



‘SEX-EQUAL’ STABLE MATCHINGS

ABSTRACT. The paper defines a measure on the set of stable matchings in the marriage problem. This measure is based on the minimization of the envy difference between the sets of men and women, while preserving stability and selects stable matchings with the least conflict of interest between both groups of agents. The solution concept proposed is called Sex-equal Matching (*SEM*) and the paper also provides an algorithm to compute the set of *SEM*.

KEY WORDS: Matching Markets, Fair Distribution, No-envy

1. INTRODUCTION

This paper addresses the problem of finding intermediate allocations in two-sided matching markets. With this objective in mind we define a measure on the set of stable matchings that selects those with the least conflict of interests between men and women in the marriage problem.

The marriage problem was first proposed by Gale and Shapley (1962). In the classical stable marriage problem there are two sets of agents: men and women. A matching is a one to one mapping between the two sets. In their seminal paper Gale and Shapley also introduce what is today known as the Gale–Shapley Algorithm. This algorithm was used to prove the existence of stable matchings, i.e. matchings that cannot be improved by an individual or a couple. Gale and Shapley also showed that their algorithm leads to a stable matching that gives all men (or women if the role of sexes is reversed) the best partner that they can have in any stable matching. This matching is called man-optimal (woman-optimal).

The Gale–Shapley Algorithm can be used to find either the man-optimal or the woman-optimal stable matching, but no other stable matchings. These two matchings treat the two sexes very asymmetrically: the optimal matching for one sex is pessimal for the other.

For this reason we are interested in finding other stable matchings that are more equitable than the man or woman-optimal matchings.

The stable matchings set is a complete lattice under the common preferences of men and dual to the common preferences of women. The lattice structure results in the existence of a conflict of interest between men and women (see Knuth, 1976; Roth and Sotomayor, 1990). Any change from one stable matching to another will weakly improve the situation of agents on one side of the market at the expense of the ones on the other side.

Since the conflict of interests in the model is between men and women, it seems natural to focus on justice between groups rather than to try to implement an individual idea of justice. The concept implemented in this paper combines the ideas of stability and minimal envy between groups. These two concepts together provide a criterion to identify and compute stable matchings. The solution proposed is called Sex-equal Matching (*SEM*) and is based on the minimization of the envy difference between men and women while preserving stability. The set of *SEM* balance agent's aspirations provides a reasonable criterion to avoid the conflict in the model. In most instances, the *SEM* solution also selects stable matchings in the middle of the lattice.

The paper provides an algorithm to compute the *SEM*. The proposed *SEM* Algorithm uses the Gale–Shapley Algorithm. It also uses Gusfield and Irving's (1989) (from now on G&I, 1989) Minimal Differences Algorithm and techniques to generate a rotation digraph. The *SEM* Algorithm uses rotations to evaluate stable matchings without computing them. The Algorithm avoids the evaluation of every stable matching by using a stop rule based on the structure of the problem.

The paper is organized as follows. Section 2 introduces the model and the definition of *SEM*. Section 3 sketches the Sex-equal Algorithm and introduces the concept of rotation. An example illustrates the major steps to compute the *SEM*. Section 4 formally describes the *SEM* Algorithm and shows that the *SEM* Algorithm computes the set of Sex-equal matchings. This section also analyzes how the *SEM* set is related to other intermediate solution concepts in the literature. Section 5 concludes.

2. THE MODEL

In a marriage problem there are two finite disjoint sets. Let $W = \{w_1, w_2, \dots, w_m\}$ denote the set of women and $M = \{m_1, m_2, \dots, m_n\}$ the set of men. Here m_i 's preferences P_{m_i} are described by a linear order on $W \cup \{m_i\}$. Given two women $w_j, w_h \in W$, the expression $w_j P_{m_i} w_h$ means that m_i prefers to be matched to w_j rather than w_h ; $m_i P_{m_i} w_h$ means that m_i prefers to stay single rather than being matched to w_h . Similarly, each woman w_j 's preferences P_{w_j} are described by a linear order on $M \cup \{w_j\}$. The marriage problem is fully described by a triplet (M, W, P) where P is a preference profile containing a full description of the agents' preferences.

