Normality operators and Classical Recapture in Extensions of Kleene Logics

ROBERTO CIUNI, *FISPPA Department, Section of Philosophy, University of Padova, Piazza Capitaniato 3, Padova, Italy.*
E-mail: roberto.ciuni@unipd.it

MASSIMILIANO CARRARA, *FISPPA Department, Section of Philosophy, University of Padova, Piazza Capitaniato 3, Padova, Italy.*
E-mail: massimiliano.carrara@unipd.it

Abstract

In this paper, we approach the problem of *classical recapture* for LP and K_3 by using *normality operators*. These generalize the consistency and determinedness operators from *Logics of Formal Inconsistency* and *Underterminedness*, by expressing, in any many-valued logic, that a given formula has a classical truth value (0 or 1). In particular, in the first part of the paper we introduce the logics $LP^{\circledast}$ and $K_3^{\circledast}$, which extends LP and K_3 with *normality operators*, and we establish a *classical recapture result* based on the two logics. In the second part of the paper, we compare the approach in terms of *normality operators* with an established approach to *classical recapture*, namely *minimal inconsistency*. Finally, we discuss technical issues connecting $LP^{\circledast}$ and $K_3^{\circledast}$ to the tradition of *Logics of Formal Inconsistency* and *Underterminedness*.

Keywords: Kleene logics, Classical Recapture, Normality Operators, Minimal Inconsistency, Logics of Formal Inconsistency, Logics of Formal Undeterminedness

1 Introduction

The philosophical applications of many-valued logics usually come with a story of 'normality': there are a number of 'abnormal phenomena' for which we need many-valued reasoning—logical paradoxes, partial information, vagueness, among others—but as long as the situation is *normal*—that is, no abnormal phenomena is at stake—classical logic is perfectly in order as it is.

This view motivates the following question: *how can we* recapture *Classical Logic* CL *in many-valued logic?* That is, how can we secure inference of classical conclusions, under the assumption that we are facing no abnormality? [2, 6, 7, 23, 24] The question fits the broad conceptual view on many-valued reasoning by the 'normality story' above: if detour from classical reasoning is motivated by specific phenomena, then we do not need to take it when these phenomena are not around. Numerous as these may be, we may assume that they are not ubiquitous. For instance, take semantical and set-theoretical paradoxes; to put it with [23, p.235]: '*paradoxical sentences seem to be a fairly small proportion of the sentences we reason with... It would seem plausible to claim that in our day-to-day reasoning we (quite correctly) presuppose that we are* *not dealing with paradoxical claims*'. [6, p.326] generalizes this remark to absence of *any* abnormality: '*The basic thought is that classical logic is 'right' (in some sense) for the broad array of 'normal' cases; however, various 'abnormal' [...] phenomena motivate a slightly weaker logic*'. In a nutshell, if one follows the 'normality story' above, then one may just want to resort to classical reasoning *when possible*—that is, in 'normal situations.'

Recapture of classical reasoning proves desirable even if we do not endorse the 'normality story' above. Indeed, the many-valued strategy in dealing with abnormal phenomena consists in *weakening* classical logic by dropping those laws or rules that lead to undesirable conclusions when abnormal phenomena are at stake. These are, typically, the *Law of Excluded Middle* $\vDash \phi \vee \neg\phi$ and *Ex Contradiction Quodlibet* $\phi \wedge \neg\phi \vDash \psi$. However, every many-valued systems comes with further failures. In particular, some desirable laws or rules are falsified. For instance, the *Law of Identity* $\vDash \phi \supset \phi$ is not valid in the strong Kleene logic K_3 [17, 18], but this looks like a basic validity, and we might want to have it available *when possible*. Also, *Modus Ponens* $\phi, \phi \supset \psi \vDash \psi$ is not valid in the Logic of Paradox LP [23, 25], but again, this is the most basic inference rule of our reasoning, and we might want to use it, when possible. In sum, ability to display classical reasoning when possible is a desideratum. In order to fulfill it, we need to answer the question about *classical recapture*.

An established approach to classical recapture is 'minimal inconsistency' by [24], which resorts to a *non-monotonic* consequence relation that preserves the valid inferences of LP and recapture classical reasoning, under the proviso that the premise-set has a classical model.

In this paper, we extend the Kleene logics[1] LP and K_3 via the normality operator $\circledast$, thus yielding the logics $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$. In both logics, $\circledast\phi$ expresses that ϕ has a *classical value* (0 or 1). We show that the normality operator guarantees *recapture* of classical reasoning (Theorem 3.3), and we compare our approach to *minimal inconsistency*. We focus on Kleene logics and their extensions since these are the most widespread formalism in philosophical logic.

The proposal of this paper has close connections with the *Logics of Formal Inconsistency* (from now on, LFIs) by [10, 12, 20] and the *Logics of Formal Undeterminedness* (from now on, LFUs) by [11, 28]. LFIs stem from [12] and have been later developed in a number of works, including [10, 20]. They are a family of *paraconsistent* systems that control the behavior of *contradictions*—and sort them out—by internalizing the notion of *consistency*[2] in the language, and expressing whether a formula is *consistent* or not.[3] LFUs [5, 11, 28] dualize the project of LFIs, and they control—and sort out—*undetermined* formulas by internalizing the notion of *undeterminedness* in the language. Definition 2.2 from Section 2 makes it clear that our normality operator generalizes the consistency and determinedness operators. However, in the three-

[1] We choose this label for the sake of simplicity, since the connectives of LP and K_3 are interpreted on *Kleene algebras*. However, notice that a paraconsistent logic such as LP was not part of the *philosophical* and *mathematical* project by Kleene, which instead aimed at the construction of paracomplete formalisms.

[2] The original proposal by [12] included an *inconsistency* operator $\bullet$ that helps express that a formula is *inconsistent*. In many logics in the LFI family, this operator is the dual of a consistency operator $\circ$—with $\circ\phi$ expressing that ϕ is consistent. Since [20], the use of $\circ$ as a primitive has become standard in LFIs.

[3] LFIs include a wide variety of systems. Some of the, as the C-systems introduced by [12], lack a traditional truth-functional semantics, but have been given a *non-deterministic many-valued semantics* by [4]. Others, like the logics $\mathsf{LFI1}$ and $\mathsf{LFI2}$ from [10] and $\mathsf{LFI3}$ from [21], have a straightforward many-valued semantics, which bring them closer to the Kleene logics that we face in this paper.

valued case, the normality operator comes to coincide with the consistency operator by LFIs or with the determinednedness operator from LFUs—see Section 2, where we go through the connections between the three operators and we explain why we keep the normality operator distinct from the other two. Additionally we discuss the connections between the present approach, LFIs, and LFUs.

The paper proceeds as follows. Section 2 introduces the logics $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ and presents the main failures of classical inferences in these logics. Section 3 establishes a classical recapture result (Theorem 3.3) for $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ and presents some concrete examples of how classical inferences can be recaptured in these two logics. Section 4 introduces the approach by *minimal inconsistency* and compare it with our approach based on normality operators. Section 5 discusses the relations between our proposal and the tradition of LFIs and LFUs. In particular, we discuss the relations between Theorem 3.3 and the Derivability Adjustment Theorems from [11, 20], the relations between $\mathsf{LP}^{\circledast}$ and $\mathsf{LFI1}$, and some other issues. Section 6 sums up the results of the paper.

2 Extensions of Kleene logics with Normality Operators

In this section we define *normality operators* and we introduce the two logics $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$, whose language deploy normality operators. Also, we discuss the relations between these logics and the Kleene logics LP and K_3, as well as some relevant failures of classical laws and inference rules.

