Knowing-How and the Deduction Theorem

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Abstract:

In his seminal address delivered in 1945 to the Royal Society Gilbert Ryle considers a special case of knowing-how, viz., knowing how to reason according to logical rules. He argues that knowing how to use logical rules cannot be reduced to a propositional knowledge. We evaluate this argument in the context of two different types of formal systems capable to represent knowledge and support logical reasoning: Hilbert-style systems, which mainly rely on axioms, and Gentzen-style systems, which mainly rely on rules. We build a canonical syntactic translation between appropriate classes of such systems and demonstrate the crucial role of Deduction Theorem in this construction. This analysis suggests that one's knowledge of axioms and one's knowledge of rules

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under appropriate conditions are also mutually translatable. However our further analysis shows that the epistemic status of logical knowing-how ultimately depends on one's conception of logical consequence: if one construes the logical consequence after Tarski in model-theoretic terms then the reduction of knowing-how to knowing-that is in a certain sense possible but if one thinks about the logical consequence after Prawitz in proof-theoretic terms then the logical knowledge-how gets an independent status. Finally we extend our analysis to the case of extra-logical knowledge-how representable with Gentzen-style formal systems, which admit constructive meaning explanations. For this end we build a typed sequential calculus and prove for it a "constructive" Deduction Theorem interpretable in extra-logical terms. We conclude with a number of open questions, which concern translations between knowledge-how and knowledge-that in this more general semantic setting.

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1 Knowing-How without Anti-Intellectualism

1.1 Intellectualism and Anti-Intellectualism

In November 1945 Gilbert Ryle gave his Presidential Address to the Aristotelian Society [24], which produced a wide epistemological debate about the concept of knowledge-how. This continuing debate [4] has been recently summarized in the following words:

> There are two main camps in the debate about the constituent concepts of knowledge-how. One camp, intellectualism, argues that knowledge-how involves propositional knowledge [...], whereas the competing camp argues that knowledge-how does not involve propositional knowledge - a view called anti- intellectualism. According to anti-intellectualists, whereas propositional knowledge is a certain type of belief, knowledge-how consists in abilities, skills, or dispositions [...]. ([10], p. 2930)

Even if the titles of intellectualism and anti-intellectualism are often used in this debate as mere labels pointing to certain accounts of propositional and non-propositional knowledge, the choice of these words is evidently not arbitrary. Ryle calls the "intellectualist legend" an epistemological thesis that all knowledge is knowledge-that, i.e., a knowledge of certain proposition or class of propositions. He argues that this view of knowledge leaves aside a sort of knowledge needed for various actions such as riding a bicycle or making logical inferences (more on this last example below), i.e., knowledge-how. Ryle's pejorative use of word "intellectualist" apparently has a social motivation: intellectualists are people who know a lot but are incapable to undertake an action.

1.2 Rules and Sentences.

As we shall now argue this Ryle's terminological decision is not simply unfortunate but reflects a genuine conceptual confusion, which continues to affect the epistemological debate on knowing-how up to the present.

One issue, which is central in this debate, is the distinction between knowing a proposition and knowing how to act. Another issue, which is also widely discussed in the same debate, is the distinction between tacit and explicit knowledge. The popular example of knowing how to ride a bicycle instantiates both these features: it is a knowledge-how and it is tacit because even an experienced rider usually cannot explain in words how she rides a bicycle and cannot transfer this knowledge to another person by linguistic means. Nevertheless it is wrong, we claim, to generalize upon this and similar examples of knowing-how. The two aforementioned distinctions should be analyzed separately. The title of "(anti)-intellectualism" reflects a confusion of these two different distinctions as we shall briefly explain.

Here is an example of different sort, which has been considered by Ryle in his original paper but attracted little attention in the later literature: knowing how to reason logically. Ryle presents it via the following imaginary dialogue:

> [T]he intelligent reasoner is knowing rules of inference whenever he reasons intelligently'. Yes, of course he is, but knowing such a rule is not a case of knowing an extra fact or truth ; it is knowing how to move from acknowledging some facts to acknowledging others. Knowing a rule of inference is not possessing a bit of extra information but being able to perform an intelligent operation. *Knowing a rule is knowing how.* (The emphasis is added by the authors.) ([24] p. 7).

What knowing a rule amounts to exactly is not an easy question. Clearly, a law-like behavior of certain agency and moreover a law-like character of some natural process is not a sufficient reason for attributing such a knowledge.

One's awareness about a rule is not a sufficient reason either because one can be aware about a rule but be unable to follow it. But however these and other related epistemological questions are settled, knowing a rule can, and in our view should, be seen on a par with knowing a proposition. Thus we have got here an example of knowing how, namely that of knowing how to make logical inferences, which instantiates a sort of knowing-how that admits a representation both in natural languages (in particular, in the form of imperative sentences) and in symbolic logical calculi in the form of syntactic rules. Clearly, logic is not the only domain where explicit rules play a role. Explicit rules are abundant in games, social and political life, technology and in many other domains. Knowledge of these rules and capacity to apply them in most pragmatic contexts is at least as important as propositional knowledge. Representation of knowing-how in the form of rules and instructions is a way of making this knowledge explicit.

Thus the popular idea according to which knowledge-how has an intrinsic tacit character is misleading. The distinction between explicit and tacit knowledge is interesting and important on its own but it should not be systematically confused with the distinction between knowing-how and knowing-that even if some suggestive examples can be readily used for studying both these issues. This is a reason why, in our view, the title of "anti-intellectualism", which refers to the allegedly tacit character of knowledge-how, is not an appropriate name for the view according to which knowledge-how is epistemically significant and not reducible to propositional knowledge

1.3 Ryle on Knowing How To Reason

A close reading of the following passage by Ryle shows how exactly he mistakes the tacit character of knowing-how in some examples for its essential property.

> [A]rguing intelligently did not before Aristotle and does not after Aristotle require the separate acknowledgment of the truth or "validity"of the formula. "God hath not left it to Aristotle to make (men) rational."Principles of inference are not extra premisses and knowing these principles exhibits itself not in the recitation of formula but in the execution of valid inferences and in the avoidance, detection and correction of fallacies, etc. The dull reasoner is not ignorant ; he is inefficient. A silly pupil may know

> by heart a great number of logicians' formulae without being good at arguing. The sharp pupil may argue well who has never heard of formal logic. ([24], p. 7).

There are two different lines of argument here. One argument is that

- (i) "Principles of inference are not extra premisses and knowing these principles exhibits itself not in the recitation of formula but in the execution of valid inferences."

It has two different parts (which I paraphrase according to my understanding of Ryle):

- (ia) Rules of logical inference are not propositions.
- (ib) Knowing a rule of inference differs from knowing a proposition. Knowing a rule is not a propositional attitude. It involves a capacity to act according to this rule and detect violations of the rule.

The other argument is this:

- (ii) One can possibly be an efficient reasoner without knowing formal logic. In other words, one's knowledge how to reason in certain cases is not based on knowing formal rules.

This last argument in different words is expressed in the first two and in the last sentences of the quoted passage while the former arguments sits in the middle. So Ryle mixes the two arguments without explaining how the rule-based character of logical inference is related to the fact that it is possible to follow certain rules without being aware of them. In the paragraph, which follows the above quoted passage, Ryle briefly considers and rejects the view according to which "the intelligent reasoner who has not been taught logic knows the logicians' formulae "implicitly "but not "explicitly "." Ryle takes it for granted that the assumption " that knowledge-how must be reducible to knowledge-that " is a part of the above view and then famously dismisses this view as a "intellectualist legend ". For some reason Ryle simply does not distinguish between the case when one reasons according to logical rules without being aware about these rules and the case when one follows logical rules intentionally and explicitly.

