Intermediate logics and Visser's rules

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Abstract

Visser's rules form a basis for the admissible rules of IPC. Here we show that this result can be generalized to arbitrary intermediate logics: Visser's rules form a basis for the admissible rules of any intermediate logic L for which they are admissible. This implies that if Visser's rules are derivable for L then L has no non-derivable admissible rules. We also provide a necessary and sufficient condition for the admissibility of Visser's rules. We apply these results to some specific intermediate logics, and obtain that Visser's rules form a basis for the admissible rules of e.g. De Morgan logic, and that Dummett's logic and the propositional Gödel logics do not have non-derivable admissible rules.

Keywords: intermediate logics, intuitionistic logic, admissible rules, projective formulas

1 Introduction

It is a simple but interesting fact that all admissible rules of classical propositional logic CPC are derivable. Thus, knowing the theorems of CPC is knowing its rules. For intermediate logics this is no longer true: there are intermediate logics that have nonderivable admissible rules, i.e. admissible rules that are not derivable. Intuitionistic propositional logic IPC is the most famous example of such a logic, but there are many more. In [9] it was shown that the countably many Gabbay-de Jongh logics [5] have this property too.

A lot is known about the admissible rules of IPC. Rybakov [14] showed that admissible derivability for IPC, $\mathrel{|\!\sim}$, is decidable and Ghilardi [7] presented a transparent algorithm. In [10] a simple syntactical characterization for $\mathrel{|\!\sim}$ was given. This result implied that Visser's rules $V = \{V_n \mid \ldots n = 1, 2, 3, \ldots\}$,

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where

$$V_n \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \vee C \ / \ \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_j) \vee C,$$

form a basis for the admissible rules of IPC. Intuitively, this means that all admissible rules of IPC can be obtained from Visser's rules via derivability in IPC.

In this paper we show that this result is in fact a particular case of a more general theorem, by showing (Theorem 8) that if Visser's rule are admissible for an intermediate logic L, they are a basis for the admissible rules of L. In particular, it follows that if Visser's rule are derivable, the logic has no non-derivable admissible rules. As we will see, the latter applies to many well-known intermediate logics, like Gödel-Dummett logic LC and the Gödel logics $\mathsf{G_k}$. (This last fact was independently observed, using different methods, by Matthias Baaz.)

As for the admissibility of Visser's rules, it might not always be easy to see whether this hold or not for a given logic. However, in many cases we can make use of the necessary and sufficient condition for the admissibility of Visser's rules developed in Section 4. Namely, there we show that Visser's rule are admissible for a logic L if and only if L is sound and complete with respect to class of models that has the so-called offspring property. This characterization enables us to apply Theorem 8 to various intermediate logics and conclude that Visser's rule form a basis for the admissible rules of e.g. De Morgan logic KC.

Summarizing, we could say that if Visser's rules are admissible for L, we have a complete description of $\mid\!\sim_{\mathsf{L}}$ once we have one of $\vdash_{\mathsf{L}}$, because in these cases Visser's rules form a basis for the admissible rules. As we will see, it is even so that in these cases there exist formulas Λ_A, so-called maximal admissible consequences, such that $A \mid\!\sim_{\mathsf{L}} B \Leftrightarrow \Lambda_A \vdash_{\mathsf{L}} B$. Therefore, having Λ_A, one obtains a description of $\mid\!\sim_{\mathsf{L}}$ in terms of $\vdash_{\mathsf{L}}$. In [7] an algorithm to compute the Λ_A was presented, and based on this we have developed a proof system to derive Λ_A [11]. All this provides not only a complete description of the admissible rules of L, but also one that is computable once $\vdash_{\mathsf{L}}$ is.

What if not all of Visser's rules are admissible? We know that such logics exist: the Gabbay-de Jongh logics are an example [9]. We do not know of many general results about the admissibility relation of such logics.

In short, the general connection between Visser's rules and admissibility obtained here is as follows.

- Visser's rules are admissible $\Rightarrow$ Visser's rules form a basis (Section 3.2).
- Visser's rules are derivable $\Rightarrow$ no non-derivable admissible rules (Section 3.2).
- Disjunction property $\Rightarrow$ not all of Visser's rules are admissible, unless the logic is IPC (Section 4).

1.1 Remark

Note than when Visser's rules are admissible, then so are the rules

$$V_{nm} \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to \bigvee_{j=n+1}^{m} A_j) \vee C / \bigvee_{h=1}^{m}(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_h) \vee C.$$

As an example we will show that V_{13} is admissible for any logic for which V_1 is admissible. For simplicity of notation we take C empty. Assume that $\vdash_{\mathsf{L}} (A_1 \to B) \to A_2 \vee A_3 \vee A_4$. Then by V_1, reading $A_2 \vee A_3 \vee A_4$ as $A_2 \vee (A_3 \vee A_4)$,

$$\vdash_{\mathsf{L}} \big((A_1 \to B) \to A_1\big) \vee \big((A_1 \to B) \to A_2\big) \vee \big((A_1 \to B) \to A_3 \vee A_4\big).$$

A second application of V_1, with $C = \big((A_1 \to B) \to A_1\big) \vee \big((A_1 \to B) \to A_2\big)$, gives

$$\vdash_{\mathsf{L}} \bigvee_{i=1}^{2} \big((A_1 \to B) \to A_i\big) \vee \bigvee_{i=1,3,4} \big((A_1 \to B) \to A_i\big).$$

Therefore, $\vdash_{\mathsf{L}} \bigvee_{i=1}^{4} \big((A_1 \to B) \to A_i\big)$.

