Applying the assumption that $\mathrm{reg}(a) = \top$, we have

$$a = (\top \otimes (\top \vee \mathrm{reg}(\neg a))) \oplus (\mathrm{reg}(\neg a) \otimes (\top \wedge \mathrm{reg}(\neg a)))$$
$$= (\top \vee \mathrm{reg}(\neg a)) \oplus (\mathrm{reg}(\neg a) \otimes (\top \wedge \mathrm{reg}(\neg a)))$$
$$= (\top \vee \mathrm{reg}(\neg a)) \oplus \mathrm{reg}(\neg a) = \top \vee \mathrm{reg}(\neg a) \geq_t \top.$$

The last two equalities hold because, by the interlacing conditions, $\mathrm{reg}(\neg a) \leq_k \top \wedge \mathrm{reg}(\neg a)$ and $\mathrm{reg}(\neg a) \leq_k \top \vee \mathrm{reg}(\neg a)$.

Now, assuming $a \leq_k b$, by Proposition 4.8 (xiv) we have $a \supset b \geq_t \top$, therefore $a \backslash b = \mathrm{reg}(a \supset b) = \top$. Conversely, if $\mathrm{reg}(a \supset b) = \top$, then $a \supset b \geq_t \top$. By Proposition 4.8 (vii), this implies $a \leq_t a \otimes b$ and, since $a, b \in \mathrm{Reg}(\mathbf{B})$, also $\neg a = a \geq_t a \otimes b = \neg(a \otimes b)$. Hence $a = a \otimes b$, i.e., $a \leq_k b$.

(ii) Note that, for all $a, b, c \in \mathrm{Reg}(\mathbf{B})$, it holds that $(a \otimes b)\backslash c = a\backslash(b\backslash c)$. This is so because, by (B2), we have

$$\begin{aligned} a\backslash(b\backslash c) &= \mathrm{reg}(a \supset \mathrm{reg}(b \supset c)) = \mathrm{reg}(a \supset (b \supset c)) \\ &= \mathrm{reg}((a \otimes b) \supset c) = \mathrm{reg}((a \otimes b)\backslash c). \end{aligned}$$

Now, using (i), the proof is straightforward. Assume that $a \otimes b \leq_k c$. Then $(a \otimes b)\backslash c = a\backslash(b\backslash c) = \top$ and this implies $a \leq_k b\backslash c$. The converse implication is also immediate. □

We are now able to prove a representation theorem for Brouwerian bilattices:

Theorem 4.13 (Representation of Brouwerian bilattices)**.** *For any Brouwerian bilattice* $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$*:*

(i) $\langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, \backslash \rangle$ *is a Brouwerian lattice,*
(ii) $\mathbf{B} \cong \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, \backslash \rangle \odot \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, \backslash \rangle$,
(iii) $\langle \mathrm{Con}(\mathbf{B}), \subseteq \rangle \cong \langle \mathrm{Con}(\langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, \backslash \rangle), \subseteq \rangle$.

Proof. (i) Follows from the fact that $\langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus \rangle$ is a lattice, together with Proposition 4.12 (ii).

(ii) Let us denote by $\supset^*$ the implication defined in $\langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, \backslash \rangle \odot \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, \backslash \rangle$ as before, that is, for all $a_1, a_2, b_1, b_2 \in \mathrm{Reg}(\mathbf{B})$,

$$\langle a_1, a_2 \rangle \supset^* \langle b_1, b_2 \rangle = \langle \mathrm{reg}(a_1 \supset b_1), a_1 \otimes b_2 \rangle.$$

The isomorphism is given by the map $\iota_{\mathbf{B}} \colon B \to \mathrm{Reg}(\mathbf{B}) \times \mathrm{Reg}(\mathbf{B})$ defined as before, i.e., for all $a \in B$, $\iota_{\mathbf{B}}(a) = \langle \mathrm{reg}(a), \mathrm{reg}(\neg a) \rangle$. We know that $\iota_{\mathbf{B}}$ is a bijection and an isomorphism between the two bilattice reducts, so we just need to check that, for all $a, b \in B$,

$$\begin{aligned} \iota_{\mathbf{B}}(a \supset b) &= \langle \mathrm{reg}(a \supset b), \mathrm{reg}(\neg(a \supset b)) \rangle \\ &= \langle \mathrm{reg}(a), \mathrm{reg}(\neg a) \rangle \supset^* \langle \mathrm{reg}(b), \mathrm{reg}(\neg b) \rangle \\ &= \langle \mathrm{reg}(\mathrm{reg}(a) \supset \mathrm{reg}(b)), \mathrm{reg}(a) \otimes \mathrm{reg}(\neg b) \rangle = \iota_{\mathbf{B}}(a) \supset^* \iota_{\mathbf{B}}(b). \end{aligned}$$

This amounts to proving that $\mathrm{reg}(a \supset b) = \mathrm{reg}(\mathrm{reg}(a) \supset \mathrm{reg}(b))$ and

$$\mathrm{reg}(\neg(a \supset b)) = \mathrm{reg}(a) \otimes \mathrm{reg}(\neg b) = \mathrm{reg}(a \otimes \neg b) = \mathrm{reg}(a \wedge \neg b).$$

The first one is immediate. As to the second, notice that, using (B5), it is easy to prove that $\neg(a \supset b) \supset (a \wedge \neg b) \geq_t \top$ and $(a \wedge \neg b) \supset \neg(a \supset b) \geq_t \top$. By Proposition 4.12 (i), this implies that $\mathrm{reg}(\neg(a \supset b)) \leq_k \mathrm{reg}(a \wedge \neg b)$ and $\mathrm{reg}(a \wedge \neg b) \leq_k \mathrm{reg}(\neg(a \supset b))$ and this completes the proof.

