Pluralities as nothing over and above*

Draft

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Consider some people: Melesha, Nadia, and Dylan. What is the relation between them—taken together—and the individual people they comprise? For example:

1. *Is Melesha one of them whenever they exist?*

   In other words, is it necessarily the case that if they exist, Melesha is one of them?

2. *Must Dylan exist whenever they exist?*

   In other words, is it necessarily the case that if they exist, Dylan exists?

Or consider a natural converse.

3. *Do they exist whenever each of Melesha, Nadia, and Dylan exists?*

   In other words, is it necessarily the case that if Melesha, Nadia, and Dylan each exists, they exist?

Standard treatments of the modal logic of plurals have typically addressed the first two kinds of question.[1] The third, however, is vitally important: positions in the philosophy of mathematics, philosophical logic, and metaphysics depend on its answer. The goal of this paper is to articulate a general conception of pluralities which answers these and many other related questions. It is based on the following simple claim, which I call the *nothing over and above conception* of pluralities.[2]

> *Some things are nothing over and above the individual things they comprise*

Here's the plan. In section 1, I make the nothing over and above conception precise. I show how it can be captured both model-theoretically and axiomatically. In section 2, I use this to shed light on debates in the philosophy of mathematics, philosophical logic, and metaphysics. Section 3 is a technical appendix.

*Thanks to Peter Fritz, Øystein Linnebo, Agustin Rayo, Alex Roberts, and Gabriel Uzquiano for helpful comments and discussion.

[1] See, for example, Uzquiano [2011] and Rumfitt [2005]. Williamson [2013] is an exception. See section 2 for discussion.

[2] This conception is arguably implicit in a lot of work on plurals—for example, in Boolos [1984], especially in his discussion of the cheerios—but is made explicit in Roberts [2016], Roberts [2019], and Linnebo [2016].

1 Pluralities as nothing over and above

The nothing over and above conception says that some things are nothing over and above the individual things they comprise. In other words, it says that there is *no difference* between some things taken together and those very same things taken individually. For some, this may seem like a mysterious statement, perhaps even meaningless; for others, like a truism, trivial and inferentially inert. I will show that neither reaction is correct: the nothing over and above conception is both tractable and has important consequences for a number of debates in philosophy. This section will focus on making the conception formally precise and the next on exploring its consequences.

First, some preliminaries. I will restrict my attention to claims in the language of plural modal logic. This language, $\mathcal{L}$, can be obtained from the language of first-order logic by adding a stock of plural variables $xx, yy, zz, ...$ etc., a relation symbol $\prec$ intended to express the relation that holds between an object and some things when it is one of them—so, "$x \prec xx$" is well-formed and read "it$_x$ is one of them$_{xx}$"—and a modal operator $\Diamond$ expressing metaphysical possibility.[3] For simplicity, I will allow the identity relation to be flanked by plural variables—so, "$xx = yy$" is well-formed.[4] My goal is thus to make the nothing over and above conception formally tractable in so far as it concerns claims in $\mathcal{L}$. For all I will say, it could have consequences beyond those claims.

Let $K = \langle W, D_1, D_2, I \rangle$ be an arbitrary Kripke model where D_1 is a domain function assigning a set of 'objects' to each world, D_2 a domain function assigning a set of 'pluralities' to each world, and I an interpretation function which interprets $\prec$ and the non-logical vocabulary of $\mathcal{L}$. Since it has no accessibility relation, K will validate the modal logic $\mathsf{S5}$. Given a plurality X, I will denote the set of things 'among' X at w: $I_w(X)$ (that is, $I_w(X) = \{x : w \vDash x \prec X\}$).[5] As it stands, no assumptions have yet been made about the behaviour of pluralities. There may be none—all $D_2(w)$ could be empty—or there may be many more pluralities in $D_2(w)$ than there are subsets of $\bigcup_{w \in W} D_1(w)$ and every one of them could be empty. In particular, K may not conform to the nothing over and above conception.

[3] See Linnebo [2017] for more details.

[4] Nothing hangs on this, however, and I will show in section 2 how identity between plural variables can be explicitly defined and thus eliminated.

[5] Since Kripke models are classical, they do not allow for indeterminacy. Although this simplifies matters considerably, it is a significant omission. It will turn out that the nothing over and above conception implies a plural comprehension principle in such models (see below). But it has been argued that some of its instances are undermined by indeterminacy in quantifiers or predicates (see, for example, [Yablo, 2006, p.151-152] and [Linnebo, 2016, p.670-673]). These arguments don't target the other components of the nothing over and above conception that I will identify, however, and so a modified form of the conception could perhaps survive those arguments with a suitably restricted comprehension principle. I will ignore this complication in what follows.

1.1 A model-theoretic characterisation

I will start with a model-theoretic characterisation of the nothing over and above conception. First, I will highlight four simple constraints that I argue should hold whenever a Kripke model conforms to the conception. I will argue, in other words, that the constraints are *sound* for the conception. I will then argue that they are *complete*: that any Kripke model satisfying the constraints conforms to the conception. The models conforming to the nothing over and above conception can then be characterised as precisely those satisfying the four constraints.

The first two constraints are perhaps the most interesting and concern the behaviour of pluralities across worlds.

Recall our people—Melesha, Nadia, and Dylan—and consider the first two questions I raised in the introduction, now in light of the nothing over and above conception. Is Melesha one of them whenever they exist? And, must Dylan exist whenever they exist? If they are nothing over and above the individual things they comprise and they comprise precisely Melesha, Nadia, and Dylan, then *nothing more* could be needed for them to comprise Melesha, Nadia, and Dylan than that they exist; and *nothing more* could be needed for Melesha, Nadia, and Dylan to exist than that they exist. There is no metaphysical gap between them and the individuals Melesha, Nadia, and Dylan. So, Melesha must be one of them whenever they exist and Dylan must exist whenever they exist.

The first constraint I want to highlight generalises this insight. It says that the individual things a plurality comprises will continue to be comprised by it and exist whenever it exists: where the plurality goes, so too do the things it comprises.[6] Formally:[7]

> Downward Dependence
>
> If $X \in D_2(w')$, then $I_w(X) \subseteq D_1(w') \cap I_{w'}(X)$

Now consider the third question in light of the nothing over and above conception. Do they exist whenever each of Melesha, Nadia, and Dylan exists? If they are nothing over and above

[6] To aid readability, I will frequently use the singular "plurality". Nothing hangs on this and can always be reformulated in terms of genuinely plural locutions.