A **matching** is a function $\mu : M \cup W \rightarrow M \cup W$, such that:

1. $(\mu(m_i) \notin W \rightarrow \mu(m_i) = m_i), (\mu(w_j) \notin M \rightarrow \mu(w_j) = w_j),$
and
2. $(\mu(m_i) = w_j) \leftrightarrow (\mu(w_j) = m_i).$

DEFINITION 1. A matching μ is blocked by an individual $\sigma \in (M \cup W)$ in (M, W, P) iff $\sigma P_\sigma \mu(\sigma)$. A matching that cannot be blocked by any individual is called individually rational.

DEFINITION 2. A matching μ is blocked by a pair (m_i, w_j) in (M, W, P) iff $w_j P_{m_i} \mu(m_i)$ and $m_i P_{w_j} \mu(w_j)$.

Any individually rational matching that cannot be blocked by pairs is said to be stable.

In this model, the notion of stable matchings is equivalent to the core. Let Γ denote the set of all possible stable matchings in (M, W, P) . The set of stable matchings is nonempty and it is a complete lattice under the preferences of men and dual to the preferences of women. When agents have strict preferences the set of stable matchings has a global maximum and a global minimum according to the men's or women's preferences. These matchings are the man-optimal and woman-optimal stable matchings, denoted by μ^M and μ^W . Each one of these matchings can be obtained by the Gale–Shapley Algorithm.

Let $v(\mu)$ be a measure that computes the difference between the number of envy situations for men and women in a matching μ .

Given a pair (m_i, w_j) , m_i envies the partners of all the women w for which $w P_{m_i} w_j$. Let $r_{m_i}(m_i, w_j)$ be the position that w_j has in P_{m_i} . The number of agents envied by m_i when matched with w_j , is $r_{m_i}(m_i, w_j) - 1$. Respectively w_j envies $r_{w_j}(w_j, m_i) - 1$ agents. Given matching μ , the difference between the total envy on each side on the market is the following:

$$v(\mu) = \left[\left[\sum_{m_i \in M} r_{m_i}(m_i, \mu(m_i)) - n \right] - \left[\sum_{w_j \in W} r_{w_j}(\mu(w_j), w_j) - m \right] \right]. \quad (1)$$

The matching with the lowest difference in envy is the one that minimizes (1):

$$\arg \min_{\mu} [|v(\mu)|]. \quad (2)$$

The value of $m - n$ depends on the number of agents in each side of the market. From now on we assume $n = m$. This assumption can be easily removed by adding a constant to the value of matchings depending on the structure of the market.

It is also important that the *SEM* are stable. Stability is a necessary condition to guarantee fairness. Any unstable matching, even if it is selected by a very appealing concept of fairness, can be blocked by a pair of agents.

A Sex-equal Matching can be defined as the set of stable matchings for which the sum of male envy situations is as close as possible to the sum of female envy situations.

DEFINITION 3. *A Sex-equal Matching is a stable matching that minimizes the absolute value of the envy difference between both sexes, i.e.,*

$$SEM(\Gamma) = \arg \min_{\mu \in \Gamma} [|v(\mu)|]. \quad (3)$$

Since the number of stable matchings is finite, the set of *SEM* is always non-empty.

Let us clarify the concept with the help of a classical example by Knuth (Roth and Sotomayor, 1990, p. 37).