Preliminaries. In this paper we adopt a semantic angle, and we regard a logic S as a pair $(\mathcal{L}, \vDash_{\mathsf{S}})$, where $\mathcal{L}$ is a language and $\vDash_{\mathsf{S}}$ is the relation of (single-conclusion) S-consequence—that is, $\vDash_{\mathsf{S}} \subseteq \wp(\mathcal{L}) \times \mathcal{L}$. Given a nonempty set $\mathcal{T}$ of truth values and a logic S, we denote by $\mathcal{D}_{\mathsf{S}} \subseteq \mathcal{T}$ the set of *designated values* of S—that is, the values that must be preserved through the valid inferences of S. We define the relative notion of *logical consequence*—S-consequence—as follows:

Definition 2.1 (Logical Consequence in Many-valued Logic)
$\Sigma \vDash_{\mathsf{S}} \psi \;\Leftrightarrow\; V_{\mathsf{S}}(\Sigma) \subseteq V_{\mathsf{S}}(\psi)$

where $V_{\mathsf{S}}(\Sigma)$ is the set of those valuations v defined for S where $v(\phi) \in \mathcal{D}_{\mathsf{S}}$ for every $\phi \in \Sigma$. $\Sigma \vDash_{\mathsf{S}} \psi$ reads 'ψ is a S-consequence of Σ'. We write $\phi \vDash_{\mathsf{S}} \psi$ instead of $\{\phi\} \vDash_{\mathsf{S}} \psi$, and we write $\vDash_{\mathsf{S}} \psi$ instead of $\varnothing \vDash_{\mathsf{S}} \psi$ in order to express that ψ is a *tautology* of S. Now we give a semantical definition of the main logical operators of the paper:

Definition 2.2
For every many-valued logic S and valuation function n defined for S, a unary connective k is a *normality operator* iff (if and only if) it obeys the following truth clause:

$$n(k\phi) = 1 \;\Leftrightarrow\; n(\phi) \in \{0,1\} \text{ and } n(k\phi) = 0 \;\Leftrightarrow\; n(\phi) \notin \{0,1\}$$

Consistency, Determinedness and Normality Operators. The normality operator from Definition 2.2 is closely related to the main operators in LFIs and LFUs. In LFIs, a unary connective k is a *consistency operator* in a logic S iff [20] S is *paraconsistent*—that is, it fails *Ex Contradictione Quodlibet* (ECQ)—and k verifies the *Principle of Gentle Explosion* (PGE) in S, respectively:

$\phi, \neg\phi \nvDash_{\mathsf{S}} \psi$
$\phi, \neg\phi, k\phi \vDash_{\mathsf{S}} \psi$

In LFUs, a unary connective k is a *determinedness operator* in a logic S iff [28] S is *paracomplete*—that is, it fails the *Law of Excluded Middle* (LEM)—and k verifies the *Principle of Gentle Implosion* (GPI) in S, respectively:
$\nvDash_{\mathsf{S}} \phi \vee \neg\phi$
$\vDash_{\mathsf{S}} \phi \vee \neg\phi \vee k\phi$

It is straightforward to see that a *normality operator* satisfies PGE in a paraconsistent many-valued logic, and PGI in a paracomplete many-valued logic. This implies that, in the logics we consider here, the normality operator will collapse on one of the two operators above. This notwithstanding, we choose here to use the label 'normality operator', for two reasons. First, this allows us to refer to the operator in a unique way, no matter whether we are dealing with the paraconsistent or paracomplete case. This has practical advantages, since here we face both cases. Second, the coincidence between the three operators is lost if more than three values are at stake. In particular, the consistency operator $\circ'$ by [22] and the determinedness operator by [5] are defined in four-valued settings that are *paraconsistent* due to a non-classical designated value (b), and *paracomplete* due to a non-classical undesignated value (n). Thus, although the normality operator collapses on the consistency (or determinedness) operator is three-values are considered, it proves a distinct operator if more than three values are at stake.[4]

Syntax. Given a nonempty denumerable set $\mathcal{P} = \{p, q, r \dots\}$ of propositional variables, the language $\mathcal{L}_1(\mathcal{P})$ of $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ is defined by the following Backus-Naur form (BNF):

$$\phi ::= p \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \circledast\phi$$

where $p \in \mathcal{P}$, connectives $\neg$, $\vee$, $\wedge$ receive their usual informal readings (negation, disjunction, conjunction), and $\circledast\phi$ reads 'ϕ has a classical truth value'. We define $\phi \supset \psi$ as an abbreviation for $\neg\phi \vee \psi$. We denote *sets* of *arbitrary formulas* in $\mathcal{L}_1(\mathcal{P})$ by Δ, Γ, Σ, ... and we omit reference to $\mathcal{P}$ when possible. $var(\phi)$ is the set of propositional variables occurring in $\phi \in \mathcal{L}_1$; we use $var(\Sigma)$ as short for $\bigcup_{\phi\in\Sigma} var(\phi)$.

Semantics. Variables in $\mathcal{P}$ are assigned a *truth value* from the triple $\{0, \frac{1}{2}, 1\}$ by a valuation function $\nu : \mathcal{P} \longmapsto \{0, \frac{1}{2}, 1\}$. The function is generalized to arbitrary formulas as follows:
Definition 2.3
A valuation $\nu : \mathcal{L}_1 \longmapsto \{0, \frac{1}{2}, 1\}$ is the unique extension of $\nu : \mathcal{P} \longmapsto \{0, \frac{1}{2}, 1\}$ that is induced by Table 1.[5]

We denote by $\mathcal{V}$ the set $\{\nu, \nu', \nu'', \dots\}$ of valuations defined in accordance with Definition 2.3. The *designated values* of a logic S are the truth values that the logic

[4] The label 'classicality operator' is also used for operators that rule out *both* inconsistency and undeterminedness—see for instance [22]. We prefer the label 'normality operator', since it makes a more direct connection with the 'normality story' that motivates classical recapture.

[5] We are including $\supset$ in Table 1, for the sake of simplicity. This will help track failures of Modus Ponens, Tollens, and Transitivity of $\supset$ in LP—see below.

TABLE 1

	$\neg\phi$	$\circledast\phi$
1	0	1
$\frac{1}{2}$	$\frac{1}{2}$	0
0	1	1

$\phi \vee \psi$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\phi \wedge \psi$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\phi \supset \psi$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1

preserves through valid inferences. In particular, $\mathcal{D}_{\mathsf{K}_3^{\circledast}} = \{1\}$ is the set of *designated values* of K_3, and $\mathcal{D}_{\mathsf{LP}^{\circledast}} = \{\frac{1}{2}, 1\}$ is the set of designated values of $\mathsf{LP}^{\circledast}$.

The operator $\circledast$ obeys Definition 2.2, thus qualifying as a *normality operator*. Specific interpretations of $\circledast\phi$ depends on the logic in question. Given $\mathcal{D}_{\mathsf{LP}^{\circledast}} = \{\frac{1}{2}, 1\}$ and the behavior of $\wedge$, $\neg$ and $\circledast$ from Table 1, we have that $\mathcal{V}_{\mathsf{LP}^{\circledast}}(\circledast\phi) = \{\nu \in \mathcal{V} \mid \nu(\phi \wedge \neg\phi) = 0\}$. As a consequence, in $\mathsf{LP}^{\circledast}$, $\circledast\phi$ reads 'ϕ is *consistent*', and $\circledast$ works as a *consistency operator* like the one by [10]. This is relevant since some contradictions are *satisfiable* in $\mathsf{LP}^{\circledast}$, as we shall see below by Fact 2.6 and paraconsistency of LP.

By contrast, $\mathcal{D}_{\mathsf{K}_3^{\circledast}} = \{1\}$ implies that $\mathcal{V}_{\mathsf{K}_3^{\circledast}}(\circledast\phi) = \{\nu \in \mathcal{V} \mid \nu(\phi \vee \neg\phi) = 1\}$; that is, in $\mathsf{K}_3^{\circledast}$, $\circledast$ works as a *determinedness* operator, with $\circledast\phi$ stating that ϕ is *determined*. This is relevant, since from Fact 2.6 and paracompleteness of K_3 imply that $\mathsf{K}_3^{\circledast}$ is itself paracomplete (see below).

2.1 Kleene Logics

We obtain the language $\mathcal{L}_2$ of Kleene logics by restricting $\mathcal{L}_1$ to those formulas where operator $\circledast$ does not occur. The valuation function for interpreting $\mathcal{L}_2$ in a Kleene logic obtains accordingly:

DEFINITION 2.4 (Valuations for Kleene logics)
A valuation $u : \mathcal{L}_2 \longmapsto \{0, \frac{1}{2}, 1\}$ is the unique extension of $\nu : \mathcal{P} \longmapsto \{0, \frac{1}{2}, 1\}$ that is induced by the relevant components of Table 1.

We denote by $\mathcal{U}$ the set $\{u, u', u'', \dots\}$ of valuations defined in accordance with Definition 2.4. The logic of paradox LP can be though of as the restriction of $\mathsf{LP}^{\circledast}$ to $\mathcal{L}_2$, and the strong Kleene logic K_3 can be thought as the restriction of $\mathsf{K}_3^{\circledast}$ to $\mathcal{L}_2$. This implies that $\mathcal{D}_{\mathsf{LP}^{\circledast}} = \mathcal{D}_{\mathsf{LP}}$ and $\mathcal{D}_{\mathsf{K}_3^{\circledast}} = \mathcal{D}_{\mathsf{K}_3}$.

