Inexact knowledge 2.0

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Feb 19th 2019

Many of our sources of knowledge only afford us knowledge that is inexact. When trying to see how tall something is, or to hear how far away something is, or to remember how long something lasted, we may come to know some facts about the approximate size, distance or duration of the thing in question but we don't come to know exactly what its size, distance or duration is (Williamson 1992, 2000). In some such situations we also have some *pointed* knowledge of how inexact our knowledge is. That is, we can knowledgeably pinpoint some exact claims that we do not know. We show that standard models of inexact knowledge leave little or no room for such pointed knowledge. We devise alternative models that are not afflicted by this shortcoming.

I

Inexact knowledge is the most one obtains when trying to evaluate a quantity by inexact means. For concreteness, we will focus on the now standard case of trying to evaluate by normal eyesight what the position of an unmarked clock's single hand is (Williamson 2011, 2014). To simplify matters we assume that (it is known with certainty that) the hand has only sixty possible positions. Provided that you are neither too far nor too close, your perceptual faculties function normally, and you have exercised them with care, the following seems evident:

(1) *Inexact knowledge.* You know something non-trivial about the position of the clock's hand, but you do not know its exact position.

Say that you *rule out* a position – or that, in your epistemic situation, a position is ruled out – iff you know that the clock's hand is not at that position. Then (1) claims that you rule out some positions but not all but one.

Some issues related to so-called *guises* or *modes of presentation* arise here. Looking at the clock from some distance when its hand is at 20, there is a sense in which you only know that it is roughly around 20: for instance, you do not know that it is not at 20, and you do not know that it is not at 19.[1] But trivially,

[1] Plausibly certain positions are easier to know because they correspond to the horizontal or vertical lines, like 30. We ignore this complication here. See Williamson (2014, 979).

you do know that the hand's position is not one position before its current position. Under that description you know that the hand's position is not 19. More subtly, depending on how we think about demonstrative thought, we may think that there is a sense in which you can know the exact position of the hand: namely, that it is here, mentally 'pointing at', or attending to, the hand's current position, i.e. 20. If so, you may also be able to think demonstratively of 19 and know that the hand is not *there* (mentally 'pointing at' one position before the hand's current position, i.e. 19). Yet there is still a clear sense in which you do not know that the hand is not at 19. That sense need not involve any contrast between your visual grasp of the position and some non-perceptual concept or label like '19'. Suppose, for instance, that you are holding a second clock in your hand with sixty marks but no numerical labels. You are trying to judge which of these corresponds to the position of the hand of the clock on the wall. We may for example ask you to draw a red line over those positions you think the hand is not at. When we say that you do not know that the hand's position is not 19, we mean that you lack the kind of visual knowledge of the hand's position that would allow you to rule out that the hand's position corresponds to the 19th mark (our label, not yours) on the second clock you hold in your hand.

Provided that you are neither too far nor too close, your perceptual faculties function normally, and you have exercised them with care, there will be a further fact about you that we take to be equally evident: there will be several positions you do not rule out – say, 19 and 20 – of which you know that you do not rule them out.

(2) *Pointed knowledge of inexactness.* There are several positions of which you know that you do not know that the clock's hand is not at them.

A more intuitive way of stating the idea is: there are several positions of which you know that you do not know *whether* the hand is at them. That is strictly speaking stronger than (2) but follows from it given two fairly innocuous instances of closure.[2]

Knowing this is more than knowing (1). It is not merely that you know that several positions are not ruled out; you can actually point to several positions of which you know that they are not ruled out. Knowing this is less than knowing of *all* positions which are not ruled out that they are not ruled out: (2) only demands that this be known of several of them.

To be clear, (2) does not always obtain. If you are very close you may be unable to know whether your knowledge is inexact and *a fortiori* unable to know of several positions that you do not rule them out. But we think it rather obvious

[2] If there are two positions you do not rule out, then you do not know exactly where the hand is. So assuming a simple instance of closure, if you know that there are two positions you do not rule out, you also know that you do not know that the clock's hand *is* at these positions. Moreover, if you know neither that the hand is at a given position, nor that it is not, then you do not know whether it is at that position. So given another simple instance of closure, you know that you do not know whether the hand is at these positions. Note that these instances need only to hold in some scenario, not out of necessity.

that (2) will often obtain. If you are far enough, and it merely seems to you that the clock's hand is somewhere in the area of 20, you can know that you do not rule out 20 and know that you do not rule out 19 either. In fact, we think that even if it seemed to you that the hand was exactly at 20, you could still know your own limitations well enough to know not merely that you cannot rule out 20, but also that you cannot rule out 19.

