

Andrei Rodin

Categories without structures

1. Introduction: Renewing Foundations

Foundations of mathematics, historically speaking, is the least stable part of this science. Pythagorean theorem, for example, stands firm since its early discovery while foundations of geometry (and in particular foundations of Pythagorean theorem) several times fall down and then have been totally renewed during the same historical period. Although these changes in foundations affected the theorem there is still a sense in which the Pythagorean theorem always remained the same. (In the Conclusion of this paper I'll make this sense more precise). But foundations of mathematics underlying different versions of this theorem did not remain the same. The traditional view that mathematics is about number and magnitude and the modern view that mathematics is about sets and structures differ radically. Actually saying that different foundations differ radically is pleonastic since one describes a difference as radical exactly when it concerns foundations rather than anything else. The above observation shows that the popular architectural metaphor of science, which describes science as an edifice with a solid foundation, is completely misleading when one talks about science in a long-term historical perspective. The renewal of foundations is not only compatible with the progress of science but also helps to make this progress possible, as we shall now see.

The notion of progress in science assumes that once certain knowledge is acquired it later remains preserved and publicly available. It happens, of course, that certain beliefs, which at some point of history are generally seen as elements of current scientific knowledge, are later refuted and disqualified. However the notion of scientific progress concerns knowledge itself, not our current beliefs about what does and what does not qualify as knowledge. Blurring the difference between the true knowledge and the related beliefs would make the notion of progress incoherent. So let me now ignore the issue of belief revision and ask a different question: Where and how the ready-made knowledge is preserved and endures through human history?

According to Popper (1978) scientific knowledge and other products of human intellect live in a special metaphysical domain that he calls the Third World. The First world on Popper's account is that of physical processes and physical objects while the Second world is that of mental states. Popper's rationale behind his notion of Third Worlds is to avoid reducing knowledge to either mental states or physical processes:

Knowledge in the objective sense consists not of thought processes but of thought contents. It consists of the content of our linguistically formulated theories; of that content which can be, at least approximately, translated from one language into another. The objective thought content is that which remains invariant in a reasonably good translation. Or more realistically put: the objective thought content is what the translator tries to keep invariant, even though he may at times find this task impossibly difficult. (Italic is Popper's, underlining mine)

What is relevant to our present discussion here is not Popper's metaphysical argument but the way in which Popper thinks about thought contents in general and the content of scientific theories in particular. As a matter of course Popper doesn't identify the content with its linguistic expression. He describes the content as an invariant of linguistic translations of a given expression from one language into another. Using today's popular mathematical jargon we can say that Popper thinks here of the thought content as a linguistic pattern taken "up to translation".

I claim that Popper's notion of thought content fails to account for the long-term endurance of scientific, and in particular mathematical, knowledge. His theory better applies to the content of a religious doctrine rather than the scientific content. A teacher of religion may indeed translate a sacred text of his religion to his less educated pupils doing his best for keeping the original sense invariant. Even if the spirit of a religion generally doesn't reduce to its letter most developed religions use sacred texts as a means of preserving their identities through generations. But science proceeds very differently. A mathematical teacher - or at least a good mathematical teachers - doesn't try to transmit to her students the invariant content of some canonical text. Teaching the Pythagorean theorem today she doesn't "try to keep invariant" what Euclid has written about it some 2300 years ago but relies upon modern textbooks. If the notion of canonical text can make a sense in science at all it should be stressed that canonical scientific texts get quickly outdated, are revised, updated and periodically wholly rewritten (Note 1).

Notice the difference between this latter kind of revision and the belief revision I mentioned earlier. The Euclid's book in its original form is no longer in use in schools not because some of Euclid's propositions have been judged false by education authorities. The fact that Euclid fails to meet today's standard of mathematical rigor is not the reason for it either because elementary textbooks anyway do not meet and are not supposed to meet such a standard. The principle reason why Euclid's *Elements* are no longer used in school is this: this book no longer provides a satisfactory basis for a study of more advanced and more specific branches of mathematics. It perfectly did this job for quite a while but lost this capacity when mathematics essentially changed its shape. This dynamics is closely related to mathematical progress but it cannot be described itself as a progress. Kids learning mathematics are hardly cleverer today than they were a century ago. Their learning capacities hardly essentially increased. But today's kids should be prepared to use and further develop mathematics that has been significantly progressed during the passed century. This is why they need a new curriculum.

This new curriculum cannot be just an extension of an older curriculum because this would require an increase of pupils' learning capacities. So they should study a different mathematics to begin with. This is why the evolution of elementary mathematical curricula is not progressive in the precise sense of the term.

True, older and newer elementary mathematics textbooks typically share some material. For example older and newer geometry textbooks usually include the Pythagorean theorem. Here is the Pythagorean theorem as it appears in Euclid's *Elements* (Proposition 1.47)

(1) In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

And here is how the same theorem appears in a modern textbook (Doneddu 1965, p.209, slightly modified as explained in Note 2)

(2) If two non-zero vectors x and y are orthogonal then $(y - x)^2 = y^2 + x^2$.

A care is needed in order to interpret the two propositions correctly. Euclid speaks here not about the areas of the squares but about the squares themselves: saying that the two smaller squares (taken together) are equal

to the bigger square he means, roughly, that the bigger square can be composed out of pieces of the smaller squares. The minus sign on the left side of $(y - x)^2 = y^2 + x^2$ and the plus sign on its right side don't stand for mutually inverse operations since the former operation applies to vectors while the latter applies to real numbers. Vectors, numbers and operations with these things are construed here as structured sets. In order to interpret correctly not only the two statements but also their proofs much more should be said about their corresponding theories. In particular, a lot is to be said about structuralist set-theoretic foundations of mathematics developed in Doneddu's book and foundations of Greek (and more specifically Euclid's) mathematics. But even without going into these details it is clear that the difference between (1) and (2) is not that of linguistic surface. The two versions of Pythagorean theorem differ in their foundations, i.e. differ radically. And yet they express the same theorem!

In the Conclusion of this paper I'll come back to this example and say more about it. But what has been already said already suffices for showing that Popper's account of thought content doesn't apply to mathematics as far as this science is observed at large historical scales. Whatever the mathematical content might be it cannot be described as an invariant of linguistic translation; the notion of linguistic translation doesn't help one to account for the long-term endurance of mathematical knowledge. The same is true for science in general. Unlike religious doctrines, poems, musical symphonies and some other inhabitants of Popper's Third World scientific knowledge endures in a long run through a permanent revision rather than a mere repetition of linguistic patterns or a mere retention of translational invariants as Popper suggests. Above I described this revision as a pedagogical necessity. But it has also a philosophical aspect. I claim that the continuing questioning, revision and renewal of foundations is a distinctive way in which science endures through time and performs a progress. This "unended quest" (to use Popper's word) concerns not only new yet unexplored domains of reality; it also concerns what is already known and well established.

The renewal of foundations amounts to the dialectical refutation of older foundations and the dialectical positing of new foundations. This activity belongs to philosophy rather than to science itself. In this latter respect my view is traditional and qualifies as a form of foundationalism. But I also think that the notion of foundation does not make sense in abstraction from what it is (or supposed to be) foundation of. The historical performance of dialectically posited foundations crucially depends on what scientists (including mathematicians) do with these foundations. So my scientific foundationalism implies the need of a close cooperation between philosophy and science but definitely not the subordination of one of the two parties to the other. I subscribe under the following strong claim about the nature of scientific foundations:

A foundation makes explicit the essential general features, ingredients, and operations of a science, as well as its origins and general laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A "pure" foundation that forgets this purpose and pursues a speculative "foundations" for its own sake is clearly a nonfoundation. (Lavwere 2003)

2. Claims and Slogans

The above generalities help me to formulate the principle purpose of this paper. I shall discuss Mathematical Structuralism considering it as a particular project in foundations of mathematics but not as a speculative view on mathematics developed for its own sake. To proponents of Mathematical Structuralism in this latter sense I have nothing more specific to say except that I disagree with them on more general grounds explained in the above Introduction. I claim that Structuralism has been extremely successful in mathematics of 20th

century but now is already worked out and should be abandoned. Recent developments in mathematics call (as ever) for renewal of foundations. My more specific claim concerns Category theory. This theory emerged in the mid-twenty century within the structural mathematics (Note 3). Many proponents of Category theory believe that this theory is capable to provide better foundations for structural mathematics than the standard set-theoretic foundations used in the structural mathematics earlier (see Section 7 below). I share with these people their enthusiasm about making Category theory into new foundations of mathematics but I don't believe that mathematics built on these these new foundations can or should be structural. I claim, on the contrary, that Category theory brings about a new understanding of mathematics, which is connected to Structuralism historically and dialectically but which is itself quite different. In what follows I shall try to present and develop this new view. I shall occasionally call this new view "categorical" but leave it to others to invent a new *ism* for it. However following a fashion I shall propose a polemical slogan. In the end of his (1996) Awodey puts forward the following structuralist slogan:

The subject matter of pure mathematics is invariant form, not a universe of mathematical objects consisting of logical atoms.