Relations between Kleene Logics and logics $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$. In LP, *no formula ϕ can express that a formula ψ is consistent*, and, in K_3, *no formula ϕ can express that a formula ψ is undetermined*:

FACT 2.5
For every $\phi, \psi \in \mathcal{L}_2$:
$u(\phi) \in \mathcal{D}_{\mathsf{LP}} \not\Rightarrow u(\psi) \in \{0, 1\}$
$u(\phi) \in \mathcal{D}_{\mathsf{K}_3} \not\Rightarrow u(\psi) = \frac{1}{2}$

By contrast, in $\mathsf{LP}^{\circledast}$, we have that $\nu(\circledast\phi) \in \mathcal{D}_{\mathsf{LP}^{\circledast}} \Rightarrow \nu(\phi) \in \{0, 1\}$, and in K_3, we have that $\nu(\neg \circledast \phi) \in \mathcal{D}_{\mathsf{K}_3^{\circledast}} \Rightarrow \nu(\phi) = \frac{1}{2}$, by Definition 2.2. Thus, $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ brings a real increase in the expressive power of LP and K_3. Fact 2.6 below clarifies what this increase amounts to, exactly. In particular, $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ are *conservative extensions* of LP and K_3, respectively:[6]

Fact 2.6
Given $\mathsf{S} \in \{\mathsf{LP}, \mathsf{K}_3\}$ and $\mathsf{S}^{\circledast} \in \{\mathsf{LP}^{\circledast}, \mathsf{K}_3^{\circledast}\}$, if $\Sigma \cup \{\psi\} \subseteq \mathcal{L}_2$, then the following holds:

$$\Sigma \vDash_{\mathsf{S}^{\circledast}} \psi \Leftrightarrow \Sigma \vDash_{\mathsf{S}} \psi$$

Failures of classical laws and inferences. Kleene logics are *weaker* than classical logic CL:[7] they fail some classical laws or rules of inferences. Here is a list of their most relevant failures:

$\nvDash_{\mathsf{K}_3} \phi$, where ϕ is a classical tautology	No TAUT
$\nvDash_{\mathsf{K}_3} \phi \vee \neg\phi$	Failure of LEM
$\phi \wedge \neg\phi \nvDash_{\mathsf{LP}} \psi$	Failure of ECQ
$\vDash_{\mathsf{LP}} \phi$	for all and only classical tautologies ϕ
$\phi, \phi \supset \psi \nvDash_{\mathsf{LP}} \psi$	Failure of MP
$\neg\psi, \phi \supset \psi \nvDash_{\mathsf{LP}} \neg\phi$	Failure of MT
$\phi \supset (\psi \wedge \neg\psi) \nvDash_{\mathsf{LP}} \neg\phi$	Failure of RAA
$\phi \supset \psi, \psi \supset \zeta \nvDash_{\mathsf{LP}} \phi \supset \zeta$	Failure of Tr $\supset$
$\nvDash_{\mathsf{K}_3} \phi \supset \phi$	Failure of the Law of Identity

Due to Fact 2.6, the above failures extend to $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$. Thus, $\mathsf{LP}^{\circledast}$ is a paraconsistent logic and $\mathsf{K}_3^{\circledast}$ is a paracomplete logic. A major difference with LP and K_3, however, is that $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ can *recapture* classical inferences and laws. It is to this topic that we now turn.

We define the valuation function of *classical logic* CL as restriction of u to $\{0,1\}$. We define $\mathcal{U}_{\mathsf{CL}} = \{u \in \mathcal{U} \mid u(\phi) \in \{0,1\}$ for every $\phi \in \mathcal{L}_2\}$ and $\mathcal{D}_{\mathsf{CL}} = \{1\}$. This yields $\mathcal{U}_{\mathsf{CL}} \subset \mathcal{U}$. The set $\mathcal{U}_{\mathsf{CL}}(\Sigma)$ of *classical models* of $\Sigma \subseteq \mathcal{L}_2$ is defined as $\{u \in \mathcal{U}_{\mathsf{CL}} \mid u(\phi) = 1$ for every $\phi \in \Sigma\}$. The standard definition of *classical consequence* $\vDash_{\mathsf{CL}}$ just follows from the above and Definition 2.1, $\mathcal{D}_{\mathsf{CL}} = \{1\}$.

3 Classical Recapture and Normality Operators

Kleene logics have been applied to a wide spectrum of phenomena, including *partial functions* [17, 18], *partial information* [1], *logic programs* [14] (K_3), *semantical* and *set-theoretical paradoxes* [13, 19, 23, 25], and *vagueness* [26, 27] (K_3 and LP). To be sure, they are not the only many-valued logics to be applied to the phenomena above. For instance, the so-called weak Kleene logics K_3^w by [8] and PWK by [16] have been applied to set-theoretical paradoxes [8] and [16], semantical paradoxes, vagueness, ambiguity and denotational failure [16]. Also, the four-valued logic FDE has been designed to reason about information that can be *partial* (undetermined) or *overabundant* (inconsistent).

The 'normality story' we sketched in Section 1 implies that we may want to use classical reasoning *when possible.* The problem of how this can be done is the problem of *classical recapture.*

[6] A conservative extension of a logic S is a superlogic S' such that (i) the language of S' extends the language of S, (ii) any valid formula or rule expressed in the language of S is already in S.

[7] That is, $\Sigma \vDash_{\mathsf{S}} \psi \Rightarrow \Sigma \vDash_{\mathsf{CL}} \psi$ for $\mathsf{S} \in \{\mathsf{K}_3, \mathsf{LP}\}$ and $\Sigma \vDash_{\mathsf{CL}} \psi \not\Rightarrow \Sigma \vDash_{\mathsf{S}} \psi$.

3.1 Classical Recapture

The intuitive idea of classical recapture is that *classical reasoning* can somehow be deployed in a weaker many-valued logic S. The very starting point of classical recapture is that S itself do not offer the tools to recapture classical consequence. Thus, a further logic S' needs enter the stage. In particular, the role of S' is to specify *at which conditions we may infer the* classical *consequences of a given premise-set* Σ.

In order for this to be the case, we need S' to be *stronger* than S. This can be done in two ways [2]: either (a) we keep the initial language $\mathcal{L}$ fixed and define S' as $(\mathcal{L}, \vDash_{\mathsf{S}'})$, or (b) we increase the expressive power of S and define S' as a *conservative extension* $(\mathcal{L}', \vDash_{\mathsf{S}'})$ of S. The two options naturally suggest a formal definition of classical recapture:

Problem 3.1 (Classical Recapture)
For every many-valued logic S *weaker than* CL *and set* $\Sigma \cup \{\psi\} \subseteq \mathcal{L}$ *of formulas, the problem of classical recapture is the problem of finding a consequence relation* $\vDash_{\mathsf{S}'}$ *and a set* Σ' *of formulas such that:*

(i) $\quad \Sigma \vDash_{\mathsf{CL}} \psi \;\Leftrightarrow \Sigma' \vDash_{\mathsf{S}'} \psi$;
(ii) $\quad \Sigma \subseteq \Sigma'$;
(iii) $\quad \Sigma \vDash_{\mathsf{S}} \psi \Rightarrow \Sigma \vDash_{\mathsf{S}'} \psi$;
(iv) $\quad \Sigma' \subseteq \mathcal{L}$ or $\Sigma' \subseteq \mathcal{L}'$ for some extension $\mathcal{L}'$ of $\mathcal{L}$.

Accordingly, every solution to Problem 3.1 will have the form imposed by condition (i) above, and will fulfill conditions (ii)—(iv). If S' satisfies condition (i)—(iv), we will call it a *classical recapture logic* for S.

Also, we might want to recapture the inference of a many-valued logic S within a weaker logic S'. This can be easily done by a natural adjustment of the definition of Problem 3.1, thus yielding a more general *problem of* S*-recapture*. We will face one such case in Section 4.

To the best of our knowledge, none has given a formal definition of the *problem of classical recapture* before. With our present definition, we hope we can help settle a reasonable package of formal requirements to impose on proposals that aim at providing techniques for classical recapture.

The approach by [24] fall under option (a) above; extension of a logic S with normality operators fall under option (b). We go to this approach now, and we show that $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ are *classical recapture logics* for LP and K_3.