II

The aforementioned two facts turn out to be incompatible with standard *fixed margin models* of inexact knowledge. In standard epistemic logic we model knowledge using a set of worlds and a function K that associates each world w with a set of worlds $K(w)$ representing the worlds compatible with what you know at w; we say that p is known at w iff it holds throughout $K(w)$. In *fixed margin* models it is assumed that what you know consists of all and only those propositions that hold throughout worlds where (some known background facts obtain and) the position of the hand is within a fixed margin for error of its actual value. In their simplest incarnation they consist of a world w_i for each position i of the hand with $K(w_i)$ the set of worlds w_j such that i and j differ at most by a certain fixed margin m.[3] Thus at a world where the clock's hand is at position i the strongest thing you know about its position is that it is within a fixed margin m of i. The models capture inexact knowledge: if m is less than 30 but greater than 0 you know something non-trivial about the position of the hand without knowing exactly where it is (Williamson 2011).[4]

However the models cannot simultaneously capture pointed knowledge of inexactness. For when the margin is less than 30 the following holds whenever p_i is the proposition stating that the clock's hand is at an exact position i:

(3) $K\neg K\neg p_i \to p_i$. If you know that you do not know that the hand is not exactly at i then the hand is at i.

Note first that (3) is indeed incompatible with fact (2). If (2) holds there are two distinct exact positions i, j of which one knows that one does not rule them out. By (3) it would follow that both p_i and p_j hold. But since these are exact and distinct at most one holds: contradiction. Now to see why (3) holds in fixed margin models, suppose the clock's hand is at 20 and your margin is 5. For all you know, the hand is at 15 and you have a margin of 5. But if that was so, you would rule out every position *after* 20. For all you know the clock's hand is at 25 and you have a margin of 5. But if that was so, you would rule out every position *before* 20. So for every position before and after 20, for all you know you do rule it out. Hence if any position is such that you know that you

[3] On the clock distances are measured in modulo arithmetic: $d(i, j) = i + j \mod 60$.

[4] Since any position on the clock is less than 30 steps away from any other, a margin of 30 or more would mean that you do not know anything non-trivial about where the hand is.

do not rule it out, it is the actual position, 20. In general, for any position i on the clock and margin $0 < m < 30$, if an exact position x other than i is within m of i, there is a position y on the other side of i such that x is further than m steps away from y. Hence on a fixed margin model with margin m, at i, for every exact position x other than i that you do not rule out, you cannot rule out being at some position y at which you rule x out. Hence (3) holds.

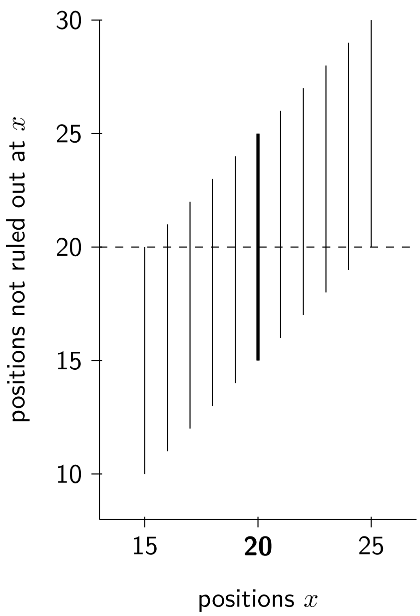


Figure 1. Intervals of inexact knowledge as a function of the value of the parameter in fixed margin models.

Figure 1 displays the intervals compatible with what you know at a world w_{20} with a fixed margin of 5. On the horizontal axis are positions where, for all you know at w_{20}, the hand might be. The actual position is in bold. On the vertical axis are positions that you know you do not rule out if the hand is at a given position on the horizontal axis. For instance, at w_{20} you know that if the hand is at 15 the positions you do not rule out are the interval $(10, 20)$. In bold are the positions you do not in fact rule out. The dotted line shows that the actual position is the only position that is not ruled out on any possibility compatible with what you know at w_{20}.

Thus the models capture (1) only if they entail (3) and reject (2). Worse, the models entail that (3) itself is known, since it holds throughout the model and whatever holds throughout the model counts as known. We take all this to be implausible. (In case you wonder whether there are even worse consequences, *e.g.* whether knowing (3) means that one can work out the actual position or whether the schema (3) holds for all propositions, the answer is no, as we explain in footnote.[5])

Every model involves some idealizations. But we do not think that this particular consequence is harmless. For the pointed knowledge that (2) affirms is integral to the very phenomenon to be modelled, *viz.* to which extent inexactness limits higher-order knowledge. In particular, the rejection of (2) suggests that the models put excessively stringent conditions on some forms of higher-order knowledge. That casts suspicion on uses of the models against other forms of higher-order knowledge such as knowing that one knows.

Now one of the idealizations fixed margin models make is that the relevant margin is known with exactitude. That is because one's margin is the same throughout the model. One might naturally be wary of that assumption: in a normal scenario, you would not know whether you are able to know the position of the hand within exactly 10 steps of its actual position, say. So it is worth noting that rescinding it will not be enough to allow for the truth of (2).