(iii) Following the proof of Proposition 3.3, we show that the isomorphism is given by the map

$$H\colon \operatorname{Con}(\mathbf{B}) \to \operatorname{Con}(\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, \backslash\rangle)$$

defined, for all $\theta \in \operatorname{Con}(\mathbf{B})$, as

$$H(\theta) = \theta \cap (\operatorname{Reg}(\mathbf{B}) \times \operatorname{Reg}(\mathbf{B})).$$

From the proof of Proposition 3.3 it follows that H is well defined and that it is an order embedding. Its inverse is

$$H^{-1}\colon \operatorname{Con}(\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, \backslash\rangle) \to \operatorname{Con}(\mathbf{B})$$

defined, for all $\theta \in \operatorname{Con}(\langle \operatorname{Reg}(\mathbf{B}), \otimes, \oplus, -\rangle)$, as follows:

$$\langle a, b\rangle \in H^{-1}(\theta) \text{ iff } \langle \operatorname{reg}(a), \operatorname{reg}(b)\rangle \in \theta \text{ and } \langle \operatorname{reg}(\neg a), \operatorname{reg}(\neg b)\rangle \in \theta.$$

We have proved that $H^{-1}(\theta)$ is an equivalence relation compatible with all the lattice operations of both orders as well as with negation. As to compatibility with the operation $\supset$, assume $\langle a, b\rangle, \langle c, d\rangle \in H^{-1}(\theta)$, that is, $\langle \operatorname{reg}(a), \operatorname{reg}(b)\rangle$, $\langle \operatorname{reg}(\neg a), \operatorname{reg}(\neg b)\rangle$, $\langle \operatorname{reg}(c), \operatorname{reg}(d)\rangle$, $\langle \operatorname{reg}(\neg c), \operatorname{reg}(\neg d)\rangle \in \theta$. By the assumptions we have

$$\langle \operatorname{reg}(\operatorname{reg}(a) \supset \operatorname{reg}(c)), \operatorname{reg}(\operatorname{reg}(b) \supset \operatorname{reg}(d))\rangle \in \theta.$$

We have seen in (ii) that $\operatorname{reg}(a \supset b) = \operatorname{reg}(\operatorname{reg}(a) \supset \operatorname{reg}(b))$ and $\operatorname{reg}(\neg(a \supset b)) = \operatorname{reg}(a \otimes \neg b)$ for all $a, b \in B$. From this we easily obtain

$$\langle \operatorname{reg}(a \supset c), \operatorname{reg}(b \supset d)\rangle, \langle \operatorname{reg}(\neg(a \supset c)), \operatorname{reg}(\neg(b \supset d))\rangle \in \theta$$

and this completes the proof. □

As in the case of bilattices with conflation, the subvarieties of Brouwerian lattices correspond to subvarieties of Brouwerian bilattices. This can be shown in general by defining translations between equations in the two signatures.

Let $\mathcal{T}$ be the set of terms in the language $\{\wedge, \vee, \otimes, \oplus, \supset, \neg, -\}$. For any term $\varphi \in \mathcal{T}$, define

$$\operatorname{reg}(\varphi) := (\varphi \vee (\varphi \otimes \neg\varphi)) \oplus \neg(\varphi \vee (\varphi \otimes \neg\varphi)).$$

Let us also define $\sim\varphi := \neg{-\varphi}$. Now, let $\mathcal{S}$ be the set of terms in the language $\{\sqcap, \sqcup, \backslash, ', 0, 1\}$ and let $\varphi(\sqcap, \sqcup, \backslash, ', 0, 1)$ be a term of $\mathcal{S}$. We define a translation $\tau\colon \mathcal{S} \to \mathcal{T}$ as follows:

$$\tau(\varphi(\sqcap, \sqcup, \backslash, ', 0, 1)) := \operatorname{reg}(\varphi(\otimes, \oplus, \supset, \sim, \bot, \top))$$

where $\varphi(\otimes, \oplus, \supset, \sim, \bot, \top)$ is obtained from $\varphi(\sqcap, \sqcup, \backslash, ', 0, 1)$ by replacing all occurrences of $\sqcap, \sqcup, \backslash, ', 0, 1$ respectively by occurrences of $\otimes, \oplus, \supset, \sim, \bot, \top$.