I counter this slogan with the following one:

The subject matter of pure mathematics is covariant transformation, not invariant form.

The rest of this paper is organized as follows. First, I briefly discuss Mathematical Structuralism, its historical origins and its relation to Set theory and Category theory. Here I explain reasons why MacLane, Awodey and some other people believe that Category theory provides a support for Mathematical Structuralism. Then I provide my critical arguments against this view arguing that the notion of category should be viewed as generalization of that of structure rather than as a specific kind of structure. Further I analyze Lawvere's paper (1966) on categorical foundations and show that the author begins this paper with a version of structuralist foundations but then proceeds with a very different project that still waits to be accomplished. I conclude with an attempt to outline the new categorical view on mathematics explicitly.

3. Mathematical Structuralism

This is how a mathematical structure is described by a working mathematician for a philosophical reader:

All infinite cyclic groups are isomorphic, but this infinite group appears over and over again - in number theory, in ornaments, in crystallography, and in physics. Thus, the "existence" of this group is really a many-splendored matter. An ontological analysis of things simply called "mathematical objects" is likely to miss the real point of mathematical existence. (MacLane "Structure in Mathematics" 1996)

For a philosophical reader who doesn't know what is the infinite cyclic group (and also for the sake of my argument) I propose a modification of MacLane's example that amounts to replacement of the words "infinite cyclic group" by the words "number three" and the word "isomorphic" by the word "equal":

All threes are equal but this number appears over and over again - in number theory, in ornaments [...]. Thus the "existence" of this number is really a many-splendored matter. [...]

Indeed the familiar number three is just as promiscuous as the infinite cyclic group or perhaps even more promiscuous. It equally "appears" (to use MacLane's word) both inside and outside mathematics: in a triple of apples, a triple of points, a triple of groups, a triple of numbers or a triple of anything else. Just like in MacLane's example there is a systematic ambiguity in our talk about numbers between the plural and the singular forms of nouns. (Notice MacLane's talk about "all infinite cyclic groups" and "this infinite group" in the same sentence; in my paraphrase I talk similarly about a number.) This analogy reveals a traditional aspect of Structuralism, which often remains unnoticed when people stress the novelty of this approach. Of course, this analogy doesn't allow for reduction of Structuralism to earlier views. But it allows one to see clearer what was really new in Structuralism. It was not the notion of "many-splendored existence" stressed by MacLane in the above quote but a more specific notion of isomorphism, which plays in the structural mathematics roughly the same role as the notion of equality (as distinguished from identity) plays in the traditional mathematics.

This point has been made clear by Hilbert in his often-quoted letter to Frege of December 29, 1899. Stressing the "many-splendored" nature of structural theories (as we would call them today) Hilbert says:

"[E]ach and every theory can always be applied to the infinite number of systems of basic elements. One merely has to apply a univocal and reversible one-to-one transformation [to the elements of the given system] and stipulate that the axiom for the transformed things be correspondingly similar (quoted by Frege 1971, underlining mine)

We see that Hilbert explicitly mentions here the reversibility condition, which implies that the given transformation is an isomorphism. Heller (forthcoming) quite rightly, in my view, recognizes Hilbert as a founder of Structuralism; however in his official definition of Mathematical Structuralism Heller doesn't mention isomorphisms explicitly:

Structuralism is a view about the subject matter of mathematics according to which what matters are structural relationships in abstraction from the intrinsic nature of the related objects. Mathematics is seen as the free exploration of structural possibilities, primarily through creative concept formation, postulation, and deduction. The items making up any particular system exemplifying the structure in question are of no importance; all that matters is that they satisfy certain general conditions—typically spelled out in axioms defining the structure or structures of interest - characteristic of the branch of mathematics in question.

The reason why Heller doesn't mention the notion of isomorphism here becomes clear from his remark concerning "axioms defining the structure or structures [notice the plural - A.R.] of interest". Take axioms defining the notion of group for example. A group is any "system" (to use Heller's word) which consists of certain "items" a, b, \dots and binary operation \oplus associating with every ordered pair of such items (possibly identical) a third item (possibly identical to one of those) from the same system such that the following axioms hold:

G1: operation \oplus is associative

G2: there exist an item I (called unit) such that for all a $a \oplus I = I \oplus a = a$

G3: for all a there exist a' (called inverse of a) such that $a \oplus a' = a' \oplus a = I$

These axioms are satisfied by many different groups. The infinite cyclic group mentioned above is just one

example (Note 4) but there are many others. Consider, for another example, a group of permutations of three letters A,B,C with the composition of permutations as group operation. This group comprises six different permutations (including the identical permutation). Since this latter group is finite it cannot be isomorphic to the infinite cyclic group. So these two groups are different in the *structural* sense. This example shows that the above axioms G1-3 describe structures of a particular *type*, not a particular structure.

In order to give a sense to the expression "type of structures" one needs to have the notion of structure beforehand. Axioms G1-3 or any other system of axioms determining some type of structure cannot help one to grasp the notion of structure unless one is already aware about the fundamental role of isomorphism. For the idea of a general description satisfied by different mathematical objects is obviously not specific for Structuralism; Euclid's axioms do the same job with respect to numbers and magnitudes. Stressing the higher importance of structures with respect to "systems", the irrelevance of "intrinsic nature" and relevance of "structural relationships" cannot clarify the notion of mathematical structure by itself.

4. Isomorphisms and "Invariant Forms"

A non-structuralist may observe that axioms G1-3 are satisfied by a number of "particular systems" (not structures so far!) called groups. Let now \mathbf{G} be a class of such systems (i.e. groups),... and consider the situation when some of these, say G and G' , are isomorphic. This actually means two things:

I1: elements of G are in one-one correspondence with elements of G' and

I2: for all elements a, b, c from G such that $a \oplus b = c$ the corresponding elements a', b', c' from G' satisfy $a' \otimes b' = c'$ where \oplus is the group operation in G and \otimes is the group operation in G' .

A one-one correspondence between elements of two given groups which satisfies I2 is called (group) isomorphism. Groups are isomorphic if and only if there exist an isomorphism between them. Notice that given two isomorphic groups there are, generally, many different isomorphisms between them. One should not confuse isomorphism as a particular correspondence between elements of two groups and isomorphism as an equivalence relation defined on some class of groups. Isomorphism in the latter sense holds between two given groups if and only if there exists an isomorphism in the former sense between these groups. As we can see this terminology is slightly confusing but it is too common to try to change it.

Since isomorphism is an equivalence relation it divides class \mathbf{G} into sub-classes containing only isomorphic groups. One may ignore differences between isomorphic groups and get through this act of abstraction various notions of groups-*qua*-structures (not to be confused with the general notion of group as a type of structure!), in particular, the notion of infinite cyclic group. To facilitate the language and provide this reasoning with some intuitive support one may talk and think about any particular structure as a thing "shared" by all members of the corresponding isomorphism class. On this basis one may claim that "[t]he items making up any particular system exemplifying the structure in question are of no importance" (as does Hellman in the above quote). This claim describes the aforementioned abstraction, which can be called *structural abstraction*. However one cannot forget about these exemplifying systems completely because this would destroy the whole reasoning bringing about the notion of mathematical structure. Noticeably Hellman needs the auxiliary notion of *system* in order to describe what is a mathematical structure. One

might think that this additional notion (no matter how one calls it) plays a role in a philosophical talk about structural mathematics but plays no role in the structural mathematics itself. In the next Section I shall argue that this is not the case.

There is yet a different way of thinking about isomorphism (this will be already the third meaning of the term by our account!), which is common in the current mathematical practice and particularly pertinent for categorical mathematics as we shall later see. One may think about a one-one correspondence between elements of groups G and G' , which satisfies condition I2, as a *transformation* $i : G \rightarrow G'$ of one group into another group. Since a one-one correspondence is a symmetric construction the choice of G as the source and G' as the target of this transformation is in fact arbitrary. In other words one and the same isomorphism-*qua*-correspondence gives rise to two isomorphisms-*qua*-transformations $i : G \rightarrow G'$ and $i : G' \rightarrow G$, which go in the opposite directions and cancel each other on both sides. This later property means precisely the following: the composition transformation $i \circ i'$ resulting from the application of transformation i' after transformation i (Note 5) sends every element of G into itself *and* composition transformation $i' \circ i$ sends every element of G' into itself (beware that none of the two conjuncts implies the other). Given these conditions each of transformations i and i' is called the *inverse* of the other. Hence this definition: a transformation is called an isomorphism when it has an inverse. See (Note 9) for a more precise categorical version of this definition.