This analytic fallacy, which lies in the core of Ryle's "anti-intellectualism" and its more recent heirs, gave to the concept of knowledge-how its anti-intellectual flavor which, in our view, it does not deserve.

In the next two Sections we consider an explicit rule-based form of knowing-how represented in the form of syntactic rules in symbolic logical calculi.

2 Two Styles of Axiomatic Thought

In this Section we introduce the standard informal distinction between the so-called Hilbert-style and Gentzen-style symbolic calculi and then explain its relevance to the epistemological debate on knowing-how and knowing-that. A part of this presentation is made in the form of historical narrative.

2.1 Hilbert-Style

The standard modern notion of axiomatic theory stems from David Hilbert's seminal work in foundations of geometry [11]. The idea here is to generate the intended theory T from a list of *axioms* A_i by inferring from the axioms further propositions called *theorems*. More precisely, axioms in this setting are *propositional forms*, which become full-fledged propositions under an *interpretation*, which is an assignment to non-logical terms of A_i certain semantic values borrowed from other theories or, less formally, simply from the "world out there" (cf. Hilbert's legendary suggestion to interpret the Pythagorean theorem in terms of tables and beer mugs).

Hilbert's 1899 work on foundations of geometry had an obvious drawback: the logical part of his axiomatic method remained underdeveloped or perhaps even not developed at all. This is a reason why in his famous lecture on "Axiomatic Thought" given in 1917 Hilbert claims that

> "[I]t appears necessary to axiomatize logic itself " ([12] p. 1113)

and then pursues this project in his later work [13, 14]. In [13] Hilbert and Ackerman consider a number of logical calculi presented via lists of axioms (which in this case are logical tautologies) and syntactic rules, which generate from the axioms further tautologies. In his later joint work with Bernays [14] Hilbert applies this upgraded form of his axiomatic method in formal theories of geometry and arithmetic. As Hintikka rightly observes Hilbert's leap from axiomatic theories of geometry and arithmetic to axiomatic theories of logic was not innocent [15]. Nevertheless such a uniform approach to logical calculi and non-logical theories makes part of the current informal idea of Hilbert-style formal system.

Hilbert never explicitly elaborated on the concept of logical inference but it is plausible that at least in his most influential [11] he had in view a prototype of the model-theoretic truth-conditional semantical concept of logical consequence later made explicit by Alfred Tarski [27]:

Definition 1 *Propositional form B is a logical consequence of propositional forms $A_1, \dots, A_n$ iff every interpretation I of the given language, which makes $A_1, \dots, A_n$ into true propositions $A_1^I, \dots, A_n^I$ makes B into true proposition B^I, in symbols $A_1, \dots, A_n \models B$.*

Notice that this conception of logical consequence does *not* involve that of rule. Under this view the syntactic rule $A_1, \dots, A_n \vdash B$ is viewed (granting its soundness with respect to the given semantics) as a mere symbolic representation the fundamental relation $A_1, \dots, A_n \models B$. (After Gödel's works we know that such a symbolic representation cannot be faithful, i.e., semantically complete, in case of sufficiently strong theories including arithmetic.) Consequence $A \models B$ is conceived of here as a fact of the matter, which concerns all possible interpretations of A and B. Rule $A \vdash B$ under this view is nothing but a symbolic trick, which allows one in appropriate circumstances to get a grasp of this fact.

2.2 Gentzen-Style

In 1935 Hilbert's associate Gerhard Gentzen published a paper [7] where he argued that

> The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. ([7], p. 68)

and proposed an alternative approach to syntactic presentation of deductive systems, which involved relatively complex systems of rules and didn't use logical tautologies. In [7] Gentzen builds in this way two formal calculi known as Natural Deduction and Sequent Calculus.

Gentzen's remark quoted above constitute a pragmatic argument but hardly points to an original epistemological view on logic and axiomatic method. However his further remark that

> The introductions [i.e. introduction rules] represent, as it were, the 'definitions' of the symbol concerned. ([7], p. 80)

is seen today by some authors as an origin of an alternative conception of logical consequence and alternative logical semantics more generally, which has been developed in a mature form only in late 1999-ies or early 2000-ies and is known today under the name of *proof-theoretic* semantics (PTS). It is instructive to compare Gentzen's idea to use syntactic rules as a form of implicit definitions with Hilbert's use of axioms as implicit definitions. The two approaches may appear to be very similar but in fact they are not. Think of usual axioms of Group Theory, which today are commonly used in mathematical courses as the definition of the group concept in Algebra. These axioms serve as a definition in the following sense: any structure, which satisfy the axioms, i.e., serves as their model, is a group. Model theory, which originates from Tarski's pioneering works, explains away the satisfaction relation in terms of truth-conditions. This is all well-known. But what kind of entity X can possibly "satisfy" a rule or a system of rules, so one could claim that the rules "define" X in some reasonable sense? How the satisfaction relation (if it can be used here at all) has to be construed in this case?

Proof theoretic semantics provides some answers to these and other related questions and develops new formal frameworks accordingly. For an overview reflecting the present state of affairs in PTS the reader is referred to [25, 5, 20]. Here we provide only few remarks, which give a general idea, prevent the reader from possible confusions and highlight some key features of PTS, which are important for our following arguments.

- Proof Theory referred to in PTS is *not* the proof theory in Hilbert's sense of the word [14] where a proof is identified with a formal derivation and then made into an object of a meta-mathematical study, but the *General Proof Theory* (GPT) due to Dag Prawitz, where, by his word "proofs are studied in their own right in the hope of understanding their nature" [21, 22]. In more a recent paper Prawitz describes the difference between the proof-theoretic and truth-conditional approaches to meaning as follows:

  > [I]n contrast to a truth-conditional meaning theory, [in PTS] one should explain the meaning of a sentence in terms of what it is to *know* that the sentence is true, which in mathematics amounts to having a proof of the sentence. ([20], p. 5-6).

This quote points to a strong conceptual link between PTS, on the one hand, and intuitionisitc and constructivist approaches in logic and foundations of mathematics, on the other hand. We shall explore this link in Section 5.

- PTS is motivated by a broad philosophical view on meaning (and hence on semantics), which is conventionally called "meaning-as-use". This view on meaning goes back to Wittgenstein and more recently has been defended and further systematically developed by Robert Brandom [2] under the name of *inferentialism*. Since PTS is a *formal* semantical approach the reference to "use" amounts here to referring to syntactic *rules*, which specify the use of symbols and symbolic expressions in logical calculi. Gentzen's original insight according to which rules of inference provide the semantic value of symbolic expressions is preserved in all existing various versions of PTS.

- PTS is not denotational. It does not assign certain entities to certain symbols. It assigns to symbols (and first of all to logical symbols, i.e., to logical constants) their *meaning*, which is not construed in this case as an entity. The procedure of such an assignment is called after Martin-Löf the *meaning explanation* and consists, roughly, of the explication of computational content of logical constructions in terms of their building blocks, which are presented in a self-explanatory canonical form [19].

- Historically the PTS approach has been motivated by the idea that formal derivability (albeit not the existence of proof) and logical consequence are the same. However there is a recent proposal according to which these two things should be properly distinguished [6]. In what follows we do not assume that under PTS formal derivation and logical consequence are the same.