It is easy to check that, given a Brouwerian (or an involutive) lattice $\mathbf{L}$ with associated Brouwerian bilattice (commutative bilattice with conflation) $\mathbf{L} \odot \mathbf{L}$, we have that, for any equation $\varphi \approx \psi$ where $\varphi, \psi \in \mathcal{S}$, it holds that $\mathbf{L} \vDash \varphi \approx \psi$ if and only if $\mathbf{L} \odot \mathbf{L} \vDash \tau(\varphi) \approx \tau(\psi)$.

Conversely, drawing inspiration from the representation theorems stated above, we may define inductively a translation $\rho\colon \mathcal{T} \to \mathcal{S} \times \mathcal{S}$ as follows. For any variable $x \in \mathcal{T}$ and terms $\varphi, \psi \in \mathcal{T}$:

$$
\begin{aligned}
\rho(x) &:= \langle x_1, x_2 \rangle, \\
\rho(\neg\varphi) &:= \langle \varphi_2, \varphi_1 \rangle, \\
\rho(-\varphi) &:= \langle \varphi_2', \varphi_1' \rangle, \\
\rho(\varphi \wedge \psi) &:= \langle \varphi_1 \sqcap \psi_1, \varphi_2 \sqcup \psi_2 \rangle, \\
\rho(\varphi \vee \psi) &:= \langle \varphi_1 \sqcup \psi_1, \varphi_2 \sqcap \psi_2 \rangle, \\
\rho(\varphi \otimes \psi) &:= \langle \varphi_1 \sqcap \psi_1, \varphi_2 \sqcap \psi_2 \rangle, \\
\rho(\varphi \oplus \psi) &:= \langle \varphi_1 \sqcup \psi_1, \varphi_2 \sqcup \psi_2 \rangle, \\
\rho(\varphi \supset \psi) &:= \langle \varphi_1 \backslash \psi_1, \varphi_1 \sqcap \psi_2 \rangle,
\end{aligned}
$$

where for any formula φ we use the convention to refer by φ_1 to the first member of the pair $\rho(\varphi)$ and by φ_2 to the second member, so that $\rho(\varphi) = \langle \varphi_1, \varphi_2 \rangle$. It is then easy to check that, for any equation $\varphi \approx \psi$ with $\varphi, \psi \in \mathcal{T}$, it holds that $\mathbf{L} \odot \mathbf{L} \vDash \varphi \approx \psi$ if and only if $\mathbf{L} \vDash \varphi_1 \approx \psi_1$ and $\mathbf{L} \vDash \varphi_2 \approx \psi_2$, where $\rho(\varphi) = \langle \varphi_1, \varphi_2 \rangle$ and $\rho(\psi) = \langle \psi_1, \psi_2 \rangle$.

The translations defined above translate equations into equations preserving satisfiability. It is not difficult to convince oneself that quasi-equations are also preserved, as both translations are defined term-wise. This implies that *the sub-quasi-varieties of Brouwerian bilattices (commutative interlaced bilattices with conflation) are in one-to-one correspondence with the sub-quasi-varieties of Brouwerian lattices (involutive lattices).*

5. Categorical equivalences

The representation theorems known since the 1990s for bounded interlaced and bounded distributive (pre-)bilattices have been used in [11] to establish categorical equivalences between a number of categories of bounded bilattices and of bounded lattices [11, Theorems 10 and 13, Corollaries 11 and 14]. The aim of this section is to exploit the new representation theorems obtained above (corresponding to unbounded bilattices, bilattices with conflation and Brouwerian bilattices) to obtain new categorical equivalences. Most of the results stated in this section can also be found in the dissertation [16] together with further categorical results not discussed here (for instance on categories corresponding to subreducts of Brouwerian bilattices).

The main categories considered in this section are listed in Table 1: interlaced pre-bilattices (**IntPreBiLat**), interlaced bilattices (**IntBiLat**), commutative interlaced bilattices with conflation (**BiLatCon**), Brouwerian bilattices (**BrBiLat**), lattices (**Lat**), involutive lattices (**InvLat**) and Brouwerian lattices (**BrLat**).

Subsection	LATTICES	BILATTICES
5.1	**Lat × Lat**	**IntPreBiLat**
5.1	**Lat**	**IntBiLat**
5.2	**InvLat**	**BiLatCon**
5.3	**BrLat**	**BrBiLat**

TABLE 1. Categorical equivalences

Note that, except for **Lat** × **Lat**, the product category of **Lat** with itself, each of the above categories corresponds to a variety of bilattices or lattices (maybe enriched with additional operations) and the morphisms are the algebraic homomorphisms in the corresponding algebraic signature. It follows that all sub-varieties of these varieties correspond to full subcategories. The exception **Lat**×**Lat** can be seen as the direct product of **Lat** with itself, both at the level of objects and of morphisms, so its objects are pairs of lattices and its morphisms are pairs of algebraic homomorphisms. We follow the convention of using the same acronym to denote a variety and its associated category, but for the category we use bold. So **Lat** is the category corresponding to the variety Lat of lattices and so on. We also write $\mathrm{Mor}(\mathbf{C})$ to refer to the morphisms of the category $\mathbf{C}$.