Thinking about an isomorphism as a reversible transformation allows one to think of a structure shared by given transformed systems as an “invariant form”, i.e. a form invariant under the given transformation. Hence Awodey's structuralist slogan quoted in the end of Section 2. This slogan describes the structural abstraction in these alternative terms: only the invariant form matters, transformed systems don't. As we shall see in what follows the notion of isomorphism-*qua*-transformation, which may seem redundant in the context introduced so far, becomes indispensable in categorical mathematics. Noticeably Hilbert in the above quote (Section 3) talks about isomorphism as transformation, not as a symmetric one-one correspondence.

5. Structures versus Abstract Objects; Collections versus Transformations

Given an equivalence relation defined on a class of mathematical objects Frege (1884) considered a possibility of replacing each obtained equivalence class by a single object through an act of abstraction (Note 6). Frege calls the result of this procedure an abstract object, not a structure, and indeed he doesn't think about this outcome as a structure. So we need a further effort for distinguishing structural abstraction from other types of mathematical abstraction. For this end let us first consider this question: What are elements of a group-*qua*-structure? For the reason that I have already explain we don't want these elements to have anything like “intrinsic nature”. So they should be just “items” or “abstract elements”; the predicate “abstract” refers here to the act of abstraction through which the notion of group-*qua*-structure is obtained. However we still need to make some assumptions about these things. We want them to be many and form (or belong to) well-distinguishable collections. Since we want to use the same notion of collection for different purposes we don't want the collected elements to hold relations. This will give us the freedom to stipulate any relation between elements by a fiat using the same notion of collection.

This is an important point where Structuralism meets Set theory. Having a notion of set in our disposal we are in a position to give the standard structural definition of group as a “structured set”, namely a set

provided with a binary operation satisfying axioms G1-3 given above. (There is a standard way to account for algebraic operations as relations that I shall not explain here.)

As we have seen the notion of isomorphism plays a crucial role in the structural abstraction. However one might expect that since an act of such abstraction is accomplished this notion will no longer play any role in the structural mathematics. But evidently this is not the case. Mathematicians think about abstract groups and other abstract structures as given in an indefinite number of isomorphic copies, not as unique objects. As I have already stressed, people think similarly about numbers in the traditional arithmetic (see Section 3). This, in my view, is the principle point where Frege's notion of abstraction fails to account for the structural abstraction as this latter notion has been developed in mathematics of 20th century. Reasoning "up to isomorphism" doesn't amount to the strict identification of isomorphic structures; it rather amounts to relaxing the notion of identity up to that of isomorphism in appropriate contexts. From a mathematical (as distinguished from logical and philosophical) viewpoint the question whether or not two structures are identical is just as pointless as the question whether or not two equal numbers are identical. A sound mathematical question about two given numbers is whether or not they are equal. A sound mathematical question about two given structures is whether or not they are isomorphic (Note 7).

Set theory makes the talk of isomorphism as transformation redundant because the notion of one-one correspondence may be analyzed set-theoretically in terms of pairs of elements. However in many important mathematical contexts the notion of transformation is widely used anyway: groups of (reversible) transformations are abundant and geometry and also in physics. As far as foundations of mathematics are concerned we have an important choice here: either to (i) consider the notion of collection as more fundamental than that of transformation and reduce the latter to the former or to (ii) consider the notion of transformation as more fundamental and reconstruct the notion of collection on this basis. The former option brings (some version of) set-theoretic foundations of mathematics. The idea of categorical (i.e. category-theoretic) foundations amounts to taking the latter option. However the project turns to be non-viable unless one takes into the account other transformations than isomorphisms.

6. Homomorphisms

Given a type of structures there is a standard way to define a general notion of map between structures of the given type. The term "homomorphism" is traditionally reserved for groups (apparently because this case has been studied first) although, as its etymology suggests, it could also be used for structures of different types like the term "isomorphism". In what follows I shall discuss only the case of groups but what I am going to say generalizes straightforwardly to structures of any other type.

The notion of group homomorphism generalizes upon that of group isomorphism in the following way: instead of one-one correspondence between elements of groups G , G' one considers a more general kind of correspondence that is allowed to be many-one but not one-many. In other words one consider a *function* (in the set-theoretic sense of the term) $f: S \rightarrow S'$ from the set S of elements of G to the set S' of elements of G' . Condition I2 from Section 4 remains the same; notice that it can be satisfied when elements a , b are different but elements a' , b' are the same.

Group homomorphism and similar general maps between structures of other types are colloquially called

"structure preserving". This is somewhat misleading because if such maps preserve anything at all this is a type of structure but not a particular structure. Think about this trivial example: for all groups G, G' there exist a homomorphism $h : G \rightarrow G'$, which sends every element of G to the unit of G' . This homomorphism "destroys all information" about G reducing its image to a single element; it doesn't provide any information about G' either. I shall now argue that homomorphisms, generally, don't allow for invariants in anything like the same sense, in which isomorphisms do so. This shows, in my view, that the usual way of thinking about homomorphisms as "imperfect isomorphisms" is based on a wrong analogy, which easily leads to serious conceptual mistakes.

Let us try to replace isomorphisms by homomorphisms in the process of structural abstraction described in Section 4 and see what happens. One might expect to get in this way a generalized notion of structure but this doesn't work. Remind the first step: given class \mathbf{G} of groups we have divided it into equivalence subclasses of isomorphic groups. Two groups are isomorphic if and only if there exist an isomorphism (i.e. a reversible transformation) between them; clearly this is an equivalence relation. Let me (for the sake of the argument) call two groups *homomorphic* if and only if there is a homomorphism between them. Although this latter relation is also an equivalence one can see the difference: since *all* groups are homomorphic (see the above example of group homomorphism) one cannot use this equivalence for dividing \mathbf{G} into equivalence subclasses! Saying that two given groups are homomorphic is tantamount to saying that the given groups are groups. So the relation of homomorphism just introduced (not to be confused with the standard notion of homomorphism as transformation) doesn't make sense.

In order to see the reason of this failure observe that the existence of homomorphism of the form $h : G \rightarrow G'$ doesn't imply the existence of homomorphism of the form $h : G' \rightarrow G$. This means that in the case of homomorphism (unlike that of isomorphism) the difference between the source and the target of the given transformation matters. But the relation of homomorphism tentatively introduced above doesn't take this difference into the account. It forgets the difference between isomorphic and non-isomorphic groups and thus confuses their structural properties offering no replacement.

A more reasonable choice of relation associated with a given homomorphism $h : G \rightarrow G'$ would be that of non-symmetrical relation $>$ such that $G > G'$ holds just in case there is a homomorphism of the form $h : G \rightarrow G'$. However since $>$ is asymmetric it is not an equivalence and so doesn't allow one to proceed further with the structural abstraction or anything similar.

We see that homomorphisms cannot do the same job as isomorphisms: the reversibility condition mentioned by Hilbert in the above quote (Section 3) turns to be crucial for the structural abstraction. One cannot reason "up to homomorphism" in anything like the same way in which people reason up to isomorphism doing structural mathematics. Since "invariant" in the given context is just another word for structure it is clear that homomorphisms, generally, don't have invariants in anything like the same sense, in which isomorphisms and groups of isomorphisms do so (Note 8).

7. Structuralists Motivations behind Category Theory

The emergence of Category theory in 1940-ies and its further development in the context of structural mathematics was related to a growing awareness of the role of general maps (not only isomorphisms). I shall

not explain here the precise mathematical context in which this theory first proved useful but only mention that the notion of category generalizes upon such examples as the class of all sets and all functions, all groups and all group homomorphisms, all topological spaces and all continuous maps (not only reversible ones!) between topological spaces. This is a simple theorem (MacLane 1996) that a class of structures of any fixed type provided with the corresponding notion of general map form a category. Generally a category comprises a class of objects and a class of composable maps (called in Category theory *morphisms*) for every ordered pair of objects, which are subject to few natural axioms. Given two different categories one defines a notion of map between categories. Such maps are called *functors*; the usual definition of functor is based on the same idea as the definition of group homomorphism given in the previous Section: a functor sends each object of the source category into an object of the target category and each morphism of the source category into a morphism of the target category in such a way that composition of morphisms is "preserved" in the same sense in which the group operation is said to be preserved by a group homomorphism. Using the notion of functor one may consider various categories of categories, i.e. categories such that their objects are themselves categories. One may also consider categories objects of which are functors. The above standard description of basic categorical concepts is structuralist in its spirit. In Section 11 I shall describe functors and categories anew from a foundational and "more categorical" viewpoint.