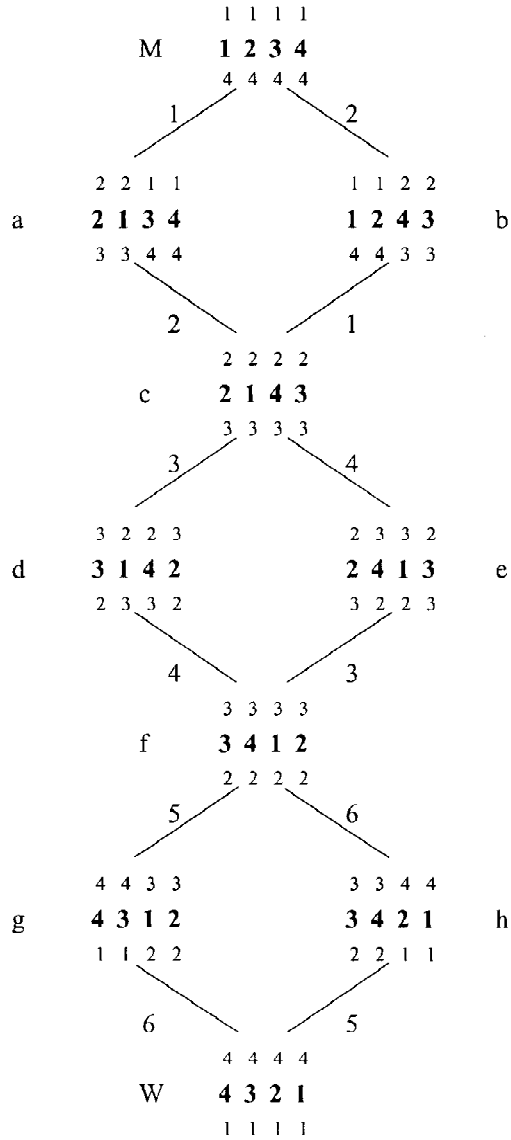


Figure 1.

EXAMPLE 1 (Knuth). Let $M = \{ m_1, m_2, m_3, m_4 \}$ and $W = \{ w_1, w_2, w_3, w_4 \}$. The preferences are as follows:

$$\begin{aligned}
 P_{m_1} &: w_1, w_2, w_3, w_4, m_1, & P_{w_1} &: m_4, m_3, m_2, m_1, w_1, \\
 P_{m_2} &: w_2, w_1, w_4, w_3, m_2, & P_{w_2} &: m_3, m_4, m_1, m_2, w_2, \\
 P_{m_3} &: w_3, w_4, w_1, w_2, m_3, & P_{w_3} &: m_2, m_1, m_4, m_3, w_3, \\
 P_{m_4} &: w_4, w_3, w_2, w_1, m_4, & P_{w_4} &: m_1, m_2, m_3, m_4, w_4.
 \end{aligned}$$

In this example there are ten stable matchings where w_1, w_2, w_3, w_4 , are matched respectively with the man denoted by the numbers in bold in Figure 1.

The numbers above and below represent $r_{m_i}(m_i, \mu(m_i))$ and $r_{w_j}(\mu(w_j), w_j)$ respectively.

For each one of these stable matchings we can compute $v(\mu)$:

$$\begin{aligned}
v(\mu^M) &= && -12 \\
v(\mu^a) = v(\mu^b) &= && -8 \\
v(\mu^c) &= && -4 \\
v(\mu^d) = v(\mu^e) &= && 0 \\
v(\mu^f) &= && 4 \\
v(\mu^g) = v(\mu^h) &= && 8 \\
v(\mu^W) &= && 12
\end{aligned} \tag{4}$$

Matchings μ^d and μ^e are *SEM* in this example. Both matchings are in the middle of the lattice of stable matchings.

3. INTUITIVE DESCRIPTION OF THE *SEM ALGORITHM*

In order to compute the set of *SEM* let us introduce some notation to relate each agent with his/her possible partners. For any m_i , $s_\mu(m_i)$ denotes the first woman w_j on P_{m_i} such that w_j strictly prefers m_i to $\mu(w_j)$ (her partner in μ); $next_\mu(m_i)$ denotes the partner in μ of the woman $s_\mu(m_i)$. This notation allows to formally define the process of changes that leads from one stable matching to another.

