ON CONSTRUCTIVE AXIOMATIC METHOD

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Abstract. The formal axiomatic method popularized by Hilbert and recently defended by Hintikka is not fully adequate to the recent practice of axiomatizing mathematical theories. The axiomatic architecture of Topos theory and Homotopy type theory do not fit the pattern of the formal axiomatic theory in the standard sense of the word. However these theories fall under a more general and in some respects more traditional notion of axiomatic theory, which I call after Hilbert *constructive*. I show that the formal axiomatic method always requires a support of some more basic constructive method.

1. Introduction

The modern notion of axiomatic method stems from works in foundations of mathematics starting in the late 19th century, most prominently from Hilbert's *Foundations of Geometry* first published in 1899 [11], [13]. Recently this method has been defended (in its semantic version) by Hintikka [18]. In this paper I argue that this notion of axiomatic method is not fully adequate to the current mathematical practice. As a remedy I describe a more general version of axiomatic method that covers some important recent instances of axiomatic thinking as well as some older instances such as Euclid's *Elements*. I attempt to ground this more general method epistemologically and show that it can be more useful in physics and other sciences than the received formal axiomatic method.

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The method that I have in mind has been already well known to Hilbert, who calls it *constructive* or *genetic* interchangeably ([17], p. 1a), and also discussed in a later literature [3], [4], [34] [2] , [41]. My contribution consists in showing how the constructive method operates in the axiomatic Topos theory [32] (in a somewhat implicit form) and the Homotopy Type theory [5] and on this basis providing certain epistemological arguments in favor of this method.

The rest of the paper is organized as follows. After making a short exposition on the constructive axiomatic method after Hilbert and Bernays [16], [17] in Section 2, I demonstrate this notion using the example of Euclid's *Elements*, Book 1 (Section 3). Although the main purpose of this paper is not historical, this historical example is very useful because it helps me to disambiguate the overloaded term "constructive" and illustrate my arguments with the familiar elementary geometry. In Sections 4 I recall some basic facts concerning Hilbert's *formal* axiomatic approach and in Section 5 I introduce the notion of Curry-Howard correspondence, which I need in what follows. In Sections 6 and 7 I treat two modern examples of axiomatic theories: Topos theory (Section 6) and HoTT (Section 7). In the concluding Section 8 I provide an epistemological discussion, which includes a critique of Hintkka's view on axiomatic method [18].

2. Constructive method versus Formal method

In the Introduction to their [16], (Eng. tr. [17]) Hilbert and Bernays specify their intended notion of axiomatic method as follows:

> The term axiomatic will be used partly in a broader and partly in a narrower sense. We will call the development of a theory axiomatic in the broadest sense if the basic notions and presuppositions are stated first, and then the further content of the theory is logically derived with the help of

[2] Piaget calls the genetic method *operational*.

> definitions and proofs. In this sense, Euclid provided an axiomatic grounding for geometry [..]. [F]or axiomatics in the narrowest sense, the *existential form* comes in as an additional factor. This marks the difference between the *axiomatic method* [in the narrow sense?] and the *constructive* or *genetic* method of grounding a theory. While the constructive method introduces the objects of a theory only as a *genus* of things, an axiomatic theory refers to a fixed system of things [..] given as a whole. [..] We will call this sharpened form of axiomatics (where the subject matter is ignored and the existential form comes in) "formal axiomatics" for short. ([17], p. 1a)

The above passage helps the authors to clarify what they mean by the *formal* axiomatic method but leaves the notion of *constructive* (aka *genetic*) method rather unclear. Is the constructive method just the same as the axiomatic method in the author's "broadest" sense of the word? Or the constructive method and the formal method are two distinct versions of the broad axiomatic method? Or perhaps the constructive method does not qualify as axiomatic at all [3]? It seems to me plain that this terminological confusion reflects a real conceptual problem which has to do with questions like: What (if any) is the constructive content of a formal axiomatic theory? In what follows I try to show that the constructive method as described in the above quote indeed qualifies as a general and basic notion of axiomatic method, i.e., as the axiomatic method in the broadest sense of the word, which covers the formal method in particular. In order to proceed let me now make my own terminological precautions. Hereafter I use the terms "constructive" and "formal" in the same sense in which Hilbert and Bernays use them in the above quote. Namely, I count as *constructive* any method of theory-building which includes a well-determined procedure for "introducing the objects of a theory" (as "genus", i.e., types); I don't specify any class of allowable procedures of this sort assuming that such specifications may vary

[3]The distinction "genetic versus axiomatic", which implies that the two things are mutually exclusive, is used in much earlier Hilbert's paper [12], (Eng. tr. [15]) and also in some later discussions [3], [4], [34], [41].

from one constructive theory to another. I distinguish the *formal* axiomatic method using Hilbert and Bernays' notion of "existential form". Existential forms in this sense are formal existential theorems and axioms like the powerset axiom of ZF. In formal theories the existential forms replace procedures used for introducing theoretical objects in constructive theories. Thus Hilbert and Bernays' "existential forms" are existential *propositional* forms, which admit truth values in their various interpretations.