The idea of categorical foundations as viewed from a structuralist perspective amounts to recovering of all the relevant properties of any structure of any given type through properties of the category of (all) structures of this given type. In the case of the category of sets this provides an alternative (category-theoretic) Set theory: one first conceives of sets as abstract objects and stipulates that they form a category; then one stipulates desired properties of this category, which make this category "into" the intended category of sets. This result (see Lawvere 1964) shows that a reasonable notion of collection (set) can be developed on the basis of that of transformation (morphism of sets) but not only the other way round.

The growing popularity of Category theory as a common (albeit certainly not unique) "language" of today's mathematics as well as the continuing efforts of building categorical foundations of mathematics are generally seen as a further step of the structuralist project briefly described above. I agree with this view as far as it does not require to preserve the basic principles of Mathematical Structuralism in the new categorical setting. In my understanding these developments diverge from Mathematical Structuralism and tend towards a very different view on mathematics and science in general. Before I shall describe this new view let me explain reasons why categorical foundations appear to many as a version of structural foundations. In the next Section I shall show that this appearance is wrong.

As I have already explained in Section 5 the notion of set plays a special role in the structural mathematics. This explains why Set theory itself is rarely seen as a structural theory on equal footing with, say, Group theory. As Hellman rightly remarks:

[D]espite the multiplicity of set theories (differing over axioms such as wellfoundedness, choice, large cardinals, constructibility, and others), the axioms are standardly read as assertions of truths about "the real world of sets" rather than receiving a structuralist treatment. (op. cit.)

The structural notion of group explained above is usually construed as a "set with a structure" or "structured

set" rather than a pure structure (whatever this might mean); the underlying set of a given group is thought of as a background supporting the structure rather than a part of this structure. This way in which mathematicians think about structures reminds of Aristotle's metaphysics of Matter and Form. The need of the set-theoretic Matter for doing structural mathematics becomes clear from our analysis given in the Section 5 but the presence of this ingredient doesn't fully comply with the philosophy of Mathematical Structuralism, which purports to make mathematical objects into pure forms (structures) and leave anything like the "background" outside mathematics. The desired "purely structural" mathematics would deal only with the "invariant Form" and require no set-theoretic Matter. Here is a historical evidence of such attitude. Talking about set-theoretic difficulties Dieudonne (under the name of Bourbaki) says in his structuralist manifesto (Bourbaki 1950):

The difficulties did not disappear until the notion of set itself disappears <...> in the light of the recent work on the logical formalism. From this new point of view mathematical structures become, properly speaking, the only "objects" of mathematics.

I don't believe that Dieudonne's claim concerning the alleged "disappearance" of sets is justified but the quote clearly demonstrates such intention.

In this context the idea of accounting for relevant properties of mathematical structures only in terms of structure-preserving maps between these structures independently of any set-theoretic background, i.e. the idea of categorical foundations, indeed looked like a further step in the structuralist direction. Hence the popular view according to which the categorical mathematics is the desired purely structural mathematics.

Remarkably Category theory did never make it into Bourbaki's *Elements* (1939 - 1983), which is the most systematic attempt to develop structural mathematics ever undertaken. This is in spite of the fact that both founders of Category theory, Eilenberg and MacLane, were eventually involved into the Bourbaki group, so all the principle members of this group were well aware about their work. This fact is often seen as a historical puzzle but in my view it is not. For categorical foundations of mathematic are not and cannot be anything like the structural foundations developed by Bourbaki in his fundamental work. The purpose of the following Sections is to justify this claim and suggest an alternative.

8. Categories versus Structures; Embodiement of Mathematical Concepts

Categories of structures like the category of groups, topological spaces, etc. capture the notion of type of structure, not the notion of singular structure. Particular structures (identified up to isomorphism) may be often also rendered as categories but in this case their morphisms are no longer structure-preserving maps. For example, a particular group (like the infinite cyclic group mentioned above) can be presented as a category with just one object such that all of its morphisms (going from this object to itself) are isomorphisms. The group operation is given by composition of morphisms; the existence of unit follows from the definition of a (general) category and the existence of inverse elements follows from the fact that all morphisms of the given category are reversible (Note 9). This construction can be pictured as follows:



This simple example shows that categorical morphisms can but should not be structure-preserving maps. Moreover the above categorical presentation of group unlike its standard set-theoretic presentation is not structuralist in its character. For the standard structuralist presentation involves this idea: an abstract group can be "exemplified" by what Hellman calls "particular systems" like systems of numbers, systems of geometrical motions and so on and so forth. Of course, when one pictures elements of a given group as loops rather than dots this does produce any conceptual change by itself. But given the above categorical presentation of a group and using standard category-theoretic means one can do somethings different than keep saying that morphisms of the given category (i.e. the given group) stand or may stand for something else than themselves. Namely, one may consider functors from the given group-category into some other categories, which in their turn present (rings or fields of) numbers, geometrical spaces, etc. This provides a much more precise idea of "standing for" in each particular case than the general structuralist rhetoric. In the structuralist setting the notion of exemplification remains meta-theoretical and escapes a precise mathematical treatment. But in the categorical setting this notion becomes a proper part of the given mathematical construction. Instead of saying that A stands for B one considers functors of the form $A \rightarrow B$ and treats these functors on equal footing with 'internal' morphisms of A and B (Note 10).

In my understanding this latter type of mathematical thinking has little if anything to do with the structural abstraction. A principle epistemic strategy of Structuralism is to capture what various "particular systems" share in common, namely their "shared structure". The corresponding categorical strategy can be described in this way: look how "particular systems" translate into each other. Unlike the structuralist strategy this categorical strategy doesn't make the particular systems less important. Given morphism $A \rightarrow B$ there is, generally, no reason to think of A and B "up to" some equivalence and dispense with A and B in favor of their shared structure or anything else. As I have already shown in Section 6 the notion of thinking "up to homomorphism" is plainly unsound.

Let us now consider the case when a category presents a type of structure rather than a singular structure. To analyze this case I shall use the notion of embodiment, which I have introduced elsewhere (see Rodin, forthcoming). As we have seen in Section 5 a mathematical structure cannot be identified with its corresponding abstract concept: something else is needed in order to make a given concept into a mathematical object. Kant would call this additional element an intuitive construction; I use the word "embodiment" for a similar purpose but in a different mathematical context. We have seen how the notion of structure allows for making a concept describing different "particular systems" into a single mathematical object (single up to isomorphism, of course). As we have seen in Section 6 this structuralist method of embodiment doesn't work for *types* of structure. While the concept "infinite cyclic group" can be embodied into a single structure, the concept "group" cannot; "the group" unlike "the infinite cyclic group" is not a

name of unique (up to isomorphism or otherwise) mathematical object. However the *category* of (all) groups is a single mathematical object like number 3, the infinite cyclic group or, say, the Euclidean plane. Each of these objects has a "many-splendored existence" (to use MacLane's word), so its singleness must be understood appropriately. But I want now to stress a different point: the way in which all isomorphic cyclic groups are made into a single object with the notion of structure and the way in which all groups are made into a single object with the notion of category are essentially different. While the former involves the structural abstraction the latter involves a different kind of abstraction, which I shall call *categorical*. Roughly, the categorical abstraction amounts to the following: one forgets about the fact that groups have elements and consider only how they map to (i.e. transform into) each other with appropriate morphisms; a relevant notion of element is recovered in this categorical setting only later on. Obviously the two kinds of abstraction are quite different. I shall say more on the categorical abstraction in the Conclusion.

A category in which morphisms (including identity morphisms) form a set (in the technical sense of the term) is called *small*. Small categories can be thought of as structures on their own. The corresponding type of structures is defined straightforwardly: one takes a set of elements called morphisms, introduces appropriate primitive relations on this set and spells out the needed axioms (see the next Section for more details). Thus small categories like groups can be thought of as structures of a specific type. Noticeably this straightforward approach doesn't go through in the case of large categories corresponding to types of structures - think again of the category of groups or the category of all small categories. Since morphisms of such categories form classes which are not sets they cannot be described as structured sets. Although this may look like a minor technical difficulty, which can be resolved by an appropriate generalization of the usual notion of structure, this difficulty provides an additional evidence that the structural approach, generally, doesn't work in Category theory. Instead of thinking of categories as structures (or generalized structures) of a particular type it seems me more reasonable to reverse the order of ideas and think of structures as categories or categorial constructions of a particular type. An immediate suggestion would be to identify structures with small categories. A more elaborated suggestion due to Lawvere (a personal talk) is to identify a structure with a functor from a small category to a large "background" category, say, that of sets.