2.3 Comparison of the Two Styles

The difference between Hilbert-style and Gentzen-style formal systems is usually described in the recent literature by saying that Hilbert-style systems are typically presented by long lists of axioms or axiom schemes and only few (typically one) rules, while Gentzen-style systems are presented by a small (possibly empty) sets of axioms and long lists of rules; it is further said that Hilbert-style systems "rely more on sets of axioms", while Gentzen-style system "rely more on sets of rules" [29]. This is a very loose and informal description - as are ways in which these titles are actually used in logicians' professional parlance. In order to be in a position to study the two axiomatic "styles" more rigorously we introduce in the next Section the concept of (propositional) *Hilbertian theory* (Def. 11) which is more narrow than what people may call a Hilbert-style propositional theory. We shall not provide a complementary syntactic definition of Gentzen-style theory because in what follows we use in its stead a general syntactic of symbolic calculus (Def. 2). Since any axiom A can be straightforwardly read as a rule of form $\vdash A$ with the empty set of premises, at the syntactic level Gentzen's approach is more general than Hilbert's. So we shall study the place of Hilbert-style theories in this more general syntactic setting.

In this paper we do not develop formal semantic approaches but refer to the two established conceptions of logical semantics mentioned above: Tarski's truth-conditional semantics and PTS. These semantic conceptions do not depend directly on syntactic details. Nevertheless both for historical and some technical reasons it is natural to associate the truth-conditional semantics with Hilbert-style and PST with Gentzen-style. This decision can be viewed as a way to make more precise the idea that Hilbert-style theories "rely more on axioms" and Gentzen-style theories latter "rely more on rules". Such semantic assumptions, once again, are stronger that the current use of titles "Hilbert-style" and "Gentzen-style" may suggest. Authors often use these titles referring only to the syntax without any semantic commitment. Our formal definitions in the next Section are also purely syntactic, so in the formal part of the paper we don't go against the common language. However in the following epistemological discussion the semantic aspects of the two axiomatic styles turn to be crucial.

2.4 Two Axiomatic Styles and the Debate on Knowing-How and Knowing-That

To conclude this Section let us explain the relevance of the distinction between the two styles of axiomatic thinking to the epistemological debate on knowledge-how. It becomes clear from Ryle's remark that knowing how to make a logical inference amounts to knowing the corresponding rule of inference ("Knowing a rule is knowing how"). We assume here that axiomatic theory T, generally, represents a piece of knowledge; in other words, we assume that theory T can be known by an epistemic agent. Since T, generally, comprises propositions (axioms and theorems) and rules of inference, we further assume that one's knowledge of T also splits, accordingly, into a propositional (knowledge-that) and a procedural (knowledge-how) parts. Since every axiomatic theory comprises at least one rule of inference (for otherwise it is not a theory but a mere collection of statements) one's knowledge of a theory always comprises a procedural part. Since Hilbert-style theories comprise few rules and a lot of axioms while Gentzen-style theories comprise few (or no) axioms and a lot of rules, the procedural and the propositional knowledge are distributed differently in the two cases. If one and the same axiomatic theory allows both for Hilbert-style and Gentzen-style presentations then a study of translations between these axiomatic presentations sheds a light on the issue of translatability of procedural knowledge into a propositional form and vise versa. The question of whether or not the procedural knowledge can in some reasonable sense be wholly reduced to the propositional knowledge in a formal axiomatic setting is treated in Section 4.

A disclaimer is here in order. In what follows we don't try to formalize knowledge-how in the same sense in which propositional knowledge-that is formalized in various systems epistemic logics where knowledge of proposition p is represented as a modal operator K applied to p. It is not clear to us whether such a strategy can be used in the case of procedural knowledge and we shall not attempt anything similar.

3 Translation between the two axiomatic Styles and the Deduction Theorem

In this Section we study mathematically the issue of syntactic translatability between Hilbert-style and Gentzen-style systems and highlight the role of Deduction Theorem. This material is by and large standard but in the view of our epistemological purpose we present it here in a more abstract form than usual. Standard definitions are given in a semi-formal manner. We begin with considering the case of propositional theories and finish with short remarks about possible generalizations to the case of first-order and other richer systems.

3.1 Hilbertian Theories

Definition 2 *Symbolic calculus* comprises:

- alphabet of symbols;
- a set of words w_i built with the alphabet;
- a set of rules r_i of form $w_i, \dots, w_k \vdash w$, which derive word w from given words $w_i, \dots, w_k$;
- set A (possibly empty) of axioms which are rules of special form $\vdash w$.

Definition 3 *Propositional language* is a calculus with a distinguished finite set of symbols called *connectives*, which includes connective "$\rightarrow$"; other symbols are called *propositional variables.*

Definition 4 *Propositional theory* is a set T of formulae closed under application of the standard *modus ponens* (MP) (other rules are allowed but not required). Elements of T are called *theorems* of the given theory. The theory is called *axiomatic* when it comprises a distinguished subset $A \subset T$ of *axioms* such that all theorems of T are derivable from the axioms via applications of MP. The notion of derivation from a set Γ of hypotheses (denoted $\Gamma \vdash_T F$ or $\Gamma \vdash F$ when there is no risk of confusion) is standard.

Definition 5 An axiomatic theory is called *Hilbertian* when it comprises as theorems all formulae of the form $K_{A,B}$ and $S_{A,B,C}$ where

$$\begin{aligned} K_{A,B} &\doteq \quad A \rightarrow (B \rightarrow A) \\ S_{A,B,C} &\doteq \quad (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \end{aligned}$$

and has exactly one rule, namely MP.

3.2 Deduction Property

Definition 6 Theory T is said to have the *Deduction Property* (DP for short) if $\Gamma, F \vdash G$ entails $\Gamma \vdash F \rightarrow G$ for all Γ, F and G.

Before we proceed let us explain the relevance of the Deduction Property to our main theme. We are interested to study this property because a theory having this property allows one to "represent" (in a sense, which is made more precise in what follows) any of its rule $A \vdash B$ by the implication $A \to B$, which is a proposition. It appears reasonable to assume that one's knowledge *how* to derive B from A is represented in this case, accordingly, by the knowledge *that* A implies B. Our next Lemma shows that the concept of Hilbertian theory and that of theory with Deduction Property are co-extensional:

Lemma 7 *An axiomatic propositional theory is Hilbertian if and only if it has the Deduction Property.*

Proof:
"$\Rightarrow$" (the "only if" part). The standard proof of the Deduction Theorem [16].

"$\Leftarrow$" (the "if" part). By the definition of derivation in propositional theories we have $A, B \vdash A$. Using the deduction property twice we get from the former formula $\vdash A \to (B \to A)$. Similarly, by using twice the deduction property from $A \to (B \to C)$, $A \to B$, $A \vdash C$ we get $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$. $\triangleleft$

Lemma 7 says that the Deductive Property is not universal and should be always expected, but a proper feature of Hilbertian theories (Definition 11) which other propositional theories do not possess. Some examples of formal systems without Deduction Property are discussed in Appendix 2.