In modern terms the relevant distinction between formal and constructive axiomatic theories can be described as follows. Formal theories are construed as collections of propositional forms related by some relation of logical inference (or several such relations) [4]. Constructive theories include procedures for building theoretical objects of other types than propositions. In the next Section I demonstrate this feature using Euclid's *Elements*, Book 1 [7], (Eng. tr. [6]) and then in Sections 6 and 7 I provide modern examples.

3. Problems versus Theorems

Euclid's theory is based on 5 Axioms (aka Common Notions), 5 Postulates and 23 Definitions. The special historical character of Euclid's Common Notions and Definitions is not relevant to the present discussion, so I leave them apart and focus on Postulates. The first three Postulates are as follows (verbatim after [6]):

- [P1:] to draw a straight-line from any point to any point.
- [P2:] to produce a finite straight-line continuously in a straight-line.
- [P3:] to draw a circle with any center and radius.

As they stand the three Postulates are not propositions and admit no truth-values. Hence they cannot be axioms in the usual modern sense of the word. In particular, the Postulates cannot be used as premises in logical inferences - if by logical inference one understands an operation that takes some propositions (premises) as its input and produces some other

[4] In Section 4 I mention the relation of formal (aka syntactic) consequence and in Section 8 more minutely discuss the relation of semantic consequence.

proposition or propositions (conclusion) as its output [5]. In fact Postulates 1-3 are themselves basic *operations*, which take certain geometrical objects as input and produce some other geometrical objects as output. The table below specifies inputs (operands) and outputs (results) for P1-3:

operation	input	output
P1	two (different) points	straight segment
P2	straight segment	(extended) straight segment
P3	straight segment and one of its endpoints	circle

The three operations are partly composable in the obvious way: the output of P1 is used as input for P2 and P3. This system of operations extended by some further basic operations assumed tacitly [6] serves Euclid for "introducing objects" of his theory. Such an introduction is systematic in the sense that it does not reduce to a simple act of stipulation: it is a procedure, which involves certain elementary operations (including P1-P3) and complex operations obtained through the composition of the elementary operations. As soon as the term *deduction* is understood liberally as a theoretical procedure, which generates some fragments of a given theory from the first principles of this theory, one can say that Euclid's geometrical constructions are deductive. The constructive deductive order is also called the *genetic* order. As we shall now see in Euclid's theory the constructive deduction is tightly related to the more familiar *logical* deduction, which operates with propositions.

Postulates and Axioms of are followed by the so-called Propositions. This commonly used title is not found in Euclid's original text where things called by later editors "Propositions" are simply numbered but not called by any common name [6]. From Proclus' *Commentary* [36], (Eng. tr. [37]) written in the 5th century A.D. we learn about the tradition dating back to Euclid's own times (and in fact even to earlier times) of distinguishing between the two sorts of "Propositions", namely *Problems* and *Theorems*. Euclid's Theorems by

[5]This fact has been stressed by A. Szabo, see [43], p. 230.

[6]Like the construction of intersection point of lines in an appropriate position, compare Prop. 1 of Book 1. The incompleteness of P1-3 has no bearing on my argument.

and large are theorems in the modern sense of the word: propositions followed by proofs. But Problems are something different: they are *complex operations* (or, if one prefers, complex *constructions*) built from elementary operations. Like Postulates Problems admit no truth-values and thus don't qualify as propositions either [7].

What I have said so far can make one imagine that Euclid's theory splits into two independent parts: one consisting of constructions from Postulates (Problems) and the other consisting of propositions proved from Axioms (Theorems). Such a split does not occur for two complementary reasons:
- (solutions of) non-trivial Problems require (propositional) proofs (which show that the obtained constructions have all required properties);
- (proofs of) non-trivial Theorems require constructions (conventionally called in such contexts "auxiliary").

This explains why the logical deductive order of Theorems and the genetic order of Problems in Euclid's theory form a joint deductive order. For a more detailed analysis of this structure see [39], ch.2. In Section 5 below I explain how the Curry-Howard correspondence supports a similar structure, which combines propositional and non-propositional forms of deduction.

All Euclid's Postulates and initial fragments (i.e., bare formulations) of Problems can be easily paraphrased into propositions. This can be done at least in two different ways. The following paraphrases of P1 are self-explanatory:
<u>P1m (modal)</u>: Given two (different) points it is always *possible* to produce a straight segment from one given point to the other given point.
<u>P1e (existential)</u>: Given two (different) points there *exists* a straight segment having these given points as its endpoint.