To conclude this Section let me stress that categories don't always represent particular structures or particular types of structure. Examples of this latter kind are today so popular only because they connect the new categorical mathematics with the older structuralist mathematics, which is nowadays widespread and well-established. But the categorical mathematics also involves concepts and constructions that have been first developed in a categorical setting, for example that of Grothendieck topology. One may expect that the further development of categorical mathematics will make such "purely categorical" concepts better known and more useful in various branches of mathematics; then the link between the categorical mathematics and its structural predecessor will become a historical and philosophical rather than mathematical issue.

9. "The category of categories"

The idea of categorical foundations amounts to taking the notions of category, functor and/or some other related categorical notions as primitive and recovering the rest of mathematics on this basis. What are possible ways of realizing this project? In which precise sense one can consider category-theoretic notions as primitive? A way to do this, which immediately suggests itself, is to use in categorical foundations a modern version of Hilbert-style axiomatic method after the example of standard set-theoretic foundations.

Consider a class of things called morphisms and three primitive relations: one that associates with every given morphism its source, one that associates with every given morphism its target and, finally, one that associates with some (ordered) pairs of morphisms a third morphism called the composition of the given two morphisms. Then we need axioms to ensure that sources and targets of morphisms behave as identity morphisms (i.e. as *objects*), that two given morphisms are composable if and only if the target of the first morphism coincides with the source of the second morphism, and some other similar axioms. Finally we should assume that the composition of morphisms is associative. For the full list of such axioms I refer the reader to (Lawvere 1966). The axiomatic theory just described this author calls the *elementary theory of abstract categories*.

This Lawvere's paper begins as follows:

In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about, and in particular in which classes and membership in classes do not play any role.

We see that Lawvere embraces Mathematical Structuralism here but at the same time rejects set-theoretic (and even more general class-based) foundations of mathematics. Since the Hilbert-style axiomatic method is essentially structural (see Section 3 above) Lawvere's method of building his *elementary theory of abstract categories* perfectly fits his announced purpose. After the introduction of the axioms of the *elementary theory* and providing some definitions on their basis Lawvere says:

By a category we of course understand (intuitively) any structure which is an interpretation of the elementary theory of abstract categories, and by a functor we understand (intuitively) any triple consisting of two categories and a rule T which assigns, to each morphism x of the first category, a unique morphism xT of the second category in such a way that ... [follow the conditions of being "structure-preserving"]

A problematic aspect of this first part of the paper concerns Mayberry's argument who claims that Lawvere's *elementary theory* like any other theory built with the Hilbert-style axiomatic method requires some primitive (non-axiomatic) notion of collection, which cannot be identified with that of category (Mayberry 2000). The argument implies that the *elementary theory* and the corresponding elementary notion of category cannot be a genuine foundation. I agree with Mayberry on this point (this follows from my understanding of the relationships between Structuralism and Set theory explained in the beginning of Section 5) but unlike Mayberry I think that such primitive notion of collection is dispensable in foundations of mathematics along with the Hilbert-style structural axiomatic method itself. In what follows I shall sketch a different version of axiomatic method that seems me more appropriate for categorical foundations. Let me now return to (Lawvere 1966).

Lawvere's *elementary theory* is a preparatory step towards another theory of categories, which Lawvere calls *basic theory*. I claim that unlike the *elementary theory* the *basic theory* is not structural, at least not in a similar sense. If I am right this shows that the main content of (Lawvere 1966) in fact doesn't agree with the structuralist agenda announced by the author in the beginning of this paper: Lawvere begins his paper with a

structural reasoning but then proceeds with a very different agenda, which can be described as genuinely categorical.

The *basic theory* begins with a re-introduction of the notion of functor:

Of course, now that we are in the category of categories, the things denoted by the capitals will be called categories rather than objects, and we shall speak of functors rather than morphisms.

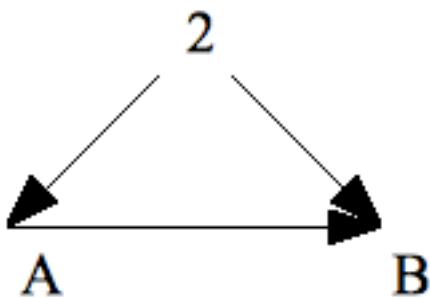
This may sound like a mere terminological convention (rather than an alternative definition) but in fact it signifies a sharp change of the viewpoint. The idea is now the following: given a preliminary notion of category (through the *elementary theory*) conceive of category \mathcal{C} of "all" categories; then pick up from \mathcal{C} an arbitrary object A (i.e. an arbitrary category) and finally specify A as a category by internal means of \mathcal{C} stipulating additional properties of \mathcal{C} when needed. More precisely it goes as follows (I omit details and streamline the argument). Stipulate the existence of terminal object 1 in \mathcal{C} , i.e. the object with exactly one incoming functor from each object of \mathcal{C} . Then identify objects (= identity functors) of A as functors in \mathcal{C} of the form $1 \rightarrow A$. Stipulate also the existence of initial object 0 , i.e. the object with exactly one outgoing functor into each object of \mathcal{C} . Then consider in \mathcal{C} object 2 of the form $0 \rightarrow 1$ and stipulate for it some additional properties among which is following: 2 is *universal generator* which means that

G (generator): for all $f, g: A \rightarrow B$ and $f \neq g$ there exist x such that $x: 2 \rightarrow A \rightarrow B$ and $xf \neq xg$

and

U (universal): if any other category N has the same property than there are y, z such that $y, z: 2 \rightarrow N$ and $yz = 2$.

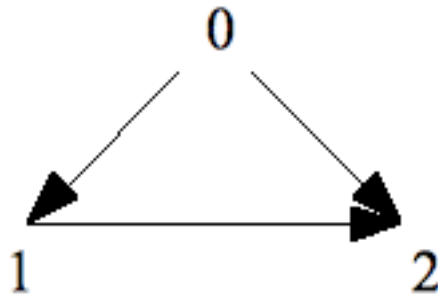
This allows Lawvere to identify functors (morphisms) of A as functors of the form $2 \rightarrow A$ in \mathcal{C} . The fact that 2 is the universal generator (as follows from the above definition it is unique up to isomorphism) assures that categories are determined "arrow-wise": two categories coincide if and only if they coincide on all their arrows. This new definition of functor also allows one to make a sense of the notion of a *component* of a given functor of the form $h: A \rightarrow B$, which in the *elementary theory* is understood as a map m sending a particular morphism f of A into a particular morphism g of B . In the basic theory m turns into this commutative triangle (Note 11):



This, once again, significantly changes the whole perspective: categories and functors are no longer built

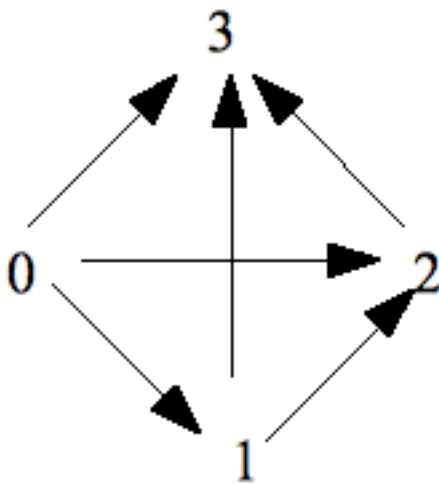
"from their elements" but rather "split them into" their elements when appropriate. Although the notion of functor as a structure-preserving map can be recovered in this new context it no longer serves for defining the very notion of functor. Rule T used by Lawvere for defining functors in the *elementary theory* disappears in the *basic theory* without leaving any trace.

Further consider this triangle which Lawvere denotes 3 :



(it should satisfy a universal property which I omit.)

3 serves for defining composition of morphisms in our "test-category" A as a functor of the form $3 \rightarrow A$ in C . Finally, in order to assure the associativity of the composition Lawvere introduces category 4 that is pictured as follows:



(The associativity concerns here the path $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$)

This construction provided with appropriate axioms makes A into an "internal model" of the *elementary theory* in the following precise sense: If F is any theorem of the *elementary theory* then "for all A , A satisfies F " is a theorem of the *basic theory* (Note 12).

10. Functorial semantics and Sketch theory

In order to see that Lawvere's basic theory unlike his elementary theory is not based on structuralist principles, and then to get an idea of non-structuralist principles behind this theory, it is instructive to take into consideration two similar approaches: *functorial semantics* developed by the same author elsewhere

(Lawvere 1963-2004) and *Sketch theory* founded by Ch. Ehresmann in 1960-ies and later developed by other people (see Wells 1993 for an overview and further references).

