MANUFACTURING A CARTESIAN CLOSED CATEGORY WITH EXACTLY TWO OBJECTS OUT OF A C-MONOID

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Abstract: A construction is described of a cartesian closed category $\mathcal{A}$ with exactly two elements out of a C-monoid $\mathcal{M}$ such that $\mathcal{M}$ can be recovered from $\mathcal{A}$ without reference to the construction.

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We answer a question of Lambek and Scott (see [LS] p.99) by proving the following:

Theorem. Let $\mathcal{M}$ be a C-monoid, with C-structure (π, π', ε, $(\_)^*$, $\langle\_,\_\rangle$). Then there exists a cartesian closed category $\mathcal{A}$ with exactly two objects U and T, such that End(U) = $\mathcal{M}$.

The construction of $\mathcal{A}$ is entirely by hand. The intuitive idea is as follows. $\mathcal{M}$ may be viewed as a collection of endomorphisms of a set U. Let $T \equiv \{*\}$ be a one-point set; then $u \mapsto \lambda *.u$ is a one-to-one correspondence between U and the set of all functions from T to U. Now if $\mathcal{A}$ is a cartesian closed category with just U and T for its objects, where T is terminal, then in $\mathcal{A}$ we must have

$$\mathrm{Hom}(U,U) \cong \mathrm{Hom}(T\times U,U) \cong \mathrm{Hom}(T,U^U) \cong \mathrm{Hom}(T,U);$$

so if we put Hom(U,U) = $\mathcal{M}$, and like to think of Hom(T,U) as $\mathrm{Hom}_{\mathbf{Sets}}(\{*\},U)$, we must have $\mathcal{M} \cong U$, as sets. Since it does not matter much what the elements of U are, we take $\mathcal{M}=U$. Then we have functions $f^\dagger \equiv \lambda *.f : \{*\} \to U$ for every $f \in U$. Composing with $\mathrm{o} \equiv \lambda u.*: U \to \{*\}$, we have

$$(\lambda *.f) \circ (\lambda u.*) = \lambda u.f : U \to U.$$

This we identify with the arrow $\lambda_U f \equiv (f\pi')^*$ in $\mathcal{M}$, described in [LS] §15. The longer definitions (notably, those of $g \circ f^\dagger$ and $\{g^\dagger, h^\dagger\}$) were forced upon us by this identification. The rest were the simplest at first sight.

Remark. By [LS] §16, the Karoubi envelope $K(\mathcal{M})$ of $\mathcal{M}$ has a full cartesian closed subcategory $K_0(\mathcal{M})$ consisting of all objects isomorphic to U (the unit of $\mathcal{M}$) or the terminal object T. Taking one representative from either isomorphism class, one gets another full subcategory, which is easily shown to be cartesian closed; and since the monoid End(U) of endomorphisms of U is isomorphic to $\mathcal{M}$, $\mathcal{M}$ can now be recovered.

This method is unsatisfactory since we are not told how to identify U in $K(\mathcal{M})$. With the approach set out below, it is not necessary for the recovery of $\mathcal{M}$ that we know which of the objects of $\mathcal{A}$ is U. We have a constructive criterion: take the object that is not terminal. If both objects are terminal, the choice is free.

We use the notation of [LS], but for one exception: we write f‘a for "f applied to a".

Proof of the theorem. Let U be the object of $\mathcal{M}$. Take some thing T distinct from U. We form $\mathcal{A}$ from $\mathcal{M}$ in a number of steps. First we add the object T to $\mathcal{M}$ as a terminal object, i.e. we also add arrows $\circ{:}U\to T$ and $1_T{:}T\to T$, and specify

$$\circ f = \circ \text{ for all arrows } f \text{ in } \mathcal{M};$$
$$1_T\circ = \circ,\ 1_T 1_T = 1_T.$$

Moreover, for each f in $\mathcal{M}$ we take a distinct new arrow $f^\dagger{:}T\to U$ with

$$f^\dagger\circ = (f\pi')^*,\ \circ f^\dagger = 1_T,\ f^\dagger 1_T = f^\dagger, \text{ and}$$
$$gf^\dagger = (g\text{‘}f)^\dagger\ (= (\varepsilon\langle g\circ(f\pi')^*,1\rangle)^\dagger) \text{ for all arrows } g \text{ in } \mathcal{M}.$$

The category $\mathcal{A}$ has now been defined. To be sure that $\mathcal{A}$ is indeed a category, the axioms for categories must be checked. The unit axioms are easy; in particular, $1_U\circ f^\dagger = (1\text{‘}f)^\dagger = f^\dagger$ by C12 ([LS] p. 96). Associativity of composition dissolves into sixteen cases

$$A \to B \to C \to D$$

with each of A,B,C,D either U or T. We write out the four least trivial.