[7] As an example of Problem consider the initial fragment of Proposition 1, Book 1: "To construct an equilateral triangle on a given finite straight-line". It is followed by (i) an appropriate construction and (ii) a proof that the obtained construction is equilateral triangle.

P1e instantiates what Hilbert and Bernays call the "existential form" used in the formal axiomatic method. The key logical feature of this paraphrase (which it shares with P1m) is the reduction of Euclid's non-propositional Postulates and Problems to certain propositions (in case of P1e - to existential propositions). Such a reduction may look trivial and even purely linguistic but in fact it is not because none of the two ways of paraphrasing is sufficient for translating Euclid's theory into a propositional form, i.e., into a theory consisting of axioms and theorems derived from the axioms according to certain fixed rules of logical inference. The difficulty is, of course, that the straightforward propositional paraphrasing does not translate *proofs* and *constructions* coherently, so in order to provide a reasonable reconstruction of Euclid's theory in the propositional form one needs a lot of further efforts [19]. For further references I shall call a procedure, which aims at replacement of all non-propositional content of a given theory by some suitable propositional content, the *propositional reduction* of this theory.

Thanks to Proclus we know that the idea of propositional reduction of Problems to Theorems is very old. But from the same source we also know about the contemporary competing idea of considering Theorems as Problems of a special sort (see [37], p. 63-64). Both proposals make equally strong echoes of more recent controversies about constructive and non-constructive approaches in mathematics. Thus there is neither historical nor obvious theoretical reason for taking for granted the common idea according to which the propositional reduction is the first necessary step in any modern logical reconstruction of ancient mathematics. Notice that Euclid's geometry exemplifies Hilbert and Bernays' notion of constructive axiomatic theory only if the non-propositional content of this theory is properly taken into the account but not reduced to some propositional form (existential, modal or other).

4. Hilbert's Views on Axiomatic Method

Since presently there exists an extensive literature, which analyses Hilbert's work on axiomatic method in historical and theoretical perspectives, in this Section I shall not try to say anything new about Hilbert but only recall some basic features of his axiomatic approach, which I later use for contrasting against them some new features of more recent axiomatic approaches. At the early stage of his life-long work on axiomatic foundation of mathematics (1893-1894) Hilbert describes his notion of well-founded axiomatic theory as follows:

> Our theory furnishes only the schema of concepts connected to each other through the unalterable laws of logic. It is left to human reason how it wants to apply this schema to appearance, how it wants to fill it with material. ([10], p. 104)

The two key features of this notion of theory are (i) its schematic character and (ii) its logical grounding. Let me first focus on (ii). Hilbert refers to the "unalterable laws of logic" as something definite and somehow known. A weaker assumption which we may attribute to Hilbert and which still allows us to make sense of his words is that the laws of logic are epistemically more reliable than any mathematical and scientific knowledge, so it makes sense to use these laws as a foundation in mathematical and scientific theories.

The fixity of logic is important for understanding the schematic character (i) of Hilbert's axiomatic theories. Axioms and theorems of non-interpreted formal theory are propositional schemes, which admit truth values (and thus turn into propositions) through an interpretation. An interpretation amounts to assigning to certain terms like "point", "straight line", etc. certain semantic values, which can be borrowed from another mathematical theory or from some extra-theoretical sources like intuition and experience. However this game of multiple interpretations does not concern *all* terms of a given theory. Some terms, namely *logical* terms, have fixed meanings, which (at least in the early versions of Hilbert's axiomatic approach) is supposed to be self-evident. The different treatment of logical and non-logical terms reflects the epistemological assumption according to which logical concepts have an epistemic priority over non-logical concepts.

In *Foundations* of 1899 [11] and other Hilbert's early axiomatic theories the "laws of logic" are taken for granted but not specified explicitly and precisely. Hilbert addresses this problem in 1917 saying that "it appears necessary to axiomatize logic itself" ([14] p. 1113). He find a solution of the problem by applying in logic symbolic methods [8]. The later mature form of his formal axiomatic method presented in [16] also involves such a symbolic setting.

This mature axiomatic method reinforced by symbolic logic has some important features, which are wholly absent in Hilbert's early conception of this method described above. While in the early version a non-interpreted axiomatic theory is understood as a "scheme of concepts" devoid of any intuitive content the later symbolic version of axiomatic method includes an additional assumption according to which this abstract scheme has its proper concrete representation, namely the symbolic representation. The symbolic representation involves a special sort of intuition, which Hilbert calls the "logico-combinatorial intuition" ([9], p. 179). A mathematical study of symbolic calculi (which include logical calculi proper and symbolic representations of formal theories based on these calculi) Hilbert isolates into a special area of mathematics, which he calls *metamathematics*. Hilbert perfectly realizes that treating the metamathematics with the same formal axiomatic method leads to a hopeless infinite regress. So his foundational project at this point becomes different and in certain respects more modest (albeit in some other respects more radical) than earlier: now he aims at isolating a limited area of elementary (and as he really hoped - only finitary) mathematics developed *constructively* and then treat the rest of mathematics on this constructive basis using appropriate non-constructive "idealizing existence assumptions"

[8] Hintinkka [18] quite rightly stresses the fact that between axiomatizing geometry (or another non-logical theory) and axiomatizing logic there is no continuity. He argues that a recursive enumeration of logical truths that Hilbert calls axiomatization is called so improperly because such a procedure doesn't allow for studying models of logic in anything like the same way in which one studies models of any other formal theory. In what follows (Sections 6 and 7) I describe a possible solution of this problem that blurs the distinction between logical and non-logical terms and allows logical terms to have non-logical interpretations.