Functorial semantics involves the presentation of mathematical theories as categories of a special sort; models of a given theory are functors from the theory to the background category of sets or another appropriate topos. The very idea of "interpretation" or "realization" of a given theory in a set-theoretic background obviously comes from the standard (due to Tarski) Model theory. Lawvere's functorial semantic can be seen as a category-theoretic version of the same basic construction. However as we shall now see this technical update comes with a significant revision of the structuralist background behind Tarski's Model theory, which this later theory inherits from Hilbert's notion of axiomatic method.

In order to determine a theoretical structure an axiomatic theory should be *categorical*, i.e. to have models that are all isomorphic. (Beware that this older sense of the term "categorical" has nothing to do with Category theory!). True, not all axiomatic theories built by the standard method satisfy this requirement; also true, non-categorical theories are usually not disqualified solely on this basis. Anyway in the standard setting the categoricity of axiomatic theory is commonly (and usually as a matter of course) viewed as an epistemic gain while the lack of categoricity is viewed as a problem. As far as one commits oneself to Structuralism such attitude is understandable: when a set of axioms fails to specify a model up to isomorphism it fails to specify a structure. Saying that a non-categorical theory determines many structures rather than one structure is somewhat misleading because such a theory, strictly speaking, doesn't specify any structure at all (cf. Section 3 above).

In the case of Lawvere's functorial semantics the structuralist pursuit of categoricity turns into an absurdity. For the purpose of this construction is to produce a workable category of models rather than just one model up to isomorphism. In the functorial setting a theory determines a category, not a structure. This makes the structuralist thinking behind the axiomatic method as expressed by Hilbert in the above quote (Section 3) irrelevant. In the new setting

The theory appears itself as a generic model. (Lawvere 1963 - 2004, Commentary of 2004)

This means that the older structuralist distinction between abstract "formal" axiomatic theories, on the one hand, and their semantic, on the other hand, doesn't apply; what distinguishes a theory from its (other) models is its generic character rather than formal or abstract character.

The setting of Sketch theory is similar to that of Lawvere's functorial semantics but in the former case generic categories are designed as "generic shapes" or "generic figures" rather than axiomatic theories. Unlike the case of functorial semantics such generic categories are not supposed to have logical properties; in some approaches sketches are not even categories but directed graphs with an additional structure. It seems natural to think of sketches as "proto-structures" but this is somewhat misleading as far as the usual notion of structure is concerned. A sketch doesn't represent a bunch of isomorphic systems but generates non-isomorphic systems (its models). These generated systems share their shape not in the same sense, in which different systems are told to share the same mathematical structure. In fact they share a shape in a more straightforward sense: the given sketch is their common source. To "have the same source" is obviously

an equivalence relation but this particular equivalence relation doesn't support anything like the structural abstraction. Unlike a shared structure a shared sketch is concrete (it is usually even supposed to be finite and easily pictured) while things generated by a sketch can be indeed described as abstract structures in the older sense because they are usually distinguished only up to isomorphism! Thus Sketch theory turns Structuralism upside down (Note 13) and in certain aspects reminds of more traditional ways of doing mathematics. Euclid's geometrical universe is generated by two generic figures, namely, the straight line and the circle, which is tantamount to saying that every geometrical object is constructed by ruler and compass. The analogy seems me straightforward.

11. Categorical foundations

Whether or not the new categorical approach to theory-building differently realized in functorial semantics, Sketch theory and the *basic theory* of (Lawvere 1966) can compete with the standard Hilbert-style structural approach depends on whether or not the categorical approach is self-sustained. The considered constructions doesn't allow one to claim that this is indeed the case: we have seen that Lawvere's *basic theory* depends on the structural *elementary theory*, functorial semantics is developed by this author similarly in two steps, and Sketch theory in its existing form uses Set theory and usually doesn't make foundational claims at all. I shall not present here an accomplished version of (non-structural) categorical foundations but try to specify what remains to be done in order to accomplish this project.

Let's see what the first structural layer of (Lawvere's 1966) serves for. First of all it helps Lawvere to introduce the notion of category, on which the *basic theory* further elaborates. In order to see that the notion of category can be first introduced without the structural *elementary theory* remind the following feature of the standard set-theoretic foundations: to build an axiomatic theory of sets like ZF one needs a pre-theoretical notion of set or collection to begin with (cf. Mayberry's argument in Section 9 above). Axioms of Set theory make this pre-theoretical notion of collection more specific and more precise saving it from known paradoxes. But they don't produce sets *ex nihilo*. Categorical foundations can and should, in my view, be built similarly: one begins with an informal intuitive notion of category and then use it for making this very notion more precise and better behaved. Such circularity is not vicious in the case of sets and it is not vicious in the case of categories either.

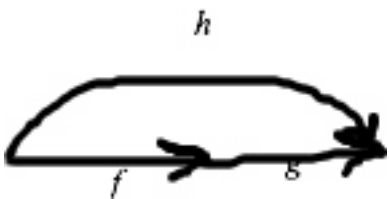
It may be argued that unlike the notion of collection that can be illustrated with everyday examples like that of a set of chairs the notion of category requires a much higher level of mathematical abstraction. I think this is a wrong impression due to the fact that popular examples of categories involve topological spaces, vector spaces and similar things about which a beginner cannot possibly know. As far as categories are supposed to provide foundations of mathematics they should be certainly introduced differently. Remind that the notion of set also first emerged in rather involved mathematical contexts; the idea that a set of chairs falls under the same concept came about later with the project of set-theoretic foundations of mathematics (Note 14). The development of so-called "naive" notion of set was as much important for this project as the development of axiomatic theories of sets like ZF; the latter part of the project wouldn't be possible without the former. Categorical foundations of mathematics require a similar job. Let me now make an attempt to develop a primitive notion of category. It may happen that pushing the project of categorical foundations further forward will bring primitive categorical notions other than the standard notions of functor and category. Here I shall stick to the standard notions and only hint to some possible alternative developments, which in my

view are worth studying.

Our basic concept will be that of *transformation* otherwise called *functor*. Spatial motion is an important example of transformation but it should be stressed that this case is special because any such motion (except a travel into a black hole) is reversible while a categorical functor is generally not (Note 15). So talking about transformations we shall also think about non-reversible transformations like that of a caterpillar's egg into a caterpillar into a butterfly. This example is helpful because the transformation in question is repeatable albeit irreversible. We need this sort of repeatability of our transformations since otherwise one cannot do with them anything like mathematics! (I'll say more about this in the Conclusion.) We shall use *arrows* for picturing transformations.



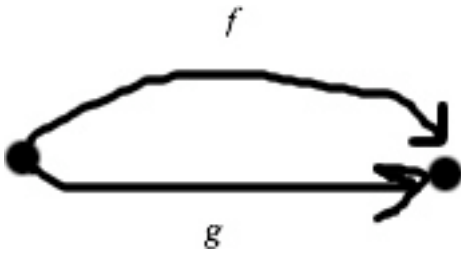
The next basic notion we need is that of *composition* of transformations: we postulate that some (but not all) transformations are composable (in a given order) and that their composition (when it exists) brings about another transformation. This operation can be equally thought of in terms of a *decomposition* of a given transformation into elements. For example, the transformation h of a caterpillar's egg into a butterfly is composed of the transformation f of the egg into the caterpillar and the transformation g of the caterpillar into the butterfly, in symbols $h=fg$. This composition can be represented by the following self-explanatory diagram:



Obviously one cannot compose f and g in the opposite order or compose any of f, g with h : such compositions are not available in the given example. Beware that the above description of composition is preliminary and incomplete as well as the description of the notion of functor given so far. A suggestive possibility, which I shall not explore here, is to treat the categorical composition as a transformation rather than introduce it as a separate primitive.

Next we introduce the notion of *stop* or “zero transformation”; we shall call stops *identity functors*, *objects* or *categories* interchangeably. We shall use *dots* to represent stops. We postulate that any transformation begins and ends with a stop, possibly the same. This strong assumption makes all our transformations in a sense discrete. Perhaps lifting this assumption may give us generalized notions of functor and category but as I have promised I shall stick here to its usual well-established version.

Transformation f from the above example begins with stop e (the egg) and ends up with stop c (the caterpillar). However pointing to the source and the target of a given transformation, generally, does not determine this transformation: given two stops there are, generally, more than one transformation between them in a chosen direction:



What matters is not only where a given transformation begins and where it ends but also how it goes. To read the above diagram correctly one shouldn't introduce into it additional points for free. The two arrows shown at this diagram are not supposed to be determined point-wise; they share all their points, namely their source and their target, and yet they are different.