3.3 Translating between the Two Styles

DP and Lemma 7 bear only onto the distribution of propositional and procedural knowledge *within* a Hilbert-style axiomatic system - while our more general goal is to understand how the procedural knowledge represented with a Gentzen-style system may (or may not) translate into a Hilbert-style system and vice versa. We warn the reader that at the time of writing we don't have the full answer to this question in the form of necessary and sufficient conditions under which the two sorts of systems are mutually translatable in a reasonable sense of the word. Below we present some partial results: we show that Hilbertian theories allow for a canonical translation into a sequential form (Lemma 9 below) and then specify a class of Gentzen-style systems which canonically translate into Hilbertian theories (Lemma 10). We need also a preliminary lemma:

Lemma 8 *All axiomatic propositional theories have the following property (rule* $(\rightarrow\vdash)$*): if* $\Gamma \vdash F$ *and* $\Gamma, G \vdash H$ *then* $\Gamma, F \rightarrow G \vdash H$*.*

Proof:

Given the two above derivations form the following sequence of formulae: $\Gamma \vdash F$, $F \rightarrow G$, $\Gamma, G \vdash H$. The sequence qualifies as a derivation $\Gamma, F \rightarrow G \vdash H$. (Some formulae may enter into this sequence more than once but the definition of derivation does not rule this possibility out.) $\lhd$

From a Hilbertian theory to its sequential presentation:
Let T be a Hilbertian theory. Consider the following sequential calculus T_G. Sequences in T_G are of form $\Gamma \Rightarrow F$. Rules of T_G comprise all structural rules (axioms of form $F \Rightarrow F$, contraction, weakening and the cut rule). For each axiom A of T there is the corresponding sequence $\Rightarrow A$ in T_G. Finally T_G has the usual rules for implication, namely:

$$\frac{\Gamma \Rightarrow F \quad \Gamma, G \Rightarrow H}{\Gamma, F \rightarrow G \Rightarrow H}\,(\rightarrow\Rightarrow) \qquad \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G}\,(\Rightarrow\rightarrow)$$

Lemma 9 $\Gamma \vdash F$ *in theory* T *if and only if sequence* $\Gamma \Rightarrow F$ *is derivable in* T_G*.*

Proof:
"$\Rightarrow$" (the "only if" part): Induction by $\Gamma \vdash F$. If F is an axiom of T or member of Γ then $\Gamma \Rightarrow F$ is derivable using structural rules. Given the cut MP is admissible:

$$\frac{\Gamma \Rightarrow A \qquad \dfrac{\dfrac{\Gamma \Rightarrow A \rightarrow B}{\Gamma, A \Rightarrow A \rightarrow B} \qquad \dfrac{\Gamma, A \Rightarrow A \quad \Gamma, A, B \Rightarrow B}{\Gamma, A, A \rightarrow B \Rightarrow B}\,(\rightarrow\Rightarrow)}{\Gamma, A \Rightarrow B}\,(Cut)}{\Gamma \Rightarrow B}\,(Cut)$$

"$\Leftarrow$" (the "if" part): Translation that replaces all entries of $\Rightarrow$ by $\vdash$ is sound with respect to all rules of T_G. See Lemmas 7,8. $\lhd$

From a sequent calculus to a Hilbertian theory:

Let sequent calculus T_G contain all structural rules and rules $(\rightarrow\Rightarrow)$, $(\Rightarrow\rightarrow)$ are admissible in T_G. T_G may also contain other rules. Consider set

$$T = \{F \mid \text{sequence} \Rightarrow F \text{ is derivable in } T_G\}$$

Lemma 10 *T is a propositional Hilbertian theory*

Proof: Axioms are all formulae of T. Since MP is admissible in T_G, T is closed with respect to MP. Formulae of forms $(K_{A,B})$ and $(S_{A,B,C})$ are elements of T because sequences $\Rightarrow K_{A,B}$ and $\Rightarrow S_{A,B,C}$ are derivable using rule $(\Rightarrow\rightarrow)$ and the structural rules. $\lhd$

Lemma 9 tells us that a Hilbertian theory admits a translation into a sequential Gentzen-style form, which preserves and reflects its deductive properties. Lemma 10 says that a sufficiently strong sequent calculus admits a translation into a Hilbertian theory.

3.4 Richer Systems

To conclude this Section we shall make few remarks about first-order and modal extensions of propositional theories. The minimal setting where the Deduction Theorem holds (i.e., which has the Deduction Property) is the *minimal* logic ML, which comprises only one connective $\rightarrow$, all axioms $K_{A,B}; S_{A,B,C}$ and one rule MP (compare Def. 11 above). So the question whether or not in a given theory T has DP is the question of whether or not T interprets ML. Hilbertian theories interpret ML fully and faithfully in the sense that every derivation in T has a pre-image in ML. But if one adds to MP some further rules then DP, generally, fails to hold as it happens, for example, in various systems of modal logic. In some such cases DP can be forced by an appropriate correction of additional rules. Thus the Deduction Theorem can be proved for usual First-Order Logic if one uses a convention according to which the usual rule of universal generalization with hypotheses

$$\frac{\Gamma \vdash P(x)}{\Gamma \vdash \forall x.P(x)}$$

applies only if Γ does not contain variable x in the free form. Without this additional requirement DP fails to hold.

For these reasons the core content DP can be fully understood and studied at the propositional level. Richer systems may have this property when they are used in a restricted form, which is essentially a way to emulate the propositional reasoning in such systems. In Appendix 2 we give an example of useful calculus without DP.

4 Reduction of Knowing-How to Knowing-That

In the last Section we have seen that Hilbertian theories are formal frameworks, which allow for a smooth passage from Hilbert-style presentation to Gentzen-style and (under appropriate conditions) vice versa. Does this property of Hilbertian theories allow for a full "reduction" of knowing-how to knowing-that? [1]

At the first glance the answer appears to be negative because any axiomatic theory deserving the name needs to comprise at least one rule (MP) and cannot reduce to a mere collection of propositions. Recall, however, that the Tarskian semantic concept of logical consequence does not involve the concept of rule. This allows one to think of rule $A \vdash B$ (granting soundness of the given theory with respect to its semantics) as a mere symbolic representation of the fundamental fact that B is a logical consequence of A (i.e., $A \models B$), which is fully explained in terms of truth-conditions. Indeed, formula $A \models B$ stands in this case for meta-theoretical proposition

($SC^{A \models B}$): All models of A are models of B,

which is obviously a proposition. Accordingly, knowledge of rule $A \vdash B$ reduces to knowledge of $A \models B$, which is propositional knowledge. This completes a sketch of the wanted propositional reduction. Now we shall discuss its details and reply some possible objections.

4.1 Ryle on Carroll Paradox

Ryle considers an alleged possibility of propositional reduction in the following form:

> [K]nowing how to reason was assumed to be analyzable into the knowledge or supposal of some propositions, namely, (1) the special

[1] Such a reduction in different contexts, mostly linguistic, was pursued and claimed to be obtained by other authors [26, 4].

> premisses, (2) the conclusion, plus (3) about the implication of the conclusion by the premisses. ([24], p. 6-7)

He rejects this possibility by pointing to an infinite regress known as Carroll Paradox [3]: in order to obtain the conclusion (2) from premisses (1) and (3) one needs to apply a new rule R_1 (*modus ponens*); a similar attempted reduction of knowing R_1 requires knowing a new rule R_2 and so on *ad infinitum*. If we assume that Ryle refers here to a Hilbertian framework with Deduction Property:

(DP) : $A \vdash B$ if and only if $\vdash A \to B$

then this regress can be presented as follows:

$A \vdash B$ if and only if $\vdash A \to B$
$A, A \to B \vdash B$ if and only if $A \vdash (A \to B) \to B$
$A, (A \to B) \to B \vdash B$ if and only if $A \vdash ((A \to B) \to B) \to B$
. . . .

where each application of DP increases the number of implication signs in each formula by one. Clearly this procedure does not allow for replacement of rule $A \vdash B$ by proposition $A \to B$. Here we are wholly with Ryle.