To see this, say that a world w consists in a sequence of parameters $\langle i, m, m^2, \ldots\rangle$ where i is the value of the parameter of interest, *e.g.* the position of the clock's hand, m your margin for error (at that world) on i's value, and, if higher-order uncertainty is required, m^{n+1} your margin for error on the value of m^n (letting $i = m^0$ and $m = m^1$). Fixed margin models say that a world $\langle i, m\rangle$ only has access to worlds with the same first-order margin m, implying that its value is known. We can relax this assumption by laying down that a world $\langle i, m^1, m^2, \ldots\rangle$ has access to a world $\langle j, l^1, l^2, \ldots\rangle$ iff j is within m^1 of i and l^1 is within m^2 of m^1 and, more generally, l^n is within m^{n+1} of m^n. We call these *product margin models* because the margins are applied independently: as soon as a parameter value i is within your first-order margin and a first-order margin value m^1 is within your second-order margin value, that particular combination of i and m^1 is itself left open by what you know. The overall accessibility relation is the "product" of accessibility relations along each margin order.

While product models allow for ignorance of one's margins (at every finite order), they still do not allow for (2). For they only make *more* worlds accessible to the actual world. So it still is the case that if the clock's hand is at 20 and your actual first-order margin is 10, you cannot rule out being at a world where the clock's hand is at 30 with a first-order margin of 10 or even less, so you cannot rule out being at a world where the hand's being at 19 is ruled out. The only novelty is that, because for all you know your margin is better than what it actually is, you do not even know of the hand's actual position that it is not ruled out. For instance, if the hand is at 20, your margin is 10, but for all you

[5] Knowing (3) does not mean that you can come to know the actual position by reflecting on your knowledge, for according to these models you do not know that you know that 20 is not ruled out. The schema (3) does not hold in full generality, for consider the disjunctive proposition that the clock's hand is at 19 or 21. In a (non-trivial) fixed margin model every world in $K(w_{20})$ either fails to rule out 19 or fails to rule out 21. Hence when the clock's hand is at 20 one knows that one does not rule out that the hand is at 19 or 21 even though it is not at 19 or 21.

know, you are at 30 and your margin is 9, then for all you know, you can rule 20 out. So making knowledge of margins for error inexact will not be enough to do justice to (2).

The problem is fairly general. As long as the values of the variable of interest can be described as spread enough on 'opposite sides' of the actual value, a fixed margin or product margin model will entail that any world at some distance of the actual world but within the actual world's neighbourhood is ruled out by some world within that neighbourhood but on the 'opposite' side. There are several ways of formalizing the idea but the details are not essential for our discussion.[6]

III

The key to solving the issue is to note that pointed knowledge of inexactness requires that one knows certain conditionals. Suppose that (1) and (2) hold in a case: the most you know about the hand is that it is between 10 and 30 and you do know that both 20 and 19 are compatible with what you know. It follows from what you know that: *if the hand is at* 30 *your ignorance of its position spans at least* 11 *steps*, since it is compatible with your knowledge that the hand is at 30 but incompatible with it that you rule 19 out. By contrast your knowledge does not entail that *if the hand is at* 20 *your ignorance of its position spans at least* 11 *steps*, for it is compatible with the actual world at which the hand is at 20 and your margin is only 10. So unlike in product models (and fixed margin models for that matter), your overall ignorance is not the product of your ignorance about the hand's position and your ignorance about how exact your knowledge of the position is. Instead your knowledge must *relate* the hand's position to how inexact your knowledge of its position is. That is indeed a natural thought to have. If the hand looks to you like it is around 20 it is natural for you to think that if it is in fact at 30 your eyesight is not that great.

Call an *inexactness profile* a function that associates each position of the hand with an interval of positions. Those profiles can be understood modally, materially and epistemically. A *modal* inexactness profile f is a set of counterfactuals of the form 'if the hand was at x, the most one would know about its position is that it is within $[y, z]$.'[7] If it is true, it characterizes how your knowledge *would*

[6] One version is this: suppose we have a distance measure d that represents how much two worlds differ on the variable of interest, and that margins are expressed as distances: $d(w, v)$ is a real number satisfying reflexivity, $d(w, v) = 0$ iff the variable's value at w and v is the same, symmetry, $d(w, v) = d(v, w)$, and the triangle inequality, $d(w, v) \leq d(w, r) + d(r, v)$. Say that the variables values are m-spread around w iff: for every v with $d(w, v) \leq m$ there is u with $d(w, u) \leq m$ and $d(u, v) > m$. A fixed margin or product margin model that has m as one's actual margin prevents pointed knowledge of inexactness at w.

[7] Here and below we use the standard interval notation: $[18, 42]$ is the set of possible values of the parameter between 18 and 42 inclusive.

be if the hand *was* at various positions. A *material* inexactness profile f is a set of material conditionals of the form ‘if the hand is at x, the most one knows about its position is that it is within $f(x)$’. Its truth itself is not particularly interesting — it merely turns on whether the profile gives the right value for the actual position. What is interesting is whether it is *known*. If it is known, it characterizes how your knowledge relates where the hand to what you know about where it is by ruling out certain combinations of position and knowledge of the position. Thus when a material inexactness profile is known we also call it an *epistemic* profile.[8]

A true modal profile captures a metaphysical relation; a known material profile an epistemic one. The two often coincide. Often you know that if p then q in virtue of the fact that if p had obtained q would have. But they need not. If I tell you that only one of my feet is injured, you know that if the left one isn’t the right one is, thereby creating an epistemic dependence between the two facts; but it need not be the case that if I hadn’t injured the left one I would have injured the right one.