DEFINITION 4. Let $\rho = \{ (m_0, w_0), (m_1, w_1), \dots, (m_{r-1}, w_{r-1}) \}$ be an ordered list of pairs in a stable matching μ such that for each i ($0 \leq i \leq r-1$) m_{i+1} is the $next_\mu(m_i)$, where $i+1$ is taken modulo r , i.e. $m_r = m_0$ and $w_r = w_0$. Then ρ is called a rotation (exposed) in μ ; m_i (or w_j) is in a rotation ρ if there is a pair (m_i, w_j) in the ordered list defining ρ .

Let $\rho = \{ (m_0, w_0), (m_1, w_1), \dots, (m_{r-1}, w_{r-1}) \}$ be a rotation exposed in μ . Let $\mu \setminus \rho$ be the matching in which each man not in ρ stays matched to his partner in μ , and the match for each man m_i in ρ is $w_{i+1} = s_\mu(m_i)$. The transformation from μ to $\mu \setminus \rho$ is called the **elimination** of ρ from μ .

The lines between matchings in Figure 1 represent the rotations that link them. The number in each line refers to the particular rotation that links two matchings. For example the ordered list of pairs $\{(m_1, w_1), (m_2, w_2)\}$ is a rotation, denoted ρ_1 , exposed in matching μ^M . The elimination of such rotation from μ^M yields the matching μ^a where m_1 is matched with w_2 and m_2 with w_1 while the other pairs remain unchanged. The six rotations in the example are:

$$\begin{aligned}
\rho_1 &= \{ (m_1, w_1), (m_2, w_2) \} \\
\rho_2 &= \{ (m_3, w_3), (m_4, w_4) \} \\
\rho_3 &= \{ (m_1, w_2), (m_4, w_3) \} \\
\rho_4 &= \{ (m_2, w_1), (m_3, w_4) \} \\
\rho_5 &= \{ (m_1, w_3), (m_2, w_4) \} \\
\rho_6 &= \{ (m_3, w_1), (m_4, w_2) \}
\end{aligned} \tag{5}$$

It is important to note that a rotation may be exposed in more than one matching, and hence the definition does not associate a rotation with a unique matching, for example, rotation ρ_1 is exposed in μ^M and μ^b . However, no ordered set of pairs is a rotation unless it satisfies the above definition for at least one stable matching of Γ . In any stable matching other than μ^W there is at least one rotation exposed. Moreover, the matching μ^M can be transformed into μ^W through a sequence of stable matchings, by successively finding and eliminating any exposed rotation in each successive matching.

DEFINITION 5. *The stable matching μ is said to be an immediate predecessor of the stable matching μ' if there is a rotation ρ such that $\mu \setminus \rho = \mu'$.*

DEFINITION 6. *A chain $C = \{ \mu^1, \dots, \mu^q \}$ in Γ is an ordered set of elements of Γ such that μ^i is an immediate predecessor of μ^{i+1} for each i , $1 \leq i \leq q - 1$.*

DEFINITION 7. *A maximal chain in Γ is a chain from μ^M to μ^W .*

Let $\rho = \{ (m_0, w_0), (m_1, w_1), \dots, (m_{r-1}, w_{r-1}) \}$ be a rotation. We say that ρ **moves m_i down** from w_i to w_{i+1} , and **moves w_i up** from m_i to m_{i-1} . If w is either w_i or is strictly between w_i and w_{i+1} in P_{m_i} , then ρ **moves m_i below** w . Similarly, ρ **moves w_i above** m if m is m_i or is strictly between m_i and m_{i-1} in P_{w_i} .

These definitions simply reflect what happens to each person in ρ when ρ is eliminated from a matching μ .