The paper is build up as follows. Section 2 contains the preliminaries. It is somewhat long, as the necessary and sufficient condition for the admissibility of V needs some explanation. Section 3 is devoted to the proof that if Visser's rules are admissible they form a basis. The proof itself is not complicated, but it uses a lot of machinery, which is discussed in Subsection 3.1. In Subsection 3.2 the result is derived. Section 4 presents the neccessary and sufficient condition for the admissibility of Visser's rules. In Section 5 the results are applied to specific intermediate logics.

2 Preliminaries

In this paper we will only be concerned with intermediate logics L, i.e. logics between (possibly equal to) IPC and CPC. We write $\vdash_{\mathsf{L}}$ for derivability in L. The letters A, B, C, D, E, F, H range over formulas, the letters p, q, r, s, t, range over propositional variables. We assume $\top$ and $\bot$ to be present in the language. $\neg A$ is defined as $(A \to \bot)$. We omit parentheses when possible; $\wedge$ binds stronger than $\vee$, which in turn binds stronger than $\to$.

2.1 Admissible rules

A *substitution* σ will in this paper always be a map from propositional formulas to propositional formulas that commutes with the connectives. A *(propositional) admissible rule* of a logic L is a rule A/B under which the logic is closed, i.e.

$$\forall \sigma : \vdash_{\mathsf{L}} \sigma A \text{ implies } \vdash_{\mathsf{L}} \sigma B.$$

We write $A \mathrel{|\!\sim}_{\mathsf{L}} B$ if A/B is an admissible rule of L. The rule is called *derivable* if $\vdash_{\mathsf{L}} A \to B$ and *non-derivable* if $\nvdash_{\mathsf{L}} A \to B$. When R is the rule A/B, we write $R^{\to}$ for the implication $A \to B$. We say that a collection R of rules, e.g. V, is admissible (derivable) for L if all rules in R are admissible (derivable) for L. We write $A \vdash_{\mathsf{L}}^{R} B$ if B is derivable from A in the logic consisting of L extended with the rules R, i.e. if there are $A = A_1, \ldots, A_n = B$ such that for all $1 \leq i < n$, $A_i \vdash_{\mathsf{L}} A_{i+1}$ or there exists a σ such that $\sigma B_i/\sigma B_{i+1} = A_i/A_{i+1}$ and $B_i/B_{i+1} \in R$. If X and R are sets of admissible rules of L, then R is a *basis for* X if for every rule A/B in X we have $A \vdash_{\mathsf{L}}^{R} B$. If X consists of all the admissible rules of L, then R is called a *basis for the admissible rules of* L. Thus R is a basis for the admissible rules of L if and only if $\mathrel{|\!\sim}_{\mathsf{L}} = \vdash_{\mathsf{L}}^{R}$, i.e.

$$A \mathrel{|\!\sim}_{\mathsf{L}} B \Leftrightarrow A \vdash_{\mathsf{L}}^{R} B.$$

Fact 1 If R is a basis for the admissible rules of L and all rules in R are derivable, then L has no non-derivable admissible rules.

2.2 The disjunction property

A logic L has the *disjunction property* if

$$\vdash_{\mathsf{L}} A \vee B \ \Rightarrow \vdash_{\mathsf{L}} A \text{ or } \vdash_{\mathsf{L}} B.$$

If L has the disjunction property, then $A \mathrel{|\!\sim}_{\mathsf{L}} C$ and $B \mathrel{|\!\sim}_{\mathsf{L}} C$ implies $A \vee B \mathrel{|\!\sim}_{\mathsf{L}} C$. Thus in the context of Visser's rules this implies that when the the following special instances of Visser's rules, the *restricted Visser rules*,

$$V_n^- \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \ / \ \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_j),$$

are admissible for L, then so are Visser's rules. Therefore, when considering only logics with the disjunction property, like e.g. IPC, the difference between the Visser rules and the restricted Visser rules does not play a role. However, when considering intermediate logics in all generality, as we do in this paper, we cannot restrict ourselves to this sub-collection of Visser's rules.

2.3 Kripke models

A Kripke model K is a triple $(W, \preccurlyeq, \Vdash)$, where W is a set (the set of *nodes*) with a unique least element that is called the *root*, $\preccurlyeq$ is a partial order on W and $\Vdash$, the *forcing relation*, a binary relation on W and sets of propositional variables. The pair $(W, \preccurlyeq)$ is called the *frame* of K. The notion of forcing in a Kripke model is defined as usual. We write $K \models A$ if A is forced in all nodes of K and say that A holds in K. We write K_k for the model with domain $\{k' \mid k \preccurlyeq k'\}$ which partial order and valuation are the restrictions of the corresponding relations of K to this domain.

2.4 Bounded morphisms

A map $f : (W, \preccurlyeq, \Vdash) \to (W', \preccurlyeq', \Vdash')$ is a *bounded morphism* when the following conditions hold

1. k and $f(k)$ force the same atoms,
2. $k \preccurlyeq l$ implies $f(k) \preccurlyeq' f(l)$,
3. if $f(k) \preccurlyeq l$, then there is a $k' \succcurlyeq k$ in W such that $f(k') = l$.

K' is a *bounded morphic image* of K, $K \twoheadrightarrow K'$, whenever there is a surjective bounded morphism from K to K'. It is well-known (see e.g. [2]) that when f is a bounded morphism from K to K', then for all k in K, for all formulas A: $k \Vdash A \Leftrightarrow f(k) \Vdash' A$. Thus if K' is a bounded morphic image of K, it validates exactly the same formulas as K.