[7] We could, if we like, break this constraint into two further constraints corresponding to each of the two initial questions. The first would say that a plurality comprises the same things whenever it exists,

> Downward Dependence$_1$
>
> If $X \in D_2(w')$, then $I_w(X) \subseteq I_{w'}(X)$

and the second would say that a plurality cannot exist without the things it comprises,

> Downward Dependence$_2$
>
> If $X \in D_2(w')$, then $I_w(X) \subseteq D_1(w')$

the individual things they comprise and they comprise precisely Melesha, Nadia, and Dylan, then *nothing more* could be needed for them to exist than that each of Melesha, Nadia, and Dylan exists. There is no metaphysical gap between the individuals Melesha, Nadia, Dylan and them. So, they must exist whenever each of Melesha, Nadia, and Dylan exists.

The second constraint I want to highlight generalises this insight. It says that a plurality exists whenever the things it comprises exist: where the things a plurality comprises go, so too does the plurality. Formally:[8]

Upward Dependence

If $\bigcup_{w \in W} I_w(X) \subseteq D_1(w')$, then $X \in D_2(w')$

The last two constraints I want to highlight are somewhat standard and concern the behaviour of pluralities within worlds. The first is an extensionality principle: it says that pluralities comprising the same things are identical. Clearly, if these things and those things are nothing over and above the individual things they comprise and these things comprise the same things as those, then these *are* those. Formally:

Extensionality

If $X, Y \in D_2(w)$ and $I_w(X) = I_w(Y)$, then $X = Y$.

Finally, there is a comprehension principle: it says that given a set of objects at a world, there is a plurality comprising precisely those objects. Clearly, if some things are nothing over and above the individual things they comprise and each individual thing in $\mathcal{X}$ exists, then *nothing more* could be needed for there to exist some things comprising the $\mathcal{X}$s. There is no metaphysical gap between the individual things in $\mathcal{X}$ and those very same things taken together. Formally:[9]

Comprehension

If $\mathcal{X} \subseteq D_1(w)$, then there is $X \in D_2(w)$ such that $\mathcal{X} = I_w(X)$

The four constraints are thus sound for the nothing over and above conception: any model that conforms to the conception must satisfy them. I will now argue that they are complete. But before I do that, I need a distinction.

[8]Given Downward Dependence, Upward Dependence comes to the simpler claim that if $X \in D_2(w)$ and $I_w(X) \subseteq D_1(w')$, then $X \in D_2(w')$. This is because, given Downward Dependence, when $X \in D_2(w)$, the things X comprises at w are the things it comprises in any world: $I_w(X) = \bigcup_{w' \in W} I_{w'}(X)$. I've chosen the slightly more complicated formulation in the main text because it does not rely on Downward Dependence in this way.

[9]It follows from Comprehension that there is an empty plurality. Nothing will depend on this, however. If we like, we can modify Comprehension so that it only applies to non-empty sets and require that every plurality is non-empty. See Linnebo [2017] for discussion of this point.

Although the above constraints tell us which things a plurality comprises when it exists—namely, by Downward Dependence, whatever it comprises in any world—they do not tell us what it comprises in worlds where it does not exist. Consider again our people—Melesha, Nadia, and Dylan—and take a world w in which Melesha and Nadia exist but Dylan does not. By Downward Dependence, they don't exist in w. But is Melesha still one of them in w?

It seems to me that the nothing over and above conception is silent on this question. Nevertheless, there appear to be three non-ad-hoc answers. First, when a plurality fails to exist, it comprises nothing. Formally:

Nothing

If $X \notin D_2(w)$, then $I_w(X) = \emptyset$

Second, it comprises everything it would otherwise comprise. Formally:

Everything

If $X \notin D_2(w)$, then $I_w(X) = \bigcup_{w' \in W} I_{w'}(X)$

Finally, it comprises the things it would otherwise comprise which also exist. Formally:

Existence

If $X \notin D_2(w)$, then $I_w(X) = \bigcup_{w' \in W} I_{w'}(X) \cap D_1(w)$

In what follows, I will assume that one of these answers is correct, though I will not take a stance on which.

To show that our four constraints are complete for the nothing over and above conception, I will employ a "squeezing" argument.[10] It rests on three claims which jointly establish that the informal notion of conforming to the nothing over and above conception coincides with the formal notion of satisfying the four constraints.

The first claim is that a Kripke model conforms to the nothing over and above conception only if it satisfies the constraints: in other words, conforming to the nothing over and above conception implies satisfying the constraints. This was established above. The second claim is that every model of a certain kind conforms to the nothing over and above conception: in other words, being a model of that kind implies conforming to the conception. The final claim is that every model satisfying the constraints is isomorphic to a model of that kind: in other words, satisfying the constraints implies being (isomorphic to) a model of that kind. The informal notion of conforming to the nothing over and above conception is thus sandwiched between two equivalent formal notions. It follows that all three notions coincide. I will now

[10] See, for example, Smith [2010].

argue for the last two claims.

For the second claim, consider a Kripke model $K = \langle W, D_1, I \rangle$ for a first-order language. As Boolos [1985] effectively showed, we can use K to interpret the language of plural modal logic without adding domains of pluralities. The idea is simple. We take a *plural assignment function* to be a relation R that associates with each singular variable "x" a unique object in $\bigcup_{w \in W} D_1(w)$ and with each plural variable "xx" 0 or more objects from $\bigcup_{w \in W} D_1(w)$. We then define truth at a world by adding the following clauses to the usual first-order ones.

- $w \vDash_R$ "$\exists xx\varphi$" $\quad\leftrightarrow\quad \exists R' =_{\text{“}xx\text{”}} R(\forall x(R'(\text{“}xx\text{”}, x) \rightarrow x \in D_1(w)) \wedge w \vDash_{R'}$ "φ"), where $R' =_{\text{“}xx\text{”}} R$ means that the plural assignment R' is just like R except perhaps in what it associates with "xx".
- $w \vDash_R$ "$xx = yy$" $\quad\leftrightarrow\quad \forall x(R(\text{“}xx\text{”}, x) \leftrightarrow R(\text{“}yy\text{”}, x))$

together with one of the following (depending on which of Nothing, Everything, or Existence we want to adopt),

(N) $w \vDash_R$ "$x \prec xx$" $\quad\leftrightarrow\quad R(\text{“}xx\text{”}, x) \wedge w \vDash_R$ "Exx"[11]

(Ev) $w \vDash_R$ "$x \prec xx$" $\quad\leftrightarrow\quad R(\text{“}xx\text{”}, x)$

(Ex) $w \vDash_R$ "$x \prec xx$" $\quad\leftrightarrow\quad R(\text{“}xx\text{”}, x) \wedge w \vDash_R$ "Ex"

Call this the *Boolos model* over K. In Boolos models, there is nothing more to the xx on a given assignment than the individual things assigned to the variable "xx". Boolos models are therefore precisely models in which pluralities are nothing over and above the individual things they comprise; they conform to the nothing over and above conception.