In this section we prove that each of the four categories of bilattices given in Table 1 is categorically equivalent to the category of lattices listed on the same row of the table. Since categorical equivalences are obviously extended to full subcategories, we will obtain that for each sub-(quasi-)variety of Lat, InvLat and BrLat there is an equivalent (quasi-)variety of interlaced bilattices (maybe enriched with additional operations).

For the sake of clarity, we give the proof of each one of these equivalences in a separate subsection. However, in all four cases the functors that allow us to prove the equivalence are defined essentially in the same way. Thus we will use the results obtained in the first subsection in order to shorten the subsequent proofs.

5.1. Lattices. Our next aim is to prove that the categories **IntPreBiLat** and **Lat**×**Lat** are naturally equivalent, and also that **IntBiLat** and **Lat** are naturally equivalent.

Let us first consider the case of pre-bilattices. If $\mathbf{L_1}$ and $\mathbf{L_2}$ are lattices, let $\mathsf{B}(\langle \mathbf{L_1}, \mathbf{L_2} \rangle)$ denote the interlaced pre-bilattice $\mathbf{L_1} \odot \mathbf{L_2}$. Conversely, given an interlaced pre-bilattice $\mathbf{B}$, let $\mathsf{L}^2(\mathbf{B}) = \langle \langle B, \otimes, \oplus \rangle / {\sim_1}, \langle B, \otimes, \oplus \rangle / {\sim_2} \rangle$, where $\sim_1$ and $\sim_2$ are defined as in Section 3. By Theorem 3.1, there is an isomorphism $\iota_{\mathbf{B}} \colon \mathbf{B} \cong \mathsf{B}(\mathsf{L}^2(\mathbf{B}))$ defined, for all $a \in B$, as

$$\iota_{\mathbf{B}}(a) = \langle [a]_1, [a]_2 \rangle. \tag{5.1}$$

It is also easy to see that, given a pair of lattices $\mathbf{L_1}$ and $\mathbf{L_2}$, in the product category **Lat** × **Lat** there is an isomorphism $\langle \nu_{\mathbf{L_1}}, \nu_{\mathbf{L_2}} \rangle$ between $\langle \mathbf{L_1}, \mathbf{L_2} \rangle$ and

$\mathsf{L}^2(\mathsf{B}(\langle \mathbf{L_1}, \mathbf{L_2}\rangle))$, where $\nu_{\mathbf{L_1}} \colon \mathbf{L_1} \cong \langle B, \otimes, \oplus\rangle/{\sim_1}$ and $\nu_{\mathbf{L_2}} \colon \mathbf{L_2} \cong \langle B, \otimes, \oplus\rangle/{\sim_2}$ are defined, for all $\langle a_1, a_2\rangle \in L_1 \times L_2$, as

$$\nu_{\mathbf{L_1}}(a_1) = [\langle a_1, a_2\rangle]_1 \quad \text{and} \quad \nu_{\mathbf{L_2}}(a_2) = [\langle a_1, a_2\rangle]_2. \tag{5.2}$$

Note that the definition of $\nu_{\mathbf{L_1}}(a_1)$ is independent of the element a_2, for it holds that $[\langle a_1, a_2\rangle]_1 = [\langle a_1, b\rangle]_1$ for any $b \in L_2$, and similarly $[\langle a_1, a_2\rangle]_2 = [\langle b, a_2\rangle]_2$ for any $b \in L_2$.

In order to establish a categorical equivalence, we define two functors,

$$F \colon \mathsf{\textbf{Lat}} \times \mathsf{\textbf{Lat}} \to \mathsf{\textbf{IntPreBiLat}} \text{ and } G \colon \mathsf{\textbf{IntPreBiLat}} \to \mathsf{\textbf{Lat}} \times \mathsf{\textbf{Lat}},$$

as follows. For all $\langle \mathbf{L_1}, \mathbf{L_2}\rangle \in \mathsf{Lat} \times \mathsf{Lat}$, let $F(\langle \mathbf{L_1}, \mathbf{L_2}\rangle) = \mathsf{B}(\langle \mathbf{L_1}, \mathbf{L_2}\rangle)$. For all $\langle \mathbf{L_1}, \mathbf{L_2}\rangle, \langle \mathbf{M_1}, \mathbf{M_2}\rangle \in \mathsf{Lat} \times \mathsf{Lat}$ and for all

$$\langle h_1, h_2\rangle \colon \langle \mathbf{L_1}, \mathbf{L_2}\rangle \to \langle \mathbf{M_1}, \mathbf{M_2}\rangle \in \mathrm{Mor}(\mathsf{\textbf{Lat}} \times \mathsf{\textbf{Lat}}),$$

let $F(\langle h_1, h_2\rangle) \colon \mathsf{B}(\langle \mathbf{L_1}, \mathbf{L_2}\rangle) \to \mathsf{B}(\langle \mathbf{M_1}, \mathbf{M_2}\rangle)$, for all $\langle a_1, a_2\rangle \in \mathsf{B}(\langle \mathbf{L_1}, \mathbf{L_2}\rangle)$, be given by

$$F(\langle h_1, h_2\rangle)(\langle a_1, a_2\rangle) = \langle h_1(a_1), h_2(a_2)\rangle.$$