2.5 Extension properties

For Kripke models $K_1, \ldots, K_n$, $(\sum_i K_i)'$ denotes the Kripke model which is the result of attaching one new node at which no propositional variables are forced, below all nodes in $K_1, \ldots, K_n$. $(\sum \cdot)'$ is called the *Smorynski operator*. Two models K, K' are *variants* of each other, written KvK', when they have the same set of nodes and partial order, and their forcing relations agree on all nodes except possibly the root. A class of models U has the *extension property* if for every finite family of models $K_1, \ldots, K_n \in U$, there is a variant of $(\sum_i K_i)'$ which belongs to U. U has the *weak extension property* if for every model $K \in U$, and every finite collection of nodes $k_1, \ldots, k_n \in K$ distinct from the root, there exists a model $M \in U$ such that

$$\exists M_1 \big((\sum_i K_{k_i})' v M_1 \ \wedge \ (M_1 \twoheadrightarrow M) \big).$$

U has the *offspring* property if for every model $K \in U$, and for every finite collection of nodes $k_1, \ldots, k_n \in K$ distinct from the root, there exists a model $M \in U$ such that

$$\exists M_1 \exists M_0 \big((\sum_i K_{k_i})' v M_1 \ \wedge \ (M_1 + K)' v M_0 \ \wedge \ (M_0 \twoheadrightarrow M) \big).$$

A logic L has the extension (weak extension, offspring) property if it is sound and complete with respect to some class of models that has the extension (weak extension, offspring) property. Note that for all three properties the class of models involved does not have to be the class of *all* models of L. However, we might as well require that, because we will see in Section 4 that if a logic has the offspring property, then so does the class of all its models. Since the class of all models of a logic is closed under submodels and bounded morphic images, this also implies that for logics

extension property $\Rightarrow$ offspring property $\Rightarrow$ weak extension property.

The reason that we have chosen the definition of offspring property as given above, not the most elegant one, is that it will turn out particularly useful for the application to various frame complete logics discussed in the last section. There are quite natural classes of models that satisfy the offspring property, e.g. the class of linear models, as the reader may wish to verify for himself.

If we would not restrict our models to rooted ones, the extension property and the weak extension property would be equivalent, at least for logics. Since we require our Kripke models to be rooted, there is a subtle difference between the two:

Fact 4 If a logic L has the extension property, it has the disjunction property.

As there are logics that do not have the disjunction property, but that have the weak extension property, the latter is indeed stronger. We will see examples of such logics in Section 5.

2.6 Projective formulas

We define $n(A)$ to be the maximal nesting of implications in A. Recall that a substitution σ is a *unifier* of A in IPC if $\vdash_{\mathsf{IPC}} \sigma A$.

In [6], Ghilardi introduced the notion of a projective formula: a formula is called *projective* if there exists a substitution σ such that

$$\vdash_{\mathsf{IPC}} \sigma A, \text{ and for all atoms } p \ (A \vdash_{\mathsf{IPC}} \sigma(p) \leftrightarrow p).$$

We call such a σ a *projective unifier* for A. A projective approximation Π_A of $A(\bar{p})$ is a set of formulas such that for all $B \in \Pi_A$,

1. all atoms in B are among the atoms $\bar{p}$ of A, $n(B) \leq n(A)$, B is projective and $B \vdash_{\mathsf{IPC}} A$, and
2. for all formulas C satisfying 1., there is a $B \in \Pi_A$ such that $C \vdash_{\mathsf{IPC}} B$.

Observe that if σ is a projective unifier for A, then $A \vdash_{\mathsf{IPC}} \sigma B \leftrightarrow B$, for all formulas B. This implies that for any projective formula A, for all formulas B we have that

$$A \mathrel{|\!\sim}_{\mathsf{L}} B \Leftrightarrow A \vdash_{\mathsf{L}} B. \tag{1}$$

For if $A \mathrel{|\!\sim}_{\mathsf{L}} B$, then $\vdash_{\mathsf{L}} \sigma B$ for any projective unifier σ of A. Whence $A \vdash_{\mathsf{L}} B$, as $A \vdash_{\mathsf{IPC}} \sigma B \leftrightarrow B$. Note that (1) implies $\bigvee \Pi_A \mathrel{|\!\sim}_{\mathsf{L}} B \Leftrightarrow \bigvee \Pi_A \vdash_{\mathsf{L}} B$.

Example 3 Examples of projective formulas are p, $\neg p$, and $A \to p$. Their projective unifiers are resp. $\sigma(p) = \top$, $\sigma(p) = \bot$ and $\sigma(p) = (A \to p) \to p$, where σ is the identity on all atoms distinct from p. For the first two, this is easy to see. To see that the last substitution is a unifier for $A \to p$, note that

$$\sigma(A \to p) = \sigma(A) \to ((A \to p) \to p) \leftrightarrow (\sigma(A) \wedge (A \to p) \to p).$$

Observe that indeed $(A \to p) \vdash \sigma(B) \leftrightarrow B$, as is required of a projective unifier. Hence $(\sigma(A) \wedge (A \to p) \to p)$ is equivalent to $((A \to p) \wedge A \to p)$, which is a tautology of IPC.

In [6], Ghilardi showed that projective formulas are exactly the formulas which class of models has the extension property. This implies that e.g. $p \vee q$ is not a projective formula. Nor are the formulas $\bigwedge_{i=1}^{n}(p_i \to q_i) \to p_{n+1} \vee p_{n+2}$ that occur in Visser's rules projective.

3 Visser's rules as a basis

We will show that once Visser's rules are admissible for a logic they form a basis, Theorem 8. The first subsection recalls the theorems that lead to the mentioned result. First, we discuss results on projective formulas and admissible rules of IPC, and how they may be connected to the admissible rules of other intermediate logics.