Ryle's further remarks that

> [K]nowing . . . a rule [of logical inference] is not a case of knowing an extra fact or truth ; it is knowing how to move from acknowledging some facts to acknowledging others. ([24], p. 7)

This time his claim is inconsistent with our claim according to which knowing rule $A \vdash B$ reduces to knowing inference $A \models B$ interpreted after Tarski as proposition $SC^{A \models B}$. Indeed, $SC^{A \models B}$ (and hence $A \models B$) is a description of a state of affairs, which doesn't involve anything like a "move" from premisses to conclusion. In this semantics framework the very name of "rule of inference" is, strictly speaking, oxymoronic or at leas incoherent. In this framework the idea of reasoning as an action in time, which comprises earlier steps and later steps and is a subject of certain rules, is relevant only to the syntax: rule $A \vdash B$ and all symbolic manipulations, which are subject to this rule, is nothing but a partial symbolic representation of the relation $A \models B$, which is a fact of the matter. One may still argue that knowing how to manipulate with symbols according to syntactic rules requires a skill, which does not necessarily belong to one's knowledge of $A \models B$, and which may help one to come to knowing $A \models B$. But even if one agrees with these claims they hardly constitute a strong objection to the thesis of epistemic reducibility of knowing-how to knowing-that because in this semantic framework the symbolic skills have only an auxiliary role and no independent epistemic value. So we get here a version of Platonic epistemology where practical skills are subsumed to the knowledge of truth.

4.2 Further Objections

Let us now consider some other possible objections to the proposed reduction. $SC^{A \models B}$ is not a theorem of the same theory T where $A \vdash B$ belongs. $SC^{A \models B}$ belongs to the model theory M_T of T. One may argue that this opens an infinite regress: in order to know a theory one needs to know another theory and so on. One may also argue that in order to make use of $SC^{A \models B}$ one needs to apply certain logical rules at the meta-theoretical level, which once again opens Carroll's infinite regress in a new form. In order to block such objections one should not think of M_T as a theory on equal footing with T; in fact our argument doesn't require that M_T has any deduction structure at all. We can describe the proposed propositional reduction in a clearer form by replacing the usual syntactic notion of theory T by a semantic conception of theory T_S which extends the syntax of T with a single symbol $\models$ and rule

$$(\vdash\models): \qquad \frac{A \vdash B}{\vdash A \models B};$$

at the semantic level M_T comprises *all* models of T. Expressions of form $A \models B$ stand in T_S for propositions $SC^{A \models B}$. Such expressions are sterile in the sense that they cannot be used in derivations along with usual formulas (the use of syntactic rules in T_S is restricted accordingly); their sole role is to make explicit the model-theoretic semantics of derivations. We allow T_S to comprise true sentences of form $\Gamma \models B$ (and, in particular, $\models B$: Gödel sentences) when T does not provide corresponding derivations $A \vdash B$ ($\vdash B$). So we have a theory and its rudimentary model theory in one pocket. The reduction of knowing rules to knowing propositions along the above lines proceeds in T_S without referring to any other theory. Arguably Hilbert in 1899 had a similar concept of theory in his mind [11].

Thus we submit that Tarski's truth-conditional semantic conception of logical consequence provides a conceptual reduction of logical knowledge-how, i.e., knowledge how to make logical inferences, to propositional knowledge-that. But this claim does not imply that Ryle was wrong saying that knowing how to make logical inferences does not reduce to propositional knowledge. He likely would not accept Tarski's concept of logical consequence and insist that the concepts of rule and rule-following are fundamental in logic and in reasoning. The above quote from Ryle suggests that he has rather an inferentialist conception of logical consequence in his mind. Thus the above analysis does not solve the epistemological problem but allows us to locate it properly. What is at stake in the debate on knowing-how and knowing-that as far as it concerns logic is the conception of logical consequence rather than anything else.

4.3 Two Conceptions of Logical Consequence and Their Epistemological Implications

The model-theoretic conception of logical consequence has a number of features, which makes it vulnerable to an epistemological critique. Prawitz argues that it involves a form of circularity and is uninformative ([22], p. 67-68). A part of the problem is that the extension of expression "all models of theory T" is not precisely defined. Should one think here only about the "real world models" developed in natural sciences, models borrowed from other parts of mathematics or models built with metaphysical speculations about the ultimate "logishe Aufbau der Welt"? If one determines some domain D of all possible models of T using another theory S then the above semantic construction of T_S is no longer self-sustained because now its semantic part is essentially determined by theory S (which, generally, cannot be incorporated into T_S as above on pain of inconsistency). In that case a critic who argues that the model-theoretic conception of logical consequence opens a regress appears to be right. There are two ways of preventing this regress from being infinite. One option, is to give theory S an exceptional epistemic status of being *the* universal theory of most general features of the world (or even of all possible worlds). The traditional name of a theory, which may fit this description, is metaphysics. Another option is to diversify one's concept (but not the general conception, which remains model-theoretic in all such versions) of logical consequence by making it dependent on S. In this way one can conceive of, for example, of Quantum Logic as an interpreted logical calculus (or a family of such calculi), which represents symbolically the corresponding special semantic notion of logical consequence that draws on Quantum Physics.

All these versions of the model-theoretic conception of logical consequence have their supporters. But observe that they share this common feature: they leave the idea of an epistemic *act* (such as acknowledgement, rejection, verification, falsification or questioning a proposition) outside of logic proper and place it into the disciplinary domains of Psychology, Sociology and other disciplines, which study the "context of discovery". This is a reason why one who believes that the concept of epistemic act is fundamental in epistemology and regards logic as a normative discipline that tells one how to perform such acts properly, is not willing to accept the model-theoretic conception of logical consequence in any of its various versions. And she does not need to do that because the proof-theoretic semantics (PTS) provides a viable alternative approach, which accords with her epistemological expectations. Like the model-theoretic approach the PTS approach exists both in formal and informal pre-mathematical versions. The informal version of PTS due to Martin-Löf involves a *meaning explanation*, which proceeds in a natural language but can be also translated into a computer code in the form of a program compiler [19]. So the meaning explanation makes explicit the computational content of the given syntax in a form, which is understandable for human or acceptable for machine (in the sense that it translates the given syntax into a series of executable commands).

A formal meaning explanation uses an extra theory M for explaining the meaning of symbolic expressions in the target theory T. From a formal point of view nothing prevents one from using for this purpose a theory M, which interprets rules in T in terms of appropriate truth-conditions [28]. This brings one back to a version of model-theoretic conception of logical consequence in T. In this case the resulting logical semantics does not any longer qualify as PTS as described in Section 2 above. In order to avoid this conceptual confusion it would be reasonable, in our view, to reserve the term "meaning explanation" to PTS-type semantics and stick to the standard terminology in model-theoretic approaches.

Thus even if recent advances in logical semantics do not allow us to resolve the epistemological debate on knowing-how and knowing-that, they provide certain means for representing knowing-how (in logic) without reducing it to a variety of knowing-that. This comprises not only syntactic means, which help to represent knowing-how in the form of syntactic rules but also semantic means such as PTS, which provides this syntax with a procedural semantics.

5 Constructive Theories

The explicit form of knowing-how, i.e., knowing how to follow certain formal rules, is not limited to logic. In this Section we extend our analysis beyond the "logical" knowing-how, i.e., knowing how to make logical inferences. Notice that in the preceding part of the paper we did not apply any formal criterion of logicality. The syntactic part of our above analysis did not involve anything (except some traditional names), which made it specific to logic. Now we shall consider some non-logical interpretations of the same or similar syntactic constructions. For a suggestive example, which demonstrates this approach, think of Kolmogorov's *calculus of problems* **CP** [17]. Syntactically **CP** is identical to the standard intuitionistic propositional logic but has a different intended semantics (known as BHK semantics). This semantics is not logical or at least not logical in a narrow sense of the word: formulae represent here problems rather than propositions. Following [23] we call hereafter formal theories, which comprise rules for non-propositional objects, *constructive* theories.