For the final claim, it is straightforward to prove that every Kripke model satisfying the four constraints together with one of Nothing, Everything, or Existence is isomorphic to the Boolos model over its first-order part (theorem 1). That completes the argument.

Model-theoretically, then, the nothing over and above conception breaks down into four components: Downward Dependence, Upward Dependence, Extensionality, and Comprehension. More precisely, given one of Nothing, Everything, or Existence, a model conforms to the nothing over and above conception just in case it satisfies those components.

Let me finish this section by noting an important feature of the model-theoretic explication of the nothing over and above conception: it implies that plural reality supervenes on first-order

[11] I use Exx as an abbreviation for $\exists yy(xx = yy)$ and similarly for Ex.

reality. More precisely, we can show that any models which (1) satisfy the four constraints—together with one of Nothing, Everything, or Existence—and (2) have the same first-order part are isomorphic (theorem 2). This means that on the model-theoretic explication, the nothing over and above conception provides a complete account of plural reality: keeping first-order reality fixed, any claim in $\mathcal{L}$ is either true in all of its models or false in all of them.

1.2 An axiomatic characterisation

I will now look at the extent to which the nothing over and above conception can be captured axiomatically. I will start by finding object language correlates of the model-theoretic constraints from the last section.

First, the successes. It is easy to see that Downward Dependence can be axiomatised by the following principle.[12]

(stability) $$x \prec xx \to \Box(Exx \to Ex \wedge x \prec xx)$$

More precisely, it is straightforward to show that a model satisfies Downward Dependence precisely when it validates stability.

And it turns out that given stability and a single instance of Comprehension, Upward Dependence and Extensionality can be jointly axiomatised by the following strong extensionality principle which says that pluralities comprising the same things across worlds are identical.

(ext) $$\Box\forall x(\Diamond(x \prec xx) \leftrightarrow \Diamond(x \prec yy)) \to xx = yy$$

More precisely, we can show that when a model validates:

$$\exists xx\forall x(x \prec xx \leftrightarrow \Diamond(x \prec yy))$$

it satisfies Downward Dependence, Upward Dependence, and Extensionality precisely when it

[12]Let me briefly explain the choice of terminology here. Just as Downward Dependence can be factored into Downward Dependence$_1$ and Downward Dependence$_2$, stability can be factored into the following two principles.

(i) $$x \prec xx \to \Box(Exx \to x \prec xx)$$

(ii) $$x \prec xx \to \Box(Exx \to Ex)$$

The literature is inconsistent on the terminology for these principles, especially the first, but there are roughly two camps. Parsons [1983] and Fine [1981] use "rigidity" for the set-theoretic version of (i). Parsons [1983] uses "full rigidity" for the conjunction of the set-theoretic versions of both, and suggests that they should hold for pluralities. Williamson [2013] and Linnebo [2016] follow them, using "rigidity" for (i). As Linnebo [2013] points out, though, the problem with this terminology is that it may engender confusion with the well-known semantic notion made famous by Kripke. For this reason, Linnebo [2013] opts for the term "stability", which I follow here. (Linnebo [2013] calls (ii) "inextensibility", though his formulation differs from mine because he is working in a weaker modal logic.) Thanks to an anonymous referee for pushing me on this point.

validates stability and ext (theorem 3).[13]

That leaves Comprehension. Unfortunately, it is well-known that Comprehension cannot be axiomatised, since any reasonable theory has models in which it fails.[14] Nonetheless, it does admit of an *approximation* in the form of a comprehension schema which says of each condition φ in a fixed language that there are some things comprising each and every φ. Formally:

(comp) $$\exists xx \forall x(x \prec xx \leftrightarrow \varphi)$$

As we move to richer and richer languages, comp approaches Comprehension closer and closer even though it is never reached.

For our purposes, the difference between Comprehension and comp is not idle. It is natural to see Comprehension as modelling a certain *combinatorial* aspect of pluralities: the idea that any combination of objects gives rise to a plurality.[15] The schema comp delivers such pluralities when the relevant combination of objects is circumscribed by a condition in some language. But there may be combinations of objects that are not circumscribed by any condition in any language. It is plausible, for example, that there is a combination of set-theoretic ordered pairs that associates each set with one and only one of its members (Pollard [1988]). By the combinatorial idea, there should thus be a plurality of such pairs. But there is no reason to think that any condition in any language will circumscribe such a plurality. If this is right, there will be claims—like those asserting the existence of these 'choice pluralities'—that follow from the nothing over and above conception but are not provable from stability, ext, and comp.[16]

[13]Other standard extensionality principles fail to capture Upward Dependence in this way. Clearly, a model satisfies Extensionality precisely when it validates:

(*) $$\forall xx, yy(\forall x(x \prec xx \leftrightarrow x \prec yy) \rightarrow xx = yy)$$

But there are models of Comprehension, stability and (*) that do not model ext and thus in which Upward Dependence fails. Indeed, there are models of Comprehension, stability and both of the following stronger extensionality principles that do not model ext (theorem 6).

(**) $$\Box \forall x(x \prec xx \leftrightarrow x \prec yy) \rightarrow xx = yy$$

(***) $$\Box \forall x \Box(x \prec xx \leftrightarrow x \prec yy) \rightarrow xx = yy$$

[14]More precisely, if Comprehension holds in every model of T, then T proves, for some n, that there are necessarily at most n things.

[15]Compare this with the combinatorial conception of set (Bernays [1935]). I am tempted to think that both this combinatorial idea and Upward dependence have a common source in a more general upward dependence idea. If that is right, then there are really three ideas behind the nothing over and above conception: a general form of upward dependence, a downward dependence idea, and an extensionality idea.

[16]It is a standard result in set theory that the existence of choice pluralities is not provable from comp and the extensionality principle (*) from footnote 13. It follows immediately that their existence is also not provable from stability, ext, and comp.