3.2 Recapture via Normality

Here we establish a recapture result based of $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$. First, we need an auxiliary notion:

Definition 3.2 (Normal Counterpart)
Given a set $\Sigma \subseteq \mathcal{L}_2$, we say that the set $\Sigma^{\circledast} = \{\circledast\phi \in \mathcal{L}_1 \mid \phi \in \Sigma\}$ is the *normal counterpart of* Σ in $\mathcal{L}_1$.

Informally, the normal counterpart of a set of formulas is the set obtained by replacing each $\phi \in \Sigma$ with $\circledast\phi$. Thus, the normal counterpart of $\{\phi, \phi \supset \psi\}$ is $\{\circledast\phi, \circledast(\phi \supset \psi)\}$.

Since $\{\phi\}^{\circledast}$ is $\{\circledast\phi\}$, we will just denote by $\circledast\phi$ the normal counterpart of a singleton $\{\phi\}$. If both Σ and $\Sigma^{\circledast}$ appear as premises, this implies that the formulas in Σ are *classically true*. Now we can establish our *classical recapture theorem* for $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$:

THEOREM 3.3 (Recapture via Normality)
If $\Sigma \cup \{\psi\} \subseteq \mathcal{L}_1$, then:

$$\Sigma \vDash_{\mathsf{CL}} \psi \;\Leftrightarrow\; \begin{cases} \Sigma^{\circledast}, \Sigma \vDash_{\mathsf{LP}^{\circledast}} \psi \\ \Sigma, \circledast\psi \vDash_{\mathsf{K}_3^{\circledast}} \psi \end{cases}$$

PROOF. We start with $\mathsf{LP}^{\circledast}$. We prove that, if $\Sigma \cup \{\psi\} \subseteq \mathcal{L}_2$, the following are equivalent:

1. $\mathcal{V}_{\mathsf{LP}^{\circledast}}(\Sigma \cup \Sigma^{\circledast}) \subseteq \mathcal{V}_{\mathsf{LP}^{\circledast}}(\psi)$
2. $\{u \in \mathcal{U}_{\mathsf{LP}}(\Sigma) \mid u[\Sigma] = 1\} \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$
3. $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{CL}}(\psi)$

$(1 \Rightarrow 2)$. The implication follows from the left-to-right direction of Fact 2.6, $\nu \in \mathcal{V}(\Sigma \cup \Sigma^{\circledast}) = \{\nu \in \mathcal{V} \mid \nu[\Sigma] = 1\}$, and $\{u \in \mathcal{U}_{\mathsf{LP}}(\Sigma) \mid u[\Sigma] = 1\} \subseteq \mathcal{U}_{\mathsf{LP}}(\Sigma)$.
$(2 \Rightarrow 3)$. From the definition of $\mathcal{U}_{\mathsf{CL}}$ and $\mathcal{U}_{\mathsf{CL}}(\Sigma)$, we have that $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \{u \in \mathcal{U}_{\mathsf{S}}(\Sigma) \mid u[\Sigma] = 1\}$. If $\{u \in \mathcal{U}_{\mathsf{S}}(\Sigma) \mid u[\Sigma] = 1\} \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$, we have $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$. From this, it follows that $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{CL}}(\psi)$—otherwise, we would have $\{u \in \mathcal{U}_{\mathsf{S}}(\Sigma) \mid u[\Sigma] = 1\} \cap \{u \in \mathcal{U}_{\mathsf{S}} \mid u(\psi) = 0\}$, which contradicts $\{u \in \mathcal{U}_{\mathsf{S}}(\Sigma) \mid u[\Sigma] = 1\} \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$.

$(3 \Rightarrow 2)$. Assume $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{CL}}(\psi)$, and suppose that there is a $u \in \mathcal{U}$ such that $u[\Sigma] = 1$ and $u(\psi) = 0$. We can build a corresponding classical model $u' \in \mathcal{U}_{\mathsf{CL}}$ such that $u'[\Sigma] = u[\Sigma]$, $u'(\psi) = u(\psi)$, and $u(p) \neq \frac{1}{2}$ for every $p \in var$. We have $u'[\Sigma] = 1$ and $u'(\psi) = 0$, which contradicts the initial assumption.
$(2 \Rightarrow 1)$. The implication follows from the fact that, for every $\phi \in \mathcal{L}_2$ and $\nu \in \mathcal{V}$, we can build a valuation $u \in \mathcal{U}$ such that $u(p) = \nu(p)$. Since $\nu \in \mathcal{V}(\Sigma \cup \Sigma^{\circledast}) = \{\nu \in \mathcal{V} \mid \nu[\Sigma] = 1\}$, this implies that, if $\{u \in \mathcal{U}_{\mathsf{S}}(\Sigma) \mid u[\Sigma] = 1\} \subseteq \mathcal{U}_{\mathsf{S}}(\psi)$, then $\mathcal{V}_{\mathsf{S}}(\Sigma \cup \Sigma^{\circledast}) \subseteq \mathcal{V}_{\mathsf{S}}(\psi)$. This proves $1 \Leftrightarrow 3$, which suffices to prove $\Sigma, \Sigma^{\circledast} \vDash_{\mathsf{LP}^{\circledast}} \psi \;\Leftrightarrow\; \Sigma \vDash_{\mathsf{CL}} \psi$.

We go to $\mathsf{K}_3^{\circledast}$. We prove that, if $\Sigma \cup \{\psi\} \subseteq \mathcal{L}_2$, the following are equivalent:

1. $\mathcal{V}_{\mathsf{K}_3^{\circledast}}(\Sigma \cup \{\circledast\psi\}) \subseteq \mathcal{V}_{\mathsf{K}_3^{\circledast}}(\psi)$
2. $(\mathcal{U}_{\mathsf{K}_3}(\Sigma \cup \{\psi \vee \neg\psi\}) \subseteq \mathcal{U}_{\mathsf{K}_3}(\psi)$
3. $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{CL}}(\psi)$

$(1 \Rightarrow 2)$. The implication follows from the left-to-right direction of Fact 2.6, $\mathcal{V}_{\mathsf{K}_3^{\circledast}}(\Sigma \cup \{\circledast\psi\}) = \{\nu \in \mathcal{V} \mid \nu[\Sigma \cup \{\psi \vee \neg\psi\}] = 1\}$, and $\{u \in \mathcal{U} \mid u[\Sigma \cup \{\psi \vee \neg\psi\}] = 1\} = \mathcal{U}_{\mathsf{K}_3}(\Sigma \cup \{\psi \vee \neg\psi\})$.
$(2 \Rightarrow 3)$. Assume $(\mathcal{U}_{\mathsf{K}_3}(\Sigma \cup \{\psi \vee \neg\psi\}) \subseteq \mathcal{U}_{\mathsf{K}_3}(\psi)$. From $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{K}_3}(\Sigma \cup \{\psi \vee \neg\psi\})$, we have $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{K}_3}(\psi)$. Suppose $\mathcal{U}_{\mathsf{CL}}(\Sigma) \cap \mathcal{U}_{\mathsf{CL}}(\psi) \neq \varnothing$. This means that there is a $u \in \mathcal{U}_{\mathsf{CL}}(\Sigma)$ such that $u(\psi) = 0$. Since $\mathcal{U}_{\mathsf{CL}} \subseteq \mathcal{U}_{\mathsf{K}_3}$, and $\mathcal{D}_{\mathsf{CL}} = \mathcal{D}_{\mathsf{K}_3}$ we have that $u(\psi) = 0$ for some $u \in \mathcal{U}_{\mathsf{K}_3}(\Sigma)$. But this contradicts the initial hypothesis.

$(3 \Rightarrow 2)$. From $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{CL}}(\psi)$, $\mathcal{U}_{\mathsf{CL}} \subseteq \mathcal{U}_{\mathsf{K}_3}$, and $\mathcal{D}_{\mathsf{CL}} = \mathcal{D}_{\mathsf{K}_3}$, it follows that $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{K}_3}(\psi)$. Suppose there is a $u \in \mathcal{U}_{\mathsf{K}_3}(\Sigma \cup \{\psi \vee \neg\psi\})$ such that $u(\psi) \notin \mathcal{D}_{\mathsf{K}_3}$.