([17], p. 19). So at that point the formal axiomatic method is no longer seen by Hilbert as self-sustained: it needs a support of constructive methods operating at the metatheoretical level. Hintikka [18] doesn't follow Hilbert here but rather elaborates on the early version of his formal axiomatic method. Namely, Hintikka construes his basic notion of logical inference as the *semantic consequence* rather than the syntactic *formal consequence* studied by Hilbert and Bernays in their metamathematics. In Section 8 I argue that Hintikka's semantic version of formal axiomatic method similarly needs a constructive support.

5. Curry-Howard Correspondence and Cartesian Closed Categories

The present Section is a preliminary to the following two Sections 6,7 where I treat modern examples of axiomatic theories. In a nutshell the idea of a Curry-Howard correspondence is given in Kolmogorov's 1932 paper [20] where the author establishes that his newly proposed *calculus of problems* has exactly the same structure as the intuitionistic propositional calculus published in 1930 by Heyting [9]. It turns out that this correspondence is extendible onto a large class of symbolic calculi including those, which have been developed independently and apparently for very different purposes. So there were established a number of correspondences (i.e., of more and less precise isomorphisms) between (proof-related) logical calculi (propositional, first-order or higher-order), on the one hand, and computational calculi (the simply-typed lambda calculus, type systems with dependent types, polymorphic type systems), on the other hand [42]. This led to the so called "proofs-as-programs and propositions-as-types" paradigm in logic and Computer science, which can be called constructive in the relevant sense. Indeed, when Hilbert and Bernays distinguish between the broadly constructive and the formal versions of the axiomatic method they firmly assume that propositions and theoretical objects (of the same contentual theory) belong to

[9] I am not making any priority claim for Kolmogorov here but simply use his 1932 article as a convenient reference. A relevant historical material can be found in [2] and [40] but a more focused historical study on the idea of Curry-Howard correspondence still waits to be written. Such a study might show that the Curry-Howard correspondence is not just an "amazing" mathematical phenomenon ([42], p. 5) but a case of genuine convergence of several different lines of research in logic and foundations of mathematics.

distinct domains of things. They don't treat this distinction formally but simply take it for granted. Within the propositions-as-types paradigm this distinction is made formally and explicitly: propositions are represented as types, namely as the types of their corresponding proofs. So within this paradigm propositions are treated as types of objects along with other (non-propositional) types of objects belonging the *same* theory [10]. Within this paradigm building proofs is a special case of building theoretical objects in general. Recall that a theory qualifies as constructive in the relevant sense when it includes procedures for building objects other than propositions.

The Curry-Howard correspondence shows that certain symbolic calculi of different kinds share a common structure. It is natural to ask whether this shared structure can be presented in some invariant way, which would not depend on particularities of syntactic presentations of these symbolic calculi. This problem has been solved in 1963 by means of mathematical Category theory by Lawvere [24], who observed that certain categories called Cartesian closed categories (CCC) "serve as a common abstraction of type theory and propositional logic" ([26], p. 1). In 1968-1972 this observation has been developed by Lambek into what is now known under the name of Categorical Logic. The three way correspondence between (i) logical calculi (propositions), (ii) computational calculi (types) and (iii) (objects of) CCC and some other appropriate categories) is called in the literature the *Curry-Howard-Lambek* correspondence [27].

Let me now tell a bit more about a relevant part of Lawvere's work that touches upon the issue of axiomatic method directly. Lawvere discovered CCC during his work on an alternative axiomatization of Set theory [11]. His idea was that the non-logical primitive of standard axiomatic Set theories like ZF, namely the binary membership relation $\in$, was badly chosen. Lawvere suggested to use instead the notion of function and the binary

[10]I emphasize that we are talking here about the *same* theory because propositions and proofs of given theory can be also made into objects otherwise, namely, by considering them as objects of a metatheory of the given theory. But this is a different matter.