Where does this leave us? I have shown how to axiomatise three core components of the nothing over and above conception—those modelled by Downward Dependence, Upward Dependence, and Extensionality—and I have provided an approximation for the other—namely, the combinatorial idea modelled by Comprehension. Clearly, the motivation for those constraints extends to show that stability, ext, and comp each follow from the nothing over and above conception. Any claim provable from them will therefore also follow from the nothing over and above conception. What the 'choice plurality' example shows is that the converse may fail: there may be claims that follow from the nothing over and above conception which are not provable from stability, ext, and comp. Nevertheless, there is a precise sense in which such claims only concern the combinatorial aspect of pluralities: it is an immediate consequence of the previous results that a model conforms to the nothing over and above conception just in case it validates stability, ext, and comp and satisfies Comprehension, assuming one of Nothing, Everything, or Existence. Those latter claims moreover have straightforward axiomatisations. In particular, it is easy to see that a model satisfies Nothing precisely when it validates:

(no) $$x \prec xx \rightarrow Exx$$

that it satisfies Everything precisely when it validates:

(ev) $$x \prec xx \rightarrow \Box(\neg Exx \rightarrow x \prec xx)$$

and that it satisfies Existence precisely when it validates:

(ex) $$x \prec xx \rightarrow \Box(\neg Exx \rightarrow [x \prec xx \leftrightarrow Ex])$$

Thus, assuming that a model validates one of no, ev, or ex, it conforms to the nothing over and above conception just in case it validates stability, ext, and comp and satisfies Comprehension. So we might say that any claim which follows from the nothing over and above conception follows from stability, ext, and comp when we enhance them with the combinatorial idea behind Comprehension (assuming one of no, ev, or ex). In other words, we can fully capture the nothing over and above conception in the object language partly precisely, in terms of stability, ext, and comp, and partly informally, in terms of the combinatorial idea behind Comprehension.

It is consequence of the failure of comp to fully axiomatise Comprehension that stability, ext, and comp fail to provide a complete account of plural reality: they have non-isomorphic models with the same first-order part. In spite of this, they do come surprisingly close. In particular, suppose we take two models K and K' of stability, ext, and comp together with one of no, ev, or ex and assume that comp holds in both models for pluralities in the other model. That is, assume that whenever X is a plurality in K, there is some plurality Y in K' for which:

$$\bigcup_{w \in W} I_w(X) = \bigcup_{w' \in W'} I_{w'}(Y)$$

and vice versa. Then we can show that if K and K' have the same first-order part, they are isomorphic (theorem 4).

Here is another way to put the point. Suppose two speakers adopt comp in an *open-ended* way—so that they intend it to hold no matter which language they consider—then if they agree on first-order reality and accept stability, ext, and comp (together with one of no, ev, ex), they will agree on plural reality.[17]

2 Discussion

The model-theoretic and axiomatic explications of the nothing over and above conception show that it provides a simple, strong, and explanatory account of pluralities. As such, we might think it can be used to justify the claims that follow from it.[18,19] This is significant because much of the work on the modal logic of plurals has focused on trying to justify principles like stability by deriving them from claims that are, as Linnebo [2016] points out, no more clearly justified.[20]

The explications moreover show just how far that justification extends. To see this, it will help to compare stability, ext, and comp with what I take to be the most comprehensive investigation of the principles of plural modal logic to date, namely Linnebo [2016].[21] In addition to comp, Linnebo proposes the following.[22]

(Idsc) $$\forall xx, yy(\forall x(x \prec xx \leftrightarrow x \prec yy) \rightarrow \varphi(xx) \leftrightarrow \varphi(yy))$$

(Stab$^+$) $$\forall xx, x(x \prec xx \rightarrow \Box(Exx \wedge Ex \rightarrow x \prec xx))$$

[17] Compare this with the 'internal' categoricity results in set theory. See Walsh and Button [2018] for discussion.

[18] In this respect, it plays a similar role to that played by the so-called *iterative conception of set* in set theory. See, for example, Boolos [1971].

[19] Roberts [2016], Roberts [2019], and Linnebo [2016] also suggest using the nothing over and above conception to justify certain claims in the language of plural modal logic. In addition to stability and comp, I had already outlined the importance of ext in my thesis, Roberts [2016].

[20] See, for example, Uzquiano [2011] and Rumfitt [2005].

[21] Linnebo [2016] defines the existence predicate for pluralities, Exx, as $\exists x(x \prec xx)$. This will not effect the claims I make below, but it is worth noting that once we define Exx in this way, what are usually logical and innocuous principles become substantive. For example, instances of the axiom of universal instantiation—$Exx \wedge \forall yy\varphi \rightarrow \varphi(xx)$—become highly non-trivial. Effectively, they assume no: $x \prec xx \rightarrow Exx$.

[22] As I pointed out in footnote 5, Linnebo [2016] actually gives some reasons for denying comp in full generality. But again this will not effect the claims I am going to make.

(Stab$^-$) $$\forall xx, x(x \not\prec xx \to \Box(Exx \wedge Ex \to x \not\prec xx))$$

(Dep) $$\forall xx, x(x \prec xx \to \Box(Exx \to Ex))$$

(BC) $$x \prec xx \to Ex$$ [23,24]

Although extensive, there are important claims that these principles fail to imply.

Recall our people—Melesha, Nadia, and Dylan—and consider them without Melesha. To avoid confusion, let xx be the former people and yy the latter. Are the yy among the xx, whenever the xx exist? Do the yy exist whenever the xx do? The nothing over and above conception implies positive answers to both questions. If these and those are nothing over and above the individual things they comprise and these are among those, then whenever those exist these must exist and be among them. Formally:

(sub-plurality) $$\forall xx, yy(xx \subseteq yy \to \Box(Eyy \to Exx \wedge xx \subseteq yy))$$

Indeed, it can be shown that sub-plurality follows from stability, ext, and comp (theorem 5). And it has important applications in the philosophy of mathematics. Take *modal structuralism*, the view that mathematics concerns possible structures. On the most prominent version of this view, due to Geoffrey Hellman, structures are pluralities.[25] It turns out that for such structures to be well-behaved—in particular, for a structure to satisfy the same second-order claims wherever it exists—sub-pluralities of a structure must continue to exist and be its sub-pluralities wherever that structure exists. This is precisely what sub-plurality ensures ([Roberts, 2019, p. 18]).