The notion of composition introduced earlier helps us to give this more precise definition: A stop i is a transformation such that $fi = f$ and $ig = g$ whenever compositions fi and ig exist. Now we are in a position to formulate the condition of composability of transformations more precisely: transformations f and g are composable if and only if one of these transformations begins exactly where the other transformation ends; "where" refers here to a particular stop. To complete this intuitive introduction of basic categorical concepts one needs only to add that the composition of transformations is associative. A category is just another word for a stop. Notice that in the proposed setting the notion of functor is more fundamental than that of category.

Usually a category is defined as a class of objects provided with a class of classes of maps ("map" is another word for transformation) corresponding to all ordered pairs of objects. Then follow examples of categories like the category of all sets, all groups, etc. (I have described the notion of category in this way in the beginning of Section 7.) This approach is not appropriate for categorical mathematics for two reasons: first, because the primitive notion of class is both unwanted and unnecessary here and, second, because in the categorical mathematics all categories are categories of categories rather than categories of anything else. The following analogy with the set-theoretic mathematics helps to clarify this important point. As far as the notion of set is not supposed to provide a foundation for mathematics one thinks of sets after examples of sets of numbers, sets of points and the like. But in a foundational axiomatic theory of sets like ZF there is no other sets but sets of sets and every mathematical object like a number or a point is supposed to be a set. Similarly in a foundational axiomatic theory of categories there is no other categories but categories of categories and every mathematical object is supposed to be a category. Interestingly in my proposed setting there is no place for anything like "the" category of categories, i.e. the totality of all categories. In fact Lawvere's *basic theory* doesn't require such a thing. It is sufficient for it to stipulate the existence of categories $0, 1, 2, 3, 4$ and the corresponding functors using the preliminary intuitive notions of category and functor introduced as suggested above.

The reader may have noticed that I avoid using the term "morphism" in this part of the paper. I do this for two reasons. First, because this term suggests thinking of transformations as structure-preserving map, which is irrelevant in the given context. Second, because the term "functor" is commonly used to denote a transformation of categories and all the transformations I am talking about are indeed transformations of categories.

We see that the notion of category can be introduced informally just like the notion of collection, and that the

former notion doesn't require the latter for its introduction. However this preliminary step doesn't complete the task of building categorical foundations. We still need Lawvere's *basic theory* or something similar for making our notions of functor and category more explicit. At this crucial step we cannot any longer proceed with purely intuitive considerations but need some logical means for making precise definitions and proving basic facts. I shall not try to complete this technical part of the job in this paper but only describe some possible strategies of doing this. One possible strategy is to push the informal considerations further forward and get the notion of cartesian closed category or topos, so one may use its internal logic for specifying the categories $0, 1, 2, 3, 4$ and developing the rest of the *basic theory*. Another possible solution that seems me more promising is to develop the "language of diagrams" used by Lawvere in his *basic theory* into a genuine logical syntax in the vein of (Wells&Bagchi, forthcoming) and (Diskin&Wolter, forthcoming). Both these strategies involve a new understanding of the role of logic in foundations of mathematics. In categorical foundations logic is conceived as a machinery of inference strongly dependent on the underlying mathematical setting, not as a self-standing system of universal laws of reasoning. This change of the traditional conception of logic is analogous to the change of the traditional conception of geometry occurred in 19th century when people stopped thinking about "the" geometrical space as a universal container of geometrical objects and learned to think about spaces as geometrical objects and about geometrical objects as spaces (with the notion of intrinsic geometry of a given geometrical object). In the first half of 20th century people learned to think about systems of logic as objects living in larger meta-logical frameworks. Category theory showed how one can think about objects (i.e. appropriate categories) as systems of logic (with the notion of internal language of a given category, see Lambek&Scott 1986 for details). This reciprocal move that allows one to avoid the bad infinity of meta-meta.....-logic and met-meta....-mathematics in foundations of mathematics has a great philosophical importance.

To conclude this brief discussion of categorical foundations I would like to stress once again that the success of any particular foundation of mathematics crucially depends on its contribution to the progress of mathematics and more generally to the progress of science. Saying this I don't count the renewal of foundations as a part of this progress (see the Introduction) but consider effects of this renewal on the rest of mathematics and science. Set-theoretic structural foundations of mathematics performed very well in this sense before they became an obstacle for further progressive developments. It is perhaps too early to judge categorical foundations from this viewpoint but the abundance of categorical methods in the today's research mathematics apparently promise categorical foundations a good future.

In eyes of a working mathematician the claim that every mathematical object is a category may sound like a philosophical absurdity just like the often repeated claim that every mathematical object is a set. In the following Conclusion I shall make a step toward the common sense of the working mathematician by showing how the categorical viewpoint in and on mathematics allows for seeing beyond the categorical foundations *per se*.

12. Conclusion: a categorical viewpoint in and on mathematics

I hope having convinced the reader that the project of categorical foundations requires a new philosophical view on mathematics, which the traditional Structuralism cannot possibly provide. Let me now try to summarize this new categorical view by contrasting it against the structuralist view. What matters in the categorical mathematics is how mathematical objects and constructions transform into each other, not what

(if anything) remains invariant under these transformations. So categorical mathematics is a theory of abstract transformation, not a theory of abstract form. A theory in categorical mathematics is a generic model (Lawvere) rather than a scheme (Hilbert).

The categorical view on mathematics - as distinguished from categorical foundations of mathematics in the sense of the previous Section - suggests a new understanding of the role of history of mathematics in mathematics itself. In the above Introduction I have argued that older and recent formulations of Pythagorean theorem cannot be identified through a merely linguistic translation. Here I strengthen this claim as follows: different versions of this theorem don't share any invariant content; such invariant content is not necessary in order to qualify these formulations as formulations of the same theorem. For the reader's convenience I quote here this example again:

(1) In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle. (Euclid's *Elements*, Proposition 1.47)

(2) If two non-zero vectors x and y are orthogonal then $(y - x)^2 = y^2 + x^2$. (Doneddu 1965, p.209, slightly modified as explained in Note 2)

Since there is no linguistic translation between (1) and (2) one may look for a translation of a different sort. Such translations certainly exist but (and this is crucial for my argument) they all go into one direction, namely from (1) to (2)! There are many ways in which Euclid's geometry can be interpreted in modern terms, (Doneddu 1965) is just one way of doing this among many others. But there is evidently no way to spell modern geometry in Euclid's terms. (As I have explained in the Introduction we must talk here about translation between corresponding theories, not only about translation between separate propositions.) Our history in general and our intellectual history in particular is oriented: there is no symmetry between the past and the future. Since sound translations between (1) and (2) go in one direction none of them is reversible. According to the argument given in the Section 6 this implies that no such translation allows for the identification of an invariant. Thus the existence of sound translations between theories doesn't imply that these theories share anything like an invariant content. There is no essence, no conceptual core preserved by the translation of (1) into (2). But why in this case we count them as different versions of the same theorem?

My answer is this: the Pythagorean theorem (as distinguished from its formulations by Euclid and Doneddu) is a particular component of the translation of Euclid's geometry into modern terms proposed by Doneddu, which takes (1) into (2). I have in mind the categorical notion of component of functor explained above in the Section 9. *Mutatis mutandis* a similar criterion of identity can be applied to theories and to particular mathematical objects. As far as we are talking about translations like (Doneddu 1965), which are made with the intent to modernize older mathematical contents, my proposed understanding of the identity of this content amounts, roughly, to the identification of this content with its conceptual history. From this point of view different versions of the Pythagorean theorem produced at different times with different foundations can be described as different temporal stages of the same conceptual entity (i.e. the same collective intellectual process) persisting through time. Noticeably this persisting entity doesn't reduce to the set of its temporal stages because such reduction forgets about translations between these stages, which from my categorical viewpoint are the most essential. A mathematical notion is not a set of its temporal stages but a category of

translations between these stages. I would like also to stress that in spite of the tendency to consensus alternative foundations of mathematics and alternative theories based on these different foundations can be well developed by contemporaries. So what has been just said about translations between mathematical theories produced at different epochs also applies to alternative contemporary approaches (Note 16).

The reader may wonder if promoting categorical foundations of mathematics and insisting on the importance of the pedagogical aspect of foundations (in the above Introduction) I suggest another radical reform of school mathematics similar to that suggested by enthusiasts of set-theoretic structural mathematics in 1960ies. This latter reform undertaken simultaneously in many countries and known in the US under the name of New Maths was an attempt to comply the elementary mathematical education with the current views on foundations: kids were supposed to learn about sets and elementary structures rather than solve traditional geometrical problems with compasses and ruler and the like (Adler 1972). Later this reform was commonly recognized as a pedagogical failure (Kline 1973).