In Williamson’s clock models the worlds are meant to be both metaphysical and epistemic possibilities. They represent both what you *would* know if the clock *was* at various positions and what, at various positions, you know about what your knowledge of the position *is* if the clock *is* at various positions. Thus the inexactness profile in those models is both modal and epistemic.

What matters for pointed knowledge of inexactness are epistemic profiles. This is what we focus on in the models below. Thus our worlds are intended as epistemic possibilities. They may but need not match the metaphysical ones: as far as our discussion is concerned it does not matter whether or not the models also capture how your knowledge of the position modally relates to the position.

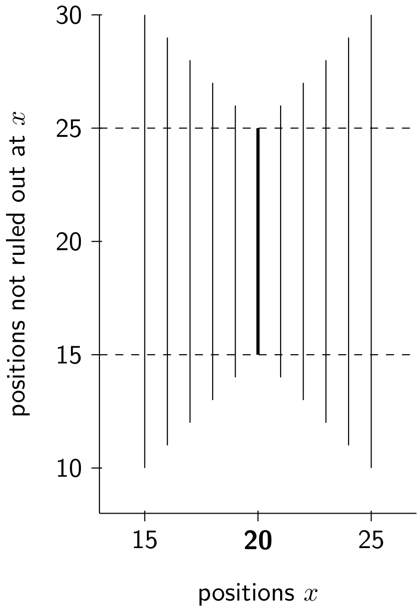
At a first pass we assume that you know that a certain (material) profile obtains. Thus we associate each world w with an inexactness profile f_w such that at w, you know that if the hand is at i the most you know about its position is that it is within $f_w(i)$. For instance, if $f_w(30) = [18, 42]$ then at w you know that if the hand is at 30 you rule out exactly the positions below 18 and above 42. We impose the factivity condition that i is in $f(i)$ for all i. We say that $K(w)$ is the set of worlds v whose parameter value i_v is within $f_w(i_w)$ and that share the same epistemic dependence function ($f_v = f_w$). Inexactness is imposed by requiring that $f_w(i)$ include at least one other position than i itself for every w.

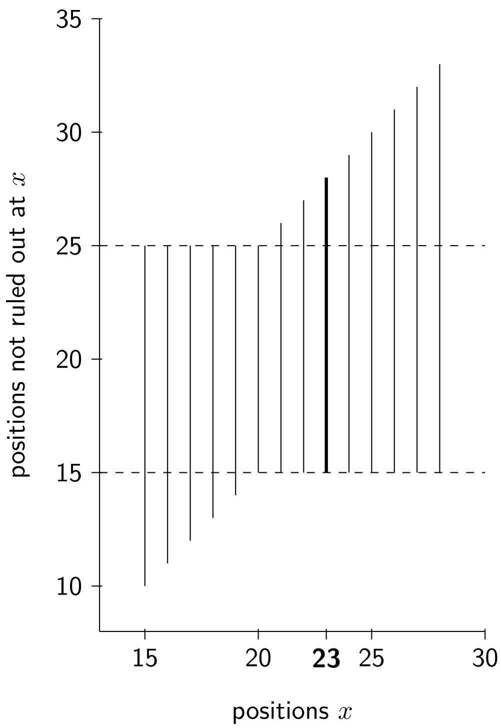
[8] Note that which material profiles are *known* to obtain is much more restricted than which material profiles obtain. If i is the actual position and $[a, b]$ is the most of what you know about it then any material profile f with $f(i) = [a, b]$ obtains. For its conditionals for positions other than i are all true simply because their antecedent is false. However, given a modest step of closure, you cannot know that two material profiles f and g obtain unless they agree not just on what they say about the actual world but also about any position you do not rule out. For if they both obtain but disagree about a position i, it follows that i does not obtain, since it is impossible that the most you know about where the hand is is both that it is within $f(i)$ and within $g(i)$. By closure, if you know that these profiles obtain but do not know that i does not, they agree on i.

In fixed margin models one's (known) inexactness profile has a constant size. Thus we would get fixed margin models if we imposed that every f_w is of the form $f_w(x) = [x - m, x + m]$. What pointed knowledge of inexactness requires instead is a profile where $f_w(i)$ *widens* as i gets further away from some positions in a way that ensures that these positions are always included within $f_w(i)$.