But sub-plurality does not follow from Linnebo's principles (theorem 6).[26] More generally,

[23] This is the first-order "being constraint" for $\prec$. See Williamson [2013] for a general discussion of the being constraint. The plural being constraint is no. Given Linnebo's definition of "Exx", no follows from BC.

[24] Linnebo also suggests the following "traversability principle":

(UnivTrav-C) $$\forall xx \Box(Exx \to \forall x(x \prec xx \leftrightarrow \bigvee_{a \prec xx} x = a))$$

where "$\bigvee_{a \prec xx} x = a$" is intended to express an infinite disjunction of claims of the form "$x = a$" where a is one of the xxs. But this is redundant. It is easy to see that Stab$^+$, Dep, and BC already imply stability and that stability in turn ensures UnivTrav-C on its intended reading. Nonetheless, see footnote 28 for a criticism of the formalism used in this principle.

[25] See Hellman [1989] and Roberts [2019] for details.

[26] Another natural claim that Linnebo's principles fail to imply is the following restricted version of the plural Barcan formula.

$$Eyy \wedge \Diamond \exists xx \subseteq yy \varphi \to \exists xx \subseteq yy \Diamond \varphi$$

It is straightforward to modify the proof of theorem 5 to show that this also follows from stability, ext, and comp, and it is easy to see that it fails at w_0 of the Kripke model of Linnebo's principles constructed in theorem 6 when $\varphi =$ "$xx = zz$", "yy" is assigned $\{0, 1\}$, and "zz" is assigned 3. Since that model validates the unrestricted first-order Barcan formula, this also shows that the first-order Barcan formula does not imply it is easy to see that sub-plurality is subsumed by the idea behind Upward Dependence and that it is precisely this idea that Linnebo's principles fail to capture. In fact, there are simple Kripke models of Linnebo's principles in which the first-order domains are constant but the plural domains are completely disjoint.[27,28]

Let me end with two more applications of the idea behind Upward Dependence.

Perhaps the most well-known use of pluralities in the philosophy of mathematics is in securing categoricity results. It is often claimed, for example, that plural formulations of arithmetic have exactly one model up to isomorphism: in so far as there are non-standard models, they can be rejected as illegitimate. More generally, plural quantification is taken to be *determinate*: given any first-order domain, there is only one legitimate corresponding plural domain. Florio and Linnebo [2016] have challenged this orthodoxy. They show that there are non-standard *plurality-based* models—where the domain of the plural quantifiers consists of pluralities—and argue that there is no reason to dismiss such models as illegitimate. More precisely, they argue that there is a symmetry between plural and set quantification: extant reasons to dismiss their non-standard models of plural quantification are also reasons to dismiss non-standard models of set quantification. The detour through pluralities in order to secure categoricity would thus be redundant. I will now argue that the nothing over and above conception, and in particular the idea behind Upward Dependence, can be used to break this symmetry.

It will help to start with an argument for the determinacy of plural quantification that Florio and Linnebo consider and reject. The argument, which they attribute to Hossack [2000], is based on the claim that plural quantification is *ontologically innocent*. On one natural way of spelling this claim out, it says that pluralities in the range of plural quantifiers are not an ontological commitment over and above the ontological commitments of the first-order quantifiers. Ontological innocence is thus determined *locally*, plurality by plurality: if each plurality in the domain of a plural quantifier is nothing over and above some individual

the restricted plural Barcan formula without ext. It is again straightforward to modify the proof of theorem 5 to show that it does follow given ext.

[27]To see this, take any Kripke model with a constant first-order domain in which the plural domains are disjoint copies of its powerset. If Y is the copy of X at w, then we let $I_w(Y) = X$. That will already satisfy all of Linnebo's principles with the exception of BC. To make sure the model also satisfies BC we just require that $x \prec X$ is trivially false at worlds at which X does not exist. Formally, we require that when $X \notin D_2(w)$, $w \vDash x \not\prec X$. In other words, we adopt Nothing.

[28]In a footnote, Linnebo also suggests the following principle as an alternative to Dep:

$$\forall xx \Box (Exx \leftrightarrow \bigwedge_{a \prec xx} Ea)$$

Once this is added, it turns out that on a natural model theory for "$\bigwedge_{a \prec xx}$", we do get Upward Dependence. The problem is that although this new resource is tolerably clear in the model theory, it is completely unclear how to axiomatise it.

things in the domain of the first-order quantifier, then the plural quantifier is ontologically innocent. This is clearly in keeping with the nothing over and above conception. But as Florio and Linnebo point out, it is not enough to ensure determinacy. Any plurality-based model respects ontological innocence in this sense and since Florio and Linnebo have shown that there are non-standard plurality-based models, ontological innocence in this sense does not ensure determinacy.

This is certainly right. But there are other equally natural ways to spell out the claim that plural quantification is ontologically innocent. For example, we could say that a plural quantifier is ontologically innocent if the pluralities in its range conform to the nothing over and above conception. Ontological innocence would then be determined *globally*: if the pluralities in the range of a plural quantifier respect the nothing over and above conception as a whole, then the plural quantifier is ontologically innocent. Once we modify ontological innocence in this way, however, we can rule out Florio and Linnebo's non-standard plurality-based models. In particular, we can show that any two models with the same first-order domain that conform to the nothing over and above conception have the same plural domain. The argument is simple and relies crucially on Upward Dependence.

I will follow Florio and Linnebo in using plural predicate variables $\mathbf{D}, \mathbf{D}', \mathbf{D}''$ and so on to circumscribe plural domains. Now suppose we have two models $\mathcal{M}$ and $\mathcal{M}'$ with the same first-order domain D but distinct plural domains $\mathbf{D}$ and $\mathbf{D}'$ respectively, and suppose that both models conform to the nothing over and above conception. Say that the xxs are in the plural domain of $\mathcal{M}$ but not $\mathcal{M}'$: $\mathbf{D}(xx)$ but not $\mathbf{D}'(xx)$. By Downward Dependence applied to $\mathcal{M}$, each of the xxs is in the domain D. Thus, by Upward Dependence applied to $\mathcal{M}'$, the xxs must exist according to $\mathcal{M}'$, which is to say $\mathbf{D}'(xx)$. Contradiction![29]

In short, pluralities are so closely tied to the things they comprise that they are not an extra ontological commitment beyond those things. Similarly, none of those individual things is an extra ontological commitment beyond them. But they are also so closely tied to the things they comprise that the existence of each of those things *does* incur an ontological commitment to them. Similarly, they incur an ontological commitment to each of those individual things. For pluralities, ontological commitment and ontological innocence cut both ways. This highlights a fundamental difference between sets and pluralities. A set *is* an extra ontological commitment beyond its members; it *is* something over and above them.[30,31] Florio and Linnebo's non-standard plurality-based models can thus be ruled out as illegitimate because they

[29] Note that this argument does not rely on Comprehension and as a consequence is insensitive to the worries about Comprehension mentioned in footnote 5.