3.1 Maximal admissible consequences

The important point in the proof of Theorem 8 is that for various logics there exist formulas λ_A, called maximal admissible consequences, such that $A \mathrel{|\!\sim} B \Leftrightarrow \lambda_A \vdash B$. In this section we explain the connection between such formulas and bases of admissible rules.

Definition 1 For a formula A, let $AC_A^{\mathsf{L}} = \{B \mid A \mathrel{|\!\sim}_{\mathsf{L}} B\}$ be the set of *admissible consequences of* A *in* L. A formula λ_A^{L} is called a *maximal admissible consequence (mac)* of A in L if

$$\forall B\,(A \mathrel{|\!\sim}_{\mathsf{L}} B \Leftrightarrow \lambda_A^{\mathsf{L}} \vdash_{\mathsf{L}} B).$$

We omit the superscript when L is clear from the context. In the case of IPC, we write Λ_A for λ_A^{IPC}. A formula A is called *stable for admissibility in* L, or *stable* for short, if it is a maximal admissible consequence of itself, i.e. if

$$\forall B\,(A \mathrel{|\!\sim}_{\mathsf{L}} B \Leftrightarrow A \vdash_{\mathsf{L}} B).$$

The name maximal admissible consequence stems from the fact that such λ_A is maximal in AC_A^{L}, or equivalently that it axiomatizes AC_A^{L}, i.e.

$$AC_A^{\mathsf{L}} = \{B \mid A \mathrel{|\!\sim}_{\mathsf{L}} B\} = \{B \mid \lambda_A \vdash_{\mathsf{L}} B\}.$$

Note that the mac's of a formula A in L (if any) are unique up to provable equivalence in L. Therefore, when A has a mac in L we speak of *the* mac of A in L and denote it by λ_A^{L}. The following fact provides a straightforward equivalent for the existence of mac's.

Fact 4 A formula λ_A is a mac of A in L if and only if

1. $A \mathrel{|\!\sim}_{\mathsf{L}} \lambda_A \vdash_{\mathsf{L}} A$, and

2. λ_A is stable, i.e. $\forall B\, (\lambda_A \mathrel{|\!\sim}_{\mathsf{L}} B \Leftrightarrow \lambda_A \vdash_{\mathsf{L}} B)$.

Proof We assume that A has a mac λ_A in L and show that 1. and 2. hold. We leave the other direction to the reader. We have $\forall B\, (A \mathrel{|\!\sim} B \Leftrightarrow \lambda_A \vdash B)$ by assumption. Thus $A \mathrel{|\!\sim} \lambda_A \vdash A$ follows, which is 1. For 2., the direction from right to left is trivial. For the other direction, assume $\lambda_A \mathrel{|\!\sim} B$. Then $A \mathrel{|\!\sim} B$ by 1. and the fact that $\mathrel{|\!\sim}$ is clearly transitive. Thus $\lambda_A \vdash B$ by the definition of λ_A. $\Box$ The following fact expresses the relation between mac's and bases for admissible rules.

Fact 3 1. If λ_A is a mac of A in L, R a set of rules such that $A \vdash^R_{\mathsf{L}} \lambda_A$, then $\forall B\, (A \mathrel{|\!\sim}_{\mathsf{L}} B \Rightarrow A \vdash^R_{\mathsf{L}} B)$.

2. If all formulas A have a mac λ_A in L and R is a set of admissible rules of L, then:

$$\forall A\, (A \vdash^R_{\mathsf{L}} \lambda_A) \Leftrightarrow (R \text{ is a basis for the admissible rules of } \mathsf{L}).$$

Proof For the first part, assume $A \mathrel{|\!\sim}_{\mathsf{L}} B$. By the definition of mac's, $\lambda_A \vdash_{\mathsf{L}} B$ follows. Thus $A \vdash^R_{\mathsf{L}} \lambda_A \vdash_{\mathsf{L}} B$, which gives $A \vdash^R_{\mathsf{L}} B$. For the second part it suffices to show that for all A we have

$$(A \vdash^R_{\mathsf{L}} \lambda_A) \Leftrightarrow \forall B (A \mathrel{|\!\sim}_{\mathsf{L}} B \Leftrightarrow A \vdash^R_{\mathsf{L}} B).$$

For the direction from left to right, assume $A \vdash^R_{\mathsf{L}} \lambda_A$. $(A \mathrel{|\!\sim}_{\mathsf{L}} B \Rightarrow A \vdash^R_{\mathsf{L}} B)$ follows from 1. $(A \vdash^R_{\mathsf{L}} B \Rightarrow A \mathrel{|\!\sim}_{\mathsf{L}} B)$ follows from the assumption that the rules R are admissible for L. The direction from right to left follows from $A \mathrel{|\!\sim}_{\mathsf{L}} \lambda_A$, see Fact 4. $\Box$

Thus by the above fact, one approach to finding a basis for the admissible rules of an intermediate logic L is to first check whether

(a) for every A there exists a mac λ_A of A in L.

And if so, to provide

(b) a set of rules R, admissible for L, such that $A \vdash^R_{\mathsf{L}} \lambda_A$ for all A.

By the previous fact it then follows that R is a basis for the admissible rules of L.

In this paper we will follow this procedure. We will see that there are many logics for which these two properties (a) and (b) hold, e.g. for the logics KC, LC, $\mathsf{G_k}$. The central point here is that (a) and (b) fold for IPC: it turns out that for all these logics the mac of a formula A is always the same, namely Λ_A, the mac of A in IPC. That is, in Corollary 7, it is shown that in any intermediate logic L for which Visser's rule are admissible, Λ_A is a mac of A and $A \vdash^V_{\mathsf{L}} \Lambda_A$. This implies that (a) and (b) hold for L, and whence that Visser's rules form a basis for the admissible rules of L. Corollary 7 therefore not only allows us to establish the basis for the admissible rules of many logics, but moreover shows that once Visser's rules are admissible, this basis is *always the same*, namely the collection of Visser's rules (Theorem 8).

