Ultimate V
Draft

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1 Introduction

Potentialism is the view that the universe of sets is inherently potential. It comes in two main flavours. *Height potentialism* is based on the idea that sets are potential relative to their elements: once the elements exist, the set *can* exist. Take some people: Nadia, Dylan, and Melesha. Since each of them exists, the height potentialist claims that there could have been a set of them, that the set {Nadia, Dylan, Melesha} could have existed. Once we have that set, we can repeat the process. Given Nadia, Dylan, and Melesha and their set, the height potentialist claims that there could have been a set of *those* things, that the set {Nadia, Dylan, Melesha, {Nadia, Dylan, Melesha}} could have existed. Continuing in this way, we get the possibility of more and more sets. So many, according to the height potentialist, that the sets thus obtained satisfy the axioms of set theory.[1]

Height potentialism is significant because it supports a compelling response to *Russell's paradox*. Russell's paradox and its variants tell us that various conditions fail to determine sets. For example, they tell us that the condition of being a non-self-membered set fails to determine a set.[2] This raises a challenge: to provide an account of the dividing line between the conditions that determine sets and those that don't. Height potentialism does this extremely well. It tells us that conditions can determine sets precisely when their instances can all exist together: that set existence is a matter of co-existence. With the right background assumptions, this explains why there could have been enough sets for the purposes of set theory but why there couldn't have been the various problematic sets, like the set of all non-self-membered sets.[3]

Width potentialism is based on the idea that some universes of sets are potential relative to others. Take an arbitrary universe of sets: $\mathcal{U}$. The width potentialist claims that by applying the *method of forcing* within $\mathcal{U}$, we can specify other possible universes of sets: universes, for example, in which there are more subsets of the natural numbers than there are in $\mathcal{U}$. Once $\mathcal{U}$ exists, those universes *can* exist. According to this view, no universe of sets is privileged. There's no ultimate background universe of sets containing absolutely all sets or even absolutely all subsets of the natural numbers. There's no *ultimate* V. Rather, there's a broad space of equally legitimate universes, containing different sets and making different claims true.

[1] The labels "height potentialism" and "width potentialism" are sometimes used for broader classes of views than I will consider in this paper. See, for example, Hamkins and Linnebo [forthcoming]. Nothing important rests on this terminological choice.

[2] Such a set would have to be a member of itself just in case it was not a member of itself!

[3] Height potentialism is most clearly expressed by recent writers like Linnebo [2010], Linnebo [2013], Studd [2019], Parsons [1977], and Hellman [1989]. But the view arguably goes back to Putnam [1967] and Zermelo [1930], and perhaps even Cantor (see Linnebo [2013] for discussion).

Width potentialism is significant because it supports a compelling response to the *problem of independence*.[4] One of the central results of modern set theory is that its axioms leave open a number of fundamental questions. The most famous example is the continuum hypothesis (CH), which says that there is no size between the size of the natural numbers and the size of the real numbers. CH is neither provable nor disprovable from the currently accepted axioms of set theory. And despite significant efforts, set-theorists and philosophers have failed to find other well-motivated principles that prove or disprove it. Width potentialism explains this failure extremely well. According to the view, attempts to settle questions like whether CH is true or not are misplaced. CH is not an unambiguous statement for which we can marshal evidence. Rather, it is true or false only relative to a universe of sets. And in the broad space of possible universes of sets, we already know via the method of forcing how it behaves: how it is true in some universes and false in others. There is no ultimate V in which CH either holds or fails to hold.[5]

It is natural to think that height and width potentialism are just aspects of a broader phenomenon of potentialism, that they might both be true.[6] The main result of this paper is that this is mistaken: height and width potentialism are jointly inconsistent. Indeed, I'll argue that height potentialism is independently committed to an ultimate background universe of sets, an ultimate V, *up to its height*. If height potentialism is correct, then, CH gets its final and unambiguous formulation within the height potential sets.

Here's the plan. In sections 2 and 3 I outline height and width potentialism and motivate some of their basic commitments. In section 4 I use these commitments to argue that height and width potentialism are jointly inconsistent and that the height potential sets constitute an ultimate V up to their height. Section 5 considers some responses and section 6 is a technical appendix.

2 Height potentialism

In its simplest form, Russell's paradox tells us that there's no set of all and only the non-self-membered sets; no *Russell set*, r, for which

$$\forall x(x \in r \leftrightarrow x \notin x)$$

Some conditions, like the condition of being non-self-membered, fail to determine sets. Nevertheless, set theory tells us that many conditions do determine sets. The axiom of pairing, for example, says that the condition of being a or b determines a set whenever a and b are sets. We are thus faced with a challenge: to provide an account of the dividing line between the conditions that determine sets and those that don't which explains why there are many of the sets there are—enough for the purposes of set theory—but not problematic sets like the Russell set. I'll refer to this as *the explanatory challenge*.

The two standard responses to this challenge are based on the *limitation of size* and *iterative* conceptions of set. According to the limitation of size conception, set existence is a matter of size: conditions determine sets precisely when their instances aren't too many. Typically,

[4] See, for example, Koellner [2006].

[5] See Hamkins [2012] and Hamkins and Linnebo [forthcoming].

[6] See, for example, Hamkins and Linnebo [forthcoming] and Scambler [forthcoming].

when their instances are fewer than the (von Neumann) ordinals. Properly formulated, the limitation of size conception arguably explains why many of the axioms of set theory are true whilst also explaining why there aren't many of the problematic sets.[7] For example, it tells us that there is no Russell set because the non-self-membered sets are not fewer than the ordinals (since every ordinal *is* a non-self-membered set).

According to the iterative conception, the sets occur in a well-ordered series of stages. At the very first stage, we have no sets whatsoever. Then, at the second stage, we have all the sets of things at the first stage. Since there's nothing at the first stage, that means the only set we get at the second stage is the empty set: $\emptyset$. At the third stage, we have all the sets of things at the second stage. Since the empty set is the only thing at the second stage, that means the sets we get at the third stage are the set containing the empty set together with the empty set itself: $\{\emptyset\}$ and $\emptyset$. At the fourth stage, we have all the sets of *those* things. And so on. In general, at any given stage we have all and only the sets of things available at some prior stage. On this conception, set existence is a matter of co-existence at a stage: conditions determine sets precisely when their instances all occur together at some stage. Properly formulated, the iterative conception also arguably explains why many of the axioms of set theory are true whilst explaining why there aren't many of the problematic sets.[8] For example, it tells us that there is no Russell set because the non-self-membered sets do not all co-exist at any one stage (since no matter what stage we consider, there will always be a non-self-membered set beyond it.[9])

Both conceptions licence a schema of separation, which says that for any set x, there is a set of all and only the ϕs in x.

(Separation) $$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \land \phi)$$

On the limitation of size conception, the members of any set are fewer than the ordinals. So, trivially, the ϕs among them are also fewer than the ordinals and thus form a set. On the iterative conception, the members of any set co-exist at some stage. So, trivially, the ϕs among them co-exist at that very same stage and thus form a set. These observations, moreover, aren't in any way sensitive to the specifics of ϕ: they hold good for any (meaningful) ϕ whatsoever. The schema, in other words, is *open-ended.* To reject an instance of Separation in any language is to give up on the simple idea that set existence is merely a matter of size or merely a matter of co-existence at a stage.

Height potentialism offers an alternative response to the explanatory challenge. It's based on the idea that sets are *potential* relative to their elements: once the elements all exist together—once they co-exist—the set *can* exist.[10] Assuming that sets cannot exist without

[7]In particular, given two ordinals, it implies the axioms of empty set, pairing, replacement, separation, choice, and, as Levy [1968] showed, union. Burgess [2004] suggests enhancing the limitation of size conception with a plural *reflection principle* that suffices to obtain in addition the axioms of powerset and infinity.

[8]In particular, with the right formulation, it implies the axioms of empty set, pairing, union, powerset, separation, and the claim that every set is in some rank of the cumulative hierarchy: $\forall x \exists \alpha (x \in V_\alpha)$. See, for example, Boolos [1971], Boolos [1989], and Parsons [1977]. The axioms of infinity and replacement can be obtained by enhancing the conception with a natural reflection principle. See Paseau [2007] for discussion (though note that we need the so-called *complete* version of the reflection principle to obtain infinity and replacement, not the *partial* version that Paseau highlights (see Lévy and Vaught [1961] for the distinction)).

[9]Namely, the set of non-self-membered sets from the given stage.

[10]A crucial question for the height potentialist is how they understand the relevant notion of potentiality, the sense in which sets *can* exist. For some, it is a distinctively mathematical notion (Linnebo [2013], Parsons

their elements,[11] it follows that the set can exist when, and only when, the elements can co-exist. What emerges, then, is a picture on which set existence is a matter of co-existence. Not co-existence at a stage, like the iterative conception, but co-existence simpliciter. More precisely, it's a picture on which *possible* set existence is a matter of *possible* co-existence.

It is now standard to make height potentialism precise using plural quantification,[12] given a conception of pluralities as nothing over and above the things they comprise.[13] As Roberts [2022] shows, this conception breaks down into three component ideas: *upward dependence*, *downward dependence*, and *extensionality*. Upward dependence says that some things taken individually suffice for those very same things taken together. It says, in other words, that when you have the individual things, you thereby have the plurality. Consider our people: Nadia, Dylan, and Melesha. The idea is that since each of them individually exists—since Nadia exists, and Dylan exists, and Melesha exists—*they* must exist. More generally, upward dependence implies the plural comprehension schema, which says that there are some things comprising all and only the ϕs.

(Plural Comp) $$\exists xx \forall x(x \prec xx \leftrightarrow \phi)$$

The thought is that since each individual ϕ exists, the plurality of ϕs thereby also exists. As with Separation, this doesn't seem to be in any way sensitive to the specifics of ϕ: it seems to hold good for any (meaningful) ϕ whatsoever. The schema is open-ended.

Downward dependence is the converse of upward dependence. It says that some things taken together suffice for those very same things taken individually. Again, consider our people. The idea is that when *they* exist, so too must Nadia, and Dylan, and Melesha. When you have the plurality, you thereby have the individual things. Extensionality simply says that pluralities comprising the same things are identical.