It is not difficult to check that F is indeed a functor. The functor G is defined, for all $\mathbf{B} \in \mathsf{IntPreBiLat}$, as $G(\mathbf{B}) = \mathsf{L}^2(\mathbf{B})$. For all $\mathbf{B}, \mathbf{C} \in \mathsf{IntPreBiLat}$ and $k \colon \mathbf{B} \to \mathbf{C} \in \mathrm{Mor}(\mathsf{\textbf{IntPreBiLat}})$, let $G(k) \colon \mathsf{L}^2(\mathbf{B}) \to \mathsf{L}^2(\mathbf{C})$ be defined as $G(k) = \langle G(k)_1, G(k)_2\rangle$, where $G(k)_1([a]_1) = [k(a)]_1$ and $G(k)_2([b]_2) = [k(b)]_2$ for all $\langle [a]_1, [b]_2\rangle \in \mathsf{L}^2(\mathbf{B})$.

Using [4, Proposition 3.6], it is easy to check that $a \sim_1 b$ implies $k(a) \sim_1 k(b)$ for any $a, b \in B$ and any homomorphism $k \colon \mathbf{B} \to \mathbf{C}$. The same holds for $\sim_2$. Thus, the above definition is sound.

Denoting by $I_{\mathbf{C}}$ the identity functor on a given category $\mathbf{C}$, we are now able to prove the following analogue of [11, Theorem 10]:

Theorem 5.1. *The families* $\iota \colon I_{\mathsf{\textbf{IntBiLat}}} \to FG$ *and* $\nu \colon I_{\mathsf{\textbf{Lat}} \times \mathsf{\textbf{Lat}}} \to GF$ *of morphisms defined in* (5.1) *and* (5.2) *are natural isomorphisms, therefore the categories* $\mathsf{\textbf{Lat}} \times \mathsf{\textbf{Lat}}$ *and* $\mathsf{\textbf{IntPreBiLat}}$ *are naturally equivalent.*

Proof. Let ι, ν, F, G be defined as above. Assume

$$\langle h_1, h_2\rangle \colon \langle \mathbf{L_1}, \mathbf{L_2}\rangle \to \langle \mathbf{M_1}, \mathbf{M_2}\rangle \in \mathrm{Mor}(\mathsf{\textbf{Lat}} \times \mathsf{\textbf{Lat}})$$

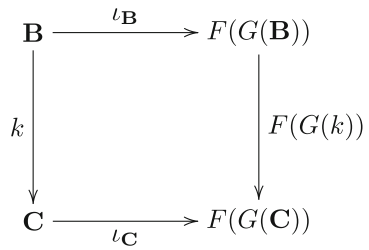
and $\langle a_1, a_2\rangle \in L_1 \times L_2$. We have to prove that the following diagram commutes:

$$\begin{array}{ccc}
\langle \mathbf{L_1}, \mathbf{L_2}\rangle & \xrightarrow{\langle \nu_{\mathbf{L_1}}, \nu_{\mathbf{L_2}}\rangle} & G(F(\langle \mathbf{L_1}, \mathbf{L_2}\rangle)) \\
{\scriptstyle \langle h_1, h_2\rangle}\big\downarrow & & \big\downarrow{\scriptstyle G(F(\langle h_1, h_2\rangle))} \\
\langle \mathbf{M_1}, \mathbf{M_2}\rangle & \xrightarrow[\langle \nu_{\mathbf{M_1}}, \nu_{\mathbf{M_2}}\rangle]{} & G(F(\langle \mathbf{M_1}, \mathbf{M_2}\rangle))
\end{array}$$

Applying our definitions, we have

$$\begin{aligned}
&G(F(\langle h_1, h_2\rangle)) \cdot \langle \nu_{\mathbf{L_1}}, \nu_{\mathbf{L_2}}\rangle(\langle a_1, a_2\rangle)\\
&= G(F(\langle h_1, h_2\rangle))\langle[\langle a_1, a_2\rangle]_1, [\langle a_1, a_2\rangle]_2\rangle\\
&= \langle[F(\langle h_1, h_2\rangle)(\langle a_1, a_2\rangle)]_1, [F(\langle h_1, h_2\rangle)(\langle a_1, a_2\rangle)]_2\rangle\\
&= \langle[\langle h_1(a_1), h_2(a_2)\rangle]_1, [\langle h_1(a_1), h_2(a_2)\rangle]_2\rangle\\
&= \langle[\langle h_1(a_1), h_2(a_2)\rangle]_1, [\langle h_1(a_1), h_2(a_2)\rangle]_2\rangle\\
&= \langle \nu_{\mathbf{M_1}}, \nu_{\mathbf{M_2}}\rangle(\langle h_1(a_1), h_2(a_2)\rangle)\\
&= \langle \nu_{\mathbf{M_1}}, \nu_{\mathbf{M_2}}\rangle \cdot \langle h_1, h_2\rangle(\langle a_1, a_2\rangle).
\end{aligned}$$