The main Corollary 7 follows from two theorems below: Theorem 3 by Ghilardi [6] implies that in IPC every formula A has a mac Λ_A that moreover is stable in any intermediate logic L (Corollary 4). Theorem 6 by the author [10] states that Visser's rules are a basis for the admissible rules of IPC. Hence it follows that $\Lambda_A \vdash_{\mathsf{IPC}} A \vdash^{V}_{\mathsf{IPC}} \Lambda_A$. By Fact 4, these two theorems together imply that Λ_A is a mac of A in any logic in which V is admissible, which is the content of Corollary 7. All this will be proved below and in the next subsection.

Theorem 3 (Ghilardi [6]) Every formula A has a finite projective approximation Π_A. For every unifier σ of A there is formula $B \in \Pi_A$ such that σ is a unifier for B too.

Corollary 4 Every formula A has a mac Λ_A in IPC. Moreover, Λ_A is stable in any intermediate logic L. The disjunction of any projective approximation of A can be taken for Λ_A.

Proof Let Π_A be a finite projective approximation of A, which exists by the previous theorem. We show that we can take $\bigvee \Pi_A$ for Λ_A. First, we show that Λ_A is a mac of A in IPC:

$$\forall B\,(A \mathrel{|\!\sim}_{\mathsf{IPC}} B \Leftrightarrow \bigvee \Pi_A \vdash_{\mathsf{IPC}} B).$$

The direction from left to right. Assume $A \mathrel{|\!\sim} B$. Whence $\vdash \sigma B$. Thus $\bigvee \Pi_A \mathrel{|\!\sim} B$. Recall from Section 2.6 that $\bigvee \Pi_A \mathrel{|\!\sim} B$ implies $\bigvee \Pi_A \vdash B$. For the other direction, assume $\bigvee \Pi_A \vdash_{\mathsf{IPC}} B$ and $\vdash_{\mathsf{IPC}} \sigma A$. By Theorem 3 there is a formula $C \in \Pi_A$ such that σ is a unifier of C, i.e. $\vdash_{\mathsf{IPC}} \sigma C$. Hence $\vdash_{\mathsf{IPC}} \sigma(\bigvee \Pi_A)$, and thus $\vdash_{\mathsf{IPC}} \sigma B$. This proves $A \mathrel{|\!\sim} B$.

It remains to show that Λ_A is stable in any intermediate logic L, that is

$$\forall B\ (\bigvee \Pi_A \mathrel{|\!\sim}_{\mathsf{L}} B \Leftrightarrow \bigvee \Pi_A \vdash_{\mathsf{L}} B).$$

Assume $\bigvee \Pi_A \mathrel{|\!\sim}_{\mathsf{L}} B$. Pick a projective formula $C \in \Pi_A$ and a projective unifier σ for C, i.e. $\vdash_{\mathsf{IPC}} \sigma C$ and $C \vdash_{\mathsf{IPC}} B \leftrightarrow \sigma B$ (Section 2.6). Thus $\vdash_{\mathsf{L}} \sigma C$ and $C \vdash_{\mathsf{L}} B \leftrightarrow \sigma B$. Since $\bigvee \Pi_A \mathrel{|\!\sim}_{\mathsf{L}} B$, we have $\vdash_{\mathsf{L}} \sigma B$. Thus $C \vdash_{\mathsf{L}} B$. As we have shown this for arbitrary $C \in \Pi_A$, $\bigvee \Pi_A \vdash_{\mathsf{L}} B$ follows. $\Box$

Corollary 5 If $A \mathrel{|\!\sim}_{\mathsf{L}} \Lambda_A$, then Λ_A is a mac of A in L, i.e. $\lambda^{\mathsf{L}}_A = \Lambda_A$.

Proof By Fact 4 it suffices to show that $A \mathrel{|\!\sim}_{\mathsf{L}} \Lambda_A \vdash_{\mathsf{L}} A$ and that Λ_A is stable in L. The last part follows from Corollary 4. The first part follows from $\Lambda_A \vdash_{\mathsf{IPC}} A$, which again follows from Corollary 4 and Fact 4. $\Box$

As mentioned in the introduction, Ghilardi in [7], constructed an algorithm to compute Λ_A. Based on this, we have developed a proof system that given a formula A derives Λ_A [11]. Although we will not use these results here, we mention them because they show that and how one can obtain the Λ_A "in practice".

3.2 When Visser's rules are admissible

Theorem 6 ([10]) $A \mathrel{|\!\sim}_{\mathsf{IPC}} B$ if and only if $A \vdash^V_{\mathsf{IPC}} B$.

Corollary 7 If V is admissible for L, then Λ_A is a mac of A in L and $A \vdash^V_{\mathsf{L}} \Lambda_A$.

Proof We have $A \mathrel{|\!\sim}_{\mathsf{IPC}} \Lambda_A$ by Corollary 4 and Fact 4. It follows from Theorem 6 that $A \vdash^V_{\mathsf{IPC}} \Lambda_A$. As V is admissible for L, this gives $A \mathrel{|\!\sim}_{\mathsf{L}} \Lambda_A$. Corollary 5 implies that Λ_A is a mac for A in L. As $A \vdash^V_{\mathsf{IPC}} \Lambda_A$ clearly implies $A \vdash^V_{\mathsf{L}} \Lambda_A$, the result follows. □

As explained above, this leads to the following characterization of the admissible rules for logics for which V is admissible.