Intuitively, upward and downward dependence jointly ensure that co-existence is equivalent to the existence of a plurality. Upward dependence ensures that when the possible ϕs co-exist, they determine a plurality (by Plural Comp); and downward dependence ensures that when there's a plurality of all the possible ϕs, they co-exist. So, plural quantification gives us a precise and perspicuous way to talk about co-existence. In particular, we can use it to formulate the central height potentialist claim that sets are potential relative to the co-existence of their elements as the claim that necessarily any plurality could have formed a set.

(Collapse$^\Diamond$) $$\Box \forall xx \Diamond \exists x(x \equiv xx)$$

where $x \equiv xx$ abbreviates $\forall y(y \in x \leftrightarrow y \prec xx)$.

[1983b], Hellman [1989], and Reinhardt [1980]); for some, an interpretational notion (Linnebo [2018], Studd [2019] and Uzquiano [2015]); and for others, a logical notion (Hellman [1989] and Berry [2018]). I won't take a stand on this issue here, since my arguments will by and large rest on assumptions that hold for all of the corresponding views.

[11] See the discussion of stability in sections 4 and 5.

[12] See, for example, Linnebo [2010], Linnebo [2013], and Studd [2019]. Another option would be to use so-called backtracking operators, $\uparrow$ and $\downarrow$. We could then formulate the claim that possible set existence is a matter of possible co-existence as the following schema (bracketing the caveat discussed below that ϕ must be possibly stable).

$$\Diamond \exists x \Box \forall y(y \in x \leftrightarrow \phi) \quad \leftrightarrow \quad \Diamond \uparrow \Box \forall y(\phi \rightarrow \downarrow Ex)$$

[13] For convenience, I will frequently use the singular "plurality" and talk of objects being elements or members of pluralities. Nothing rests on this, however, and everything I say can be reformulated using genuinely plural locutions.

With the right background assumptions,[14] Plural Comp and Collapse$^\Diamond$ allow us to prove that the dividing line between the conditions that can determine sets and those that can't is indeed the possible existence of a plurality (which is to say possible co-existence). There's a caveat. Say that a condition is *stable* if it doesn't change its instances: if, for any possible object, when it has that object as an instance, it necessarily has it as an instance; and when it doesn't, it necessarily doesn't. Formally, say that ϕ is stable when: $\Box\forall x(\Box\phi \vee \Box\neg\phi)$. Then, what Plural Comp and Collapse$^\Diamond$ entail is that a condition can determine a set when and only when there could have been a plurality of all its possible instances *and* it's possibly stable. Formally:

$$\Diamond\exists y\Box\forall x(x \in y \leftrightarrow \phi) \quad \leftrightarrow \quad [\Diamond\exists xx\Box\forall x(\phi \to x \prec xx) \wedge \Diamond\Box\forall x(\Box\phi \vee \Box\neg\phi)]$$

This dividing line, in turn, can arguably be used to explain why many of the axioms of set theory are true in the height potential sets,[15,16] whilst also explaining why there couldn't have been many of the problematic sets. For example, it tells us that there couldn't have been the Russell set because there couldn't have been a plurality of all the possible non-self-membered sets (since no matter what sets we consider, there could have been a non-self-membered set not among them.[17])

Just as the limitation of size and iterative conceptions license a non-modal schema of separation, height potentialism licences a modal schema of separation, which says that if ϕ is possibly stable and x is any possible set, then there could have been a set of all and only the ϕs in x.

$$(\text{Separation}^\Diamond) \qquad \Diamond\Box\forall z(\Box\phi \vee \Box\neg\phi) \quad \to \quad \Box\forall x\Diamond\exists y\forall z(z \in y \leftrightarrow z \in x \wedge \phi)$$

Intuitively, since the members of any possible set x can co-exist, the ϕs among them can also co-exist, trivially, and thus could have formed a set. When ϕ is stable, the ϕs among x's members are necessarily the ϕs among them. So, there could have been a set containing all and only the ϕs in x. Again, this thought isn't sensitive to the specifics of ϕ: it holds good for any (meaningful) ϕ whatsoever. Like Separation, Separation$^\Diamond$ is open-ended. To reject an instance of Separation$^\Diamond$ in any language is to give up on the simple idea that possible set existence is merely a matter of possible co-existence. In more precise, plural, terms: the instance of Separation$^\Diamond$ for ϕ is implied by the corresponding instance of Plural Comp together with Collapse$^\Diamond$ (theorem 2). Since Plural Comp is open-ended, so too is Separation$^\Diamond$. Given the other background assumptions, to reject an instance of Separation$^\Diamond$ in any language is to give up either on the simple idea that plural existence is merely a matter of individual co-existence—that is, to give up on the nothing over and above conception—or the idea that any possible things could have formed a set.

[14] See, for example, Linnebo [2013] or Roberts [2016].

[15] More precisely, we say that a statement in the language of set theory is true in the height potential sets when its *modalisation* is true, where the modalisation of a statement ϕ is the result of replacing occurrences of $\exists x$ and $\forall x$ in ϕ it with $\Diamond\exists x$ and $\Box\forall x$ respectively. This has the effect of forcing its quantifiers to range over all potential sets. See Studd [2019], Linnebo [2010], and Linnebo [2013].

[16] In particular, we get the modalisations of the axioms of pairing, union, and separation. See Linnebo [2013] for details. The modalisations of the axioms of foundation, infinity, powerset, and replacement require further assumptions (see footnote 22).

[17] Namely, by Collapse$^\Diamond$, the set of non-self-membered sets among them.

The strongest argument in favour of height potentialism is that it does better than its competitors at answering the explanatory challenge.[18] In particular, the usual claim is that height potentialism better explains why crucial conditions do or do not determine sets. For example, on pain of contradiction, we know that the ordinals don't form a set. According to the limitation of size conception:

> the explanation is that [the ordinals] are too many to form a set, where being too many is defined as being as many as [the ordinals]. Thus, the proposed explanation moves in a tiny circle. [Linnebo, 2010, p.154]

In contrast, the height potentialist's proposed explanation is non-circular: the ordinals cannot form a set because they cannot all co-exist. Similar criticisms are levelled against the iterative conception.[19] As the key constraint on set existence, then, co-existence is claimed to be more explanatory than both size or co-existence at a stage. Other things being equal, that's a good reason to prefer height potentialism.[20]

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[18]See, for example, Linnebo [2010] and Studd [2019].

[19]I find these much less persuasive, however. Linnebo [2010], for example, argues that the unexplanatoriness of the limitation of size conception is inherited by the iterative conception. He shows that given plausible assumptions, the iterative conception *implies* the limitation of size claim that conditions determine sets precisely when their instances are fewer than the ordinals. The problem with this argument is that although the proponent of the iterative conception may be committed to such a claim, they need not think it carries any *explanatory* weight. It may be that Zara is fond of all and only the pluralities that form sets. But it doesn't follow that Zara's fondness for a plurality explains why it does or does not form a set. Explanation is a highly intensional notion and equivalent statements will not always explain the same things (or be explained by the same things). As should be clear from the above discussion, I find the most natural explanatory notion for the iterative conception to be co-existence at a stage. The ordinals don't form a set because there is no stage at which they co-exist.

Studd [2019] focuses on a different example. For him, the most natural explanatory notion for the iterative conception is the bounding of ranks in the cumulative hierarchy. The finite ordinals *do* form a set because there is some ordinal greater than their ranks. But since each ordinal *is* its own rank, the proposed explanation is that they form a set because there is some ordinal greater than them. Which is to say: because they form a set. As before, the proposed 'explanation moves in a tiny circle'. But, again, I see no reason for the proponent of the iterative conception to take bounding of ranks, rather than co-existence at a stage, to be the right explanatory notion.

For my money, a better focus would be the treatment of impure sets. The height potentialist's account applies equally well to pure and impure sets. It tells us that no matter what condition we consider, it can determine a set if its instances can all co-exist, regardless of whether those instances are sets or non-sets. The iterative conception, on the other hand, has a notoriously hard time incorporating sets of non-sets. See, for example, Menzel [2014].

[20]It's worth noting one possible issue for the height potentialist that doesn't immediately arise for the limitation of size or iterative conceptions. The height potentialist has subtly shifted focus from the question when conditions determine sets to the question when they *could* determine sets. The limitation of size and iterative conceptions answer the first question, whereas height potentialism answers the second. Since the limitation of size and iterative conceptions are non-modal, the second question doesn't immediately arise for them. Height potentialism, however, faces both. It faces the question when conditions determine sets *within* worlds—the *intra-world* question—and the question when conditions determine sets *across* worlds—the *inter-world* question. And although it answers the inter-world question, it does not, as it stands, answer the intra-world question. One particularly tricky instance of that question concerns the actual world. Namely: when do conditions *actually* determine sets? Relatedly, why are there the sets there are, actually—whatever sets they may be—rather than others? For the height potentialist, there could have been many sets other than there actually are. Why are those sets merely possible, rather than actual? I call this the *problem of actuality.* It's unclear how the height potentialist should respond to the problem of actuality, and without a satisfactory response, it may be that the balance of explanatory virtue doesn't tip so heavily in their favour after all.

As it stands, height potentialism fails to imply as many principles of set theory as the limitation of size or iterative conceptions. For example, it fails to imply the axioms of foundation and powerset, both of which are easy consequences of the iterative conception.[21] It's therefore incumbent on the height potentialist to supplement their account, so that its explanatory successes aren't overshadowed by a loss of strength. And indeed, this is precisely what they do. Both Linnebo [2013] and Studd [2019], for example, supplement Plural Comp and $\mathsf{Collapse}^{\Diamond}$ with principles that suffice to prove that all of the theorems of ZFC are true in the height potential sets.[22]

In summary, for the height potentialist, the dividing line between the conditions that can determine sets and those that can't is possible co-existence. In plural terms: the dividing line is the possible existence of a plurality. The open-ended modal separation schema, $\mathsf{Separation}^{\Diamond}$, is an immediate consequence. Together with supplementary principles, height potentialism can be used to explain why there could have been enough sets to make the axioms of ZFC true but why there couldn't have been problematic sets like the Russell set.

3 Width potentialism

Width potentialism is the view that by applying the method of forcing within a given universe of sets, we can specify other possible universes of sets. The idea, in other words, is that some universes of sets are potential relative to others: once a given universe of sets exists, universes specified by applying the method of forcing within it *can* exist.