[11]Lawvere also had other important motivations for introducing CCC, see [22].

operation (i.e., ternary relation) of composition of functions. The resulting axiomatic theory is known as the Elementary Theory of Category of Sets (ETCS) [21], [25], [32]. This proposal may appear radical to one who has habituated oneself to ZF and its likes but as it stands this proposal does not require any modification of the standard formal axiomatic method. The new choice of primitives allowed Lawvere to see that the condition of being CCC makes part of the wanted axiomatic description of the category of sets (Set). In this context the further observation that CCC provides a structural description of Curry-Howard correspondence appears as a bonus.

Let us see more precisely what happens here. The standard axiomatic approach requires to fix logic first and then use it for sorting out intuitive ideas about sets, numbers, spaces and whatnot. If one encounters then some difficulties of a logical nature one may try to modify the assumed logical principles and do this again. But unless one is axiomatizing *logic* to begin with one always expects to get as an outcome of axiomatization a logically transparent theory of some given non-logical subject-matter but not a new theory about logic. However in the case of ETCS something similar happens: the axiomatization reveals the fact that Set is equipped with its proper *internal* logically-related structure, namely CCC, which is not transported from the background logic but emerges as a specific feature of category Set.

This fact does not prevent one from construing ETCS as a standard formal axiomatic theory [32] but nevertheless suggests a reconsidering of the place and the role of logic in this theory. One can use the above observation for arguing that ETCS is after all a logical theory, which shows that the Set theory is in fact a part of logic. However this way of thinking about ETCS would commit one to qualify Lawvere's axiomatic Topos theory, which I present in the next Section, also as a logical theory. Although there is indeed a sense in which Topos theory qualifies as logical there is also an obvious sense in which this theory is geometric [12]. Thus the relationships between logical and non-logical concepts in both cases require a further examination.

6. Topos theory

The concept of *topos* first appeared in Algebraic Geometry in the circle of Alexandre Grothendieck around 1960 as a far-reaching generalization of the standard concept of topological space and didn't have any special relevance to logic. In his seminal paper [23] Lawvere provided an axiomatic definition of topos called today *elementary* topos [13]. Like ETCS the axiomatic theory of elementary topos does not bring by itself any new notion of axiomatic theory. However any systematic exposition of topos theory contains a chapter on the *internal logic* of a topos. In standard textbooks the internal logic is introduced as an extra feature on the top of the basic topos construction [32], [28]. As usual it has a syntactic part (Mitchel-Bénabou language) and a formal semantic part, which interprets the Mitchel-Bénabou syntax in terms of constructions available in the base topos (Kripke-Joyal semantics). Kripke-Joyal semantics assigns to symbols and syntactic expressions, which have an intuitive logical meaning (logical connectives, quantifiers, truth-values, etc.), explicit semantic values, which otherwise can be called *geometrical* (since the base topos is a generalized space). This is not something wholly unprecedented in the history of the 20th logic: think, for example, of Tarski's topological of (Classical and Intuitionistic) propositional logic [44]. However this feature of Kripke-Joyal semantics makes it quite unlike a notion of semantics derived from the idea of interpreting a formal theory by assigning explicit semantic values only to its non-logical elements (compare Section 4 above).

[12]Throughout this paper I use the word "geometric" in a broad sense that covers all space-related concepts including the topological ones.

[13]The title "elementary" reflects the fact that Lawvere's definition (unlike Grothendieck's original definition) is expressible in the standard first-order formal language [32]. The concept of elementary topos is slightly more general than that of Grothendieck topos: there are elementary toposes, which are not Grothendieck toposes.

Internal logic L_T can be used for developing further axiomatic theories "internally" in given topos T. It also provides an additional "internal" description of T itself ([32], Ch. 16). If one looks at Lawvere's [23] where the axioms for elementary topos first appeared in the press, one can see that namely the *internal* logical analysis of topos concept allows Lawvere to formulate these axioms: he observes that the internal logic of general topos and the internal logic of Set share the same CCC structure and thus the wanted axiomatic Topos theory is obtained through a simple generalization of ETCS:

> [A] Grothendieck "topology" appears most naturally as a modal operator, of the nature "it is locally the case that", the usual logical operators, such as $\forall$, $\exists$, $\Rightarrow$ have natural analogues which apply to families of geometrical objects rather than to propositional functions. [..] [I]n a sense logic is a special case of geometry. ([23], p. 329)

This logical analysis of Grothendieck's geometric topos concept is quite unlike Hilbert's analysis of geometric concepts, which relies onto the "unalterable laws of logic". Unlike Hilbert, Lawvere doesn't use logic as a ready-made tool for sorting out intuitive geometric concepts but rather makes explicit a logical structure "naturally" associated with certain geometric concepts. The popular word "natural" refers here to the fact that logic and geometry in a topos share a common categorical structure, which includes the CCC structure.