For any man m_i and for any woman w_j in (M, W, P) there is at most one rotation that moves m_i down to w_j , and w_j up to m_i (G&I, 1989, Lemma 3.2.1)

The order in which each rotation may be eliminated at each time is contained in agents' preferences. An order between rotations can be established using the two precedence relation that follows. Assume that (m, w) is in a rotation ρ :

- (i) If ρ' is the (unique) rotation that moves m to w , then ρ' is a **type 1 predecessor** of ρ .
- (ii) If ρ moves m below w , and $\rho' \neq \rho$ is the (unique) rotation that moves w above m , then ρ' is a **type 2 predecessor** of ρ .

In words, a rotation ρ_i is a type 1 predecessor of ρ_j if ρ_i joins a couple (m, w) that ρ_j separates. Therefore ρ_i must be eliminated before ρ_j is exposed. To illustrate (ii), consider a case where a pair (m, w) must be formed (by the elimination of rotation ρ_i) to avoid that another rotation elimination ρ_j induces an unstable matching. Then, rotation ρ_i is a type 2 predecessor of ρ_j and ρ_i must be eliminated before ρ_j appears exposed.

Let $\Pi(\Gamma)$ be the set of all the rotations in Γ . $\Pi(\Gamma)$ is ordered by the precedence relations of type 1 and 2. The relation of precedence in $\Pi(\Gamma)$ is transitive and asymmetric.¹ Using the two precedence relations we can define the directed, acyclic graph $G(\Gamma)$, called rotation digraph, whose edges correspond to a subset of the pairs of $\Pi(\Gamma)$. $G(\Gamma)$ tells us the order in which the rotations are exposed and can be eliminated.

There is an efficient way of finding the rotations on $\Pi(\Gamma)$. This is the Minimal Differences Algorithm (for a detailed discussion see G&I, 1989). The minimal differences algorithm can be implemented in $O(n^2)$.

The digraph $G(\Gamma)$ can be constructed from the preference list in $O(n^2)$ time. The procedure is summarized in G&I (1989) Lemma 3.3.2.

In Knuth's example, ρ_1 and ρ_2 are type 1 predecessors of ρ_3 and ρ_4 . At the same time ρ_3 and ρ_4 are type 1 predecessors of ρ_5 and ρ_6 . There are no type 2 predecessors in this case.

Let α_i be the change in the value of v between a matching and its predecessor due to the elimination of the rotation ρ_i than links them,

$$\alpha_i = v(\mu \setminus \rho_i) - v(\mu), \quad (6)$$

Changes produced by the elimination of ρ_i involve always the same partners. The value of this change is independent of the stable matching μ at which the rotation is exposed.

In the example, it can be seen that the difference between the value of $v(\mu^M)$ and the value of $v(\mu^a)$ is the same as the difference between $v(\mu^b)$ and $v(\mu^c)$. This difference is generated by ρ_i . In the example α_i is equal to 4 for all $i = \{1, \dots, 6\}$. Notice that α_i is always positive. Also note that, given $v(\mu^M)$ and α_1 , $v(\mu^M \setminus \rho_1)$ can be obtained without computing μ^a .

$$v(\mu^M \setminus \rho_1) = v(\mu^M) + \alpha_1 = -12 + 4 = v(\mu^a) = -8. \quad (7)$$

Hence if $v(\mu^a) = v(\mu^M \setminus \rho_1)$, then $v(\mu^a) > v(\mu^M)$.

Once we have found the rotations in a matching market we can compute the α_i associated to each rotation. After that, every matching along all maximal chains can be evaluated without eliminating the rotation or computing the matching. This is because all the information about which rotations (α_i) may be eliminated (added) at each time is contained in agents' preferences and all rotations can be ordered by using type 1 and type 2 precedence relations on $G(\Gamma)$.