In fixed margin models one's (known) inexactness profile has a constant size. What pointed knowledge of inexactness requires is an inexactness profile where $f_w(i)$ *widens* as i gets further away from some positions in a way that ensures that these positions are always included within $f_w(i)$. For instance, letting the actual value $i_w = 20$, you might know at w that if the clock's hand is at 20 the most you know is that its position is within $[15, 25]$ and that if the clock's hand is at 22 the most you know is that its position is within a wider interval that properly includes $[15, 25]$ and that if it is at 25 the most you know is that its position is within an even wider interval that still includes $[15, 25]$. We will give two models where that occurs; call them *widening margin models*. We think both have merit. In the first, the inexactness profile known at w is such that as the hand gets further away from a certain optimal position o_w at which you know that the position is within some margin $c_w > 0$ of o_w, your ignorance of its position grows *in both directions*: you know that if it is at 25 the most you know is that the hand's position is within $[10, 30]$, and more generally $f_w(x) = [o_w - c_w - |o_w - x|, o_w + c_w + |o_w - x|]$. In the second, the inexactness profile known at w is such that as the hand gets further away from o_w, your ignorance of its position grows *in the same direction*: you know that if the hand is at 25 the most you know is that its position is within $[15, 35]$, and more generally $f_w(x) = [o_w - c_w - |o_w - x|, o_w + c_w]$ if $x \leq o_w$ and $f_w(x) = [o_w - c_w, o_w + c_w + |o_w - x|]$ otherwise.[9] The two models are illustrated below.

[9] In both cases the full model consists of one world w for each triple i, o, c with $c > 0$.





Figures 2 and 3. Intervals of inexact knowledge as a function of the value of parameter in a bidirectional and in a unidirectional widening margin model, *respectively.*

Figures 2 and 3 display the possibilities compatible with what you know at a world w with $o_w = 20$ and $c_w = 5$ on the two kinds of models. In the first model the actual position is the one that is optimal in the light of what you know at w, $i_w = o_w = 20$. In the second the actual value $i_w = 23$ is not the one that is optimal in the light of what you know at w ($o_w = 20$). As before, on the horizontal axis are positions of the hand and on the vertical one intervals that you know that you do not rule out if the hand is at a given position. The actual position and the interval of positions you actually do not rule out are in bold. In the second model, for instance, the hand is at 23 and the interval of positions you actually fail to rule out is $[15, 28]$, but you know that if the hand is at 20 the most you know about its position is that it is within $[15, 25]$.

It is easy to check that the models satisfy (1). Since $\langle i, g\rangle$ is in $K(w)$ iff $i \in f_w(i_w)$ and $g = f_w$, the interval $f_w(i_w)$ is indeed the range of positions that you do not rule out at w. Moreover, since $c > 0$ the range $f_w(i_w)$ contains more than one position for each w, so the exact hand position is not known.

When $o_w = i_w$ the world w has the particular feature that it is optimal by its own lights: in the light of your knowledge, your knowledge of the hand's position is most exact if the clock's hand is at a certain position i, and that position happens to be the actual position i_w. That should not always be the case. Suppose your view of the clock is skewed so that the hand looks closer to 20 than it actually is. It is natural to think that in the light of your knowledge, your knowledge of the hand's position is most exact if the hand is at 20. In the models this is borne out at worlds where o_w differs from i_w: in our second model, for instance, for all you know at w the hand is at $o_w = 20$, and you know that if this is so your knowledge of the hand places its position within an interval of 10. Unfortunately the hand is in fact at 23 and you merely know its position up to a 13-point interval.

The crucial feature of the models is the *common core* of their profiles: for each f_w, there is a core interval of positions $[x, y]$ (including more than one position) such that $[x, y]$ is included in $f_w(i)$ for every i. This ensures that in the light of your knowledge at w there is no epistemic possibility in which you rule out the positions between x and y. Since $f_w(i)$ must include i for every i, this requires the intervals $f_w(i)$ to widen as i gets further away from the $[x, y]$ interval. While this property is enough for pointed knowledge of inexactness, our models have the stronger property of *relative optimality*: the common core coincides with what is known at a given world.[10]

[10] The common core property is that for every w there is an extended interval $[a, b]$ such that $[a, b] \subseteq f_w(i)$ for all i. Relative optimality is the property that for every w there is an o_w such that $f_w(o_w) \subseteq f_w(i)$ for all i. Provided $f_w(o_w)$ is extended the latter entails the former but not the other way round.

IV

In our widening margin models, knowledge and ignorance are not luminous: the models validate neither the positive introspection schema 4 ($K\phi \rightarrow KK\phi$) nor the negative introspection schema 5 ($\neg K\phi \rightarrow \neg KK\phi$) of modal logic. At an optimal world, for all you know you might be at a suboptimal world at which you know less about the position of the hand; so there are things you know but do not know that you know. At a suboptimal world for all you know you could be at an optimal one at which you know more about the position of the hand, so there are things you fail to know without knowing that you do not know them.