One may argue that this feature of Lawvere's axiomatic approach can matter only when we are talking about the way and the context in which his axioms for the elementary topos have been first obtained. The argument goes through if one uses the standard notion of formal axiomatic theory for distinguishing the final result from the context of its discovery. However I believe that this case requires a reconsidering of this standard notion. Lawvere's reasoning suggests the view that the internal logic of topos is its genuine logical foundation rather than an extra feature. Under this view, Lawvere's axiomatic approach qualifies as broadly constructive because along with rules for internal logical operations Lawvere also uses "natural analogues" of these rules, which apply to geometric operations in a topos. Since the resulting theory includes rules for non-logical operations it qualifies as constructive in the relevant sense. Since CCC structure makes part of the topos structure, the above remarks about the constructive character of Curry-Howard-Lambek correspondence (under the *propositions as types* interpretation, see Section 5) also apply here [14]. A precise constructive axiomatic architecture for Topos theory remains a work in progress. In the next Section I describe a more recent theory that already uses such an architecture officially.

7. Homotopy Type theory and Univalent Foundations

Homotopy Type theory (HoTT) is a recently emerged field of mathematical research, which has a special relevance to philosophy and logic because it serves as a basis for a new tentative axiomatic foundations of mathematics called the *Univalent Foundations* (UF)[5]. In this paper I do not attempt to review HoTT and UF systematically but only to describe a special character of its axiomatic architecture.

HoTT emerged through a synthesis of two lines of research, which earlier seemed to be quite unrelated: geometrical Homotopy theory and logical Type theory. The key idea is that of modeling types (including the type of propositions) and terms (including proofs) in Type theory by spaces and their points in Homotopy Theory. Beware that along with basic spaces Homotopy theory also considers *path spaces* where "points" are paths in the basic spaces, spaces of "paths between paths" called *homotopies*, spaces of "paths between paths between paths" and so on. All these higher-order spaces are also used for interpreting types.

Like in Topos theory, in HoTT geometry and logic are glued together with some category-theoretic concepts. The central categorical concept used in HoTT is that of $\omega - groupoid$. In the standard category theory a groupoid is defined as a category where all morphisms between objects are reversible, i.e., are isomorphisms. ω-groupoid is a higher-categorical

[14] For a discussion on constructive aspects of Category theory and Topos theory see [33].

generalization of this concept (called in this context 1-groupoid) where usual morphisms are equipped by morphisms between morphisms (called 2-morphisms), further morphisms between 2-morphisms (called 3-morphisms) and so on up to the first infinite ordinal ω). An analogy between spaces equipped with paths between points, homotopies between those paths, etc. is straightforward. It allows for mixing the geometrical and the categorical languages and talk interchangeably, e.g., about "spaces of paths", "groupoids of paths" and "groupoids" *simpliciter*.

The axiomatic HoTT uses Constructive Type theory with depended types due to Martin-Löf [31] (MLTT) for turning the tables at this point. The notion of ω-groupoid aka space aka homotopy type is taken as primitive while the notions of proposition, set, (one-dimensional) groupoid, category, etc. are construed as derived notions with MLTT. Types (and, in particular, propositions) and terms (and, in particular, proofs) in MLTT are interpreted, correspondingly, as spaces and points of these spaces. I skip further details. The obtained interpretation of MLTT in the categorically construed Homotopy theory directly translates all constructions in MLTT into geometric constructions. Then one may consider some further axioms such as the Axiom of Univalence (AU), which gives its name to the Univalent Foundations. In this paper I consider HoTT (with or without AU) only as an axiomatization of modern Homotopy theory. The idea of UF according to which HoTT can be used as a basis for developing the rest of mathematics has no bearing on my argument but implies that the special character of HoTT axiomatic architecture may be of general importance for logic and mathematics.

The constructive character of HoTT is already fully present in MLTT. Unlike ZF and other standard formal axiomatic theories, MLTT is *not* a system of formal propositions aka propositional forms. MLTT construes propositions as particular types among other non-propositional types. It comprises rules, which apply to types and their terms in general and to propositions and their proofs in particular. In MLTT, "the mathematical activity of *proving a theorem* is identified with a special case of the mathematical activity of *constructing an object* - in this case, an inhabitant of a type that represents a proposition"

([5], p. 17). Given that in MLTT the object construction does *not* reduce to proving propositions this theory qualifies as constructive in the relevant sense (Section 2). Notice that MLTT supports the ancient view according to which all Theorems are Problems of a special sort (Section 3).

The homotopical interpretation of MLTT (namely, the ω-groupoid interpretation) brings geometry into the picture and thus makes this case more similar to Euclid's. For example, MLTT validates the following rule (as a special case of a more general rule that I shall not discuss):

(F): Given types A, B to produce type $A \to B$ of functions with domain A and codomain B.