5.1 Constructive Deduction Theorem

Let T be a Hilbertian theory. We associate now with T a typed sequential calculus CT, which is more apt to standard PTS-style constructive interpretations than the sequential calculus T_G from Section 3.3 above. We prove the deductive equivalence between T and CT (Lemma 12) and, finally, prove for CT a "constructive version" of Deduction Theorem (Theorem 13), which gives us some insights about extra-logical forms of knowing-how.

Definition 11 CT comprises:

- Types of CT are all formulae of T;
- With each axiom A of T associate constant c^A, which we interpret as the trivial derivation of A in T. In the cases of axioms $(K_{A,B})$ and $(S_{A,B})$ we use the established notation and denote the corresponding constants as $k^{A\to(B\to A)}$ and $s^{(A\to(B\to C))\to((A\to B)\to(A\to C))}$ omitting the upper index when this cannot cause a confusion.
- Terms of CT correspond to derivations in T; these terms are built from variables and constants with a single binary operation (multiplication),

which is an application of rule MP. Each such term determines a unique binary tree such that its internal nodes are marked by MP and its leaves (??) correspond either to T-derivations of axioms or to variables. Rules of CT specify when this tree is the correct tree of derivation from hypotheses in T.

- Sequences of CT are expressions of form

$$x_1{:}F_1, \dots, x_n{:}F_n \vdash t{:}F,$$

  where $x_1, \dots x_n$ are mutually different variables, $F_1, \dots F_n, F$ are types (formulae) and t is a term. Sequences determine the same trees but comprise an additional markup: they put label F to the root and attach mark F_i to each leave x_i , which signifies that x_i is a variable over derivations of formula F_i). The obtained tree can get new isolated nodes marked by variables, which are not elements of term t; leaves, which are not in the list $x_1, \dots x_n$ may remain unmarked.

- Axioms and rules of CT:

  - $x_1{:}F_1, \dots, x_n{:}F_n \vdash c^A{:}A$, where A is an axiom of T ??,
  - $x_1{:}F_1, \dots, x_n{:}F_n \vdash x_i : F_i$,
  - $\dfrac{x_1{:}F_1, \dots, x_n{:}F_n \vdash u{:}(F \to G) \quad x_1{:}F_1, \dots, x_n{:}F_n \vdash v{:}F}{x_1{:}F_1, \dots, x_n{:}F_n \vdash (u \cdot v){:}G}$.

Lemma 12 *Every derivable sequence $x_1 : F_1, \dots, x_n : F_n \vdash t : F$ in CT corresponds to a unique derivation $F_1, \dots F_n \vdash F$ in* **T**. *Each derivation $F_1, \dots F_n \vdash F$ in* **T** *corresponds to a unique term t such that its associated sequence $x_1{:}F_1, \dots, x_n{:}F_n \vdash t{:}F$ is derivable in CT.*

Proof: Induction by derivations.
Remark: When variable x_i is not an element of t the introduction and the elimination of declaration $x_i : F_i$ to/from the given context does not affect the derivability of the sequence. These operations correspond to introduction and elimination of hypothesis, which is not used in the derivation.

Theorem 13 (*"Constructive" Deduction Theorem* or CDP) If sequence $x_1{:} F_1, \dots, x_n{:}F_n, x{:}F \vdash t : G$ is derivable in CT, then there exists term u such that sequence $x_1{:}F_1, \dots, x_n{:}F_n \vdash u{:}(F \to G)$ is also derivable.

Proof: This follows immediately from Lemma 12 and the standard Deduction Theorem. In Appendix 1 we provide a direct proof, which is instructive because it makes explicit the computational content of Theorem 13.

The standard PTS-style constructive reading of Theorem 13 is as follows: if in the given context one is in a position to produce from a given token x of type F a new token t of type G then one is also in a position to produce in the same context a token u of type $F \to G$, i.e., a *method* of producing tokens of G from tokens of F. In this framework "method" u is an object on equal footing with tokens of other types such as F and G. Here is a dummy example: if one knows how to produce porridge from oat one also knows how to produce a method of cooking porridge from oat - say, in the form of written recipe. This property of Hilbertian systems can be very useful in applications but at the same it would be unreasonable to expect that everyone who knows how to cook porridge also knows how how to write cooking books!

5.2 Tacit Knowledge Revisited

The constructive Deduction Property sheds some light on the issue of the allegedly "tacit" character of knowing-how in many practical examples such as that of riding a bicycle. So far we called knowledge-how explicit when it involved knowing certain explicitly written rules. Let us now change this vocabulary and assume for the sake of the argument that in a given symbolic calculus, which is supposed to represent some bulk of knowledge, all syntactic rules are hidden from view while all its formulae (words) are observable. Let us now call one's knowing of rule

$$\Gamma, v\!:\!V \vdash w\!:\!W$$

explicit only if it translates into the the form

$$u\!:\!(V \to W)$$

(in the sense that the given calculus is Hilbertian and hence $\Gamma, v\!:\!V \vdash w\!:\!W$ entails $\Gamma \vdash u\!:\!(V \to W)$; otherwise we call this knowledge *tacit* . Using these terms we shall now call tacit one's knowledge how to cook porridge if this person is unable to write a recipe, and call the same knowledge-how explicit if this person also has this extra capacity. At the syntactic level the difference between the two forms of knowing-how corresponds to the difference between the calculi, which do have and do not have the Deduction Property. We can see no good reason to expect that all symbolic calculi, which represent certain useful knowledge-how in the form of system of rules, have the Deduction Property. In Appendix 2 we give an example of a useful calculus without this property.

5.3 Which Rules are Logical?

Let us now come back to the question of whether or not a non-logical knowledge-how can be reduced to knowledge-that. In the last Section we have seen that the task cannot be completed by reducing the number of rules to minimum; we have also seen that it can be completed (if one seeks for a conceptual rather than a formal reduction) with Tarski's semantic conception of logical consequence. This suggests the following strategy for the non-logical case. Given symbolic calculus C interpreted in certain constructive terms, first, translate it syntactically into a Hilbertian form C_H and, second, interpret the single syntactic rule of C_H as usual, i.e., as the *logical* rule of *modus ponens*. Finally use Tarski's logical semantics as in the former case. This strategy, if successful, reduces a given constructive theory, which comprises rules applied to objects other than propositions, to a standard Hilbert-style axiomatic theory, where all well-formed formulae are interpreted as propositions and where all rules are logical rules applied to propositions. For a historical example of such a propositional rendering of a given constructive theory think once again of Hilbert's semi-formal axiomatization of Euclidean geometry, which translates Euclid's *rules* (of how to produce a straight segment and other rules) into a convenient propositional form [23]. The obtained axiomatic theory falls under the analysis given above: as soon as one accepts Tarski's conception of logical consequence one's knowledge of how to make geometrical constructions in a strong sense reduces to a propositional knowledge.