(i) Suppose we have

$$U \xrightarrow{\ g\ } U \xrightarrow{\ \circ\ } T \xrightarrow{\ f^\dagger\ } U .$$

Then $f^\dagger\circ(\circ g) = f^\dagger\circ = (f\pi')^* = (f\pi')^*g$ by C9, [LS] p. 96

$$= (f^\dagger\circ)g.$$

(ii) If we have

$$U \xrightarrow{\ \circ\ } T \xrightarrow{\ f^\dagger\ } U \xrightarrow{\ g\ } U ,$$

$$g\circ(f^\dagger\circ) = g\circ(f\pi')^* = (\varepsilon\langle g\circ(f\pi')^*\pi,\pi'\rangle)^* = (\varepsilon\langle g\circ(f\pi')^*,1\rangle\pi')^* \quad \text{(using C9)}$$
$$= ((g\text{‘}f)\pi')^* = (g\text{‘}f)^\dagger\circ = (gf^\dagger)\circ.$$

(iii) Given

$$T \xrightarrow{f^\dagger} U \xrightarrow{g} U \xrightarrow{h} U \quad ,$$

we find $h\circ(gf^\dagger) = h\circ(g`f)^\dagger = (h`(g`f))^\dagger = ((hg)`f)^\dagger$ by C10, [LS] p. 96

$$= (hg)f^\dagger.$$

(iv) In a diagram

$$T \xrightarrow{f^\dagger} U \xrightarrow{\circ} T \xrightarrow{g^\dagger} U \quad ,$$

we have $g^\dagger\circ(\circ f^\dagger) = g^\dagger = ((\lambda_u g)`f)^\dagger$ (cf. [LS] Cor. 15.3)

$$= ((g\pi')^*`f)^\dagger = (g\pi')^* f^\dagger = (g^\dagger\circ)f^\dagger.$$

The next step is to define the cartesian structure.

$U\times U = U,\ U\times T = T\times U = U,\ T\times T = T.$

$\pi_{U,U} = \pi,$ $\qquad \pi'_{U,U} = \pi',$
$\pi_{U,T} = 1_U,$ $\qquad \pi'_{U,T} = \circ,$
$\pi_{T,U} = \circ,$ $\qquad \pi'_{T,U} = 1_U,$
$\pi_{T,T} = 1_T,$ $\qquad \pi'_{T,T} = 1_T.$

We write $\{f,g\}$ for the pair of f and g in $\mathcal{A}$, and set

$\{f,g\} = \langle f,g\rangle$ if f,g belong to $\mathcal{M}$;
$\{f,\circ\} = f,\ \{\circ,f\} = f$ for f in $\mathcal{M}$;
$\{\circ,\circ\} = \{1_T,\circ\} = \{\circ,1_T\} = \circ,\ \{1_T,1_T\} = 1_T;$
$\{f^\dagger,g^\dagger\} = (\langle\lambda_u f,\lambda_u g\rangle`1)^\dagger,\ \{1_T,f^\dagger\} = f^\dagger,\ \{f^\dagger,1_T\} = f^\dagger.$

We must check if these definitions satisfy the additional axioms for a cartesian category, the equations E3 of [LS] p. 52. A number of these checks are trivial. We shall write out one case of E3a, and three cases of E3c.

(ad E3a.) $\pi_{U,U}\{f^\dagger,g^\dagger\} = \pi\circ(\langle\lambda_u f,\lambda_u g\rangle`1)^\dagger = (\pi`(\langle\lambda_u f,\lambda_u g\rangle`1))^\dagger$

$= ((\pi\langle\lambda_u f,\lambda_u g\rangle`1)^\dagger$ by C10 ([LS] p. 96)

$= ((\lambda_u f)`1)^\dagger = f^\dagger$, by [LS] Cor. 15.3.

(ad E3c.) (i) If $k{:}U\to T\times U$, then in fact $k{:}U\to U$, and

$$\{\pi_{T,U}k,\pi'_{T,U}k\} = \{\circ k,k\} = \{\circ,k\} = k.$$

(ii) Let g,h be arrows of $\mathcal{M}$. Then

$$(*) \quad \lambda_u.g`h = ((g`h)\pi')^* = (\varepsilon\langle g\circ(h\pi')^*,1\rangle\pi')^* = g\circ(h\pi')^*,$$

since by C9 $(h\pi')^*\pi' = (h\pi')^* = (h\pi')^*\pi$. Now if $k{:}T\to U\times U$, then in fact $k = f^\dagger{:}T\to U$ for some $f{:}U\to U$, and we have

$$\{\pi_{U,U}k,\pi'_{U,U}k\} = \{\pi f^\dagger,\pi' f^\dagger\} = \{(\pi`f)^\dagger,(\pi''f)^\dagger\} = (\langle\lambda_u.\pi`f,\ \lambda_u.\pi''f\rangle`1)^\dagger$$

$$= (\varepsilon\langle\langle\lambda_u.\pi`f,\ \lambda_u.\pi''f\rangle(\pi')^*,1\rangle)^\dagger = (\varepsilon\langle\langle\pi\circ(f\pi')^*,\pi'\circ(f\pi')^*\rangle,1\rangle)^\dagger \text{ by } (*)$$

$$= (\varepsilon\langle(f\pi')^*,1\rangle)^\dagger = (1`f)^\dagger = f^\dagger = k, \text{ using C12 ([LS] p. 96).}$$