Theorem 8 If V is admissible for L, then V is a basis for the admissible rules of L, i.e. $\mathrel{|\!\sim}_{\mathsf{L}} = \vdash^V_{\mathsf{L}}$ when V is admissible.

Proof By 2. of Fact 3 and Corollary 7. □

Corollary 9 If V is admissible for L then all admissible rules of IPC are admissible for L.

Proof By Corollary 7 and Theorem 4

$$A \mathrel{|\!\sim}_{\mathsf{IPC}} B \Leftrightarrow \Lambda_A \vdash_{\mathsf{IPC}} B \Rightarrow \Lambda_A \vdash_{\mathsf{L}} B \Leftrightarrow A \mathrel{|\!\sim}_{\mathsf{L}} B.$$

□ Note that the last corollary follows already from the fact that V is a basis for the admissible rules of IPC.

Corollary 10 If V is derivable for L then L has no non-derivable admissible rules.

Proof By Corollary 8 and Fact 1. □ Note that this theorem implies that CPC has no non-derivable admissible rules, as stated in the introduction, a fact that can also be derived directly from the definition of admissible rules.

In Section 5 we will apply the results above to specific intermediate logics and obtain characterizations of their admissible rules. We conclude this section by some general facts on admissible rules for the case that Visser's rules are not admissible, before we proceed in the next section with a semantic criterion for the admissibility of V.

3.3 General remarks

For completeness sake we include the following known facts for logics for which Visser's rules are not admissible. They only provide necessary conditions for admissibility.

Fact 11 If $A \mathrel{|\!\sim}_{\mathsf{L}} B$, then $\mathsf{CPC} \vdash A \to B$.

Proof Suppose $A \mathrel{|\!\sim}_{\mathsf{L}} B$. This means that for all σ, $\vdash_{\mathsf{L}} \sigma A$ implies $\vdash_{\mathsf{L}} \sigma B$. Suppose the variables that occur in A and B are among $p_1 \dots p_n$. Consider $\sigma \in \{\top, \bot\}^n$. Note that for such σ, $\vdash_{\mathsf{CPC}} \sigma A$ iff $\vdash_{\mathsf{IPC}} \sigma A$ iff $\vdash_{\mathsf{L}} \sigma A$. Whence for all $\sigma \in \{\top, \bot\}^n$, if $\vdash_{\mathsf{CPC}} \sigma A$ then $\vdash_{\mathsf{CPC}} \sigma B$. Thus $\vdash_{\mathsf{CPC}} A \to B$. $\square$

Corollary 12 If $A \mathrel{|\!\sim}_{\mathsf{L}} B$, then the logic that consists of L extended with the axiom scheme $(A \to B)$ is consistent.

Fact 13 If $A \mathrel{|\!\sim}_{\mathsf{L}} B$ then $\Lambda_A \vdash_{\mathsf{L}} B$.

ProofBy Corollary 4 and Fact 4. $\square$

4 Semantic criterion for Visser's rules

In this section we give a semantic criterion for the admissibility of V. Both statement and proof are similar to analogues but weaker results on intermediate logics with the disjunction property in [9], where the following has been proved.

Theorem 14 ([9]) For any intermediate logic L with the disjunction property, if Visser's rules are admissible for L, then its class of models has the extension property.

Here we find, Theorem 18, that in leaving out the disjunction property one can obtain a similar criterion for the admissibility of V, namely the offspring property, which is not only sufficient but also necessary. The offspring property holds for many intermediate logics, as we will see in Section 5. This in contrast to the extension property, which only holds for IPC:

Theorem 15 (Folklore, proof in [9]) If the class of models of an intermediate logic has the extension property, it is the logic IPC.

As an aside, let us mention that this implies the following.

Corollary 16 If a logic has the disjunction property, not all Visser's rules are admissible.

Here we set out to prove that the offspring property is a necessary and sufficient condition for the admissibility of Visser's rules, and that the weak extension property is a necessary and sufficient condition for the admissibility of the restricted Visser rules. As we will see below, it is not so difficult to show that the conditions are sufficient. The proofs that they are also necessary are more involved and are based on the following idea, part of which is already present in [10]. We explain it for the case of the weak extension property, as the proof for the offspring property if similar. In will be shown that when the restricted Visser rules are admissible, the class of all models of L has the weak extension property. Thus, since we consider the class of all models, it suffices to show that given a model K of L and nodes $k_1, \ldots, k_n$ in K distinct from the root, some variant M_1 of $(\Sigma K_{k_i})'$ is a model of L. In order to do so, we consider the k_i's as saturated sets x_i (namely as the set of formulas that are forced at k_i). Then we show that the intersection of these n saturated sets contains a saturated set x such that there are no saturated sets properly between x and any x_i. If exactly those atoms are forced at the root of $(\Sigma K_{k_i})'$ that are elements of x, then one can show that this model is a model of L. In the case that Visser's rules are admissible, one has to repeat the same trick to construct a variant of $(M_1+K)'$. The main ingredient of the proof of Theorem 18 is the next lemma that shows the existence of the mentioned saturated set.

Definition 2 A set x is called L-saturated if it does not contain $\bot$, is closed under derivability in L, and $x \vdash_{\mathsf{L}} A \vee B$ implies $x \vdash_{\mathsf{L}} A$ or $x \vdash_{\mathsf{L}} B$. A saturated set x is called a *tight* predecessor of saturated set $x_1, \ldots, x_n$ if $x \subseteq x_1 \cap \ldots \cap x_n$, and for all L-saturated sets $x \subset y$ there is some $i \leq n$ such that $x_i \subseteq y$. A node k in a Kripke model K is called a *tight* predecessor of the nodes $k_1, \ldots, k_n$ in K, if $k \preccurlyeq k_i$ for all i, and for all nodes $k \prec l$ in K there is some $i \leq n$ such that $k_i \preccurlyeq l$. Note that in the canonical model of a logic both definitions of tight predecessor coincide.