Assume now $k\colon \mathbf{B} \to \mathbf{C} \in \mathrm{Mor}(\textsf{IntPreBiLat})$ and $a \in B$. We have to prove that the following diagram commutes:



Applying again the definitions, we obtain

$$F(G(k)) \cdot \iota_{\mathbf{B}}(a) = F(G(k))\langle[a]_1, [a]_2\rangle = \langle[k(a)]_1, [k(a)]_2\rangle = \iota_{\mathbf{C}} \cdot k(a).$$

We have thus proved that ι and ν are natural transformations. Since, as we have noted, $\iota_{\mathbf{B}}\colon \mathbf{B} \to F(G(\mathbf{B}))$ and $\nu_{\mathbf{L}}\colon \mathbf{L} \to G(F(\mathbf{L}))$ are isomorphisms, we conclude that ι and ν are natural isomorphisms. □

Let us now consider the case of bilattices. As we have seen above, in the presence of negation we can establish an isomorphism between a bilattice $\mathbf{B}$ and the product bilattice $\mathbf{L} \odot \mathbf{L}$ where $\mathbf{L} = \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus\rangle$. Given an interlaced bilattice $\mathbf{B}$, we may then set $\mathsf{L}(\mathbf{B}) = \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus\rangle$. Conversely, given a lattice $\mathbf{L}$, we denote by $\mathsf{B}(\mathbf{L})$ the interlaced bilattice $\mathbf{L} \odot \mathbf{L}$. The isomorphism $\iota_{\mathbf{B}} : \mathbf{B} \cong \mathsf{B}(\mathsf{L}(\mathbf{B}))$ is then defined, for all $a \in B$, as

$$\iota_{\mathbf{B}}(a) = \langle \mathrm{reg}(a), \mathrm{reg}(\neg a)\rangle. \tag{5.3}$$

Given a lattice $\mathbf{L}$, we have an isomorphism $\nu_{\mathbf{L}} : \mathbf{L} \cong \mathsf{L}(\mathsf{B}(\mathbf{L}))$ given, for all $a \in L$, by

$$\nu_{\mathbf{L}}(a) = \langle a, a\rangle. \tag{5.4}$$

We now define the functors $F\colon \textsf{Lat} \to \textsf{IntBiLat}$ and $G\colon \textsf{IntBiLat} \to \textsf{Lat}$ as follows. For every $\mathbf{L} \in \mathsf{Lat}$, set $F(\mathbf{L}) = \mathsf{B}(\mathbf{L})$, and for all $h\colon \mathbf{L} \to \mathbf{M} \in \mathrm{Mor}(\textsf{Lat})$, $F(h)\colon \mathsf{B}(\mathbf{L}) \to \mathsf{B}(\mathbf{M})$ is given, for all $a, b \in \mathsf{B}(\mathbf{L})$, by $F(h)(\langle a, b\rangle) = \langle h(a), h(b)\rangle$. Note that F preserves injections and surjections, i.e., if $h\colon L \to M$ is injective (surjective), then so is $F(h)\colon \mathsf{B}(\mathbf{L}) \to \mathsf{B}(\mathbf{M})$. For any $\mathbf{B} \in$

$\mathsf{IntBiLat}$, we set $G(\mathbf{B}) = \mathsf{L}(\mathbf{B})$, and for every $\mathbf{B}, \mathbf{C} \in \mathsf{IntBiLat}$ and $k\colon \mathbf{B} \to \mathbf{C} \in \mathrm{Mor}(\mathbf{IntBiLat})$, the functor $G(k)\colon \mathsf{L}(\mathbf{B}) \to \mathsf{L}(\mathbf{C})$ is defined as $G(k)(a) = k(a)$.

We are now able to state an analogue of [11, Theorem 13].

Theorem 5.2. *The families* $\iota\colon I_{\mathsf{IntBiLat}} \to FG$ *and* $\nu\colon I_{\mathsf{Lat}} \to GF$ *of morphisms defined in* (5.3) *and* (5.4) *are natural isomorphisms, therefore the categories* $\mathbf{Lat}$ *and* $\mathbf{IntBiLat}$ *are naturally equivalent.*

Proof. Let ι, ν, F, G be defined as above. Assume $h\colon \mathbf{L} \to \mathbf{M} \in \mathrm{Mor}(\mathbf{Lat})$ for some $\mathbf{L}, \mathbf{M} \in \mathsf{Lat}$ and $a \in L$. We have to prove that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{\ \nu_{\mathbf{L}}\ } & G(F(\mathbf{L})) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle G(F(h))} \\ \mathbf{M} & \xrightarrow[\ \nu_{\mathbf{M}}\ ]{} & G(F(\mathbf{M})) \end{array}$$

Applying our definitions, we have

$$\begin{aligned} G(F(h)) \cdot \nu_{\mathbf{L}}(a) &= G(F(h))(\langle a, a\rangle) = F(h)(\langle a, a\rangle) \\ &= \langle h(a), h(a)\rangle = \nu_{\mathbf{M}} \cdot h(a). \end{aligned}$$