Let $\Omega_{i,\dots,j}$ be the value of the last matching in the chain $C = \{\mu^M, \dots, \mu^t\}$ as a result of eliminating rotations $\{\rho_i, \dots, \rho_j\}$ from μ^M . Let $\Omega_{i,\dots,j}^-$ denote the last negative element in each maximal chain and let $\Omega_{i,\dots,j}^+$ denote the first positive or zero element in each maximal chain.

$$\Omega_{i,\dots,j} = v(\mu^M) + \alpha_i + \dots + \alpha_j = v(\mu^t). \quad (8)$$

For the sake of convenience I will denote $\Omega = v(\mu^M)$. Using Knuth's example we can show how each $\Omega_{i,\dots,j}$ can be easily computed

knowing Ω and the α_i involved.

$$\begin{aligned}
\Omega &= v(\mu^M) = -12 \\
\Omega_1 &= v(\mu^M) + \alpha_1 = -8 \\
\Omega_2 &= v(\mu^M) + \alpha_2 = -8 \\
\Omega_{12}^- &= v(\mu^M) + \alpha_1 + \alpha_2 = -4 \\
\Omega_{123}^+ &= v(\mu^M) + \alpha_1 + \alpha_2 + \alpha_3 = 0 \\
\Omega_{124}^+ &= v(\mu^M) + \alpha_1 + \alpha_2 + \alpha_4 = 0 \\
\Omega_{1234} &= v(\mu^M) + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4
\end{aligned} \tag{9}$$

In (9) all maximal chains have changed their signs from positive to negative. The *SEM* will be found among the last negative and the first no negative matchings. The matchings with the smallest absolute value between them are the set of *SEM*.

$$\begin{aligned}
&\min [|\Omega|, |\Omega_1|, |\Omega_2|, |\Omega_{12}|, |\Omega_{123}|, |\Omega_{124}|, |\Omega_{1234}|] \\
&= \min [|\Omega_{12}^-|, |\Omega_{123}^+|, |\Omega_{124}^+|] = 0
\end{aligned} \tag{10}$$

In this case we have two *SEM*, corresponding to Ω_{123}^+ and Ω_{124}^+ . After the rotations ρ_1, ρ_2, ρ_3 and ρ_1, ρ_2, ρ_4 are eliminated from μ^M , μ^d and μ^e are obtained as *SEM*.

The enumeration process described above may become exponential if each stable matching is computed more than once. This possibility is avoided when we record already exposed in Ω 's subindex. We can avoid reaching a matching more than once by excluding rotations with a number smaller than the last rotation eliminated in a particular matching. For example, in Ω_2 rotation ρ_1 is excluded from elimination. Proceeding in this way no matching will be computed more than once. This simple process allows us to enumerate each possible stable matching, if needed, without reaching each matching more than once.

If there is no change in signs along the maximal chains, $v(\mu^M)$ and $v(\mu^W)$ have the same sign. In this case, μ^M is the unique *SEM* if $v(\mu^M) \geq 0$, and μ^W is the unique *SEM* if $v(\mu^W) \leq 0$.

4. THE ALGORITHM

This section presents an algorithm to summarize the steps used to compute the *SEM* on Section 3. After the presentation of the *SEM* Algorithm and the paper's main results the section ends with a brief comparison between the *SEM* and other intermediate solution concepts.

The SEM Algorithm

- **Step 1:** Compute $v(\mu^M)$ using the Gale–Shapley Algorithm.
 - * If $v(\mu^M) \geq 0$ the algorithm stops. μ^M is the unique SEM.
 - * If not, continue to the next step.
- **Step 2:** Compute $v(\mu^W)$ using the Gale–Shapley Algorithm.
 - * If $v(\mu^W) \leq 0$ the algorithm stops. μ^W is the unique SEM.
 - * If not, continue to the next step.
- **Step 3:** Determine the preorder $\Pi(\Gamma)$ and the digraph $G(\Gamma)$ using the minimal difference algorithm and the G&I procedure to compute $G(\Gamma)$.
- **Step 4:** Evaluate each rotation.
- **Step 5:** Compute the change in signs at each maximal chain.
- **Step 6:** Compute the set

$$\Omega_{1,\dots,j}^* = \arg \min [\left| \Omega_{1,\dots,k}^- \right|, \left| \Omega_{1,\dots,k+1}^+ \right|, \dots, \left| \Omega_{1,\dots,t}^- \right|, \left| \Omega_{1,\dots,t+1}^+ \right|], \quad (11)$$

- **Step 7:** Compute the set of *SEM* by eliminating from μ^M the rotations corresponding to the subindex of $\Omega_{1,\dots,j}^*$.