Since $u(\psi \vee \neg\psi) = 1$, we have $u(\psi) = 0$. We can then build a model $u' \in \mathcal{U}$ such that $u'(p) \neq \frac{1}{2}$, $u'(\Sigma) = u(\Sigma)$, and $u'(\psi) = u(\psi)$. But this contradicts $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{CL}}(\psi)$.
$(2 \Rightarrow 1)$. From the left-to-right direction of Fact 2.6, we have $\Sigma, \psi \vee \neg\psi \vDash_{\mathsf{K}_3} \psi \Rightarrow \Sigma, \psi \vee \neg\psi \vDash_{\mathsf{K}_3^{\circledast}} \psi$. From this and $u(\psi \vee \neg\psi) = 1 \Leftrightarrow u(\circledast\psi) = 1$, we have $\Sigma, \psi \vee \neg\psi \vDash_{\mathsf{K}_3} \psi \Rightarrow \Sigma, \circledast\psi \vDash_{\mathsf{K}_3^{\circledast}} \psi$. This proves $1 \Leftrightarrow 3$, which in turn suffices to prove $\Sigma, \circledast\psi \vDash_{\mathsf{K}_3^{\circledast}} \psi \Leftrightarrow \Sigma \vDash_{\mathsf{CL}} \psi$. ∎

Remark 3.4
We can recapture classical reasoning in $\mathsf{LP}^{\circledast}$ by assuming that the premise-set is consistent, and in $\mathsf{K}_3^{\circledast}$ by assuming that the conclusion is consistent. The statements in the theorem have the form required by condition (*i*) in Problem 3.1 and $\Sigma \cup \Sigma^{\circledast}$ and $\Sigma \cup \{\circledast\psi\}$ satisfy condition (*iv*). Notice that condition (*ii*) is fulfilled, and condition (*iii*) is fulfilled—due to Fact 2.6. Thus, Theorem 3.3 provides a solution to Problem 3.1. In particular, $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ are *classical recapture logics* for LP and K_3, respectively: in using them, we can deploy all the inferential power of the two Kleene logics, and we can specify conditions at which we can reason classically.

Let us see how *recapture via normality* recovers the classical inferences or laws that fail in LP and K_3—see Section 2:

$$(\phi \wedge \neg\phi), \circledast(\phi \wedge \neg\phi) \vDash_{\mathsf{LP}^{\circledast}} \psi \quad \text{Recapture of ECQ}$$

Notice that recapture of ECQ is just another version of PGE. Similarly, LEM and every classical tautology can be recaptured in $\mathsf{K}_3^{\circledast}$:

$\circledast\phi \vDash_{\mathsf{K}_3^{\circledast}} \phi \vee \neg\phi$	Recapture of LEM
$\circledast\phi \vDash_{\mathsf{K}_3^{\circledast}} \phi$, where ϕ is a classical tautology	Recapture of TAUT

Again, recapture of LEM is just another version of PGI. Recapture of the Law of Identity is an instance of Recapture of TAUT:

$$\circledast\phi \vDash_{\mathsf{K}_3^{\circledast}} \phi \supset \phi \qquad \text{Recapture of Law of Identity}$$

Finally, the following inferences are valid in $\mathsf{LP}^{\circledast}$ if all the formulas involved belong to $\mathcal{L}_2$:

$\phi, \phi \supset \psi, \circledast\phi, \circledast(\phi \supset \psi) \vDash_{\mathsf{LP}^{\circledast}} \psi$	Recapture of MP
$\neg\psi, \phi \supset \psi, \circledast\neg\psi, \circledast(\phi \supset \psi) \vDash_{\mathsf{LP}^{\circledast}} \neg\phi$	Recapture of MT
$\phi \supset (\psi \wedge \neg\psi), \circledast(\phi \supset (\psi \wedge \neg\psi)) \vDash_{\mathsf{LP}^{\circledast}} \neg\phi$	Recapture of RAA
$\phi \supset \psi, \psi \supset \zeta, \circledast(\phi \supset \psi), \circledast(\psi \supset \zeta), \vDash_{\mathsf{LP}^{\circledast}} \phi \supset \zeta$	Recapture of Tr $\supset$

4 Minimal Inconsistency and Recapture via Normality

Minimal inconsistency [24] applies to LP and it is based on the insight that, given the premises from which we have to reason, we should assume that the situation is *no more inconsistent* that we are forced to assume by the syntactic structure of the premises. Below, we introduce minimal inconsistency, we compare it with recapture via normality and we discuss what we consider benefits of the latter over the former.

Minimal Inconsistency. For every $u \in \mathcal{U}_{\mathsf{LP}}$, we define the set

$$u! = \{p \in \mathcal{P} \mid u(p \wedge \neg p) \in \mathcal{D}_{\mathsf{LP}}\}$$

of the propositional variables that form a contradiction and are assigned value $\frac{1}{2}$ in u [24, p.325]. The definition of $u!$ plays a precise role in minimally inconsistency, namely that of comparing valuations in terms of *how much inconsistent they are.* The comparison follows the intuitive principle that *the greater* $u!$ *is, the more* inconsistent u *is.* This intuition is encoded in Priest's definition of a *consistency ordering* [24, p.325]:[8]

Definition 4.1 (Consistency ordering)
$u \leq u' \;\Leftrightarrow\; u! \subseteq u'!$
(u is *less or equally inconsistent* than u' iff $u!$ is contained in $u'!$).

The strict part of the relation ($u < u'$) equates with $u! \subset u'!$. The relation $<$ ($\leq$) is a strict (weak) *partial order*; in particular, $<$ relates valuations in $\mathcal{U}$ from *more* to *less* consistent, to the effect that $u < u'$ expresses that u *is less inconsistent than* u'; this is the case when $u! \subset u'!$.

Definition 4.2 (Minimally inconsistent model, [24])
For every $u \in \mathcal{U}_{\mathsf{LP}}$, u is a *minimally inconsistent (mi-) model* of Σ iff $v \in \mathcal{U}_{\mathsf{LP}}(\Sigma)$ *and* $u' < u \Rightarrow u' \notin \mathcal{U}_{\mathsf{LP}}(\Sigma)$ (iff any *less* inconsistent u' falsifies Σ).

In a nutshell, the mi-models of Σ are the valuations $u \in \mathcal{U}$ satisfying both (1) $u \in \mathcal{U}_{\mathsf{LP}}(\Sigma)$, and (2) $u(p) = \frac{1}{2} \Rightarrow p \wedge \neg p \in \mathsf{sub}(\Sigma)$.[9] We denote by $\mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma)$ the set $\{u \in \mathcal{U}_{\mathsf{LP}}(\Sigma) \mid u' < u \Rightarrow u' \notin \mathcal{U}_{\mathsf{LP}}(\Sigma)$ for every $u' \in \mathcal{U}_{\mathsf{LP}}\}$ of the *mi-models of* Σ. *Minimally Inconsistent Consequence* is defined as preservation of *designated values* in mi-models [24, p.325]:

Definition 4.3 (Minimally Inconsistent Consequence)
$\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi \;\Leftrightarrow\; \mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$
(ψ is a *minimally inconsistent (mi-) consequence* of Σ iff all the mi-models of Σ are *models* of ψ).[10]

mi-consequence is *non-monotonic*, since extending Σ with further information may lead to *loose* previously drawn consequences. As an example [24]: q is a mi-consequence of $\{p, p \supset q\}$, but it is not an mi-consequence of $\{p \wedge \neg p\} \cup \{p, p \supset q\}$, since there is a mi-model u of $\{p, p \supset q, p \wedge \neg p\}$ where $u(q) = 0$. Thus, we have $\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi \not\Rightarrow \Sigma, \phi \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$. The following is a fact that we will presuppose in what follows:[11]

Fact 4.4 (Fact 1, [24])
$\Sigma \vDash_{\mathsf{LP}} \psi \;\Rightarrow\; \Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi \;\Rightarrow\; \Sigma \vDash_{\mathsf{CL}} \psi$

Minimal inconsistency and recapture. Here we show that minimal inconistency offers a particular way to recapture of classical consequence. More precisely: *with minimal inconsistency, we can reason* classically *whenever the premise-set has* a classical model*; otherwise, we must resort to* LP*-reasoning*:

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[8] The definition we use here is just the special case of Priest's definition for propositional logic.

[9] Here $\mathsf{sub}(\phi)$ denotes the set of subformulas of ϕ. As for the rationale of clause (2): take $\Sigma = \{p, p \wedge \neg p\}$, and assume $u! = \{p, r\}$ for some $r \notin var(\Sigma)$. We have $u \in \mathcal{U}_{\mathsf{LP}}(\Sigma)$, but we also have another valuation u' of Σ such that $u'! = \{p\}$. We have $u' < u$. In a nutshell, *no minimally inconsistent model of* Σ *assigns a non-classical value to variables out of* $var(\Sigma)$.