It is worth noting, however, that the models presented so far uphold a restricted version of these schemas where ϕ is itself a claim about what one fails to know:

- ($4_{\neg K\psi}$) $K\neg K\psi \rightarrow KK\neg K\psi$
- ($5_{\neg K\psi}$) $\neg K\neg K\psi \rightarrow K\neg K\neg K\psi$

These follow from relative optimality and exact knowledge of one's inexactness profile. By the latter, all the worlds that are accessible from w, and the worlds accessible from those worlds, and so on, share the same profile f_w. By the former, each profile f_w comes with an optimal value o_w such that one's knowledge at the world $\langle o_w, f_w\rangle$ is at least as strong as one's knowledge at any world with that profile. Hence (a) if anything is known at a world $\langle i, f_w\rangle$ it is known at $\langle o_w, f_w\rangle$. Now for ($4_{\neg K\psi}$) suppose that $K\neg K\psi$ holds at some world w. Then $\neg K\psi$ holds at any world accessible from w, hence in particular at the f_w-optimal world $\langle o_w, f_w\rangle$. By (a) it holds at any worlds $\langle i, f_w\rangle$, hence at all worlds accessible from w and at worlds accessible from those. So $KK\neg K\psi$ holds at w. For ($5_{\neg K\psi}$) suppose that $\neg K\neg K\psi$ holds at some world w. Then $K\psi$ holds at some world $\langle i, f_w\rangle$ accessible from w. By (a) it follows that $K\psi$ holds at the f_w-optimal world $\langle o_w, f_w\rangle$. But then any world $\langle j, f_w\rangle$ accessible from w has access to a world where $K\psi$ holds, namely $\langle o_w, f_w\rangle$. So $\neg K\neg K\psi$ holds at all worlds accessible from w, hence $K\neg K\neg K\psi$ holds at w.

There are of course well-known objections to introspection principles that would apply to ($4_{\neg K\psi}$) and ($5_{\neg K\psi}$) as well. Some are orthogonal to issues of inexactness. For instance, it is arguably possible for someone to fail to know that they do not know ϕ without believing that they fail to know that they do not know ϕ (perhaps because they have not considered the matter). If knowledge requires belief, this provides a counterexample to ($5_{\neg K\psi}$). These kinds of objections can be addressed by focusing on suitably idealized subjects or suitably idealized knowledge-like notions (like being in a position to know). They may also be simply set aside as they concern other limits on knowledge than the ones we are interested in here, namely those arising from inexactness. Other objections are more central to the matter at hand. For instance, one may use margin for error considerations to argue that inexact sources can generate a continuous transition from cases in which one knows that one does not know ϕ to cases in which one does not know that one does not know ϕ so that the first cases of the latter type are ones in which one is not in a position to know that one does not know that one does not know ϕ, thereby contradicting ($5_{\neg K\psi}$) (Williamson 2000). Those who are convinced by such objections should turn to the models we introduce in the next section, which do not validate ($4_{\neg K\psi}$) and ($5_{\neg K\psi}$). However, see (Rosenkranz, n.d.) for a detailed discussion and an argument that, when K is interpreted as being in a position to know, ($4_{\neg K\psi}$) and ($5_{\neg K\psi}$) hold, and for an argument that the standard considerations against introspection principles 4 and 5 do not extend to these instance (see also Rosenkranz 2018).[11]

Some of our models are isomorphic to Williamson's appearance-reality models (Williamson 2013). In those models a world is given by a pair of values $\langle r, a\rangle$ where r is the value of the parameter of interest (*e.g.* the position of the clock's hand) and a captures how the parameter appears: for instance, the world $\langle 30, 25\rangle$ represents a situation in which the hand is at 30 but appears to you to be at 25. One's knowledge of its position is as good as it can be, but still inexact, if appearances match reality—at $\langle x, x\rangle$ the most one knows about the position of the hand is $[x - c, x + c]$, for some fixed value c. One's ignorance grows in both directions around the apparent value as the gap between reality and appearance grows: at $\langle r, a\rangle$ the most one knows about the position of the hand is $[a - c - |a - r|, a + c + |a - r|]$. Finally, the appearances are themselves exactly known. Thus $\langle r', a'\rangle$ is accessible from $\langle r', a'\rangle$ iff $a' = a$ and r' is within $[a - c - |a - r|, a + c + |a - r|]$. That is equivalent to the sub-model generated by a world w with $o_w = a$ and $c_w = c$ in our first kind of widening margin models. The isomorphism ensures that Williamson's appearance-reality models also satisfy (2) and avoid (3). Exact knowledge of appearances in these models corresponds to exact knowledge of one's inexactness profile in ours. Thus the models also satisfy the restricted luminosity principles ($4_{\neg K\psi}$) and ($5_{\neg K\psi}$).

It is good news that we arrive at the structure of appearance-reality models via a wholly different route than Williamson's. Our models are motivated by the phenomenon of pointed knowledge of inexactness, not by considerations on how knowledge and justified belief depend on appearances. It is also good that those models can be motivated without introducing appearances. There is inexact knowledge that does not turn on appearances, at least not in the straightforward perceptual sense. For instance, when looking at a shopping cart full of goods one can gain some knowledge about how much the sum of goods costs. One's knowledge of the price is inexact and subject to margins but those have little to do with one's perceptual acuity. Some philosophers would say that these cases involve non-perceptual monetary seemings (Huemer (2001)), but those who do not cannot appeal to appearance-reality models here.[12] What our models

[11] Thanks to an anonymous referee here for alerting us to the need to adding these remarks.