Lemma 17 Let L be an intermediate logic for which the restricted Visser's rules are admissible. Then for all n, for all L-saturated sets $x_1, \ldots, x_n$ for which there is a L-saturated set $x_0 \subseteq x_1 \cap \ldots \cap x_n$, there exists a tight predecessor x of $x_1, \ldots, x_n$. If Visser's rules are admissible for L, we can moreover construct x in such a way that there also exists a tight predecessor x' of x, x_0.

Proof In the proof, saturated means L-saturated, $\vdash$ stands for $\vdash_{\mathsf{L}}$. Let $x_0, x_1, \ldots, x_n$ be as in the lemma. We first prove the second part of the lemma, i.e. when all Visser's rules, not only the restricted ones, are admissible. First we construct x, then x'. Let

$$\Delta_0 = \{A \mid \exists B \notin x_0 \, (\vdash A \vee B)\},$$

$$\Delta_1 = \{(A \to B) \mid A \notin x_1 \cap \ldots \cap x_n \text{ and } B \in x_1 \cap \ldots \cap x_n\}.$$

Note that $\Delta_0 \subseteq x_0$, as x_0 is saturated. Consider $\Delta = \Delta_0 \cup \Delta_1$. Clearly, $\Delta \subseteq x_1 \cap \ldots \cap x_n$. Now we construct a sequence of sets $z_0 \subseteq z_1, \ldots$, where $z_0 = \{C \mid \Delta \vdash C\}$, such that x will be the union of the z_i.

The explanation behind the set's Δ_i is as follows. Since x has to be such that we can construct a tight predecessor x' of x, x_0, we should at least be able to construct a saturated set in $x \cap x_0$. This implies that the following has to hold for x:

$$\vdash \bigvee_{i=1}^{m} D_i \;\Rightarrow\; \exists i \leq m (D_i \in x \cap x_0). \tag{2}$$

Observe that when $\Delta_0 \subseteq x$, this indeed is the case. For assume $\vdash \bigvee_{i=1}^{m} D_i$. If $D_i \in x_0$ for all i, then clearly $D_i \in x_0 \cap x$ for some i, because x is saturated. Therefore, assume not all D_i belong to x_0. Note that some $D_i \in x_0$. W.l.o.g.

assume that there is a number $1 \leq k < m$ such that $D_1, \ldots, D_k \in x_0$ and $D_{k+1}, \ldots, D_m \notin x_0$. Whence $\bigvee_{i=k+1}^m D_i \notin x_0$. Thus by the definition of Δ_0, $\bigvee_{i=1}^k D_i$ belongs to Δ_0, and thus to x. The saturatedness of x implies that whence $D_h \in x$ for some $h \leq k$, which proves (2). The set Δ_1 is put in x in order to make x a tight predecessor of the x_i. The exact use of this set will get clear later on in the proof when we prove that for all $x \subset y$ there is some i such that $x_i \subseteq y$.

We proceed with the construction of the z_i. Let $C_0, C_1, \ldots$ enumerate all formulas, with infinite repetition. Define the property $*(\cdot)$ on sets via

$$*(z) \quad \text{iff} \quad \forall A_1, \ldots, A_m \big(z \vdash \bigvee_{i=1}^m A_i \Rightarrow \exists i \leq m \ (A_i \in x_1 \cap \ldots \cap x_n)\big).$$

Define z_i as follows.

$$z_{i+1} = \begin{cases} z_i & \text{if not } *(z_i \cup \{C_i\}) \\ z_i \cup \{C_i\} & \text{if } C_i \text{ no disjunction and } *(z_i \cup \{C_i\}) \\ z_i \cup \{D, C_i\} & \text{if } C_i = D \vee E, *(z_i \cup \{C_i\}) \text{ and } *(z_i \cup \{D, C_i\}) \\ z_i \cup \{E, C_i\} & \text{if } C_i = D \vee E, *(z_i \cup \{C_i\}) \text{ and not } *(z_i \cup \{D, C_i\}) \end{cases}$$

We show that $*(z_i)$ holds with induction to i.

For $i = 0$, assume $\Delta \vdash \bigvee_{h=1}^m A_h$. Whence there are $k, l \in \omega$, $(B_i \to D_i) \in \Delta_1$ and $E_j \in \Delta_0$ such that

$$\vdash \bigwedge_{i=1}^k (B_i \to D_i) \wedge \bigwedge_{j=1}^l E_j \to \bigvee_{h=1}^m A_h.$$

By assumption there exists for all $j \leq l$ formulas $E'_j \notin x_0$ such that $\vdash E_j \vee E'_j$. Thus by elementary logic

$$\vdash \big(\bigwedge_{i=1}^k (B_i \to D_i) \to \bigvee_{h=1}^m A_h\big) \vee \bigvee_{j=1}^l E'_j.$$

Let $B = \bigwedge_{i=1}^k (B_i \to D_i)$. As V_{km} is admissible for L by Remark 1.1, an application of V_{km} (with $\bigvee_{j=1}^l E'_j$ for C) gives

$$\vdash \bigvee_{i=1}^k (B \to B_i) \vee \bigvee_{h=1}^m (B \to A_h) \vee \bigvee_{j=1}^l E'_j.$$

As x_0 is a saturated set it follows that it contains $(B \to B_i)$ for some $i \leq k$, or $(B \to A_h)$ for some $h \leq m$, or E'_j for some $j \leq l$. Since the E'_j do not belong to x_0, only the first two possibilities remain. Since $x_0 \subseteq x_1 \cap \ldots \cap x_n$ it follows that some $(B \to B_i)$ or some $(B \to A_h)$ belongs to $x_1 \cap \ldots \cap x_n$. Since $B \in x_1 \cap \ldots \cap x_n$ and $B_i \notin x_1 \cap \ldots \cap x_n$, it follows that it has to be one of the $(B \to A_h)$. Thus $A_h \in x_1 \cap \ldots \cap x_n$ which is what we had to show.