We can fruitfully ignore the finer details of the method of forcing,[23] but the basic idea is that we use the axioms of ZFC to construct models—call them *forcing models*—that describe possibilities for the universe of sets. For example, there are forcing models that describe possibilities according to which the continuum hypothesis (CH) is true and forcing models that describe possibilities according to which it's false. There are forcing models that describe possibilities according to which there are new sets of natural numbers. In fact, for any infinite set s, there are forcing models that describe possibilities according to which there are new subsets of s. Moreover, for any set s, there are forcing models that describe possibilities according to which s is countable. Following the literature, I'll assume for now that whatever universes are, they satisfy the axioms of ZFC.[24] We can therefore construct these forcing models within any universe.

So far, so standard. Width potentialism goes beyond these mathematical facts by claiming that the possibilities described by forcing models in possible universes are actualised in other possible universes. More precisely, it says that:

For any possible universe $\mathcal{U}$ and any possibility described by some forcing model

[21] At least, on the formulation I gave earlier.

[22] Studd [2019] obtains the modalisation of the powerset axiom by adopting a stronger version of $\mathsf{Collapse}^{\Diamond}$, whereas Linnebo [2013] simply adds it as a further assumption. Similarly, Studd [2019] secures foundation via a natural modal principle, whereas Linnebo [2013] assumes it. Both adopt a modal reflection principle to obtain the modalisations of the axioms of infinity and replacement.

[23] The interested reader can consult Jech [2003] or Kunen [2011].

[24] As will become apparent in the next section, this is overkill. For example, the first argument I give that height and width potentialism are jointly inconsistent merely requires that the height potential sets count as a possible universe according to the width potentialist and that they are rich enough to define a forcing model in which there's a new subset of one of their sets.

in $\mathcal{U}$, there could be a universe $\mathcal{U}'$ witnessing that possibility.

I'll refer to this as the *central width potentialist claim.* So, assuming there's at least one possible universe of sets, it follows from the central width potentialist claim that there could be a universe in which CH is true and there could be a universe in which it's false. For any possible universe $\mathcal{U}$, it similarly follows that there could be a universe $\mathcal{U}'$ containing a set of natural numbers not in $\mathcal{U}$. In fact, for any infinite set s in $\mathcal{U}$, it follows that there could be a universe $\mathcal{U}'$ containing a subset of s not in $\mathcal{U}$. Moreover, for any set s in $\mathcal{U}$, it follows that there could be a universe $\mathcal{U}'$ in which s is countable.

As I mentioned in the introduction, the primary philosophical motivation for width potentialism is that it provides an attractive response to the problem of independence. The problem, recall, is that it's completely unclear how we should settle a number of fundamental questions in set theory. CH, for example, is neither provable nor disprovable in ZFC. And despite significant efforts, set-theorists and philosophers have failed to find well-motivated principles beyond those of ZFC that manage to prove or disprove it. The width potentialist's response the problem has two parts: one negative and one positive. The negative part says that statements like CH are not unambiguous and well-defined. There is no ultimate background universe of sets, no ultimate V, in which we can query whether CH is true or not. The very questions on which the problem of independence rests are, in other words, illegitimate.

But what does it mean to say there's no ultimate V? Some examples will help. Suppose it turned out that the collection of absolutely all sets satisfied the axioms of ZFC (no matter what possible universe they come from, or indeed if they come from any possible universe at all). That collection would, I take it, count as an ultimate V. After all, it would comprise absolutely all sets and have all of the set theoretic structure required by the axioms of ZFC. Now consider a collection of sets $\mathcal{C}$ satisfying the axioms of ZFC and closed under subsets: that is, for every x in $\mathcal{C}$ and any $y \subseteq x$, y is also in $\mathcal{C}$. I take it that $\mathcal{C}$ too would count as an ultimate V, in the following sense.

The core assumption of modern set theory is that every set is in some rank of the *cumulative hierarchy.*[25] The ranks of the cumulative hierarchy correspond to the stages of the iterative conception, but are explicitly definable within ZFC. We define the first rank, V_0, to be the empty set. Then, for any ordinal α, we define $V_{\alpha+1}$ to be the set of all subsets of V_α; the powerset of V_α. Finally, when λ is a limit ordinal, we define V_λ to be the union of the ranks V_α for $\alpha < \lambda$. The claim that every set is in some rank is then: $\forall x \exists \alpha (x \in V_\alpha)$.

With this core assumption in place, what sets there are is a function of what subsets there are of each V_α and what ordinals there are. As [Reinhardt, 1974, p. 190] puts it:

> ...a general and universal framework... [has now] been provided for set theory by

[25]This claim is provable in ZF and, given the axioms of separation and extensionality, it suffices to prove every other axiom of ZF with the exception of the axioms of infinity and replacement. The axioms of infinity and replacement, in turn, are naturally seen as assumptions about how far the ranks extend, which is to say, how far the ordinals extend. The axiom of infinity tells us that the ordinals extend so far that there's at least one infinite ordinal and the axiom of replacement tells us that the ordinals extend further than any ordinals that can be put in one-one correspondence with the members of any set. Large cardinal hypotheses—which have been suggested as additions to ZF—are also naturally viewed as claims about how far the ordinals extend. Since most set theories that are currently taken to be well-motivated are extensions of ZF with large cardinal hypotheses, this means that most set theories currently taken to be well-motivated can be seen as a combination of the core claim that every set is in some rank together with some claim (or claims) about how far those ranks extend.

> the clarification of the intuitive idea of the cumulative hierarchy (due chiefly to Zermelo) (i.e., the sets in the series [V_0, V_1,...] ...). We might say this reduces all structural questions to questions about "arbitrary subset of" and "arbitrary ordinal".

Beyond the core claim, in other words, the only questions that remain concern the *width* of the V_αs and their *height*. Since our hypothesised collection $\mathcal{C}$ contains absolutely all subsets of any of its members, it settles the first question in the most inclusive way possible. It tells us that each of its V_αs are maximal in width: that, for any α in $\mathcal{C}$, its $V_{\alpha+1}$ contains absolutely all subsets of its V_α. We might say, therefore, that $\mathcal{C}$ is an ultimate V *up to its height*; that is, up to the height of its ordinals. Since CH is a statement concerning sets in a relatively small rank of the cumulative hierarchy—in particular, it's a statement concerning the elements of $V_{\omega+2}$—it would get its ultimate and unambiguous formulation in $\mathcal{C}$. Indeed, we can push the point a little further. If it turned out that there was a set of absolutely all real numbers, a set of absolutely all sets of real numbers, and a set of absolutely all functions from sets of real numbers to the real numbers, CH would get its ultimate and unambiguous formulation in those sets. At the very least, then, in denying that there's an ultimate V, the width potentialist should reject these scenarios.

The positive part of the width potentialist's response to the problem of independence says that the versions of the fundamental questions that *do* remain have available answers. For example, the most salient version of the question whether CH holds is how it behaves across possible universes. But, as Hamkins puts it:

> On the [width potentialist] view, consequently, the continuum hypothesis is a settled question; it is incorrect to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in [possible universes], about how to achieve it or its negation in combination with other diverse set-theoretic properties.

Forcing models within possible universes describe all sorts of possibilities for CH, and the central width potentialist claim tells us that universes witnessing those possibilities are possible. This process of describing possibilities via forcing models and then actualising them in other possible universes provides a rich picture of the behaviour of CH across possible universes.

The width potentialist must be careful to avoid other equally legitimate versions of the fundamental questions. For example, suppose there are collections of sets that are just like possible universes, but *aren't* counted as such by the width potentialist. Then the question of CH's behaviour across possible universes would seem to be as legitimate as the question of its behaviour across these *exouniverses*. The central width potentialist claim, however, is silent on the latter question. It tells us that possibilities described by forcing models in possible universes are witnessed by other possible universes, but it tells us nothing about the possibilities described by forcing models in exouniverses. The width potentialist should thus deny that there are such exouniverses. They should claim, in other words, that whatever looks for all intents and purposes like a possible universe of sets *is* a possible universe of sets. In a slogan: *no universe left behind.* For similar reasons, the width potentialist should deny that there are individual sets outside of any possible universe. If there were, interesting questions concerning their behaviour may arise that the central width potentialist claim is again silent on. The width potentialist should claim, in other words, that absolutely every set is in some possible universe. In a slogan: *no set left behind*.[26]

In summary, the width potentialist's central claim is that possibilities described by forcing models in possible universes are actualised in other possible universes. To support an adequate response to the problem of independence, I've argued that this should be supplemented with three further claims. First, that there is no ultimate V, even up to height. Second, that what looks for all intents and purposes like a possible universe is a possible universe (no universe left behind). And finally, that absolutely every set is in some possible universe (no set left behind).

4 Ultimate V

You may have already noticed the tension between height and width potentialism. On the one hand, the height potentialist's central claim—that possible set existence is a matter of possible co-existence—entails that the height potential sets are maximal in width. In particular, the open-ended schema $\mathsf{Separation}^{\Diamond}$ implies that any subset of a height potential set must itself correspond to a height potential set. On the other hand, the width potentialist rejects width maximalism in various ways. The purpose of this section is to make this tension as precise as possible and draw out its consequences.

I'll work in the language of set theory supplemented with three modal operators: $\Diamond$, $\blacklozenge$, and @. Call this $\mathcal{L}^*_{\in}$. We can think of $\Diamond$ and $\blacklozenge$ as expressing possibility in the height and width potentialist senses respectively, and we can think of @ as *rigidly* referring to the actual circumstance.[27] Given this interpretation, $@\Diamond\phi$ and $\Diamond\phi$ are materially, though not necessarily, equivalent. Similarly, for $@\blacklozenge\phi$ and $\blacklozenge\phi$. What's $\Diamond$-possible or $\blacklozenge$-possible is what's @$\Diamond$-possible or @$\blacklozenge$-possible, though not necessarily so. And since the compound operators @$\Diamond$ and @$\blacklozenge$ turn out to be logically better behaved than $\Diamond$ and $\blacklozenge$,[28] I will focus on what's @$\Diamond$-possible and @$\blacklozenge$-possible rather than what's $\Diamond$-possible or $\blacklozenge$-possible. I will use "could$_{\Diamond}$" and its cognates to express @$\Diamond$-possibility and its cognates, and similarly for "could$_{\blacklozenge}$" and @$\blacklozenge$-possibility.