Under the homotopical interpretation this rule becomes

(HF): Given spaces A, B to produce space $A \to B$ of continuous maps from space A to space B,

which has exactly the same form as Euclid's First Postulate allowing for producing a straight line from a given point to another given point [15]. Claiming that HoTT is constructive theory I need to pay a particular care for disambiguating the term "constructive" properly. MLTT is constructive in a strong sense, which makes this theory *computable*. It is not known to the date whether or not HoTT with AU is constructive in the same strong sense; prima facie it is not ([5], p. 11). Anyway HoTT with AU *is* constructive in the weaker sense of being constructive, which I borrow from Hilbert and Bernays and use throughout this paper.

When one says that (the appropriately construed) Homotopy theory provides a *model* of MLTT, one uses the word "model" not in the same sense in which one usually talks, say, about various models of Lobachevskian geometry like Poincaré model, Beltramy-Klein model, etc. A major difference is this. Usual interpretations of a formal theory

[15]For another analogy between HoTT and Euclid's geometry see [5], p. 56-57.

of Lobachevskian geometry concern only the *non-logical* elements of this formal theory. But the homotopic interpretation of MLTT also interprets geometrically such types as the propositional function types, which represent implication and thus qualify as logical. Technically speaking, the distinction between logical and non-logical elements of syntax can be easily ignored, so one can use the same word "model" in all these cases. However as I have argued in Section 4 this distinction is fundamental for the standard Hilbertian notion of axiomatic method. When one systematically blurs this distinction the standard notion of formal axiomatic theory no longer applies. This doesn't mean however that HoTT in its present form doesn't qualify as an axiomatic theory at all because it perfectly instantiates the notion of *constructive* axiomatic theory on a par with Euclid's geometry [16].

8. What is Constructive Axiomatic Method?

In the preceding Sections I considered some older and some new recent trends in the axiomatic method. In the present final Section I would like to evaluate these trends from a more critical and more theoretical viewpoint. Are new developments in the axiomatic method justified logically and epistemologically? Do they change the standard Hilbert-style axiomatic method in a right direction? What (if any) are logical and epistemological advantages of the new constructive axiomatic approach? As my point of departure I take Hintikka's recent paper [18] where the author defends a modern version of Hilbert's *formal*

[16] In addition to the ω-groupoid model discussed earlier in this Section HoTT with AU has some other "natural" models [1]. Studying and comparing these models from an unified viewpoint largely remains an open research problem. However Voevodsky makes it clear that he considers the ω-groupoid model as basic for UF ([45], p. 7). Even if Voevodsky's reason for preferring this model is prima facie pragmatic, it has strong epistemological consequences. The idea of reconstructing the world of today's everyday mathematics using homotopy types as building blocks represents what Marquis calls the "geometrical point of view" in foundations of mathematics [29], [30]. Since HoTT and UF include logic as its proper part one may argue that this point of view is in fact more general than the more familiar Quinean "logical point of view" [38]. I leave a further discussion on this foundational issue for another occasion.

axiomatic method. I shall try to show that this formal method is not self-sustained and needs to be supported by a modern constructive method like one used in HoTT.

Hintikka:

> What is crucial in the axiomatic method [..] is that an overview on the axiomatized theory is to capture all and only the relevant structures as so many models of the axioms. ([18], p. 72)

Where these structures come from? Hintikka gives the following answer:

> The class of structures that the axioms are calculated to capture can be either given by intuition, freely chosen or else introduced by experience (ib., p. 83)

One may wonder how a mathematical structure or a structure of some different sort can be "given" or "introduced" without being construed axiomatically beforehand. Should we take at this point a Platonistic view according to which mathematical structures in some form exist independently of our axiomatic descriptions of these structures? Hintikka's answer is different. He explicitly rejects the notion of intuition as an intellectual analogue of sense-perception and insists that the intuition (along with the free choice and the experience) plays rather an active role. Defending his semantic view on logical inference (on which I comment below), Hintikka says

> [N]ew logical principles are not dragged [..] by contemplating one's mathematical soul (or is it a navel?) but by active thought-experiment by envisaging different kinds of structures and by seeing how they can be manipulated in imagination. [..] [M]athematical intuition does not correspond on the scientific side to sense-perception, but to experimentation. (ib., p. 78)

These elements together give the following picture. Building an axiomatic theory is a complicated two-way process; it is a game with Nature (and perhaps also with Society) where raw empirical and intuitive data effect one's axiomatic construction in progress while this construction in its turn effects back one's choice of further data, which become in this way less raw and more structured. Asking where the process starts exactly is the chicken or the egg kind of question. Hintikka's IF logic with its intended game-theoretic semantics provides a precise mathematical model for games of this sort [35].

My concern is about the kind of games that we need to play with Nature for doing science and mathematics. Although yes-no questioning games indeed play an important role in science and perhaps also in mathematics I claim that this is only the top of an iceberg. The main body of this iceberg is filled by mathematical and empirical constructive activities such as designing new experiments. If we consider applications of mathematics outside the pure science we may also mention the role of mathematical in engineering. In order to design a bridge or a particle accelerator one usually plays with certain mathematical *models* of these things, not with formal axioms.