In Section 3.3 we have studied how Hilbert-style and Gentzen-style formal systems translate into each other syntactically. However a propositional translation of constructive theories, which we discuss here, involves semantics and cannot by fully analyzed from a syntactic viewpoint. We know how standard first- and higher-order logical calculi combine logical semantics (semantic values of logical constants) with extra-logical semantics (interpretations of non-logical symbols). It is important for our argument, that such a semantic combination admitted by well-formed first-order formulae also admits the standard interpretation of these formulae as propositions, which accords with the idea that the syntactic rules of the given calculus represent *logical* rules. But a constructive theory (in the relevant sense of the word) comprises rules, which are applied to extra-logical objects, such as Euclid's rule of how to produce a line from a given point to another given point. Let us call such rules extra-logical rules for short. How a constructive theory can be presented in a form of interpreted symbolic calculus where all rules are logical? We know that in some cases constructive rules can be reasonably replaced by appropriate first-order sentences as in Hilbert's version of Euclidean geometry. However no schematic semantic procedure, which reduces extra-logical rules to logical rules and in some appropriate sense preserve semantics, is known to the authors. Epistemological motivations and implications of such semantic translations also need to be better understood. There is a general consensus that Hilbert's axiomatic presentation of Euclidean geometry, which gets rid of Euclid's extra-logical rules and in this way makes Euclid's geometrical proofs "purely logical", somehow makes this mathematical theory more rigorous, but the total score of epistemic gains and losses involved into this procedure remains rather unclear. Evidently these and other related questions cannot be answered without using a formal criterion of logicality and describing the extra-logical semantic contents of constructive theories in more precise terms. We leave this issue for a further research.

6 Conclusion

Following Ryle's remark that "Knowing a rule is knowing how" [24] we argued, on the contrary to a popular opinion, that knowing-how does not have an intrinsically tacit character but in many cases allows for an explicit representation in the form of formal rules. This holds both for natural languages, which allow one to formulate rules and related deontic expressions, and formal languages where rules are represented symbolically and play an important role in the architecture of formal calculi. Leaving natural languages aside we reviewed two "styles" of building formal systems one of which (Hilbert-style) employs few rules and an many axioms while the other (Gentzen-style) employs complex systems of rules and may use no axiom. Using some historical indications we construed the difference between the two styles not only syntactically but also semantically by associating Tarski's model-theoretic conception of logical consequence (and Model theory more generally) with the Hilbert-style and the proof-theoretic conception of logical consequence (and Proof-theoretic Semantics more generally) with the Gentzen-style.

In this context we introduced a syntactic definition of Hilbertian theory (Def. 11), which reflects and narrows the informal idea of "Hilbert-style axiomatic theory", and proved a lemma (Lemma 7) that says that the concept of Hilbertian theory is co-extensional with that of theory having Deduction Property, i.e., a theory for which the Deduction Theorem holds. The Deduction Property is of interest in the context of epistemological discussion on knowing-how and knowing-that because there is a sense (which has been made precise in the paper) in which it represents rule $A \vdash B$ (and, as we assume, the associated knowledge of how to follow this rule) in the form of (knowledge of) implication $A \rightarrow B$. We also studied here from a syntactic viewpoint how Hilbertian theories can be translated into sequent calculi (i.e., in the Gentzen-style systems) and vice versa (Lemmas 9 and 10)

These preparatory steps allowed us to attack the question of whether or not knowing-how in a reasonable sense reduces to knowing-that. First, we considered the special case of logical knowing-how, i.e., knowing how to make logical inferences. Our conclusion here is this: while the syntactic translations between different forms (or "styles") of symbolic presentations by themselves don't provide such a reduction, the model-theoretic conception of logical consequence allows for seeing syntactic rules as mere symbolic representations of the meta-theoretical propositions, in terms of which the logical consequence is defined in this case, and in that sense provides the wanted reduction. (A purported syntactic reduction, which relies on the Deduction Property and represents inferences as implication does not complete the task at the pain of Carroll Paradox). This does not resolve the epistemological problem but shows that in the chosen framework it concerns the conception of logical consequence rather than syntactic details. The alternative proof-theoretic conception, which we consider in this paper, does not allow for a similar reduction of knowing-how to knowing-that and make the concepts of rule and rule-following (and hence also the concept of knowing-how) indispensable. This conclusion not only sheds a new light on the old epistemological debate but also reveals epistemological implications of theories of logical semantics, which in technical works are rarely discussed explicitly and moreover systematically.

Finally we considered a more general case of rule-based knowledge-how, which includes knowledge-how outside logic. For this purpose we proposed a canonical translation of Hilbertian theories into typed sequential calculi, which are apt for non-logical constructive interpretations, some of which have been earlier studied by other authors. For such calculi we proved the "constructive version" of Deduction Theorem (Theorem 13) and proposed its informal interpretation in terms of knowing-how. Answering the question of possible propositional reduction of knowing-how to knowing-that in this more general case requires a more nuanced formal study, which may show how logical and non-logical semantics combine and interact in such cases. We leave this for a further research.

As a final remark we would like to stress that formal systems, which admit proof-theoretic semantics, are natural candidates for the role of representational tools for the procedural knowledge. Given the importance of procedural knowledge in the Society one may expect that such systems can have more applications in Knowledge Representation than they have presently. There is apparently a general bias towards the Hilbert-style approach in thinking about knowledge and reasoning in many areas from Philosophy to Information Engineering. However philosophical questions concerning the nature of knowing-how are answered there is no reason to understate this type of knowledge in the development of Knowledge Representation Systems and other relevant applications.

Appendix 1: Direct Proof of Theorem 13

We construct term u using the induction by steps of derivation of the sequence

$$x_1{:}F_1, \ldots, x_n{:}F_n, x{:}F \vdash t{:}G.$$

Case 1: Axiom of the form $(t : G = c^A : A)$. Then $u = k^{A\to(F\to A)} \cdot c^A$. Observe that term c^A contains no variable. Hence the sequence

$$x_1{:}F_1, \ldots, x_n{:}F_n \vdash c^A{:}A$$

is also derivable. Then apply the rule

$$\frac{x_1{:}F_1, \ldots, x_n{:}F_n \vdash k^{A\to(F\to A)}{:}(A \to (F \to A)) \quad x_1{:}F_1, \ldots, x_n{:}F_n \vdash c^A{:}A}{x_1{:}F_1, \ldots, x_n{:}F_n \vdash (k^{A\to(F\to A)} \cdot c^A){:}(F \to A)}$$

Case 2: Axiom of the form ($t : G = x_i : F_i$, where x is not one of the x_i). Proceed as in Case 1; $u = k^{F_i\to(F\to F_i)} \cdot x_i$.

Case 3: $(t\!:\!G = x\!:\!F)$. In this case declaration $x\!:\!F$ cannot be eliminated from the context because t contains x. But in this case $F = G$, and so the wanted term u is a Hilbertian proof of formula $F \to F$:

$$
\begin{array}{ll}
(F \to ((F \to F) \to F)) \to ((F \to (F \to F)) \to (F \to F)) & \text{scheme}(S_{\dots}) \\
F \to ((F \to F) \to F) & \text{scheme}(K_{\dots}) \\
(F \to (F \to F)) \to (F \to F) & (MP) \\
F \to (F \to F) & \text{scheme}(K_{\dots}) \\
F \to F & (MP)
\end{array}
$$

Thus we obtain the wanted term $u = (s^{(\dots)} \cdot k^{(\dots)}) \cdot k^{(\dots)}$ where the upper indexes are the first, the second and the forth lines of the Hilbertian derivation.