[26] Of course, by Gödel's incompleteness theorem, the width potentialist cannot completely avoid independence. Indeed, they face relatively interesting instances. For example, it's hard to see how they might settle the question whether there are universes where the axiom of choice fails but where there are Reinhardt cardinals. The working hypothesis, presumably, is that these questions are either not as stubborn as whether CH holds in an ultimate V, or not as important. See, for example, [Hamkins et al., 2012, p.16].

[27] Although natural, we need not think of @ this way. The axioms for @ that I'll appeal to hold whenever it expresses truth in some univocal point of evaluation for the other two operators (see footnote 29). For example, for all those axioms say, it could express truth from the perspective of a possibility corresponding to the minimal non-empty rank $V_1 = \{\emptyset\}$. The axioms, moreover, do not presuppose that what's actual is possible in either sense. That is, they do not presuppose:

$$@\phi \to @\Diamond\phi$$

or:

$$@\phi \to @\blacklozenge\phi$$

[28] See lemmas 1 and 2 in the appendix.

The background modal logic, $\mathsf{K}_@$, is a positive free version of K for the three modal operators together with axioms ensuring the rigidity of @.[29,30] Let $\Diamond^*$ abbreviate the disjunction of $@\Diamond$ and $@\blacklozenge$. That is, let:

$$\Diamond^*\phi =_{df} @\Diamond\phi \vee @\blacklozenge\phi$$

Beyond the axioms of $\mathsf{K}_@$, I'll make use of three non-logical assumptions. The first two concern the nature of sets. As I argue in the next section, they hold whenever $\Diamond$ and $\blacklozenge$ respect the fact that sets are fundamentally extensional in nature. The first is a generalisation of the axiom of extensionality. It says that possible$_{\Diamond^*}$ sets with the same possible$_{\Diamond^*}$ elements are identical. Formally:

(Extensionality$^{\Diamond^*}$) $$\Box^*\forall x\Box^*\forall y(\Box^*\forall z(\Diamond^*(z \in x) \leftrightarrow \Diamond^*(z \in y)) \rightarrow x = y)$$

The second says that possible$_{\Diamond^*}$ sets cannot$_{\Diamond^*}$ exist without their elements. Formally:

(Stability$^{\Diamond^*}$) $$\Box^*\forall x\Box^*\forall y\Box^*(x \in y \rightarrow \Box^*(Ey \rightarrow Ex \wedge x \in y))$$

The third and final assumption is the obvious analogue in the current setting of the modal separation schema Separation$^{\Diamond}$. It says that if ϕ is stable$_{\Diamond}$ and x is any possible$_{\Diamond}$ set, then there could$_{\Diamond}$ be a set of the ϕs in x. Formally:

(Separation$^{\Diamond^*}$) $$@\Box\forall z(@\Box\phi \vee @\Box\neg\phi) \rightarrow @\Box\forall x@\Diamond[Ex \wedge \exists y\forall z(z \in y \leftrightarrow z \in x \wedge \phi)]$$

for $\phi \in \mathcal{L}^*_{\in}$. Just as Separation$^{\Diamond}$ is provable from Plural Comp and Collapse$^{\Diamond}$, given suitable background assumptions, Separation$^{\Diamond^*}$ is provable from Plural Comp and the obvious analogue of Collapse$^{\Diamond}$ in the current setting (see theorem 2).

With these assumptions in place, we can prove a simple but important result: the possible$_{\Diamond}$ sets are closed under their possible$_{\blacklozenge}$ subsets.[31]

Theorem 1. *In* $\mathsf{K}_@$*, it is provable from* Extensionality$^{\Diamond^*}$*,* Stability$^{\Diamond^*}$*, and* Separation$^{\Diamond^*}$ *that:*

(Subset Closure$^{\Diamond}$) $$@\Box\forall x@\blacksquare\forall y(y \subseteq x \rightarrow @\Diamond Ey)$$

[29] See definition 1 in the appendix.

[30] Given @, we're forced to work in a free quantificational logic. The reason is that "Ex" is a theorem of classical logic, and so, by necessitation and universal generalisation, it would follow in $\mathsf{K}_@$ that actually necessarily everything actually exists—$@\Box\forall x@Ex$—which is inconsistent with height potentialism.

[31] So far, I've made no assumptions about what possible$_{\Diamond}$ sets are possible$_{\blacklozenge}$ and vice versa. For all I've said, they might be completely disjoint. Consequently, Subset Closure$^{\Diamond^*}$ may not be as strong as we'd like. Suppose, for example, that when sets don't exist, they have no elements. Formally: $\neg Ex \rightarrow y \notin x$. Now if x couldn't$_{\blacklozenge}$ exist, then it follows that its only possible$_{\blacklozenge}$ subset would be the empty set (assuming the empty set possibly$_{\blacklozenge}$ exists). But there could$_{\blacklozenge}$ still be non-empty subsets of x in the sense that their possible$_{\Diamond^*}$ elements are possible$_{\Diamond^*}$ elements of x. Formally, there could$_{\blacklozenge}$ be non-empty sets y for which:

$$\Box^*\forall z(z \in y \rightarrow \Diamond^*(z \in x))$$

Let $y \subseteq^* x$ abbreviate this claim. The corresponding version of Subset Closure$^{\Diamond^*}$ would then be:

$$@\Box\forall x@\blacksquare\forall y(y \subseteq^* x \rightarrow @\Diamond Ey)$$

It's easy to modify the proof of theorem 1 to show that this version of Subset Closure$^{\Diamond}$ also follows from Extensionality$^{\Diamond^*}$, Stability$^{\Diamond^*}$, and Separation$^{\Diamond^*}$. Similarly, for any of the claims I make below, $\subseteq$ can be replaced with $\subseteq^*$.

Proof. See the appendix. □

Let me now sketch some of the ways theorem 1 leads to inconsistency between height and width potentialism.

The height potential sets look for all intents and purposes like a possible universe of sets. In particular, they have all of the set-theoretic structure required by the axioms of ZFC.[32] The width potentialist should thus count them as a possible universe of sets. No universe left behind, not even a height potential one. It follows from the central width potentialist claim that some possible universe has more sets of natural numbers than there are in the height potential sets. This means, in the current terminology, that there could$_\blacklozenge$ be a subset of some possible$_\lozenge$ set that cannot$_\lozenge$ exist. Formally:

$$@\lozenge\exists x @\blacklozenge\exists y(y \subseteq x \land \neg @\lozenge Ey)$$

which is just the negation of Subset Closure$^\lozenge$. Contradiction!

Let $\omega^\lozenge$ denote the possible$_\lozenge$ set of possible$_\lozenge$ natural numbers and for any possible$_\lozenge$ set x let $\mathcal{P}^\lozenge(x)$ denote its possible$_\lozenge$ powerset. It follows from Subset Closure$^\lozenge$ that $\mathcal{P}^\lozenge(\omega^\lozenge)$ contains all possible$_\blacklozenge$ subsets of $\omega^\lozenge$.[33] Formally:

(*) $$@\blacksquare\forall x(x \subseteq \omega^\lozenge \to @\lozenge(x \in \mathcal{P}^\lozenge(\omega^\lozenge)))$$

Since the width potentialist should count every possible$_\lozenge$ set as a possible$_\blacklozenge$ set—no set left behind, not even a height potential one—there should be a possible universe containing both $\omega^\lozenge$ and $\mathcal{P}^\lozenge(\omega^\lozenge)$. By Stability$^{\lozenge^*}$, $\omega^\lozenge$ must be infinite whenever it exists. So it follows from the central width potentialist claim that there could be a universe containing a new subset of $\omega^\lozenge$. This means, in the current terminology, that:

$$@\blacklozenge\exists x(x \subseteq \omega^\lozenge \land @\blacklozenge(E(P^\lozenge(\omega^\lozenge)) \land \neg Ex))$$

which, by Stability$^{\lozenge^*}$, contradicts (*).[34]

[32] Again, this is overkill. The argument here merely requires that the height potential sets count as a possible universe according to the width potentialist and that the relevant forcing models can be constructed within them. The discussion in the previous section suggests that the height potentialist sets should be counted as a possible universe even if they merely satisfy the axioms of extensionality, foundation, and infinity together with the core claim that every set is in some rank of the cumulative hierarchy. It turns out, moreover, that we can use that theory to construct forcing models that describe possibilities according to which there are new sets of natural numbers.

[33] Here I rely on the fact that, by Stability$^{\lozenge^*}$:

$$\Box^*\forall x\Box^*\forall y(y \subseteq x \to \Box^*(Ex \to y \subseteq x))$$

See the proof of theorem 1 in the appendix.

[34] It's worth noting a much stronger, though slightly more complicated, consequence of the no set left behind policy. First, it can be shown that Subset Closure$^\lozenge$ implies that every well-founded width potential set is a height potential set. To see this, recall that it follows from the central width potentialist claim that any set in any possible universe is countable in some other possible universe. In the current terminology that means:

$$@\blacksquare\forall x @\blacklozenge[Ex \land \exists f(f \text{ is a one-one function from } x \text{ to } \omega)]$$

Therefore, any possible$_\blacklozenge$ set x can$_\blacklozenge$ be coded via its membership structure as a set of natural numbers. But theorem 1 tells us that any such code will possibly$_\lozenge$ exist, assuming that the height and width potential sets agree on the natural numbers. If x's membership structure is well-founded and c is a relation coding it, then it

Although natural, the interpretations of $\Diamond$, $\blacklozenge$, and @ we've been working with are not mandatory. Theorem 1 tells us that the height potential sets comprise all of their possible$_\blacklozenge$ subsets on *any* interpretation of $\blacklozenge$ that obeys $\mathsf{Extensionality}^{\Diamond^*}$ and $\mathsf{Stability}^{\Diamond^*}$. If those principles hold whenever $\blacklozenge$ respects the extensional nature of sets, then absolutely any set whatsoever should possibly$_\blacklozenge$ exist on at least *some* interpretation of $\blacklozenge$ that obeys them. After all, if an object cannot$_\blacklozenge$ exist on any interpretation of $\blacklozenge$ that respects the nature of sets, it ain't a set! So, theorem 1 tells us that the height potential sets comprise absolutely all of their subsets. It tells us, in other words, that the height potential sets are maximal in width and thus constitute an ultimate V up to their height. Consequently, CH gets its ultimate and unambiguous formulation in the height potential sets. Clearly, this contradicts the width potentialist's claim that there is no ultimate V, not even up to height.