Since such activities qualify as instances of Hintikka's "active thought-experiment" I hardly diverge from Hintikka up to this point. The divergence comes next. I don't grant Hintikka's view according to which the mathematical thought experimentation is, generally, a spontaneous ruleless activity, which should be studied by "empirical psychologists" rather than logicians and epistemologists (ib. p. 83). I observe that constructive axiomatic theories like Euclid's geometry, Newton's mechanics, Lawvere's axiomatic Topos theory and Voevodsky's HoTT-UF greatly increase one's capacities of mathematical thought experimentation by providing basic elements (points and straight lines in Euclid) and precise rules (Postulates in Euclid) for it. I can see that the spontaneity and the rulelessness may play a creative role in mathematics and science just like elsewhere but I claim that *typically* a constructive axiomatic organization of science and mathematics makes the thought experimentations in these fields richer and more powerful. Compare playing with pieces of wood with playing chess. Such an organized rather than spontaneous mathematical thought experimentation is typically used for building mathematical models of physical phenomena, designing bridges, accelerators etc.

Thus the real question is not how liberally one can use the word "axiomatic" but how exactly an axiomatic theory controls its contents. Formal theory T motivated by certain intended model M controls M through the *truth-evaluation* of its axioms and theorems in M. For the sake of my argument I after Hintikka understand the relation of logical inference in T as the *semantic* consequence relation. Having granted this I claim that this method of axiomatic control is not self-sustained because the concept of semantic consequence relation is highly sensitive to one's basic semantical setting. As long as the semantic consequence relation is discussed with respect to intuitive structures coming from the air (intuition, free choice, experience) it remains itself a very imprecise intuitive notion. I agree with Hintikka that this fact does not mean that one has here a choice between appealing to irrational resources and giving up the semantical view on logical consequence (ib., p. 77-78). In order to construe the relation of semantic consequence with a mathematical precision one should fix some *formal semantics*, which allows for doing the truth-evaluation properly (as in the case of Kripke-Joyal semantics for topos logic) [17]. In other words one needs to build a basic mathematical model (in the sense of "model" used in science rather than Model theory) of the appropriate class of intended structures suggested us by intuition, experience and what not. The mere yes-no questioning games cannot solve this problem because, recall, we are looking now for a mathematical framework allowing us to do the truth-evaluation properly. Unless we have got such a framework we are not in a position to give to yes-no questions definite answers. The wanted setting cannot come from the air but can be built by constructive methods [18]. Recall that Hilbert realized the need to support his formal axiomatic approach by constructive methods considering this issue

[17] In order to construe a notion of semantic consequence for given formal language L one needs to

- fix a formal semantics M for that language

- take a collections A_T of well-formed formulas of L (which may express axioms of a given theory T formalized with L)

- specify the class $M(A_T)$ of models of W_T by evaluating formulas from A_T in M

Formula ϕ is called a semantic consequence of A_T, in symbols $A_T \models \phi$, iff ϕ is a tautology in $M(W_T)$.

[18] In the last quote Hintikka says that the notion of logical (=semantic) consequence can be made more precise by introducing some "new logical principles" suggested by the intuitive thought-experimentation

from a very different perspective: he meant to apply constructive methods for proving the (formal) consistency of axiomatic theories syntactically. Now we see that Hintikka's semantic approach to axiomatization does not allow one to get rid of constructive methods either.

The above analysis suggests a view on the truth-evaluation as an advanced rather than basic feature of mathematical and other theories. Unlike Topos theory, HoTT in its existing form has no resources for doing truth-evaluation internally. There is however a "general consensus" that an internal truth-evaluation for HoTT can be construed within a higher-order topos structure in which HoTT would play the role of internal language ([5], p. 12). This recent example illustrates the thesis that the constructive axiomatic method is more general and more basic than the formal version of this method, which requires the truth-evaluation. This thesis is plausible since mathematics and science not only seek for truths and logical relations between those truths (*knowledge-that*) but also for effective methods of doing this and that *knowledge-how* [8]. From a historical viewpoint it is obvious that the knowledge-how is a more primitive form of knowledge, which can exist outside any scientific context. When science is brought into the picture there is an unfortunate tendency to isolate the relevant knowledge-how either in a special domain of *applied* science (and applied mathematics) or in social, psychological, educational, pragmatic and other contexts of *doing* science and leave scientific and mathematical theories outside. In this paper I have shown that such a separation of knowing *that* and knowing *how* cannot work for axiomatic mathematical theories because in this case the two types of knowledge are interlaced already at the atomic level of theoretical reasoning. The case of experimental sciences prima facie appears similar but requires a separate study, which I leave for a future research.

with mathematical structures. Talking about constructive methods I have in mind constructive principles specific for a given theory like Euclid's Postulates, which, generally, don't qualify as logical.

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