(iii) If $k{:}T\to T\times U$, then $k = f^\dagger$ for some $f{:}U\to U$, and

$$\{\pi_{T,U}k,\pi'_{T,U}k\} = \{\circ f^\dagger, f^\dagger\} = \{1_T, f^\dagger\} = f^\dagger = k.$$

The last step is the specification of exponents and evaluation. We define

$$U^U = U^T = U,\ T^U = T^T = T;$$

$$\varepsilon_{U,U} = \varepsilon;\ \varepsilon_{T,T} = 1_T;\ \varepsilon_{T,U} = \circ;\ \varepsilon_{U,T} = 1_U.$$

Cartesian closed categories associate to each $f{:}A\times B\to C$ an arrow $\Lambda^A_{C,B}(f){:}A\to C^B$. Usually one writes f^* for $\Lambda^A_{C,B}(f)$, since the indices A,B,C tend to be clear from the context. In our category $\mathcal{A}$, however, many products cannot be distinguished (recall $U\times U = U\times T = T\times U$), and because of this the type of f does not contain enough information. Thus we must specify the operations $\Lambda^A_{C,B}$ instead of just $(\_)^*$.

$$\Lambda^U_{U,U}(f) = f^*$$

$$\Lambda^T_{U,U}(f) = f^\dagger$$

$$\Lambda^U_{T,U}(\circ) = \circ$$

$$\Lambda^U_{U,T}(f) = f$$

$$\Lambda^U_{T,T}(\circ) = \circ$$

$$\Lambda^T_{U,T}(f^\dagger) = f^\dagger$$

$$\Lambda^T_{T,U}(\circ) = 1_T$$

$$\Lambda^T_{T,T}(1_T) = 1_T$$

We finish by checking a few cases of the evaluation laws E4 ([LS] p. 53).

(ad E4a.) (i) $\varepsilon_{U,U}\{\Lambda^{T}_{U,U}(f)\pi_{T,U},\pi'_{T,U}\} = \varepsilon\langle f^{\dagger}\circ,1_U\rangle = \varepsilon\langle(f\pi')^{*},1_U\rangle = f$

by the corollary to the functional completeness theorem, [LS] 15.3.

(ii) $\varepsilon_{U,T}\{\Lambda^{U}_{U,T}(f)\pi_{U,T},\pi'_{U,T}\} = 1_U\{f,\circ\} = f.$

(iii) $\varepsilon_{U,T}\{\Lambda^{T}_{U,T}(f^{\dagger})\pi_{T,T},\pi'_{T,T}\} = 1_U\{f^{\dagger},1_T\} = f^{\dagger}.$

(iv) $\varepsilon_{T,U}\{\Lambda^{T}_{T,U}(\circ)\pi_{T,U},\pi'_{T,U}\} = \circ\{\circ,1_U\} = \circ.$

(ad E4b.) (i) Suppose $k{:}T\to U^U$, then $k = f^{\dagger}$ for some $f{:}U\to U$. Then

$$\Lambda^{T}_{U,U}(\varepsilon_{U,U}\{k\pi_{T,U},\pi'_{T,U}\}) = \Lambda^{T}_{U,U}(\varepsilon\{f^{\dagger}\circ,1_U\}) = \Lambda^{T}_{U,U}(\varepsilon\langle(f\pi')^{*},1\rangle)$$

$$= \Lambda^{T}_{U,U}(f) = k \qquad \text{(cf. (i) ad E4a.).}$$

(ii) Suppose $k{:}T\to U^T$; again k is of the form $f^{\dagger}$, and

$$\Lambda^{T}_{U,T}(\varepsilon_{U,T}\{k\pi_{T,T},\pi'_{T,T}\}) = \Lambda^{T}_{U,T}(\{f^{\dagger},1_T\}) = \Lambda^{T}_{U,T}(f^{\dagger}) = k.$$

(iii) If $k{:}T\to T^U$, then $k = 1_T$, and

$$\Lambda^{T}_{T,U}(\varepsilon_{T,U}\{k\pi_{T,U},\pi'_{T,U}\}) = \Lambda^{T}_{T,U}(\circ\{\circ,1_U\}) = \Lambda^{T}_{T,U}(\circ) = k.$$

The proof is complete.

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