[30] I am unsure whether the members are also an extra ontological commitment beyond the set. It is typically thought that sets metaphysically cannot exist without their members (see footnote 12). But it is not clear that this extends to the associated claim about ontological commitment.

[31] Another respect in which sets differ from pluralities may be the generality with which upward and downward dependence principles like stability and ext hold. For example, it strikes me as plausible that on the nothing over and above conception, stability and ext hold when $\Box$ is interpreted as *logical* necessity. But the analogous principles for sets are clearly false.

do not conform to the nothing over and above conception, whereas non-standard models of set quantification cannot analogously be ruled out as illegitimate. In general, if plural quantifiers are ontologically innocent in the sense I am advocating, plural quantification is determinate.

Williamson [2013] is a sustained defence of *necessitism*—the view that existence is necessary—against *contingentism*—the view that it is not. One part of that defence concerns the kind of plural modal logic that each view allows for. He argues, in particular, that the contingentist lacks an adequate way to express very natural principles concerning pluralities. For example, it looks like the contingentist cannot easily formulate a "plural analogue of identity" or a principle that ensures statements like the following.

$$(1) \qquad \forall xx, x(\forall y(y \prec xx \leftrightarrow y = x) \rightarrow \Box(Ex \rightarrow Exx))$$

In order to do so, Williamson claims, the contingentist would have to avail themselves of so-called *backtracking operators* [Williamson, 2013, p. 250-252]. Such operators effectively allow worlds to be cross referenced: to refer in one world back to what is true in another. The worry is that these operators may really be hidden quantifiers over possible worlds.[32] But as Williamson argues, it is unclear whether the contingentist can countenance such quantification. For the contingentist, modal operators must be understood primitively.

What my axiomatisation of the nothing over and above conception shows is that the contingentist can both formulate a notion of identity for pluralities and justify the relevant consequences of Upward Dependence.[33] In particular, the extensionality principle ext shows that "$\Box\forall x(\Diamond(x \prec xx) \leftrightarrow \Diamond(x \prec yy))$" can be treated as a plural analogue of identity without the need for backtracking operators. More precisely, although I have stated ext as a principle concerning a primitive relation of identity for pluralities, it could just as well be replaced by a principle which says that it *is* an identity relation. Formally:

$$(2) \qquad \Box\forall x(\Diamond(x \prec xx) \leftrightarrow \Diamond(x \prec yy)) \rightarrow \varphi(xx) \leftrightarrow \varphi(yy)$$

The theory stability + ext + comp with the identity relation holding between pluralities is provably equivalent to stability + 2 + comp with identity confined to first-order terms.[34] Furthermore, the proof that sub-plurality follows from stability + ext + comp extends to show (1) and other similar claims also follow (see theorem 5).[35]

[32] See Melia [1992].

[33] The Kripke models I used have variable domains and all the proof theoretic results are carried out in a simple positive free version of S5 modal logic. They are thus perfectly compatible with contingentism.

[34] Technically, they are bi-interpretable.

[35] Williamson also claims that the necessitist has a simpler account of pluralities than the contingentist. It is certainly true that for the necessitist, stability and ext have slightly simpler formulations. In particular, stability is equivalent to:

$$x \prec xx \rightarrow \Box x \prec xx$$

3 Technical appendix

In this appendix, I provide proofs for the main results of the paper. Unless otherwise stated, I will assume that we have uniformly settled on adopting Existence, Ex, or ex. The proofs can then be easily modified for the other principles.

Theorem 1. *Let K be a model of* Downward Dependence*,* Upward Dependence*,* Extensionality*,* Comprehension*, and* Existence*. Then K is isomorphic to the Boolos model of its first-order part.*

Proof. Let K be as in the theorem and let K' be its first-order part. Let a be a variable assignment over K and let A_a be the following Boolos assignment:

$$\langle \text{“}x\text{”}, y\rangle \in A_a \leftrightarrow a(\text{“}x\text{”}) = y$$

$$\langle \text{“}xx\text{”}, y\rangle \in A_a \leftrightarrow y \in \bigcup_{w\in W} I_w(a(\text{“}xx\text{”}))$$

By a simple induction on the complexity of formulas we show that:

$$w \vDash_a \varphi \leftrightarrow w \vDash_{A_a} \varphi$$

The only non-trivial cases are those for “$x \prec xx$”, “$xx = yy$”, and “$\exists xx\varphi$”. For “$x \prec xx$”, first note that by Ex, $w \vDash_{A_a}$ “$x \prec xx$” iff $\langle \text{“}xx\text{”}, a(\text{“}x\text{”})\rangle \in A_a$ and $a(\text{“}x\text{”}) \in D_1(w)$. By the definition of A_a, that is in turn equivalent to $a(\text{“}x\text{”}) \in \bigcup_{w\in W} I_w(a(\text{“}xx\text{”})) \cap D_1(w)$. Now if $a(\text{“}xx\text{”}) \in D_2(w)$, then $I_w(a(\text{“}xx\text{”})) = \bigcup_{w\in W} I_w(a(\text{“}xx\text{”})) \cap D_1(w)$ by Downward Dependence; but also if $a(\text{“}xx\text{”}) \notin D_2(w)$, then $I_w(a(\text{“}xx\text{”})) = \bigcup_{w\in W} I_w(a(\text{“}xx\text{”})) \cap D_1(w)$ by Existence. So, in either case, $w \vDash_{A_a}$ “$x \prec xx$” iff $w \vDash_a$ “$x \prec xx$”