We will prove that the *SEM* Algorithm computes the set of *SEM* with the help of some intermediate results.

LEMMA 1. *The value of α_i is positive for every rotation ρ_i on $\Pi(\Gamma)$.*

Proof. There is only one rotation that links every pair of stable matchings μ and $\mu \setminus \rho_i$. The α_i associated to each rotation ρ_i , is independent of the μ at which ρ_i is exposed.

In a rotation elimination only the pairs in ρ_i change partners. By definition of a rotation, each elimination increases $r_m(m, w)$ and decreases $r_w(m, w)$. By definition of α_i , its value is precisely the change in the difference between $r_m(m, w)$ and $r_w(m, w)$. Therefore, the value of α_i is positive. \square

A stable matching belongs to the set of *SEM* if $v(\mu)$ is zero. If not, our task is to find the set of stable matchings each of which has a value nearest to zero knowing that $v(\mu^M) < 0$ and $v(\mu^W) > 0$. Since $v(\mu^M) < 0$ and $v(\mu^W) > 0$, there must be a change of sign in the set of stable matchings. Clearly the situation where $v(\mu^W) < 0$, and $v(\mu^M) > 0$ is impossible by Lemma 1.

THEOREM 2 [G&I, 1989, Th. 2.2.4]. *Every rotation of Γ appears once on every maximal chain of Γ .*

THEOREM 3 [G&I, 1989, Th. 2.2.5]. *Every stable matching μ' can be generated by a sequence of rotation eliminations, starting from μ^M , and every sequence leading to the same stable matching contains exactly the same rotations.*

PROPOSITION 4. *If $v(\mu^M) \geq 0$ the μ^M is the unique SEM. If $v(\mu^W) \leq 0$, μ^W is the unique SEM.*

Proof. This result is a direct consequence of the lattice structure of the set of stable matchings, Theorem 3 and Lemma 1. \square

THEOREM 5. *The SEM Algorithm generates the set of Sex-equal matchings.*

Proof. If $v(\mu^M)$ is positive or $v(\mu^W)$ is negative or any of them is zero, then the *SEM* is computed by the *SEM* Algorithm in Step 1 or Step 2. The proof follows from Proposition 4.

For the proof of the remaining cases I will proceed by contradiction. Let us suppose that the statement of the theorem is not true and $v(\mu^M) < 0$ and $v(\mu^W) > 0$.

Case 1. There is matching μ^i an a maximal chain where either its predecessor μ^{i-1} is such that $v(\mu^{i-1})$ has a different sign, or its successor μ^{i+1} is such that $v(\mu^{i+1})$ has a different sign, so μ^i is in one of the changes of sign on a maximal chain, $v(\mu^i) \in [\left| \Omega_{1, \dots, k}^- \right|,$

$|\Omega_{1,\dots,k+1}^+|$]. If the theorem is not true, then $|v(\mu^i)| < |v(\mu^*)|$ where μ^* is a matching obtained by the *SEM* Algorithm. However, μ^* is precisely a matching that has been formed eliminating from μ^M the rotations $\Omega_{1,\dots,j}^*$. Then

$$|v(\mu^*)| = \arg \min_{\mu \in \Gamma} [|\Omega_{1,\dots,k}^-|, |\Omega_{1,\dots,k+1}^+|, \dots, |\Omega_{1,\dots,t}^-|, |\Omega_{1,\dots,t+1}^+|].$$

A contradiction.

Case 2. There is a matching μ^i in a maximal chain where neither its predecessor μ^{i-1} nor its successor μ^{i+1} is such that $v(\mu^{i-1})$ or $v(\mu^{i+1})$ has a different sign.