[10] Notice that the valuations of ψ needs not be mi-models of ψ.

[11] Notice that none of the converses of these implications hold [24].

Proposition 4.5
$\mathcal{U}_{\mathsf{CL}}(\Sigma) \neq \varnothing \ \Rightarrow \ (\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi \ \Leftrightarrow \ \Sigma \vDash_{\mathsf{CL}} \psi)$

Proof. If $\mathcal{U}_{\mathsf{CL}}(\Sigma) \neq \varnothing$, then $\mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma) = \mathcal{U}_{\mathsf{CL}}(\Sigma)$, by Definition 4.2. As a consequence, $\mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{CL}}(\psi)$ iff $\mathcal{U}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$. But this equates with $\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi \Leftrightarrow \Sigma \vDash_{\mathsf{CL}} \psi$ *if* $\mathcal{U}_{\mathsf{CL}}(\Sigma) \neq \varnothing$. ∎

What if $\mathcal{U}_{\mathsf{CL}}(\Sigma) = \varnothing$*?* Well, we can have different cases here. For instance, the classically valid inference from $p, p \supset q, r \wedge \neg r$ to q is valid also in $\mathsf{LP}^{\mathrm{mi}}$. Indeed, the mi-models of $\{p, p \supset q, r \wedge \neg r\}$ assign 1 to p and $p \supset q$, and $\frac{1}{2}$ to r (only). As a consequence, such models will also assign 1 to q as well. By contrast, the inference from $p, p \supset q, p \wedge \neg p$ to q fails in $\mathsf{LP}^{\mathrm{mi}}$—see example above—while it is classically valid. The latter suffices to show that the three conditions $\mathcal{V}_{\mathsf{CL}}(\Sigma) = \varnothing$, $\mathcal{V}_{\mathsf{CL}}(\Sigma) \subseteq \mathcal{V}_{\mathsf{CL}}(\psi)$ and $\mathcal{V}_{\mathsf{LP}}(\Sigma) \nsubseteq \mathcal{V}_{\mathsf{LP}}(\psi)$ may be compatible with $\mathcal{V}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma) \nsubseteq \mathcal{V}_{\mathsf{LP}^{\mathrm{mi}}}(\psi)$, for an adequate choice of Σ and ψ. This implies that, if $\mathcal{V}_{\mathsf{CL}}(\Sigma) = \varnothing$, then condition (i) from Problem 3.1 may not be met.

In a nutshell, minimal inconsistency secures classical reasoning if our premise-set has a classical model, but it does not secure classical reasoning otherwise. We will then say that mi-consequence provides a *partial solution* to the problem of classical recapture, and that it gives rise to a *partial classical recapture logic* for LP.

A corollary of Theorem 3.3 and Proposition 4.5, is that *recapture* via normality *and* via minimal inconsistency *coincide when the premise-set has a classical valuation*:

Corollary 4.6
If $\Sigma \cup \{\psi\} \subseteq \mathcal{L}_2$ and $\mathcal{U}_{\mathsf{CL}}(\Sigma) \neq \varnothing$, then
$\Sigma^{\circledast}, \Sigma \vDash_{\mathsf{LP}^{\circledast}} \psi \ \Leftrightarrow \ \Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$

This is no surprise: all successful methods of *classical recapture* spots exactly the same premise-set/conclusion pairs, namely the pairs in the relation of classical consequence. Once again, a comparison of the two approaches must hinge on extra-logical features of the two approaches. We go to this issue in the discussion below (Section 4.2).

4.1 Addition of Information in a minimally inconsistent setting

Here, we present some facts that help get a more general view on the exact role played by addition of inconsistent information to a set of initial premises. An interesting consequence of such an interaction is that, if the information (that is, the set of propositional variables) of a given premise-set Σ is already contained in an $\mathsf{LP}^{\mathrm{mi}}$ conclusion of Σ, then the inference is also valid in LP.

Fact 4.7
For every $u \in \mathcal{U}$, if $u(\phi) \in \mathcal{D}_{\mathsf{LP}}$ and $\{p_1, \dots, p_k\} \subseteq var(\phi)$, then $u'(\phi) \in \mathcal{D}_{\mathsf{LP}}$ for every $u' \in \mathcal{U}$ such that $u'(p) = \frac{1}{2}$ for some (every) $p \in \{p_1, \dots, p_k\}$ and $u'(q) = u(q)$ for every $q \notin \{p_1, \dots, p_k\}$.

Proof. By a straightforward induction on the complexity of ϕ. ∎

Let $X_{u,\phi}$ be the set $\{u' \in \mathcal{U} \mid u[p_1, \dots, p_k] = \frac{1}{2}$ for some $\{p_1, \dots, p_k\} \subseteq var(\phi)$ and $u'(q) = u(q)$ for every $q \notin \{p_1, \dots, p_k\}\}$ of the valuations in $\mathcal{U}$ that coincide with u except for some variables in ϕ. A corollary of Fact 4.7 above is:

Corollary 4.8
For every $u \in \mathcal{U}$ and $u' \in X_{u,\phi}$, $u(\phi) \in \mathcal{D}_{\mathsf{LP}} \;\Rightarrow\; u'(\phi) \in \mathcal{D}_{\mathsf{LP}}$

Fact 4.7 and Corollary 4.8 help us prove two interesting facts about minimal inconsistency. First, addition of information that is *irrelevant* w.r.t. the initial premises does not force us to drop previously held consequences:

Fact 4.9
$var(\Sigma) \cap var(\Sigma') = \varnothing \;\Rightarrow (\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi \;\Rightarrow\; \Sigma, \Sigma' \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi)$

Proof. Assume $var(\Sigma) \cap var(\Sigma') = \varnothing$ and $\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$ as the initial hypothesis. We distinguish two cases. *Case 1)*: $\mathcal{U}_{\mathsf{CL}}(\Sigma') \neq \varnothing$. $\mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma \cup \Sigma') \subseteq \mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma)$. Since $\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$, this implies $\mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma \cup \Sigma') \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$, which in turn equates with $\Sigma, \Sigma' \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$. *Case 2)*: $\mathcal{U}_{\mathsf{CL}}(\Sigma') = \varnothing$. We distinguish two subcases. *Case 2.1)*: $var(\Sigma') \cap var(\psi) = \varnothing$. For every $u \in \mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma \cup \Sigma')$, there is a $u' \in \mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma)$ such that $u'(\psi) = u(\psi)$ and $u'(\phi) = u(\phi)$ for every $\phi \in \Sigma$. This implies that, if $\Sigma, \Sigma' \nvDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$, then $\Sigma \nvDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$. But this contradicts the initial hypothesis. *Case 2.2)*: $var(\Sigma') \cap var(\psi) = \varnothing$. From Fact 4.7, if follows that if $\mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma) \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$, then$\mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma \cup \Sigma') \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$. Given the initial assumption, this implies $\Sigma, \Sigma' \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$. ∎

Notice that addition of *relevant inconsistent information* is a *necessary* condition for monotonicity in minimal inconsistency, but it is not a *sufficient* condition. For instance, we have $p, p \supset q \vDash_{\mathsf{LP}^{\mathrm{mi}}} q$—see above—and also $p, p \supset q, q \wedge \neg q \vDash$. Indeed, all the LP-models satisfying $q \wedge \neg q$ (and *a fortiori*, all the mi-models of $\{p, p \supset q, q \wedge \neg q\}$), also satisfy q. The next fact reveals a class of cases where addition of relevant information does not force nonmonotonicity:

Proposition 4.10
$\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi \;\Rightarrow\; (var(\Sigma') \subseteq var(\psi) \;\Rightarrow\; \Sigma, \Sigma' \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi)$