For “$xx = yy$”, first note that if $a(\text{“}xx\text{”}) = a(\text{“}yy\text{”})$, then $\bigcup_{w\in W} I_w(a(\text{“}xx\text{”})) = \bigcup_{w\in W} I_w(a(\text{“}yy\text{”}))$ and thus $w \vDash_{A_a}$ “$xx = yy$”. Conversely, suppose $w \vDash_{A_a}$ “$xx = yy$” and thus $\bigcup_{w\in W} I_w(a(\text{“}xx\text{”})) = \bigcup_{w\in W} I_w(a(\text{“}yy\text{”}))$. Now, let w' be such that $a(\text{“}xx\text{”}) \in D_2(w')$. Then, by Downward Dependence, $\bigcup_{w\in W} I_w(a(\text{“}xx\text{”})) = \bigcup_{w\in W} I_w(a(\text{“}yy\text{”})) \subseteq D_1(w')$ and so by Upward Dependence $a(\text{“}yy\text{”}) \in D_2(w')$. It follows from Extensionality and Downward Dependence that $a(\text{“}xx\text{”}) = a(\text{“}yy\text{”})$ as required.

for the necessitist, and ext is equivalent to:

$$\forall x(x \prec xx \leftrightarrow x \prec yy) \rightarrow xx = yy$$

But if the necessitist and the contingentist both motivate their respective principles using the nothing over and above conception, then they have precisely the same account of pluralities: namely, the nothing over and above conception! As I see it, the crucial difference between the contingentist and the necessitist with respect to plurals comes down to which of no, ev, or ex they adopt. For the necessitist, they all follow from stability. The contingentist, on the other hand, has a genuine choice, and it is unclear what principled reasons they could give for choosing one over the others.

For "$\exists xx\varphi$", assume that $w \vDash_a$ "$\exists xx\varphi$". Then $w \vDash_{a'} \varphi$ for some a' that differs from a at most in what it assigns to "xx" where that thing exists at w: that is, $a'(\text{"}x\text{"}) \in D_1(w)$. Thus $w \vDash_{A_{a'}} \varphi$ by the induction hypothesis and $\bigcup_{w\in W} I_w(a(\text{"}xx\text{"})) \subseteq D_1(w)$ by the definition of $A_{a'}$. So, $w \vDash_{A_a}$ "$\exists xx\varphi$". Conversely, suppose $w \vDash_{A_a}$ "$\exists xx\varphi$". Then $w \vDash_{A'} \varphi$ for some A' that differs from A_a at most in what it relates to "xx" where those things all exist at w: that is, $\{x : A'(\text{"}xx\text{"}, x)\} \subseteq D_1(w)$. Now, by Comprehension, there will be an $X \in D_2(w)$ for which $I_w(X) \cap D_1(w) = \{x : A'(\text{"}xx\text{"}, x)\}$. By Downward Dependence, $I_w(X) \cap D_1(w) = I_w(X) = \bigcup_{w\in W} I_w(X)$. Thus, where a' is an assignment just like a expect possibly in that it assigns X to "xx", $A' = A_{a'}$. It follows that $w \vDash_a$ "$\exists xx\varphi$". □

Theorem 2. *Let K and K' be models of* Downward Dependence*,* Upward Dependence*,* Extensionality*,* Comprehension*, and* Existence*. Assume, moreover, that they have the same first-order part. Then, they are isomorphic.*

Proof. Let X be a plurality in K and assume that it exists at w. By Downward Dependence, $I_w(X) = \bigcup_{w\in W} I_w(X) \subseteq D_1(w)$. By Comprehension, there is some Y in K' that exists at w for which $I'_w(Y) \cap D_1(w) = \bigcup_{w\in W} I_w(X)$. Since Y exists at w it follows from Downward Dependence that $I'_w(Y) \cap D_1(w) = \bigcup_{w\in W} I'_w(Y)$ and thus that $\bigcup_{w\in W} I'_w(Y) = \bigcup_{w\in W} I_w(X)$. By Downward Dependence, Upward Dependence, and Extensionality, there is at most one plurality comprising a given set of things: that is, for Z, Z' in a given model of Downward Dependence, Upward Dependence, and Extensionality, if $\bigcup_{w\in W} I_w(Z) = \bigcup_{w\in W} I_w(Z')$, then $Z = Z'$. (This is precisely what ext says.) So, Y is unique. In general, for X in K, let $f(X)$ be the unique such Y.

By induction on the complexity of formulas we prove that:

$$w \vDash_K \varphi(\vec{X}) \leftrightarrow w \vDash_{K'} \varphi(f(\vec{X}))$$

For "$x \prec xx$": if $X \in D_2(w)$, then $f(X) \in D'_2(w)$ and $I_w(X) = \bigcup_{w\in W} I_w(X) = \bigcup_{w\in W} I'_w(Y) = I'_w(Y)$ by Upward Dependence and Downward Dependence. Thus, $w \vDash_K$ "$x \prec X$" iff $w \vDash_{K'}$ "$x \prec X$". If $X \notin D_2(w)$, then $Y \notin D'_2(w)$ and $I_w(X) = \bigcup_{w\in W} I_w(X) \cap D_1(w) = \bigcup_{w\in W} I'_w(Y) \cap D_1(w) = I'_w(Y)$ by Downward Dependence, Upward Dependence, and Existence. Thus, again, $w \vDash_K$ "$x \prec X$" iff $w \vDash_{K'}$ "$x \prec X$".

By Downward Dependence, Upward Dependence, and Extensionality, $w \vDash$ "$X = Y$" iff $X = Y$ iff $\bigcup_{w\in W} I_w(X) = \bigcup_{w\in W} I_w(Y)$ which in turn is equivalent to $\bigcup_{w\in W} I'_w(f(X)) = \bigcup_{w\in W} I'_w(f(Y))$ and thus to $f(X) = f(Y)$ and to $w \vDash$ "$f(X) = f(Y)$". In other words, f is one-one.

To complete the proof, we just need to show that f onto. So let Y be a plurality in K' and suppose that $Y \in D'_2(w)$. Then by Downward Dependence $I'_w(Y) = \bigcup_{w\in W} I'_w(Y) \subseteq D_1(w)$. So, by Comprehension, there is some $X \in D_2(w)$ such that $I_w(X) \cap D_1(w) = I'_w(Y)$. Thus, by Downward Dependence, $\bigcup_{w\in W} I_w(X) = I_w(X) = I_w(X) \cap D_1(w) = I'_w(Y) = \bigcup_{w\in W} I'_w(Y)$.

So, $f(X) = Y$. □

Theorem 3. *Suppose that K models* stability *and the following instance of* comp*:*

$$\exists xx \forall x(x \prec xx \leftrightarrow \Diamond(x \prec yy))$$

Then, K models ext *just in case it satisfies* Extensionality *and* Upward Dependence*.*

Proof. Let K be as in the theorem, and assume that it models ext. Let $X, Y \in D_2(w)$ be such that $I_w(X) \cap D_1(w) = I_w(Y) \cap D_1(w)$. By stability, $\bigcup_{w \in W} I_w(X) = I_w(X) = I_w(X) \cap D_1(w) = I_w(Y) \cap D_1(w) = I_w(Y) = \bigcup_{w \in W} I_w(Y)$. So, by ext, $X = Y$. In other words, K satisfies Extensionality.