We can also vary the interpretation of $\Diamond$. Suppose, for example, that we interpret $\Diamond$ to mean @ and we adopt the iterative or limitation of size conceptions concerning the actual sets. Then the assumptions of theorem 1 will be as plausible as they were when $\Diamond$ expressed height potentiality. In particular, $\mathsf{Separation}^{\Diamond^*}$ will then be equivalent to the non-modal Separation, which we saw in section 2 was an immediate consequence of both the iterative and limitation of size conceptions. Assuming the actual sets satisfy the axioms of ZFC, the above arguments show that both conceptions are also inconsistent with width potentialism. In so far as the *height actualist*—who thinks there couldn't$_\Diamond$ have been more sets than there actually are—adopts one of those conceptions, their view too will be inconsistent with width potentialism.

This may strike you as strange. After all, height potentialism, the iterative conception, and the limitation of size conception seem to represent our best responses to the explanatory challenge raised by Russell's paradox. So width potentialism had better be consistent with one of them! The issue, as I see it, is that the incompatibility between height and width potentialism is downstream from a more fundamental disagreement about what the central explanatory challenge is. For the height potentialist, a crucial target has been the axioms of ZFC. One consequence of theorem 1 is that whatever the reason those axioms hold within a given possible universe of sets, it isn't because height potentialism, or the iterative conception, or the limitation of size conception, holds there. For the width potentialist, the crucial target is not the fact that universes satisfy the axioms of ZFC, but rather the plethora of universes. For them, set theory is like geometry and universes of sets are like arbitrary models of set theory.[35] And just as there is nothing informative to say about why an arbitrary model of geometry satisfies some given geometrical axioms, so too there is nothing informative to say about why a universe of sets satisfies the axioms of ZFC. The height and width potentialist, in other words, are engaged in different and, if the arguments of this paper are correct, ultimately

is a standard result in ZFC that there's a set y whose membership structure is isomorphic to c. $\mathsf{Extensionality}^{\Diamond^*}$ then implies that $x = y$. So every possible$_\blacklozenge$ set with a well-founded membership structure possibly$_\Diamond$ exists. This argument assumes that we have some univocal notion of well-foundedness. Plural quantification provides one way to do this. Since pluralities are nothing over and above the things they comprise, the subpluralities of a set are the same whenever the set exists. As a consequence, on the plural formulation of well-foundedness, a structure will be well-founded whenever it exists or non-well-founded whenever it exists.

Now, if absolutely every set is well-founded and occurs in some possible universe, then the height potential sets comprise absolutely all sets. They would then be an ultimate V. Not merely up to their height, but simpliciter.

[35] See, for example, Hamkins [2012].

incompatible projects.[36,37]

5 Extensionality, stability, and separation

In this section, I'll look at whether we can resist the conclusion of theorem 1—that is, Subset Closure$^{\Diamond^*}$—by rejecting Extensionality$^{\Diamond^*}$, Stability$^{\Diamond^*}$, or Separation$^{\Diamond^*}$. I'll first argue that Extensionality$^{\Diamond^*}$ and Stability$^{\Diamond^*}$ are true on any interpretations of $\Diamond$ and $\blacklozenge$ that respect the extensional nature of sets. I'll then look at a proposal due to Stephen Yablo according to which Separation$^{\Diamond^*}$ should be restricted to suitably determinate conditions. As I show, this restriction turns out to suffice for theorem 1.

5.1 Extensionality and stability

Sets are typically taken to be *extensional* in nature. In particular, they're taken to be completely characterised by what members they have. As Linnebo [2010] puts it:

> [T]he nature of a set is exhausted by what elements it has. Once you specify the elements of a set, you have specified everything that is essential to it. Every other property of the set flows from its having precisely these elements.[38]

[36]It's worth noting that if we separate the central height potentialist claim that set existence is a matter of co-existence from the goal of explaining why the axioms of ZFC are true, it becomes perfectly consistent with width potentialism. For example, one natural model of width potentialism given by the class of all countable transitive models of ZFC. With the right background assumptions, that class is easily shown to obey the claim that set existence is a matter of co-existence. Similarly, a version of the limitation of size conception holds over the model too: namely, some things in the model form a set in some universe precisely when they are countable. So, it is an open question whether some version of either of these claims would help with the width potentialist's project of explaining the plethora of possible universes. It is much harder to see how a version of the iterative conception might play such a role. Relatedly, Studd [2019] advocates a stronger version of Collapse$^{\Diamond}$ according to which any things *must* form a set. The arguments above show that this stronger version *is* inconsistent with width potentialism, since it implies that the height potential sets satisfy the core claim that every set is in some rank of the cumulative hierarchy (see footnote 32).

[37]Faced with these results, it seems that the width potentialist must simply reject height potentialism. But it turns out that a thoroughgoing rejection of height potentialism is harder than it might at first seem. Let me explain.

Modal structuralism is a nominalist friendly form of height potentialism that interprets (or re-interprets) claims in the language of set theory as claims about logically or metaphysically possible ZFC2 structures, where a structure is just some mereological sums of ordinary objects that behave suitably like set-theoretic ordered pairs (see Hellman [1989] and Roberts [2019] for details). In effect, claims about sets are interpreted as claims about pluralities. With suitable modifications, we can obtain a modal structural version of theorem 1 which says that every possible$_{\blacklozenge}$ set corresponding to a subset of some modal structural set is represented in the modal structural sets. So, the modal structural sets would, in other words, comprise an ultimate V up to their height too. Arguably, then, the modal structural interpretation of CH would be something very much like its ultimate and unambiguous formulation. It therefore seems like a thoroughgoing width potentialist should reject modal structuralism. The problem is that, understood metaphysically, modal structuralism is merely a view about how many ordinary objects are metaphysically possible; in particular, it's the view that there are enough to constitute the various ZFC2 structures required by its axioms. Similarly, when understood in terms of logical possibility. It is one thing to make a claim about the sets, understood as sui generis mathematical objects, but quite another to make a claim about the metaphysical and logical possibilities for pluralities of objects in general. Thoroughgoing width potentialism seems to be hostage to metaphysical and logical fortune.

[38]Of course, *being a set* is an essential property of any set that doesn't in general depend on which elements it has. The empty set, for example, has the same elements as any non-set. The claim is thus that once you have specified the elements of a set, you have specified everything that is essential to it *over and above its being a set.* See [Fine, 1981, p. 179-180].

There are two ideas here. The first idea is that sets are *at least* characterised by what members they have. Once you know what members a set has, you know what set it is; you can distinguish it from all other sets. Usually, this is taken to motivate the axiom of extensionality, which is the claim that sets with the same elements are identical.[39]

(Extensionality) $$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$$

But it also motivates Extensionality$^{\lozenge^*}$. If possible$_{\lozenge^*}$ sets have the same possible$_{\lozenge^*}$ elements, the only possible$_{\lozenge^*}$ difference in membership between them would concern *when* they have their members rather than *what* members they have. The difference, in other words, would be intensional rather than extensional. If $\lozenge$ and $\blacklozenge$ respect the extensional nature of sets, then, Extensionality$^{\lozenge^*}$ should be true.

The second idea is that sets are *at most* characterised by what members they have. The thought is that there's *nothing more* to a set beyond what members it has. In so far as it has other properties, it has them in virtue of its having the the members it has. Its other properties 'flow from its having precisely those elements'. Extensionality$^{\lozenge^*}$ tells us that sets are uniquely singled out by their possible$_{\lozenge^*}$ members. But it allows that sets go beyond them in various ways. For example, it's consistent with Extensionality$^{\lozenge^*}$ that the singleton of the empty set, $\{\emptyset\}$, contains $\emptyset$ when and only when there's a greatest ordinal. In that case, there would be more to $\{\emptyset\}$ than merely *what* members it has. There would also be non-trivial facts about *when* it has them, facts that don't flow from its having $\emptyset$ as its sole member. Its membership behaviour, in other words, would be non-trivially intensional. To rule this out, we should at the very least require that sets don't change their elements across possibilities where they exist. And this is precisely what Stability$^{\lozenge^*}$ says.[40,41] If $\lozenge$ and $\blacklozenge$ respect the extensional nature of sets, then, Stability$^{\lozenge^*}$ should also be true.[42]

[39] Extensionality is universally adopted by philosophers working on modal set theory. See, for example, Linnebo [2013], Linnebo [2018], Studd [2019], Parsons [1983b], Fine [1981], and Reinhardt [1980].

[40] It has very little to say about what members a set has when it *doesn't* exist, however. To rule out all non-trivial intensional membership behaviour, then, Stability$^{\lozenge^*}$ will also need to be supplemented with assumptions covering that case. See Roberts [2022] for discussion of the same issue in the case of pluralities and some natural principles we can add.

[41] Versions of Stability$^{\lozenge^*}$ have also been unanimously adopted by philosophers working on modal set theory. See the references in footnote 39.

[42] I have implicitly assumed so far that height and width potentialism concern a univocal and *absolute* notion of set. The things that are said to be possible$_{\lozenge}$ are the same kind of thing that are said to be possible$_{\blacklozenge}$: namely, sets! It's against this background that it even makes good sense to talk about *the* nature of sets. But we may not think that there is an absolute notion of set. Or, we may simply not want to work with such a notion. The modal structuralist, for example, works with a *relative* notion of set-according-to-a-ZFC2-structure. And if $\lozenge$ and $\blacklozenge$ do concern the possibility of sets in different senses of set, Extensionality$^{\lozenge^*}$ and Stability$^{\lozenge^*}$ may simply be false. Nevertheless, theorem 1 can be used by the relativist to obtain a similar conclusion without a detour through the nature of sets. To see this, suppose that Extensionality$^{\lozenge^*}$ and Stability$^{\lozenge^*}$ merely hold for the @$\lozenge$-possible sets (that is, they hold when we replace $\lozenge^*$ throughout with @$\lozenge$). Now suppose there could be a set y—in some sense of "could" and some sense of set—of elements of a possible$_{\lozenge}$ set x such that *no possible*$_{\lozenge}$ *set has the same elements as* y. I claim that it must be possible—in at least some sense of "possible"—that (1) there's a set z with the same elements as y which does not possibly$_{\lozenge}$ exist, (2) everything in the transitive closure of x according to @$\lozenge$ exists and has its @$\lozenge$-members, and (3) nothing else exists. Let $\blacklozenge$ access one such possibility. It is not hard to see that Extensionality$^{\lozenge^*}$ and Stability$^{\lozenge^*}$ will then hold *trivially*. It follows from theorem 1 that z must possibly$_{\lozenge}$ exist after all. Contradiction! Theorem 1 tells us, in other words, that any set in *any* sense whose elements are contained in a possible$_{\lozenge}$ set is co-extensive with a possible$_{\lozenge}$ set. To me, that's a perfectly good statement of the claim that the possible$_{\lozenge}$ sets are width maximal and thus—assuming they satisfy enough set theory—comprise an ultimate V up to their height. Of course, this result is also available to an absolutist who would rather avoid a detour through the nature of sets.