Proof. Assume $\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$ and $var(\Sigma') \subseteq var(\psi)$ as the initial hypothesis. For every $\Sigma \subseteq \mathcal{L}$, take the set $\mathfrak{I}_\Sigma = \{p \in var(\Sigma) \mid u(p) \neq \frac{1}{2} \Rightarrow u(\phi) = 0 \text{ for some } \phi \in \Sigma\}$ of the variables from Σ whose inconsistency is indispensable for Σ to be designated. We have that $\mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma) = \{u \in \mathcal{U}_{\mathsf{LP}}(\Sigma) \mid u(p) = \frac{1}{2} \;\Leftrightarrow\; p \in \mathfrak{I}_\Sigma\}$. From this an the initial hypothesis, we have $\{u \in \mathcal{U}_{\mathsf{LP}}(\Sigma) \mid u(p) = \frac{1}{2} \;\Leftrightarrow\; p \in \mathfrak{I}_\Sigma\} \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$. From this and Corollary 4.8, it follows that $\{u \in \mathcal{U}_{\mathsf{LP}}(\Sigma \cup \Sigma') \mid u(p) = \frac{1}{2} \;\Leftrightarrow\; p \in \mathfrak{I}_{\Sigma \cup \Sigma'}\} \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$. Indeed, take a $u \in \mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma)$ such that $p \neq \frac{1}{2}$ for every $p \in var(\Sigma') \smallsetminus var(\mathfrak{I}_\Sigma)$, and take any $u' \in \mathcal{U}$ where $u'(p) = \frac{1}{2}$ for every $p \in \mathfrak{I}_{\Sigma \cup \Sigma'}$ and $u'(q) = u(q)$ for every $q \notin \mathfrak{I}_{\Sigma \cup \Sigma'}$. Clearly, $u' \in \mathcal{U}_{\mathsf{LP}^{\mathrm{mi}}}(\Sigma \cup \Sigma')$. Also, $u' \in X_{u,\psi}$. From this and Corollary 4.8, $u'(\psi) \in \mathcal{D}_{\mathsf{LP}}$. Since u' is arbitrary, we have $\{u \in \mathcal{U}_{\mathsf{LP}}(\Sigma \cup \Sigma') \mid u(p) = \frac{1}{2} \;\Leftrightarrow\; p \in \mathfrak{I}_{\Sigma \cup \Sigma'}\} \subseteq \mathcal{U}_{\mathsf{LP}}(\psi)$ and, as a consequence, $\Sigma, \Sigma' \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$. ∎

Finally, the next result individuates a condition at which mi-consequence coincides with LP-consequence:

Fact 4.11
$var(\Sigma) \subseteq var(\psi) \;\Rightarrow (\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi \;\Leftrightarrow\; \Sigma \vDash_{\mathsf{LP}} \psi)$

Proof. Assume $var(\Sigma) \subseteq var(\psi)$ and $\Sigma \vDash_{\mathsf{LP}^{\mathrm{mi}}} \psi$ as the initial hypothesis. ($\Leftarrow$) It follows immediately from Fact 4.4. ($\Rightarrow$) Remind the definition $\mathfrak{I}_\Sigma = \{p \in var(\Sigma) \mid u(p) \neq \frac{1}{2} \Rightarrow u(\phi) = 0 \text{ for some } \phi \in \Sigma\}$ of the set of the variables from Σ whose inconsistency is indispensable for Σ to be designated, and take the class $\mathfrak{K}_\Sigma = \{u \in \mathcal{U}_{\mathsf{LP}}(\Sigma) \mid u(q) = \frac{1}{2}$ for some $q \in var(\Sigma) \smallsetminus \mathfrak{I}_\Sigma\}$. Since $var(\Sigma) \subseteq var(\psi)$, from Corollary 4.8 we have that $u(\psi) \in \mathcal{D}_{\mathsf{LP}}$ for every $u \in \mathfrak{K}_\Sigma$. But since $\mathfrak{K}_\Sigma$ suffices to generate all the LP-models of Σ, this implies $\Sigma \vDash_{\mathsf{LP}} \psi$. ∎

4.2 Discussion on Recapture via Normality and Minimal Inconsistency

Here, we compare *recapture via normality* and *minimal inconsistency* on the ground of the symmetry (asymmetry) between evidence of inconsistency and consistency.

Suppose we are reasoning in $\mathsf{LP}^{\mathrm{mi}}$ and our assumptions is the premise-set $\{\phi, \phi \supset \psi\}$. By Proposition 4.5, we can reason classically from it. Now suppose that we hit 'against evidence for inconsistency' relative to $\{\phi, \phi \supset \psi\}$. Suppose that, after hitting against such evidence, we assume $\{\phi \wedge \neg\phi, \phi, \phi \supset \psi\}$. Again by Proposition 4.5, we will withdraw the conclusion ψ. Since we are facing evidence-based reasoning of an agent, we can take premise-sets to be *information sets*, from which the agent draws her conclusions. Information-sets can of course be extended by new information, and this is what happens in our example: we have extended our initial information-set by including the information that ϕ is inconsistent. In $\mathsf{LP}^{\mathrm{mi}}$, *information can so increase as to exclude that we are facing a* consistent *situation.* However, things are different if we receive evidence that we are positively facing a consistent situation. By Fact 2.5, no formula in LP^{+} implies that $\phi \wedge \neg\phi$ is *classically false. This implies that, in the setting by* [24], *no increase of information can ever* exclude *that we are facing an* inconsistent *situation.*

The above brings a puzzling asymmetry in, namely: when reasoning via $\mathsf{LP}^{\mathrm{mi}}$, we can increase our information by receiving evidence of *inconsistency*, but not by receiving evidence of *consistency*. As for the chance for evidence of consistency, we believe there are some reasonable examples: while the logical structure of the Liar sentence itself is evidence (quite conclusive) of an inconsistent situation, the logical structure of (say) 'I am playing football' is evidence (though maybe not conclusive) of a consistent situation. And we would like to state this. However, if our language is $\mathcal{L}_2$, we cannot. Also, take the application of paraconsistent logic to *vagueness*. Empirical information could give evidence of the fact that 'Jude Law is bald' (j) expresses a borderline case of baldness and is, then, an inconsistent sentence. However, suitable empirical information would also give evidence that 'Ben Kingsley is bald' (b) *does not* express a borderline case, and in this case, we would like to state that *that sentence is consistent*—that is, either classically true, or classically false.

The import of the above fact can be better grasped with an epistemic analogy: suppose that reasoners only make given assumptions (*i.e.*, assert the formulas in a premise-set) if they have received *persuasive* information—that is, information that leads you to attribute probability 1 to the formulas in question. Then, the asymmetry above equates with the fact that reasoners can be *sure* that an inconsistency is at stake, but they cannot be sure that the situation is perfectly consistent.

By contrast, recapture via normality does not bring any asymmetry in. Again, we can add $\phi \wedge \neg\phi$ after receiving evidence of inconsistency of ϕ. However, we can also add $\circledast\phi, \circledast(\phi \supset \psi)$ if we get evidence that we are facing a normal case. In particular, in $\mathsf{LP}^{\circledast}$ we can use $j \wedge \neg j$ to state that 'Jude Law is bald and not bald' (hence, a borderline case of baldness) and $b \wedge \circledast b$ to state that 'Ben Kingsley is bald and is not a borderline case of baldness.' By running the epistemic analogy again: with recapture via normality, reasoners can be *sure* that an inconsistency is at stake, but they can also be sure that the situation is perfectly consistent. Indeed, information can increase both because evidence of inconsistency and evidence of consistency.[12]

5 Extensions of Kleene Logics, Normality Operators, Logics of Formal Inconsistency and Undeterminedness

In this section, we go through the connections between $\mathsf{LP}^{\circledast}$ and LFIs, Theorem 3.3 and Derivability Adjustment Theorems [11, 20], definability issues and semantical closure.

Theorem 3.3 and Derivability Adjustment Theorems. Results similar to Theorem 3.3 appear in the tradition of LFIs, where they are usually referred to as *Derivability Adjustment Theorems*. [20, Theorem 3.11] establishes a similar *classical recapture theorem* for the paraconsistent logic bC [9], and [20, Theorem 3.46] proves a particular recapture that involves a satisfaction-preserving translation τ from CL to the paraconsistent logic Ci [20]. We briefly comment on the differences between Theorem 3.3 and these results.

First, *Derivability Adjustment Theorems* involve the derivability relation—whence the name—and their proofs display a proof-theoretical approach, while the proof of Theorem 3.3 relies on semantical considerations. Second, Theorem 3.11 from [20] relies on adding the normal counterpart of *some* set Δ of well-formed formulas, and not necessarily the normal counterpart of the premise-set Σ itself. Third, Theorem 3.46 from [20] relies on translating CL into Ci. By contrast, no translation is needed for our purposes. Fourth, the wide majority of *Derivability Adjustment Theorems* above apply to *paraconsistent logics*, and give no recipe for recapture in paracomplete logics, while Theorem 3.3 does. One exception is [11, Theorem 7.2.20], which applies to the paracomplete logic K_3^w from [8]. That result differs from Theorem 3.3 since it focuses on propositional variables from the conclusion, not on the conclusion itself. This focus is not necessary for Theorem 3.3. Fifth and final, none of the Derivability Adjustment Theorems comes with a comparison with other current recapture techniques—such as minimal inconsistency.