If μ^i is not involved in the change of sign of its maximal chain, $|v(\mu^i)|$ cannot be smaller than $|v(\mu^*)|$. This is because going through a maximal chain from μ^M to μ^W , $v(\mu)$ is always increasing (see proof of Lemma 1). So $|v(\mu^i)|$ must be larger than the value of at least some of the matchings that are in the change of sign on a maximal chain $[|\Omega_{1,\dots,k}^-|, |\Omega_{1,\dots,k+1}^+|, \dots, |\Omega_{1,\dots,t}^-|, |\Omega_{1,\dots,t+1}^+|]$. In that case, $|v(\mu^i)|$ cannot be smaller than the value of the matching that is in $\arg \min_{\mu \in \Gamma} [|\Omega_{1,\dots,k}^-|, |\Omega_{1,\dots,k+1}^+|, \dots, |\Omega_{1,\dots,t}^-|, |\Omega_{1,\dots,t+1}^+|]$. Then $|v(\mu^i)| > |v(\mu^*)|$. A contradiction.

Given that the algorithm find all the changes in sign in all the maximal chains it is clear that all the *SEM* are found by the *SEM* Algorithm. \square

Let us compare the concept of *SEM* and the procedure described in this section with some of the related concepts and algorithms in the literature. An early attempt to design an algorithm with the purpose of finding equitable allocations was the Fair Optimal Solution by S. Selkow (Knuth, 1976). This algorithm minimizes the regret of the most unhappy person. The matchings μ^c , μ^d , μ^e and μ^f are Fair Optimal Solutions of the Example 1.

G&I (1989) proposed several algorithms to find intermediate matchings. The first one is the Egalitarian Stable Marriage. This matching treats each individual equally and minimizes the number of agents that other agents in the market prefer to their partners in a stable matching. In Example 1 no matching can be selected

according to this criteria, since each stable matching has the same value according to the Egalitarian Stable criterion. This concept is generalized by G&I (1989) to a weighted version called Optimal Stable Marriage Problem in which each agent provides a real number weight instead of a rank-ordered preference list. Finally, G&I (1989) propose the Parametric Stable Marriage. This algorithm takes into account the asymmetry between the man-optimal and the woman-optimal stable matching, generalizing the Egalitarian and the Optimal stable marriage problems, and identifying a continuum of compromises between the preferences of men and women.

Following a different approach Roth and Vande Vate (1990) propose the Randomized Matching Mechanism. This mechanism cannot achieve every stable matching and, in particular, has difficulties reaching those matchings that have less conflict of interest (see Ma, 1996, for a proof).

The *SEM* Algorithm follows the spirit of G&I's (1989) solutions since it relies heavily on the structural properties of the marriage problem without using linear programming techniques. However, the concepts of fairness behind *SEM* and the ones in G&I (1989) are very different. Whereas G&I (1989) use an utilitarian idea of fairness and relies on individuals to define their Egalitarian and Egalitarian Stable Marriage, the *SEM* solution relies on fairness inter groups and no envy.

5. FINAL REMARKS

This paper presents an approach to approximate equitable allocation in matching markets. This approach takes into account the conflict of interests between groups in these markets. The criterion proposed in the paper combines stability and envy considerations to select fair matchings within the set of stable allocations. An algorithm is proposed to compute the set of *SEM*. This algorithm makes intensive use of the structure of the set of stable allocations and works in polynomial time.

The main shortcoming of this solution concept are its multiplicity and the absence of any structure linking the different solutions in the set. This problem complicates the attempt to select a particular *SEM* when there are multiple *SEM* available.

Both the concept and the algorithm proposed in this paper can be easily generalized to be used in the Colleges' Admissions Problem.

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NOTE

1. The precedence relations of type 1 and 2 are defined over rotations, not over matchings. The relation of immediate predecessor (Definition 5) is defined over matchings.

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