5.2 Separation and determinacy

As I argued in section 2, $\mathsf{Separation}^{\Diamond^*}$ is an immediate consequence of the central height potentialist claim that possible$_\Diamond$ set existence is a matter of possible$_\Diamond$ co-existence. In more precise, plural terms, $\mathsf{Separation}^{\Diamond^*}$ is provable from Plural Comp and $\mathsf{Collapse}^\Diamond$ (theorem 2). But some have denied instances of Plural Comp. For them, plural existence requires more than mere co-existence. The nothing over and above conception is mistaken. If they're right, then the potentialist could adopt a fall back position according to which possible$_\Diamond$ set existence is a matter of the possible$_\Diamond$ existence of a plurality but where plural existence is a matter of co-existence plus some extra requirement. The idea, in other words, would be to keep $\mathsf{Collapse}^\Diamond$ and articulate some well-motivated restriction on Plural Comp. In this section, I want to look at one such proposal due to Stephen Yablo.[43]

For Yablo [2006], the existence of a plurality of ϕs is a matter of the determinacy of the ϕs. As he puts it:

> The view once again is that plurality comprehension is mistaken.
>
> This may seem at first puzzling. The property P that (I say) fails to define a plurality can be a perfectly determinate one; for any object x, it is a determinate matter whether x has P or lacks it. How then can it fail to be a determinate matter what are all the things that have P? I see only one answer to this. Determinacy of the Ps follows from
>
> (i) determinacy of P in connection with particular candidates,
>
> (ii) determinacy of the pool of candidates.
>
> If the difficulty is not with (i), it must be with (ii). It is not the case that there are some things the [xxs] such that every candidate for being P is among them. If there were, one could go through the [xx]s one by one, asking of each whether it has P, thus arriving finally at the sought-after plurality of Ps.

In a moment, I'll show that theorem 1 only requires Plural Comp for ϕ obeying (i) and (ii) and is therefore kosher by Yablo's lights. But before I do that, I want to mention some misgivings about his proposal.

It's well known that classical treatments of determinacy force us to distinguish sharply between its being determinate that something is ϕ and something's being determinately ϕ. Formally, we are forced to distinguish sharply between:

$$\Delta\exists x\phi$$

[43] Linnebo [2016] and Florio and Linnebo [2020] articulate an alternative requirement according to which pluralities must be "circumscribed". Their notion of "circumscription", however, exclusively concerns worries of type (ii) below. As a consequence, they accept an unrestricted plural separation schema, which says, of any condition ϕ and for any things xx, that there are some things which are all and only the ϕs among the xxs. As I'll point out, this already suffices for theorem 1.

Given their restrictions on Plural Comp, Yablo, Linnebo, and Florio and Linnebo all adopt a non-modal collapse principle which says that any things form a set: $\forall xx\exists x(x \equiv xx)$. It is then hard to see what role, if any, there is for a notion of potentiality. Indeed, if there couldn't have been more pluralities of actual things than there actually are, the following simple argument shows that any possible pure set actually exists, given non-modal collapse. Assume that x is a $\in$-least possible set that doesn't actually exist. So all its members actually exist and therefore the plurality of its members also actually exists. Thus, by non-modal collapse, it actually exists, contradicting our assumption.

and:

$$\exists x \Delta \phi$$

where Δ is an operator expressing determinacy. Consider, for example, a sorites series $x_0, ..., x_n$ for F. It's a consequence of classical logic that something is the last F in the series. Since the consequences of classical logic are determinate, it's therefore determinate that something is the last F in the series. But, because it's a sorites series, nothing is determinately the last F. Formally, we have:

$$\Delta \exists x (x \text{ is the last } F \text{ in the series})$$

but not:

$$\exists x \Delta (x \text{ is the last } F \text{ in the series})$$

In the classical setting, then, the determinacy of F in connection with particular candidates is orthogonal to the determinate existence of a cut-off point for F, although not to the existence of a determinate cut-off point.

The distinction is equally important for pluralities. Consider, for example, a sorties series $xx_0, ..., xx_n$ for "surrounds the house". It's a consequence of classical logic that some things are the last things in the series surrounding the house. Since the consequences of classical logic are determinate, it's therefore determinate that some things are the last things in the series surrounding the house. But, because it's a sorites, no things are determinately the last things surrounding the house.

I find it very natural to think that Plural Comp patterns with these cases, in the sense that once we sharply distinguish Plural Comp from:

(Plural Comp$^{\Delta}$) $$\exists xx \Delta \forall x (x \prec xx \leftrightarrow \phi)$$

the determinacy of ϕ seems to be orthogonal to the (determinate) existence of a plurality of the ϕs, even though it is relevant to the existence of a determinate plurality of the ϕs.

One way to see this is by considering a Kripke model K for determinacy. The first-order domains of its points can vary as much as we like—so that the "pool of candidates" is as indeterminate as we like—and the interpretation of F can vary across points as much as we like—so that there is as much indeterminacy of F in connection with individual candidates as we like. Nevertheless, as long as we require that the plural domain of a point in K is (or is isomorphic to) the powerset of its first-order domain and truth at a point is defined in the usual classical way, Plural Comp is guaranteed.[44]

Another way to see the point is by noting that in the classical setting, Yablo's account implies many instances of Plural Comp for indeterminate ϕ. For example, let $x_0, ..., x_n$ again be a sorites series for F. It's provable in classical logic that the Fs in the series are either all and only $x_0, ..., x_n$ or all and only $x_0, ..., x_{n-1}$ or all and only $x_0, ..., x_{n-2}$, and so on. Assuming identity and distinctness are determinate, every condition of the form "$x = x_0 \vee x = x_1 \vee ... \vee x = x_m$" has determinate application with respect to particular candidates. And since its pool of candidates is finite, that too is plausibly determinate. By Yablo's lights, Plural Comp should thus hold for each of these conditions. It follows that there are some things which are all and only the Fs in the series. Moreover, if we assume that "$x \prec xx$"

[44] The situation changes drastically in non-classical settings. For example, if we use a three-valued Weak Kleene semantics that assigns value $1/2$ to $F(o)$ at a point for some o in its first-order domain, the corresponding instance of Plural Comp will also be assigned value $1/2$.

is suitably determinate, then no things are determinately the Fs in the series. Formally, we have:

$$\exists xx \forall x(x \prec xx \leftrightarrow x \text{ is an } F \text{ in the series})$$

but not:

$$\exists xx \Delta \forall x(x \prec xx \leftrightarrow x \text{ is an } F \text{ in the series})$$

The reasoning generalises. Suppose the xxs are our pool of candidates. Since they are determinate, it is natural to think that we can make good sense of infinitary reasoning about them. So assume, for the sake of argument, that we can make sense of $2^{|xx|}$-length conjunctions and disjunctions about them. Then, in classical logic, the infinitary generalisation of the above reasoning will say, for any ϕ, that the ϕs in xx are either all and only $x_0, ..., x_i, ...$ or all and only $y_0, ..., y_i, ...$ or ... Since each condition of the form "$w = x_0 \vee w = x_1 \vee ...$" has determinate application with respect to particular candidates, each condition of the form "$w \prec xx \wedge (w = x_0 \vee w = x_1 \vee ...)$" has both determinate application with respect to particular candidates and a determinate pool of candidates. So, by Yablo's account, each should define a plurality. It follows that Plural Comp holds for "$w \prec xx \wedge \phi$". At the very least, this suggests that (i) is effectively redundant and that everything turns on (ii).[45]

Regardless of these misgivings, I will now argue that theorem 1 only relies on instances of Plural Comp that satisfy (i) and (ii).

By inspecting the proof of theorem 1, it is easy to see that it relies on a single instance of Separation$^{\Diamond^*}$, which says that for any possible$_{\Diamond^*}$ set x and any possible$_{\Diamond}$ set y, there could$_{\Diamond}$ be a set of the elements of y that are possibly$_{\Diamond^*}$ in x. Formally:

(Separation$^{\Diamond^*-}$) $$\Box^* \forall x @ \Box \forall y @ \Diamond \exists z[Ey \wedge \forall w(w \in z \leftrightarrow w \in y \wedge \Diamond^*(w \in x)])$$

This, in turn, is a consequence of the following instance of Plural Comp, which says that for any possible$_{\Diamond^*}$ set x and any possible$_{\Diamond}$ set y, there are some things which are all and only the elements of y that could$_{\Diamond^*}$ be in x (theorem 2). Formally:

(Plural Comp$^-$) $$\Box^* \forall x @ \Box \forall y \exists xx \forall w(w \prec xx \leftrightarrow w \in y \wedge \Diamond^*(w \in x))$$

Now, given Stability$^{\Diamond *}$, the members of a set *are* its possible$_{\Diamond^*}$ members. Formally:

$$\Box^* \forall y, w(\Diamond^*(w \in y) \leftrightarrow w \in y)$$

It follows immediately that Plural Comp$^-$ is equivalent to:

(1) $$\Box^* \forall x @ \Box \forall y \exists xx \forall w(w \prec xx \leftrightarrow \Diamond^*(w \in y) \wedge \Diamond^*(w \in x))$$

So, we can focus all our attention on the legitimacy of (1).

[45] These observations also show that although (i) and (ii) may suffice for an instance of Plural Comp, they aren't in general necessary. But, I take it that we ultimately want an account of pluralities that tells us when an instance of Plural Comp is true. In other words, we want an account of the dividing line is between the conditions that determine pluralities and those that don't. Whatever *that* dividing line, it isn't characterised by (i) and (ii). (It's worth noting that there's a reading of (ii) on which the combination of (i) and (ii) don't imply the existence of *any* pluralities. In particular, given Yablo's comments, it's natural to read (ii) as the claim that there some things comprising *at least* the ϕs: formally, $\exists xx \forall x(\phi \rightarrow x \prec xx)$. But then it would be a precondition of (ii) that there are some things.)