[12] One may object that our models do covertly appeal to appearances since our parameter o_w (the optimal value from the point of view of w) plays a role akin to that of the apparent position at w in appearance-reality models. However, note that we introduced that parameter in terms of knowledge rather than appearances: it is the position at which, in the light of your knowledge, your knowledge of the hand's position would be best. Most importantly, the prime mover in our models is the inexactness profile f_w which captures a known relation between

emphasize is that pointed knowledge of inexactness requires an epistemic relation relating the value of the variable of interest to what you know about it, that is, knowledge of a widening inexactness profile. Whether that epistemic relation can be explained in terms of appearances, safety, reliability and so on is a further question.

An intriguing way to unify our models with the appearance-reality models would be to define appearances themselves in terms of knowledge. Roughly, the idea would be that it appears to one that p iff p would hold in the situations that, in the light of what one knows, maximize one's knowledge. Arguably, such appearances will not always involve phenomenology, which will make some reluctant to call them "appearances" at all. But perhaps this notion of appearance is after all the epistemically most relevant one. They might underwrite a principle according to which one is justified in believing what appears to one to be so, for instance. While one of us finds the perspective attractive, we will not discuss it further here.[13]

Our models generalize the appearance-reality structure in several ways. First, some of our models allow one-sided epistemic dependence: in the shopping cart scenario, for instance, it is natural to think that in the light of your knowledge a few hidden expensive items would impair your knowledge of its maximum price without impairing your knowledge of its minimum price. Second, we have laid out a general feature of known inexactness profiles f_w that ensures pointed knowledge of inexactness, namely, that for some common core interval $[x, y]$ containing at least two values, $[x, y] \subseteq f_w(i)$ for all i. That can be implemented in models that are substantially different from Williamson's (Goodman 2013; Cohen and Comesaña 2013). In the final section, we further generalize our models to drop the assumption of exact knowledge of one's inexactness profile.

Our models are also isomorphic to some normality-based models of knowledge according to which one knows p iff p holds in situations not much more abnormal than one's own situation.[14] For as Goldstein and Carter (2020) point out, reality-appearance models are themselves isomorphic to normality-based models where abnormality is a function of the distance between appearance and reality. This is again an interesting finding, as our models are motivated independently from considerations of normality. In particular, we make no assumption that our optimal worlds are more normal than suboptimal ones.[15]

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the position of the hand and your knowledge of it. We constructed some such profiles from an optimal value o_w and a minimal margin c_w but these are dispensable artefacts.

[13]Thanks to an anonymous referee for the suggestion. See Dutant (n.d.) for an account of justified belief along those lines where the appearances in question are called the "supported".

[14]See Goodman (2013), Greco (2014), Goodman and Salow (2018), Carter (2019), Beddor and Pavese (n.d.), Goldstein and Carter (2020).

[15]See Carter (2019), §6 for a discussion of whether epistemic optimality is normal. Goldstein and Carter (2020) raise a Preface-style issue for appearance-reality models that arises only when we consider inexact knowledge gathered from distinct sources, as when one tries to evaluate different parameters using distinct measuring instruments. They show that a property analogous to our relative optimality (which they call "dominance") leads to problematic results in such cases. They propose a generalization of the models that avoid the issue. A similar

V

Exact knowledge of an inexactness profile is a strong assumption. An ordinary subject might not be able to know that, say, if the hand is at 20 the most they know about its position is exactly that it is within the interval $[20, 30]$. Rather, they would know that if the hand is at 20 then the most they know about its position is that it is within some interval around $[20, 30]$, perhaps $[19, 31]$, $[19, 29]$, $[21, 30]$, and the like. Can we relax the assumption?

A straightforward way to do so is to introduce a closeness relation among inexactness profiles and say that one knows that one's inexactness profile is within a certain range of such profiles. For instance, we may say that two inexactness profiles f and g are close, $f{\sim}g$, iff their intervals always have maxima and minima that are within a fixed distance k of each other: for all i, $|\max(f(i)) - \max(g(i))| \leq k$ and $|\min(f(i)) - \min(g(i))| \leq k$. We then revise our models so that $K(w)$ is the set of worlds v whose parameter value i_v is within $f_w(i_w)$ and whose inexactness profile is close to that of w: $f_v{\sim}f_w$. In those models you do not know that a precise material profile f_w obtains but merely that some profile similar enough to f_w does. The models still provide for pointed knowledge of inexactness. When $f_w(o_w)$ is large enough (namely, more than $2k$ wide so that some positions within it are at least k units away from both its maximum and minimum), a common core of positions within it will be included not just in $f_w(i)$ for all i but also in any $g(i)$ with $g{\sim}f_w$. So at w one will know that these specific positions are not ruled out. However, in those models one's knowledge of one's ignorance is not luminous. For instance, consider a bidirectional widening margin model with $w = \langle 30, f_w \rangle$, with f_w given by $o_w = 30$ and $c_w = 5$, and let k be 1. Given that $k = 1$, at w you do not rule out being at a world v where your eyesight is better ($o_v = o_w$, $c_v = 4$), and at v you would not rule out being at a world u where your eyesight is even better ($o_u = o_w$, $c_u = 3$). Now at w you know that you do not rule out the position 26. But at v you do not know that, for at u you would rule out 26. So at w we have $K\neg K\neg p_{26}$ but not $KK\neg K\neg p_{26}$.[16] A parallel reasoning invalidates $(5_{\neg K\psi})$, because at each world for all you know your eyesight is a bit poorer than it actually is.