Let now $k\colon \mathbf{B} \to \mathbf{C} \in \mathrm{Mor}(\mathbf{Lat})$ for some $\mathbf{B}, \mathbf{C} \in \mathsf{IntBiLat}$ and $a \in B$. We have to show that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\ \iota_{\mathbf{B}}\ } & F(G(\mathbf{B})) \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle F(G(k))} \\ \mathbf{C} & \xrightarrow[\ \iota_{\mathbf{C}}\ ]{} & F(G(\mathbf{C})) \end{array}$$

In order to see this, recall that $\mathrm{reg}(a) = (a \vee (a \otimes \neg a)) \oplus \neg(a \vee (a \otimes \neg a))$. It is then obvious that $k(\mathrm{reg}(a)) = \mathrm{reg}(k(a))$ and $k(\mathrm{reg}(\neg a)) = \mathrm{reg}(\neg k(a))$. We may now apply our definitions to obtain

$$\begin{aligned} F(G(k)) \cdot \iota_{\mathbf{B}}(a) &= F(G(k))\langle \mathrm{reg}(a), \mathrm{reg}(\neg a)\rangle = \langle k(\mathrm{reg}(a)), k(\mathrm{reg}(\neg a))\rangle \\ &= \langle \mathrm{reg}(k(a)), \mathrm{reg}(\neg k(a))\rangle = \iota_{\mathbf{C}} \cdot k(a). \end{aligned}$$

Thus we obtained that ι and ν are natural transformations. Since, as we have observed, $\iota_{\mathbf{B}}\colon \mathbf{B} \to F(G(\mathbf{B}))$ and $\nu_{\mathbf{L}}\colon \mathbf{L} \to G(F(\mathbf{L}))$ are isomorphisms, we conclude that ι and ν are natural isomorphisms. □

5.2. Involutive lattices. Our aim is to prove that the category **InvLat** of involutive lattices is naturally equivalent to the category **BiLatCon** of commutative interlaced bilattices with conflation. The situation is analogous to the previous case.

Let $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg, - \rangle$ be a commutative interlaced bilattice with conflation. We define $\mathsf{L}(\mathbf{B}) := \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, - \rangle$. Conversely, for any involutive lattice $\mathbf{L}$, we denote by $\mathsf{B}(\mathbf{L})$ the commutative interlaced bilattice with conflation $\mathbf{L} \odot \mathbf{L}$. By Theorem 4.2, there is an isomorphism $\iota_{\mathbf{B}} \colon \mathbf{B} \cong \mathsf{B}(\mathsf{L}(\mathbf{B}))$ defined as before:

$$\iota_{\mathbf{B}}(a) = \langle \mathrm{reg}(a), \mathrm{reg}(\neg a) \rangle. \tag{5.5}$$

Also, given an involutive lattice $\mathbf{L}$, we have an isomorphism $\nu_{\mathbf{L}} \colon \mathbf{L} \cong \mathsf{L}(\mathsf{B}(\mathbf{L}))$ given, for all $a \in L$, by

$$\nu_{\mathbf{L}}(a) = \langle a, a \rangle. \tag{5.6}$$

The functors $F \colon \textsf{\textbf{InvLat}} \to \textsf{\textbf{BiLatCon}}$ and $G \colon \textsf{\textbf{BiLatCon}} \to \textsf{\textbf{InvLat}}$ are also defined as in the case of bilattices. For every $\mathbf{L} \in \mathsf{InvLat}$, $F(\mathbf{L}) = \mathsf{B}(\mathbf{L})$, and for all $h \colon \mathbf{L} \to \mathbf{M} \in \mathrm{Mor}(\textsf{\textbf{InvLat}})$, $F(h) \colon \mathsf{B}(\mathbf{L}) \to \mathsf{B}(\mathbf{M})$ is given, for all $a, b \in \mathsf{B}(\mathbf{L})$, by $F(h)(\langle a, b \rangle) = \langle h(a), h(b) \rangle$. For any $\mathbf{B} \in \mathsf{BiLatCon}$, we set $G(\mathbf{B}) = \mathsf{L}(\mathbf{B})$ and for every $\mathbf{B}, \mathbf{C} \in \mathsf{BiLatCon}$ and $k \colon \mathbf{B} \to \mathbf{C} \in \mathrm{Mor}(\textsf{\textbf{BiLatCon}})$, the functor $G(k) \colon \mathsf{L}(\mathbf{B}) \to \mathsf{L}(\mathbf{C})$ is defined as $G(k)(a) = k(a)$.

We have then the following:

Theorem 5.3. *The families* $\iota \colon I_{\textsf{\textbf{BiLatCon}}} \to FG$ *and* $\nu \colon I_{\textsf{\textbf{InvLat}}} \to GF$ *of morphisms defined in* (5.5) *and* (5.6) *are natural isomorphisms, therefore the categories* **InvLat** *and* **BiLatCon** *are naturally equivalent.*

Proof. Similar to the one of Theorem 5.2. □

5.3. Brouwerian lattices. We are going to prove that the category **BrLat** of Brouwerian lattices is naturally equivalent to the category **BrBiLat** of Brouwerian bilattices. This case is also analogous to the previous ones.