Since the pool of candidates for (1) comprises the members of the set y, that is clearly determinate. I will now argue that the condition "$\Diamond^*(w \in y) \wedge \Diamond^*(w \in x)$" has determinate application with respect to particular candidates. I will assume that Δ obeys the modal logic K. It follows that the compound operator $\Delta\Box^*$ also obeys K. It is natural to formulate the claim that "$\Diamond^*(w \in y) \wedge \Diamond^*(w \in x)$" has indeterminate application with respect to a particular candidate w as:

(*) $$\neg\Delta(\Diamond^*(w \in y) \wedge \Diamond^*(w \in x)) \wedge \neg\Delta\neg(\Diamond^*(w \in y) \wedge \Diamond^*(w \in x))$$

But there's a caveat. This formulation only works when it's determinate what exists (in our case, when it's determinate what possibly$_{\Diamond^*}$ exists). Suppose, for example, that when sets don't exist they aren't sets. Formally: $\neg Ex \rightarrow \neg Set(x)$. Then if it's neither determinate that x exists nor determinate that it doesn't, it will be neither determinate that it's a set nor determinate that it isn't. "$Set(x)$", in other words, would have indeterminate application with respect to particular candidates. In general, if Δ is sufficiently liberal with existence, lots of completely harmless conditions will fail to have determinate application with respect to particular candidates. To avoid this, I will require that the sets in question at least determinately possibly$_{\Diamond^*}$ exist. In particular, I will assume that:

(**) $$\Delta\Diamond^* Ey \wedge \Delta\Diamond^* Ex$$

I argued above that when $\Diamond$ and $\blacklozenge$ respect the extensional nature of sets, $\mathsf{Stability}^{\Diamond^*}$ should be true. The same argument supports the following stronger principle.

$$\Box^*\forall x\Box^*\forall y\Box^*(\Diamond^*(x \in y) \rightarrow \Box^*(Ey \rightarrow Ex \wedge x \in y))$$

I claim that when $\Diamond$ and $\blacklozenge$ *determinately* respect the extensional nature of sets, the following version of this principle for the compound operator $\Delta\Box^*$ should also be true.

(***) $$\neg\Delta\neg\Diamond^*(x \in y) \rightarrow \Delta\Box^*(Ey \rightarrow (Ex \wedge x \in y))$$

Given the right-hand-side of (*), (***) implies:

(****) $$\Delta\Box^*(Ey \rightarrow w \in y)$$

and:

(*****) $$\Delta\Box^*(Ex \rightarrow w \in x)$$

So by the left-hand-side of (*), it follows that:

$$\neg\Delta\Diamond^* Ey \vee \neg\Delta\Diamond^* Ex$$

In other words, it follows from the assumption that "$\Diamond^*(w \in y) \wedge \Diamond^*(w \in x)$" has indeterminate application that either it is not determinate that x possibly$_{\Diamond^*}$ exists or it is not determinate that y possibly$_{\Diamond^*}$ exists, contradicting (**).

6 Appendix

Definition 1. *The axioms of* $\mathsf{K}_@$ *comprise the instances in* $\mathcal{L}_\in + \{\Diamond, \blacklozenge, @\}$ *of the truth-functional tautologies, the following quantificational and identity axioms:*

(Q1) $\forall y(\forall x\phi \to \phi[y/x])$, *where* y *is free for* x *in* ϕ

(Q2) $\forall x(\phi \to \psi) \to (\forall x\phi \to \forall x\psi)$

(Q3) $\phi \leftrightarrow \forall x\phi$, *where* x *is not free in* ϕ

(I1) $x = y$

(I2) $x = y \to (\phi[x/z] \leftrightarrow \phi[y/z])$, *where* x *and* y *are free for* z *in* ϕ

the K *axiom for* $\Diamond, \blacklozenge$, *and* $@$*—that is, for* $\Diamond$, $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$*—and the following axioms ensuring the rigidity of* $@$*:*

(@1) $@\neg\phi \leftrightarrow \neg@\phi$

(@2) $@\forall x@\phi \leftrightarrow @\forall x\phi$

(@3) $\Diamond@\phi \leftrightarrow @\phi$

(@4) $\blacklozenge@\phi \leftrightarrow @\phi$

(@5) $@\Box\forall x@\Box\phi \to @\Box\forall x\phi$

(@6) $@\blacksquare\forall x@\blacksquare\phi \to @\blacksquare\forall x\phi$

The rules of inference for $\mathsf{K}_@$ *are* MP*—from* ϕ *and* $\phi \to \psi$ *infer* ψ*—*GEN*—if* ϕ *is a theorem, then so is* $\forall x\phi$*—and* NEC *for* $\Diamond, \blacklozenge$, *and* $@$*—that is, for* $\Diamond$, *if* ϕ *is a theorem, then so is* $\Box\phi$.

NEC, K, and (@1) jointly ensure that $@\Diamond$ is the dual of $@\Box$ (and similarly for $@\blacklozenge$ and $@\blacksquare$). I will use this fact without comment in what follows.

In section 4, I claimed that the compound operators $@\Diamond$ and $@\blacklozenge$ are logically better behaved than $\Diamond$ and $\blacklozenge$. What I meant by this is that they both (almost) obey the principles of the modal logic S5. The following lemmas make this precise.

Lemma 1. *If* $\mathcal{O}$ *and* $\mathcal{O}'$ *are among* $@\Diamond, @\blacklozenge$, *and* $\Diamond^*$, *then* $\mathsf{K}_@$ *proves:*

$$\mathcal{O}'\mathcal{O}\phi \leftrightarrow \mathcal{O}\phi$$

$$\neg\mathcal{O}'\neg\mathcal{O}\phi \leftrightarrow \mathcal{O}\phi$$

Proof. I'll establish the first claim for $\mathcal{O}' = @\Diamond$ and $\mathcal{O} = @\blacklozenge$. The other cases are proved similarly. By NEC and K, it follows from (@3) that:

(1) $$@\Diamond@\blacklozenge\phi \leftrightarrow @@\blacklozenge\phi$$

It is an immediate consequence of (@2) that:

(2) $$@@\blacklozenge\phi \leftrightarrow @\blacklozenge\phi$$

since we can pick x so that it is not free in ϕ. Thus:

$$@\Diamond@\blacklozenge\phi \leftrightarrow @\blacklozenge\phi$$

as required. □

Corollary 1. *If $\mathcal{O}$ is one of $@\Diamond$, $@\blacklozenge$, or $\Diamond^*$, then $\mathsf{K}_@$ proves each instance of the following axioms.*

(4) $\mathcal{O}\mathcal{O}\phi \to \mathcal{O}\phi$

(5) $\mathcal{O}\phi \to \neg\mathcal{O}\neg\mathcal{O}\phi$

(T^-) $\neg\mathcal{O}\neg(\phi \to \mathcal{O}\phi)$

This means that compound modal claims can be moved freely in and out of other compound modal contexts. I'll frequently use this fact without comment in what follows. The next result shows that we can naturally reason existentially within the scope of our compound operators.

Lemma 2. *If $\mathcal{O}$ is one of $@\Diamond$, $@\blacklozenge$, or $\Diamond^*$, then $\mathsf{K}_@$ validates the following rule of inference. If $\mathcal{O}\psi$ is provable from Γ and $\mathcal{O}(Ex \wedge \phi)$ (where x is not free in ψ or any formula in Γ), then $\mathcal{O}\psi$ is provable from Γ and $\mathcal{O}\exists x\phi$.*

Proof. I will prove the case for $@\Diamond$. The other two cases are proved similarly. Suppose $@\Diamond\psi$ is provable from Γ and $@\Diamond(Ex \wedge \phi)$ as above. Thus, Γ proves $@\Box\neg\psi \to @\Box(Ex \to \neg\phi)$. It follows by NEC, GEN, and K that:

$$@\Box\forall x@\Box\neg\psi \to @\Box\forall x@\Box(Ex \to \neg\phi)$$

Since x isn't free in ψ, this is equivalent to:

$$@\Box@\Box\neg\psi \to @\Box\forall x@\Box(Ex \to \neg\phi)$$

And thus to:

$$@\Box\neg\psi \to @\Box\forall x@\Box(Ex \to \neg\phi)$$

Finally, by (@5), we have:

$$@\Box\neg\psi \to @\Box\forall x\neg\phi$$

and thus:

$$@\Diamond\exists x\phi \to @\Diamond\psi$$

□

I'm now in a position to prove theorem 1.

Theorem 1. *In $\mathsf{K}_@$, it is provable from* Extensionality$^{\Diamond^*}$, Stability$^{\Diamond^*}$, *and* Separation$^{\Diamond^*}$ *that:*

(Subset Closure$^\Diamond$) $$@\Box\forall x@\blacksquare\forall y(y \subseteq x \to @\Diamond Ey)$$

Proof. By lemma 2, it suffices to show that $@\Diamond Ex$ and $@\blacklozenge(Ey \wedge y \subseteq x \wedge \neg@\Diamond Ey))$ are jointly inconsistent. So, assume they both hold. By lemma 1, "$\Diamond^*(z \in y)$" is possibly$_\Diamond$ stable. So, Separation$^{\Diamond^*}$ implies that:

$$@\Diamond(Ex \wedge \exists w\forall z(z \in w \leftrightarrow z \in x \wedge \Diamond^*(z \in y)))$$

By lemma 2, let w be a witness:

(1) $$@\Diamond(Ex \wedge Ew \wedge \forall z(z \in w \leftrightarrow z \in x \wedge \Diamond^*(z \in y)))$$

I will show that $\Box^*(w = y)$ and thus $@\Diamond Ey$, contradicting our initial assumptions. By $\mathsf{Extensionality}^{\Diamond^*}$, it suffices to prove:

$$\Box^*\forall t(\Diamond^*(t \in w) \leftrightarrow \Diamond^*(t \in y))$$

So, for contradiction, suppose:

(2) $$\Diamond^*(Et \wedge \Diamond^*(t \in w) \wedge \not\Diamond^*(t \in y))$$

By $\mathsf{Stability}^{\Diamond^*}$, this immediately contradicts (1). So assume instead that:

(3) $$\Diamond^*(Et \wedge \neg\Diamond^*(t \in w) \wedge \Diamond^*(t \in y))$$

By $\mathsf{Stability}^{\Diamond^*}$ and $@\blacklozenge(Ey \wedge y \subseteq x \wedge \neg @\Diamond Ey))$, it follows that $\Diamond^*(t \in x)$. So, by $\mathsf{Stability}^{\Diamond^*}$ and (1), $\Diamond^*(t \in w)$, contradicting (3). □

Now I will show that Plural Comp and $\mathsf{Collapse}^{\Diamond}$ together with natural background assumptions imply $\mathsf{Separation}^{\Diamond^*}$. First, we expand $\mathsf{K}_{@}$ to the language of plural logic in the usual way. Then, we add three new assumptions. The first is an analogue of $\mathsf{Stability}^{\Diamond^*}$ for pluralities:

(Plural $\mathsf{Stability}^{\Diamond^*}$) $$\Box^*\forall x \Box^*\forall xx \Box^*(x \prec xx \rightarrow \Box^*(Exx \rightarrow Ex \wedge x \prec xx))$$

It is easy to see that this is a consequence of downward dependence. The second principle is a consequence of upward dependence. It says if the xxs are among x's members, then they exist whenever x does. Formally:[46]

(Set-UD) $$\Box^*\forall x \Box^*\forall xx(xx \subseteq x \rightarrow \Box^*(Ex \rightarrow Exx))$$

Finally, I need a principle about the compossibility$_\Diamond$ of sets. In particular, I need a principle which says that any possible$_\Diamond$ set can$_\Diamond$ co-exist with any of its possible$_\Diamond$ subsets. For simplicity, though, I'll work with the stronger claim that any two possible$_\Diamond$ sets can$_\Diamond$ co-exist.

(Compossibility) $$@\Box\forall x @\Box\forall y @\Diamond(Ex \wedge Ey)$$

It will then be routine but tedious to modify the proof so as to use the weaker claim.

Theorem 2. *In* $\mathsf{K}_{@}$, Plural Comp, $\mathsf{Collapse}^{\Diamond}$, Plural $\mathsf{Stability}^{\Diamond^*}$, *and* Compossibility *imply each instance of* $\mathsf{Separation}^{\Diamond^*}$.

Proof. Suppose ϕ is possibly$_\Diamond$ stable. Then it is stable by lemma 1. Now, suppose that $@\Diamond Ex$. By lemma 2, it suffices to show that $@\Diamond(Ex \wedge \exists z \forall y(y \in z \leftrightarrow y \in x \wedge \phi))$. By Plural Comp:

$$@\Diamond(Ex \wedge \exists xx \forall y(y \prec xx \leftrightarrow y \in x \wedge \phi))$$

By lemma 2, let xx witness this claim.

(1) $$@\Diamond(Ex \wedge Exx \wedge \forall y(y \prec xx \leftrightarrow y \in x \wedge \phi))$$

[46]This principle is actually provable from Plural Comp and an analogue of $\mathsf{Extensionality}^{\Diamond^*}$ for pluralities. See Roberts [2022] for discussion. Nevertheless, for our purposes, it is more illuminating to work directly with Set-UD.

By $\mathsf{Collapse}^{\Diamond}$, it follows that:

$$@\Diamond(Exx \land \exists z \forall y(y \in z \leftrightarrow y \prec xx))$$

By lemma 2, let z witness this claim.

$$@\Diamond(Exx \land Ez \land \forall y(y \in z \leftrightarrow y \prec xx)) \tag{2}$$

By Compossibility and Set-UD, we have:

$$@\Diamond(Exx \land Ex \land Ez) \tag{3}$$

By $\mathsf{Stability}^{\Diamond^*}$, $\mathsf{Plural\ Stability}^{\Diamond^*}$, and the stability of ϕ, we have:

$$\Box^*([Exx \land Ex \land \forall y(y \prec xx \leftrightarrow y \in x \land \phi)] \to \Box^*(Exx \land Ex \to \forall y(y \prec xx \leftrightarrow y \in x \land \phi)))$$

For the same reason, it follows from $\mathsf{Stability}^{\Diamond^*}$ and $\mathsf{Plural\ Stability}^{\Diamond^*}$ that:

$$\Box^*([Exx \land Ez \land \forall y(y \in z \leftrightarrow y \prec xx)] \to \Box^*(Exx \land Ez \to \forall y(y \in z \leftrightarrow y \prec xx)))$$

It thus follows from (1), (2), and (3) that:

$$@\Diamond(Ex \land \exists z \forall y(y \in z \leftrightarrow y \in x \land \phi))$$

as required. □

References

S. Berry. Modal structuralism simplified. *Canadian Journal of Philosophy*, 48(2):200–222, 2018. doi: 10.1080/00455091.2017.1344502.

G. Boolos. The Iterative Conception of Set. *The Journal of Philosophy*, 68:215–232, 1971. Reprinted in Boolos [1998].

G. Boolos. Iteration Again. *Philosophical Topics*, 17:5–21, 1989. Reprinted in Boolos [1998].

G. Boolos. *Logic, Logic, and Logic*. Harvard University Press, Cambridge, MA, 1998. Edited by Richard Jeffrey.

J. P. Burgess. *E pluribus unum*: Plural Logic and Set Theory. *Philosophia Mathematica*, 12 (3):193–221, 2004.

K. Fine. First-order modal theories i–sets. *Nos*, 15(2):177–205, 1981. ISSN 00294624, 14680068. URL `http://www.jstor.org/stable/2215323`.

S. Florio and O. Linnebo. Critical plural logic. *Philosophia Mathematica*, 28(2):172–203, 2020. doi: 10.1093/philmat/nkaa020.

J. D. Hamkins. The set-theoretic multiverse. *Review of Symbolic Logic*, 5(3):416–449, 2012.

J. D. Hamkins and O. Linnebo. The modal logic of set-theoretic potentialism and the potentialist maximality principles. *Review of Symbolic Logic*, pages 1–36, forthcoming. doi: 10.1017/s1755020318000242.

J. D. Hamkins, G. Kirmayer, and N. L. Perlmutter. Generalizations of the kunen inconsistency. *Annals of Pure and Applied Logic*, 163(12):1872–1890, 2012.

G. Hellman. *Mathematics without Numbers: Towards a Modal-Structural Interpretation.* Clarendon, Oxford, 1989.

T. Jech. *Set Theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The Third Millennium Edition.

P. Koellner. On the Question of Absolute Undecidability. *Philosophia Mathematica*, 14(2): 153–188, 2006.

K. Kunen. *Set Theory.* College Publications, 2011.

A. Levy. On von neumann's axiom system for set theory. *The American Mathematical Monthly*, 75(7):762–763, 1968. ISSN 00029890, 19300972. URL `http://www.jstor.org/stable/2315201`.

A. Lévy and R. Vaught. Principles of Partial Reflection in the Set Theories of Zermelo and Ackermann. *Pacific Journal of Mathematics*, 11:1045–1062, 1961.

O. Linnebo. Pluralities and sets. *Journal of Philosophy*, 107(3):144–164, 2010.

O. Linnebo. The Potential Hierarchy of Sets. *The Review of Symbolic Logic*, 6:205–228, 2013.

O. Linnebo. Plurals and modals. *Canadian Journal of Philosophy*, 46(4-5):654–676, 2016. doi: 10.1080/00455091.2015.1132975.

O. Linnebo. *Thin Objects.* Oxford: Oxford University Press, 2018.

C. Menzel. Wide sets, zfcu, and the iterative conception. *Journal of Philosophy*, 111(2): 57–83, 2014. doi: 10.5840/jphil201411124.

C. Parsons. What is the Iterative Conception of Set? In R. E. Butts and J. Hintikka, editors, *Logic, Foundations of Mathematics and Computability Theory*, volume 9 of *University of Western Ontario Series in Philosophy of Science*, pages 335–367. Reidel, Dordrecht, 1977. Reprinted in Parsons [1983a].

C. Parsons. *Mathematics in Philosophy: Selected Essays.* Cornell University Press, Ithaca, 1983a.

C. Parsons. Sets and Modality. In *Mathematics in Philosophy: Selected Essays*, pages 298–341. Cornell University Press, Ithaca, 1983b.

A. Paseau. Boolos on the Justification of Set Theory. *Philosophia Mathematica*, 15(1):30–53, 2007.

H. Putnam. Mathematics without foundations. *Journal of Philosophy*, 64(1):5–22, 1967.

W. N. Reinhardt. Remarks on Reflection Principles, Large Cardinals, and Elementary Embeddings. In T. Jech, editor, *Axiomatic Set Theory*, pages 189–205. American Mathematical Society, Providence, 1974.

W. N. Reinhardt. Satisfaction Definitions and Axioms of Infinity in a Theory of Properties with Necessity Operator. In *Mathematical logic in Latin America*, volume 99 of *Studies in Logic and the Foundations of Mathematics*, pages 267–303. North-Holland, Amsterdam, 1980.

S. Roberts. *Potentialism and reflection.* PhD thesis, Birkbeck, University of London, London, 2016.

S. Roberts. Modal structuralism and reflection. *The Review of Symbolic Logic*, 12(4):823860, 2019. doi: 10.1017/S1755020318000138.

S. Roberts. Pluralities as nothing over and above. *Journal of Philosophy*, 119(8):405—424, 2022.

C. Scambler. Can all things be counted? *Jounral of Philosophical Logic*, forthcoming.

J. P. Studd. *Everything, More or Less: A Defence of Generality Relativism.* Oxford University Press, 2019.

G. Uzquiano. Varieties of indefinite extensibility. *Notre Dame Journal of Formal Logic*, 56 (1):147–166, 2015. doi: 10.1215/00294527-2835056.

S. Yablo. Circularity and paradox. In T. Bolander, V. F. Hendricks, and S. A. Pedersen, editors, *Self-Reference*, pages 139–157. CSLI Publications, 2006.

E. Zermelo. Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre. *Fundamenta Mathematicae*, 16:29–47, 1930. Translated in Zermelo [1996] vol. 2 pp. 1219-1233.

E. Zermelo. On Boundary Numbers and Domains of Sets: New Investigations in the Foundations of Set Theory. In W. Ewald, editor, *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, volume 2, pages 1219–1233. Oxford University Press, New York, 1996. Two volumes.