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problem (and solution) would arise in our models when using several inexactness profiles at each world (corresponding to distinct sources of inexact knowledge). Our discussion focuses instead on issues that arise with higher-order knowledge; the generalizations we propose below are orthogonal to (and as far as we can tell compatible with) theirs.

[16] At w the most you know about the hand's position is that it is within $[25, 35]$. We have $f_w(25) = [15, 35]$ and $f_w(35) = [25, 45]$, thus given $k = 1$ we have $[16, 34] \subseteq g(25) \subseteq [14, 26]$ and $[26, 44] \subseteq g(35) \subseteq [24, 36]$: you know that if the hand is at 25 what you know about its position is at best that it is within $[16, 34]$ and at worst that it is within $[14, 36]$, and you know that if the hand is at 35 the most you know about its position is at best that it is within $[26, 44]$ and at worst that it is within $[24, 46]$. So at w you know that you do not rule out 26. But at u you know that the hand's position is within $[27, 33]$, so there you rule out 26. Since u is accessible from v, at v you do not know that you do not rule out 26. Hence at w you do not know that you know that you do not rule out 26.

Note, however, that these models assume that your inexact knowledge of your (first-order) inexactness profile is given by a *fixed* second-order margin k. So these models prevent pointed knowledge of inexactness at the second order: it will be impossible for there to be two specific and distinct ranges $[a, b]$ and $[c, d]$ such that you know that you cannot rule them out as being the strongest thing you know about the hand's position. It should now be clear, however, that this can be avoided by repeating our solution at a higher-order: one should replace the constant k with a (second-order) epistemic dependence function whose range may widen as it gets away from a central value.

For fully general models, we let a world w be a sequence of parameters $\langle i_w, f_w, f_w^2, \ldots, \rangle$ where $i_w = f_w^0$ is the value of the parameter of interest, *e.g.* the position of the clock's hand, and each f_w^k a function from possible values of the parameters from order 0 to $k-1$ ($x^0, x^1, \ldots, x^{k-1}$) to a set of possible values of the $k-1$ order parameter (a set of x^{k-1}). Thus $f_w = f_w^1$ is an inexactness profile of the kind we encountered, capturing a set of conditionals along the lines of 'if the hand is at x the most I know about its position is that it is within $[a, b]$', while f_w^2 is a second-order inexactness profile capturing a set of conditionals along the lines of 'if the hand is at x and my knowledge of the hand's position has inexactness profile f_w, the most I know about that inexactness profile is that it is within $f_w^2(i, f_w)$', and so on. We impose the reflexivity constraint that f_w^{k-1} is in $f_w^k(i_w, f_w, f_w^2, \ldots, f_w^{k-1})$ for all w. We say that v is in $K(w)$ iff each of v's parameters is within w's interval for that parameter: f_v^k is in $f_w^{k+1}(i_w, \ldots, f_w^{k-1})$ for all k. The widening margin models with a fixed second-order margin on one's first-order inexactness profile are a special case of these models where the parameters k^n are constant for $n > 1$. It is thus easy to see that these models will afford inexact knowledge, pointed knowledge of inexactness and anti-luminosity at the first-order like the previous ones, but more generally at all higher levels as well.

VI

We have highligthed the phenomenon of pointed knowledge of inexactness. Standard fixed margin models of inexact knowledge, as well as product margin models, fail to capture it. The source of the phenomenon is an epistemic relation between a value of a parameter and how inexact one's knowledge of it is. The relation can be captured through (ranges of) inexactness profiles and motivates widening margin models of inexact knowledge. These turn out to be structurally analogous to Williamson's appearance-reality models but are differently motivated. We have further generalized those models to show how pointed knowledge of inexactness is compatible with various degrees of anti-luminosity.[17]

[17]We are grateful to Sam Carter, Simon Goldstein and Clayton Littlejohn for detailed discussions of the material in this paper and to an anonymous referee for their helpful

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comments. We are also grateful to David Christensen and the participants in the 2nd Workshop on Fallibility, Knowledge and Rational Belief that took place in Barcelona on 26-27 May 2016. Work on this paper has received support from the research project *Justification, its Structure and Grounds* (PGC2018-099889-B-I00), financed by the Spanish Ministry of Science, Innovation and Universities (MICINN).

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