Now assume that $\bigcup_{w \in W} I_w(X) \subseteq D_1(w)$. By the above instance of comp there is some $Y \in D_2(w)$ for which $I_w(Y) \cap D_1(w) = \bigcup_{w \in W} I_w(X)$. Now, since $Y \in D_2(w)$, $I_w(Y) \cap D_1(w) = I_w(Y) = \bigcup_{w \in W} I_w(Y)$. Thus, $\bigcup_{w \in W} I_w(X) = \bigcup_{w \in W} I_w(Y)$. By ext it follows that $X = Y$. In other words, K satisfies Upward Dependence.

Finally, and conversely, assume that K satisfies Extensionality and Upward Dependence. Then since it models stability, it also satisfies Downward Dependence. It follows from all three principles that $X = Y$ iff $\bigcup_{w \in W} I_w(X) = \bigcup_{w \in W} I'_w(Y)$. □

Theorem 4. *Let K and K' be two Kripke models with the same first-order part such that for $X \in D_2(w)$, there is $Y \in D'_2(w)$ for which:*

$$I_w(X) \cap D_1(w) = I'_w(Y) \cap D_1(w)$$

and vice versa. Assume, moreover, that they both model comp + stability + ext + ex*. Then, they are isomorphic.*

Proof. By the previous theorem, we know that K and K' both satisfy Upward Dependence, Downward Dependence, and Extensionality. Moreover, ex is valid just in case Existence holds. Finally, note that the proof of theorem 2 only relied on the assumption about comprehension in the statement of this theorem. □

Theorem 5. *In positive free* S5*,* stability + ext + comp *entail* sub-plurality*.*[36]

Proof. Assume that $\Diamond(Exx, yy$ and $xx \subseteq yy)$ and Eyy. By comp there are some zz such that $\forall x(x \prec zz \leftrightarrow \Diamond(x \prec xx))$. First, we want to show that the zz are the xx. So, suppose $\Diamond(x \prec zz)$. By stability, $x \prec zz$ and Ex and thus $\Diamond(x \prec xx)$ by the definition of zz. Now, suppose $\Diamond(x \prec xx)$. By stability, it follows that $\Diamond\Box(Exx \rightarrow Ex \wedge x \prec xx)$ and thus that $\Box(Exx \rightarrow Ex \wedge x \prec xx)$. So, by our first assumption, we have $\Diamond(x \prec yy)$. By stability and

[36] The following two results are taken from Roberts [2019]. I repeat them here for completeness.

our assumption that Eyy, it follows that Ex. Thus, $x \prec zz$ by the definition of the zz and thus $\Diamond(x \prec zz)$. By ext, the zz are identical with the xx as required, and so Exx.

Second, we show that $xx \subseteq yy$. So, suppose Ez and $z \prec xx$, then $\Diamond(x \prec yy)$ by stability and our first assumption. But then $x \prec yy$ by stability and our second assumption. Putting all of this together we get:

$$\Diamond(Exx, yy \wedge xx \subseteq yy) \to (Eyy \to Exx \wedge xx \subseteq yy)$$

and S5 allows us to transform that into:

$$Exx, yy \wedge xx \subseteq yy \to \Box(Eyy \to Exx \wedge xx \subseteq yy)$$

□

Theorem 6. *In positive free* S5*,* stability + comp *together with any of:*

(*) $$\forall xx \forall yy(\forall x(x \prec xx \leftrightarrow x \prec yy) \to xx = yy)$$

(**) $$\Box\forall x(x \prec xx \leftrightarrow x \prec yy) \to xx = yy$$

(***) $$\Box\forall x\Box(x \prec xx \leftrightarrow x \prec yy) \to xx = yy$$

fail to imply sub-plurality*.* ext *is thus strictly stronger than these principles.*

Proof. Consider an S5 Kripke model K with two worlds w_0, w_1, both with the first-order domain $\{0, 1\}$ (so the model validates first-order necessitism; formally, $\Box\forall x\Box Ex$). Let the pluralities at w_0 be $\emptyset, \{0\}, \{1\}$, and $\{0, 1\}$ and the pluralities at w_1 be $\emptyset, 3, \{1\}$, and $\{0, 1\}$. At w_0 and w_1, we let $\emptyset$ contain nothing, $\{1\}$ contain 1, and $\{0, 1\}$ contain 0 and 1. At w_0, we let $\{0\}$ contain 0, but at w_1 we make it contain nothing; and at w_1, we let 3 contain 0, but at w_0 we make it contain nothing. So, K validates $\Box\forall x(\Diamond(x \in \{0\}) \leftrightarrow \Diamond(x \in 3))$ even though $\{0\} \neq 3$. ext thus fails in the model. It is straightforward to check that K validates stability, comp and (*), (***), and (***). Moreover, at w_0, both $\{0\}$ and $\{0, 1\}$ exist and $\{0\}$ is a subplurality of $\{0, 1\}$; but, at w_1, $\{0, 1\}$ exists even though $\{0\}$ does not. So, K does not validate sub-plurality.[37,38] □

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[37] It is natural to factor sub-plurality into the following two principles.

$$\forall xx, yy(xx \subseteq yy \rightarrow \Box(Eyy \rightarrow xx \subseteq yy))$$

$$\forall xx, yy(xx \subseteq yy \rightarrow \Box(Eyy \rightarrow Exx))$$

It is easy to see from the proof of theorem 5 that stability already implies the first principle. It is thus the second that can fail without ext.

[38] It's worth noting that the model also validates no. It is straightforward to show that given either of ev or ex, $\Box\forall x(\Diamond(x \prec xx) \leftrightarrow \Diamond(x \prec yy))$ is equivalent to both $\Box\forall x(x \prec xx \leftrightarrow x \prec yy)$ and $\Box\forall x\Box(x \prec xx \leftrightarrow x \prec yy)$ and so ext is equivalent to (**) and (***). Nonetheless, if we modify the above model so that $\{0\}$ and 3 contain 0 in all worlds, we obtain a model that validates stability, comp, ev, ex, and (*) but not ext, (**), or (***).

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