Although basic elements of Category theory just like elements of Set theory can be learned at early stages of mathematical education (see Lawvere&Schanuel 1997) the categorical view on mathematics outlined above suggests a different pedagogical strategy. An average today's elementary mathematical textbook presents a mixture of patterns of mathematical reasoning coming from various historical epochs: it usually contains some elements of Euclid-style synthetic geometry, some Cartesian-style elementary symbolic algebra, some oldish analytic geometry, some elements of Set theory. In my view such a diversity of styles and contents is quite justified because older approaches continue play a role in today's mathematical practice. Pupils definitely need to learn something about older ways of doing mathematics, not just learn older mathematical contents presented in a recent fashionable form. In other words today's foundations of mathematics must have a historical dimension and include most significant older foundations. Remind Lawvere's words quoted in the Introduction concerning the role of foundations in clarifying the "origins and general laws of development" of a science.

A major problem with such historical approach in mathematics education is that it appears to be incompatible with the usual notion of axiomatic method. Facing this problem today's textbooks written after the failure of the New Maths often compromise severely against the classical standard of systematicity without suggesting any replacement.

My proposed solution of this problem is this: a modern mathematical textbook should provide few different versions of axiomatic method rather than one. Basic patterns of Euclid-style, Cartesian-style, Hilbert-style and Bourbaki-style theory-building should be definitely included. Radical differences between these approaches should be articulated rather than hidden. The controversial dialectical nature of foundations of mathematics should be stressed rather than kept as a Pythagorean secret. Categorical foundations should be treated not only as a basis of an important part of recent mathematics but also as a means of organization the above diverse contents into a systematic whole. Teaching different ways of doing mathematics the teacher should show how to translate between mathematical contents developed on different foundations without trying to smooth the differences between these foundations. Given today's overwhelming flow of rapidly updating information such a translation skill is crucially needed in any domain of activity including mathematics. A coherent translation is possible even when no invariant structure is available.

Endnotes:

Note 1:

Euclid's *Elements* are often referred to as a typical example of canonical mathematical text. It is often said that until recently people used this book as a Bible of geometry. In fact this alleged habit never existed. To see this it is sufficient to look more precisely into book titled *Euclid's Elements*, which have been published before 19th century. One finds a surprisingly diverse literature under this title. Early publishers and translators of Euclid's *Elements* tried to produce a sound mathematical textbook rather than reproduce a canonical text. They didn't hesitate to improve on earlier editions of the *Elements* when they judged this appropriate. According to today's common standard the existing early editions of the *Elements* don't qualify as different versions of the same text. Any of these people could get today a copyright as the author of his *Euclid's Elements*. The notion of being an author is certainly changed since then.

Today's canonical Greek edition of Euclid's *Elements* has been produced by Heiberg and his assistant Menge only in the end of 19th century (see Euclides 1883-1886); noticeably these people were philologists, not mathematicians. So the idea to reproduce Euclid's text literally and translate it into modern languages "keeping its content invariant" is relatively recent; it is relevant to history of mathematics rather than to mathematics itself.

It should be also stressed that since 17th century (a modernized version of) Euclid's *Elements* was no longer the only geometry textbook on the market. Arnauld and some other authors produced their own original textbooks.

Note 2:

The original version reads

Two non-zero vectors x and y are orthogonal if and only if $(y - x)^2 = y^2 + x^2$.

I use for my example only the "only if" part of the statement. Euclid counts the converse of his 1.47 as a separate Proposition 1.48. Donneddu makes both propositions into one theorem. I have modified Donneddu's theorem because this obvious difference is irrelevant to my argument.

Note 3:

By structural mathematics I understand mathematics build on foundations complying with principles of Mathematical Structuralism. A canonical example of structural mathematics is given by (Bourbaki 1939 - 1983).

Note 4:

We are now ready to spell the precise definition: an infinite cyclic group is a group with an infinite number of elements and such that any of its elements is generated by some distinguished element g and its inverse g' . A group is said to be generated by a set of its distinguished elements called *generators* when every element of

this group is a product of the generators. A canonical example of infinite cyclic group is the additive group of whole numbers, which is generated by 1 and -1.

Note 5:

Notice that the order, in which transformations are composed, matters. I use here the so-called *geometrical* notation where the composition is written in the "direct" order. According to another notation called *algebraic* the composition is written in the inverse order.

Note 6:

Frege's example is the concept of *direction* build as follows. One considers the class of all straight lines on Euclidean plane and the equivalence relation "is parallel". Then one associates a single abstract concept called *direction* with each isomorphism class of parallel lines.

Note 7:

I elaborate on this issue in (Rodin 2007)

Note 8:

I mention here groups of isomorphisms (not to be confused with isomorphisms of groups!) because they are very important in geometry and physics. I mean groups of geometrical transformations of a given space. Only *reversible* geometrical transformations, i.e. geometrical isomorphisms, of a single object (the given space) form groups (with the composition of transformations as group operation) because in this case the reversibility is equivalent to the existence of inverse elements. So the talk of invariants of groups, which is so important for structural approaches in physics, concerns only reversible transformations and doesn't apply to geometrical (or other) transformations in general. A non-mathematical reader may skip the reference to groups of isomorphisms in this part of the paper. I shall explain the idea of group of isomorphisms more clearly in categorical terms in Section 8.

Note 9:

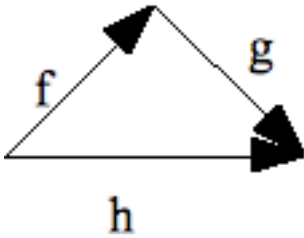
Categorical definition of isomorphism resembles the definition of reversible transformation given in the end of Section 4. However it doesn't involve a reference to elements. Think about groups G , G' as objects of a category and modify the definition of Section 4 in this way: $ii' = I_G$ and $i'i = I_{G'}$ where I_G is the identity morphism of G and $I_{G'}$ is the identity morphism of G' . The rest of the definition remains the same.

Note 10:

A further step of such categorical analysis amounts to considering the full category of functors of the given form; such a functor category provides a precise information about how A translates into B.

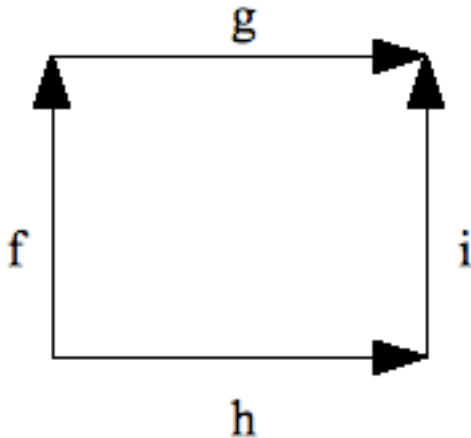
Note 11:

A categorical diagram is said to commute or be commutative when the compositions of all morphisms shown at the given diagram produce other morphisms shown at the same diagram in appropriate places, so that any ambiguity about results of the compositions is avoided. For example, saying that the triangle



is commutative is simply tantamount to saying that $fg = h$.

Morphisms resulting from composition of shown morphisms can be omitted at a commutative diagram when this doesn't lead to an ambiguity. For example, saying this this square



is commutative is tantamount to saying that $fg = hi$.

Note 12:

Isbell in his review (1967) of (Lawvere 1966) points to a technical flaw in Lawvere's proof of this theorem . This flaw is fixed, in particular, in (McLarty 1991).

Note 13:

Does this mean that Ehresmann misconceived of his own invention when he thought of Sketch theory as a general theory of structure? I don't think so. A general theory of structure should not be necessarily a structural theory and should not provide a support for Structuralism as a philosophical view about mathematics.

Note 14:

Cantor's first publication related to Set theory (1872) concerned trigonometric expansions. There are also strong reasons to trace the history of Cantor's set concept back to Riemann's concept of manifold, see (Ferreirós 1999).

Note 15:

Any trajectory in space can be followed in both directions; any spatial motion has a reverse motion. This is how we think about spatial motions no matter which mathematical notion of space (Euclidean space, Riemannian manifold, ...) if any supports this thinking. The case of a black hole involves a space-time, not merely a space, and points to a deep problem in foundations of physics, which I cannot discuss here.

Note 16:

A more precise criterion of identity of mathematical content developed along these lines would require a more precise account of what counts as a sound translation of one piece of mathematics into another. I cannot provide it in this paper and leave for a future work.

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