We believe that these differences speak for the relevance of our approach and results. In particular, the semantical methodology that we follow in this paper allows for an easier comparison with other approaches to recapture, and especially with *minimal inconsistency*, which has a strong semantical focus, and a relatively undeveloped proof

[12] Notice that $\mathsf{LP}^{\circledast}$ does not provide, per se, any *belief-revision* (or better, information-revision) *procedure*, contrary to $\mathsf{LP}^{\circledast}$ (Graham Priest brought this to our attention in private conversation. However, it is easy to extend the semantical and linguistic setting of $\mathsf{LP}^{\circledast}$ with operations analogous to *expansion*, *contraction* from AGM theory [3]. Once this is done, it is possible to use the resulting extension of $\mathsf{LP}^{\circledast}$ in order to capture belief-revision. In particular, the additional reasoning power would help up model situations where $\phi \wedge \neg\phi$ is included in the initial premise-set, and information that $\circledast\phi$ is released. In the basic $\mathsf{LP}^{\circledast}$ setting, $\{\phi \wedge \neg\phi, \circledast\phi\}$ has no model, but if we describe this situation via a revision-like operator, we will end up in a situation where $\phi \wedge \neg\phi$ and its consequences are dropped from the premise-set, in favor of $\circledast$. Of course, the new setting would also capture the opposite situation, where information of $\phi \wedge \neg\phi$ trumps previous information that $\circledast\phi$, via AGM-style revision.

theory. Also, our comparison with other approaches (Section 4) has highlighted the conceptual merits of recapture via normality. In turn, this shows that the approach followed by us and researchers in the LFI/LFU traditions is not only mathematically interesting, but also an appealing tool for conceptual applications (for instance, to revision of assumptions in presence of new information).

LP *and the Logic* LFI1 *of formal inconsistency.* The language $\mathcal{L}_3(\mathcal{P})$ of the *logic of formal inconsistency* LFI1 is defined by the following Backus-Naur form (BNF):

$$\phi ::= p \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \rightarrow \psi \mid \bullet\phi$$

The logic LFI1 has been first introduced by [10], and it obtains by setting $\mathcal{D}_{\mathsf{LFI1}} = \{\frac{1}{2}, 1\}$, and by extending a valuation function $n : \mathcal{P} \longmapsto \{0, \frac{1}{2}, 1\}$ to arbitrary formulas $\mathcal{L}_3$ in conformity with Table 3 below:

TABLE 2

	$\neg\phi$	$\bullet\phi$
1	0	0
$\frac{1}{2}$	$\frac{1}{2}$	1
0	1	0

$\phi \vee \psi$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\phi \wedge \psi$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\phi \rightarrow \psi$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

The following established fact that will be relevant below.

PROPOSITION 5.1
The following negation connective ~ is definable in $\mathsf{LP}^{\circledast}$ and LFI1:

$$\sim\phi :=_{\mathsf{LP}^{\circledast}} \circledast\phi \wedge \neg\phi \qquad\qquad \sim\phi :=_{\mathsf{LFI1}} \neg\bullet\phi \wedge \neg\phi$$

The next definability result clarifies the relations between $\mathsf{LP}^{\circledast}$ and LFI1 is established by the following proposition:

PROPOSITION 5.2
The following operators are definable in $\mathsf{LP}^{\circledast}$:

1*a*. $\bullet\phi :=_{\mathsf{LP}^{\circledast}} \neg\circledast\phi$ 1*b*. $\circledast\phi :=_{\mathsf{LFI1}} \neg\bullet\phi$
2. $\phi \rightarrow \psi :=_{\mathsf{LP}^{\circledast}} \sim\phi \vee \psi$

Definability of ~ as $\circledast\phi \wedge \neg\phi$ and $\bullet\phi$ as $\neg\circledast\phi$ is from [10]; definability of → in $\mathsf{LP}^{\circledast}$ is proved by [21].[13] Definability of → implies that a *detachable* conditional[14] is definable in $\mathsf{LP}^{\circledast}$. A consequence of Proposition 5.2 is that $\mathsf{LP}^{\circledast}$ and LFI1 are the same logic, and the Hilbert-style axiomatization of LFI1 by [10] extends to $\mathsf{LP}^{\circledast}$—with the obvious rephrasing of • into ¬⊛.

Since no current formalism in the LFU tradition is equivalent to $\mathsf{K}_3^{\circledast}$—to the best of our knowledge—we prefer to keep the label $\mathsf{LP}^{\circledast}$ instead of LFI1 for our paraconsistent

[13]The results from [10] and [21] are proved relative to the consistency operator ∘. Since ⊛ and ∘ are equivalent in any three-valued paraconsistent logic, these results immediately extend to ⊛ and $\mathsf{LP}^{\circledast}$.

[14]That is, a conditional that obeys MP.

logic, since this allows for a uniformity of presentation with $\mathsf{K}_3^{\circledast}$.[15]

Normality operator and semantical closure. A language $\mathcal{L}$ is *semantically closed* when it contains its own truth predicate Tr and names $\underline{\phi}$ for every formula $\phi \in \mathcal{L}$. Semantical closure is related to the derivation of paradoxes. One prominent example is the *Liar paradox*, consisting in the sentence 'λ iff λ is false'—in symbols, λ iff $\neg Tr(\underline{\lambda})$.[16] In CL, semantical closure and the resulting Liar paradox bring trivialization. LP and K_3 escape this, but things get more troublesome for *semantically closed* $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$: semantically closed $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ are *trivial*—see [5, 10] and [15], respectively.[17]

However, these facts concerning $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$ do not generalize to all many-valued logics including a *normality operator*. In particular, [5] proves that *infectious* logics endowed with a normality operator are *non-trivial*. These logic assign a non-classical value to any formula $\phi \in \mathcal{L}_2$ if and only if at least one subformula ψ of ϕ has a non-classical value assigned. The most prominent examples are the three-valued $\mathsf{K}_3^{w^{\circledast}}$ and $\mathsf{PWK}^{\circledast}$ from [8] and [16]. The former designates 1 only, and $\mathsf{PWK}^{\circledast}$ obtains by designating the third value as well. A consequence of the result by [5] is that the addition of a truth predicate Tr to $\mathsf{K}_3^{w^{\circledast}}$ and $\mathsf{PWK}^{\circledast}$ is harmless.

Whether $\mathsf{K}_3^{w^{\circledast}}$ and $\mathsf{PWK}^{\circledast}$ can be suitable logics for reasoning about truth, it is for a dedicated investigation to say. To the purposes of this paper, suffices it to notice that it is possible to have logics that are *semantically closed*, include *normality operators* and are *non-trivial*.

6 Conclusions

In this paper, we have introduced the logics $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$, which extend the Kleene logics LP and K_3 with a *normality operator*. This operator helps express that a formula as a classical truth value (namely, 0 or 1). We have established a *classical recapture result* (Theorem 3.3,) which secures that, if given conditions hold, we can deploy classical reasoning in $\mathsf{LP}^{\circledast}$ and $\mathsf{K}_3^{\circledast}$. We have then compared our approach with one established approach to classical recapture, namely *minimal inconsistency* by [24]. In particular, we have discussed their respective merits on the ground of their descriptive adequacy with respect to some phenomena concerning information. We have discussed the relations between $\mathsf{LP}^{\circledast}$ and the *Logics of Formal Inconsistency* from [10, 11, 20, 22], among normality operators and consistency and determinedness operators in the style of [10, 11, 20, 22, 28], and between normality operators and semantical closure.

[15] We do not go through a proof theory for $\mathsf{K}_3^{\circledast}$, especially for reasons of space and coherence with the main focus of the present paper. Notice, however, that a sound and complete natural deduction should not be difficult to obtain, given natural deduction for K_3 and the behavior of $\circledast$.

[16] Here, we are neutral on the exact nature of the 'iff'. Some logics, such as K_3, requires it to be a material biconditional; others, like LP, requires it to be interderivability or semantical equivalence.

[17] Notice that minimal inconsistency is not necessarily in a better situation, albeit for a very different reason. Indeed, as noticed by Hartry Field in private conversation, Curry's paradox has a classical model, and as a consequence, its trivializing conclusion can be inferred in $\mathsf{LP}^{\mathrm{mi}}$ by using a 'safe' instance of Modus Ponens.

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