Given a Brouwerian bilattice $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$, we define $\mathsf{L}(\mathbf{B}) := \langle \mathrm{Reg}(\mathbf{B}), \otimes, \oplus, \backslash \rangle$, where the operation $\backslash$ is defined as in Section 4.2. Conversely, for any Brouwerian lattice $\mathbf{L}$, we denote by $\mathsf{B}(\mathbf{L})$ the Brouwerian bilattice $\mathbf{L} \odot \mathbf{L}$. By Theorem 4.13, there is an isomorphism $\iota_{\mathbf{B}} \colon \mathbf{B} \cong \mathsf{B}(\mathsf{L}(\mathbf{B}))$ defined as before:

$$\iota_{\mathbf{B}}(a) = \langle \mathrm{reg}(a), \mathrm{reg}(\neg a) \rangle. \tag{5.7}$$

Also, given a Brouwerian lattice $\mathbf{L}$, we have an isomorphism $\nu_{\mathbf{L}} \colon \mathbf{L} \cong \mathsf{L}(\mathsf{B}(\mathbf{L}))$ given, for all $a \in L$, by

$$\nu_{\mathbf{L}}(a) = \langle a, a \rangle. \tag{5.8}$$

The functors $F \colon \textsf{\textbf{BrLat}} \to \textsf{\textbf{BrBiLat}}$ and $G \colon \textsf{\textbf{BrBiLat}} \to \textsf{\textbf{BrLat}}$ are also defined as in the case of bilattices. For every $\mathbf{L} \in \mathsf{BrLat}$, $F(\mathbf{L}) = \mathsf{B}(\mathbf{L})$, and for all $h \colon \mathbf{L} \to \mathbf{M} \in \mathrm{Mor}(\textsf{\textbf{BrLat}})$, $F(h) \colon \mathsf{B}(\mathbf{L}) \to \mathsf{B}(\mathbf{M})$ is given, for all $a, b \in \mathsf{B}(\mathbf{L})$, by $F(h)(\langle a, b \rangle) = \langle h(a), h(b) \rangle$. For any $\mathbf{B} \in \mathsf{BrBiLat}$, we set $G(\mathbf{B}) = \mathsf{L}(\mathbf{B})$

and for every $\mathbf{B}, \mathbf{C} \in \mathsf{BrBiLat}$ and $k\colon \mathbf{B} \to \mathbf{C} \in \mathrm{Mor}(\mathbf{BrBiLat})$, the functor $G(k)\colon \mathsf{L}(\mathbf{B}) \to \mathsf{L}(\mathbf{C})$ is defined as $G(k)(a) = k(a)$.

We have then the following:

Theorem 5.4. *The families* $\iota\colon I_{\mathbf{BrBiLat}} \to FG$ *and* $\nu\colon I_{\mathbf{BrLat}} \to GF$ *of morphisms defined in* (5.7) *and* (5.8) *are natural isomorphisms. Hence, the categories* **BrLat** *and* **BrBiLat** *are naturally equivalent.*

Proof. Similar to those of Theorems 5.2 and 5.3. □

5.4. Subcategories. As observed above, the equivalences proved in the previous sections extend to full subcategories of the above-mentioned categories which have as objects algebras belonging to sub-quasivarieties of Lat, InvLat, IntPreBiLat and so on. In this way we obtain the equivalences shown in Table 2.

Lattices	**DLat** × **DLat** **DLat**	**DPreBiLat** **DBiLat**	Bilattices
Involutive lattices	**DMLat** **KLat** **BA**	**DBiLatCon** **KBiLatCon** **CBiLatCon**	Bilattices with conflation
Brouwerian lattices	**GenBA**	**ImpBiLat**	Brouwerian bilattices

TABLE 2. Equivalences between some subcategories

On the left column we have distributive lattices (**DLat**) as a subcategory of lattices, De Morgan lattices (**DMLat**), Kleene lattices (**KLat**) and Boolean algebras (**BA**) as subcategories of involutive lattices, and the 0-free subreducts of Boolean algebras, usually called generalized Boolean algebras (**GenBA**), as a subcategory of Brouwerian lattices. On the right column we have: one subcategory of interlaced bilattices, i.e., distributive bilattices (**DBiLat**); subcategories of commutative interlaced bilattices with conflation, i.e., distributive bilattices with conflation (**DBiLatCon**), Kleene bilattices with conflation (**KBiLatCon**) and classical bilattices with conflation (**CBiLatCon**), and one subcategory of Brouwerian bilattices, namely implicative bilattices (**ImpBiLat**).

Notice that, as mentioned above, these equivalence results imply that we could define categories of bilattices that are equivalent to any of the categories associated with sub-quasi-varieties of Brouwerian lattices (for instance, all varieties corresponding to super-intuitionistic logics).

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