Application of the rule: By the inductive hypotheses the following sequences are derivable:

$$
\begin{array}{l}
x_1\!:\!F_1, \dots, x_n\!:\!F_n \vdash v\!:\!(F \to (X \to G)), \\
x_1\!:\!F_1, \dots, x_n\!:\!F_n \vdash w\!:\!(F \to X).
\end{array}
$$

Let $u = (s^{(\dots)} \cdot v) \cdot w$, where the type of the first factor is $(F \to (X \to G)) \to ((F \to X) \to (F \to G))$. This guaranties that the product is well-typed in the same context: the product $(s^{(\dots)} \cdot v)$ is of type $(F \to X) \to (F \to G)$ and term $u\!:\!(F \to G)$ is as required. This completes the proof.

Appendix 2: Primal infon logic

We prove that in the implicational fragment $\mathbf{P}_{\to}$ of Primal Infon logic $\mathbf{qP}$ introduced by Gurevich and Neeman [8, 9] the sequent $\Gamma \vdash p \to p$ is not derivable. Since $p \vdash p$ is derivable for all p this shows that $\mathbf{P}_{\to}$ does not have the Deduction Property.

The derivability relation of the natural deduction calculus for $\mathbf{P}_{\to}$ can be defined as follows (see [1]):

$$
\frac{}{\Gamma, A \vdash A}\,(Ax) \qquad \frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\,(Cut)
$$

$$
\frac{\Gamma \vdash A}{\Gamma \vdash B \to A}\,(\to I) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}\,(\to E)\,.
$$

Here A, B are formulas constructed from atomic ones using "$\rightarrow$" (primal implication) and Γ is a set of such formulas.

It has the axiom scheme (Ax) but a sequent of the form $\Gamma \vdash A \rightarrow A$ may be not derivable. For example, it is the case when $A = p \notin \Gamma$ is atomic and Γ does not contain formulas of the form

$$X_1 \rightarrow (X_2 \rightarrow \ldots (X_n \rightarrow p) \ldots).^2 \tag{1}$$

Indeed, let p be fixed. Consider a sequent rule with the conclusion $\Gamma \vdash Y$ with Y of the form (1) and with Γ that does not contain formulas of the form (1). It can be seen that some premise of the rule also has this property, with the same p. This premise cannot be derived by (Ax), so one can apply the same argument to it. Thus, there is no finite derivation for the sequent $\Gamma \vdash Y$. In particular, the sequent $\Gamma \vdash p \rightarrow p$ is not derivable.

System **qP** has a computational semantics — and in fact more than one such semantics [1, 18] — which makes it useful in Computer Science. Under the encryption semantic introduced by one of the authors of the present paper [18] the lack of Deduction Property in the above fragment of **qP** is interpreted as follows. The derivability of $\Gamma \vdash A$ means that an agent who has access to set of messages Γ has access to message A. Formula $A \rightarrow B$ stands for message B encoded by message A, so $A \rightarrow A$ stands for a message encoded by itself. Given $\Gamma, A \vdash A$ there is no reason to expect $\Gamma \vdash A \rightarrow A$ (in words, Γ provides access to message $A \rightarrow A$) unless A makes part of context Γ to begin with.

Список литературы

[1] L. Beklemishev and Yu. Gurevich. Propositional primal logic with disjunction. *Journal of Logic and Computation*, 24(1):257–282, 2014.

[2] R.B. Brandom. *Articulating Reasons: Introduction to Inferentialism.* Cambridge (Mass.): Harvard University Press, 2000.

[3] L. Carroll. What the tortoise said to achilles. *Mind*, 4(14):278–280, 1895.

[2] In case $n = 0$ it is p.

[4] Jeremy Fantl. Knowledge how. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, fall 2017 edition, 2017.

[5] N. Francez. *Proof-Theoretic Semantics (Studies in Logic, vol. 57)*. College Publications, 2015.

[6] N. Francez. On distinguishing proof-theoretic consequence from derivability. *Logique et Analyze*, 238:151–166, 2017.

[7] G. Gentzen. Investigations into logical deduction. *M.E. Szabo (Ed.) The Collected Papers of Gerhard Gentzen, North-Holland Publishing Company: Amsterdam-London, 1969, p. 68-131; original publication in German: Untersuchungen über das logische Schliessen, Mathematische Zeitschrift, 39 (1934-35), S. 176-210, 405- 431.*, 1969.

[8] Yu. Gurevich and Neeman I. DKAL: Distributed-knowledge authorization language. *Proc. of CSF 2008, IEEE Computer Society*, pages 149–162, 2008.

[9] Yu. Gurevich and Neeman I. Logic of infons: the propositional case. Technical report, Microsoft Research, July 2011.

[10] I. Harmon and Z Horne. Evidence for anti-intellectualism about know-how from a sentence recognition task. *Synthese*, 193(9):2929–2947, 2016.

[11] D. Hilbert. *Grundlagen der Geometrie*. Leipzig, 1899.

[12] D. Hilbert. Axiomatic thought. *W. Ewald (ed.), From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, 2:1105–1115, 1996.

[13] D. Hilbert and W. Ackermann. *Grundzüge der theoretischen Logik*. Berlin, Heidelberg, 1928.

[14] D. Hilbert and P. Bernays. *Grundlagen der Mathematik (in two volumes)*. Springer, 1934-1939.

[15] J. Hintikka. What is axiomatic method? *Synthese*, 183(1):69–85, 2011.

[16] S.C. Kleene. *Mathematical logic*. New York: Dover Publications, 2000 [1967].

[17] A. N. Kolmogorov. On the Interpretation of Intuitionistic Logic. *V. M. Tikhomirov (ed.) Selected Works of A. N. Kolmogorov, Springer (original publication in German: Zur Deutung der Intuitionistischen Logik. Math. Ztschr., 35, S. 58-65 (1932))*, 1:151–158, 1991.

[18] V. Krupski. Primal implication as encryption. *Hirsch E.A., Kuznetsov S.O., Pin J.-E., Vereshchagin N.K. (eds) Computer Science - Theory and Applications. CSR 2014. Lecture Notes in Computer Science, vol 8476. Springer, Cham*, pages 232–244, 2014.

[19] P. Martin-Löf. Constructive mathematics and computer programming. *Proceeding of a discussion meeting of the Royal Society of London on Mathematical logic and programming languages*, pages 167–184, 1985.

[20] Th. Piesha and P. (eds.) Shroeder-Heister. *Advances in Proof-Theoretic Semantics (Trends in Logic vol. 43)*. Springer, 2015.

[21] D. Prawitz. The philosophical position of proof theory. *Olson, R.E. and Paul, A.M. (eds.) Contemporary Philosophy in Scandinavia, Baltimore, London: John Hopkins Press*, pages 123–134, 1972.

[22] D. Prawitz. On the idea of a general proof theory. *Synthese*, 27(1/2):63–77, 1974.

[23] A. Rodin. On constructive axiomatics method. *Logique et Analyse (forthcoming)*, 2017.

[24] G. Ryle. Knowing how and knowing that: The presidential address. *Proceedings of the Aristotelian Society, New Series,*, 46:1–16, 1945-46.

[25] Peter Schroeder-Heister. Proof-theoretic semantics. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition, 2016.

[26] J. Stanley and T. Williamson. Knowing how. *Journal of Philosophy*, 98(8):411–444, 2001.

[27] A. Tarski. On the concept of logical consequence. *Logic, semantics, metamathematics: Papers from 1923 to 1938 (J. H. Woodger, translator), Oxford, Clarendon Press 1956*, pages 409–420, 1956.

[28] D. Tsementzis. Meaning explanation of hott, 2017.

[29] The Free Encyclopedia Wikipedia. Sequent calculus, 2017. [Online; accessed 12-July-2017].