For $i > 0$, the only nontrivial case is that in which $C_i = D \vee E$ and $z_{i+1} = z_i \cup \{E, C_i\}$, as the other cases follow immediately from the induction hypothesis. If $*(z_i \cup \{E, C_i\})$ we are done. Therefore, suppose not $*(z_i \cup \{E, C_i\})$. Note that not $*(z_i \cup \{D, C_i\})$, as otherwise z_{i+1} would have been $z_i \cup \{D, C_i\}$. Therefore there are $k, l \in \omega$ and $F_1, \ldots, F_l$ such that $F_j \notin x_1 \cap \ldots \cap x_n$ for all $j \leq l$, and

$$z_i \cup \{D, C_i\} \vdash \bigvee_{j=1}^{k} F_j \text{ and } z_i \cup \{E, C_i\} \vdash \bigvee_{j=k+1}^{l} F_j.$$

But this implies

$$z_i, C_i, D \vee E \vdash \bigvee_{j=1}^{l} F_j,$$

and thus $z_i, C_i \vdash \bigvee_{j=1}^{l} F_j$, which contradicts $*(z_i \cup \{C_i\})$. This completes the proof that for all i, $*(z_i)$ holds.

Let $x = \cup_i z_i$. We have to show x is a tight predecessor of $x_1, \ldots, x_n$. For $x \subseteq x_1 \cap \ldots \cap x_n$, note that $z_i \subseteq x_1 \cap \ldots \cap x_n$. We show the saturation of x. If $\perp \in x$, then $\perp \in z_i$ for some i. Because $*(z_i)$ this implies $\perp \in x_1 \cap \ldots \cap x_n$, contradicting the fact that the x_i are saturated. x is closed under derivability because $z_i \vdash A$ and $*(z_i)$ implies $*(z_i \cup \{A\})$. If $x \vdash A \vee B$, then $z_i \vdash A \vee B$ for some i. Thus $*(z_i \cup \{A \vee B\})$. The construction of the z_i guarantees that either A or B will be an element of x.

The proof that x is a tight predecessor is finished once we have shown that for all saturated sets $x \subset y$ there is some $i \leq n$ for which $x_i \subseteq y$. Arguing by contradiction assume $x \subset y$ and $x_i \not\subseteq y$ for all $i \leq n$. For all $i \leq n$ choose A_i such that $A_i \in x_i \backslash y$. Observe that x cannot be extended to a larger saturated set inside $x_1 \cap \ldots \cap x_n$, i.e. for no saturated set z it holds that $x \subset z \subseteq x_1 \cap \ldots \cap x_n$. For if so, there would be an i such that $C_i \in z \backslash x$. The fact that z is saturated and a subset of $x_1 \cap \ldots \cap x_n$ implies that $*(z)$. Thus certainly $*(z_i \cup \{C_i\})$, since $z_i \cup \{C_i\} \subseteq z$, which would imply $C_i \in x$. As $x \subset y$, this observation gives $y \not\subseteq x_1 \cap \ldots \cap x_n$. Thus there is a formula $B \in y \backslash (x_1 \cap \ldots \cap x_n)$. Hence $(B \to \bigvee_{i=1}^{n} A_i) \in \Delta \subseteq x \subset y$. Since $B \in y$, $\bigvee_{i=1}^{n} A_i \in y$, contradicting the fact that y is saturated and whence should contain one of the A_i. This completes the proof that x is a tight predecessor of $x_1, \ldots, x_n$.

Finally, we have to see that there exists a tight predecessor x' of x, x_0. We proceed in a similar way as for the construction of x: we construct a sequence of sets $y_0 \subseteq y_1 \subseteq y_2 \ldots$, where $y_0 = \{C \mid \Delta_2 \vdash C\}$, such that x' will be the union of the y_i. Here

$$\Delta_2 = \{A \to B \mid B \in x \cap x_0, A \notin x \cap x_0\}.$$

Instead of $*$ we consider the property $\star$:

$$\star(y) \quad \text{iff} \quad \forall A_1, \ldots, A_m \big(y \vdash \bigvee_{i=1}^{m} A_i \Rightarrow \exists i \leq m \ (A_i \in x \cap x_0)\big).$$

We define y_{i+1} inductively as the z_{i+1} above but with the property $\star$ instead of $*$:

$$y_{i+1} = \begin{cases} y_i & \text{if not } \star(y_i \cup \{C_i\}) \\ y_i \cup \{C_i\} & \text{if } C_i \text{ no disjunction and } \star(y_i \cup \{C_i\}) \\ y_i \cup \{D, C_i\} & \text{if } C_i = D \vee E, \star(y_i \cup \{C_i\}) \text{ and } \star(y_i \cup \{D, C_i\}) \\ y_i \cup \{E, C_i\} & \text{if } C_i = D \vee E, \star(y_i \cup \{C_i\}) \text{ and not } \star(z_i \cup \{D, C_i\}) \end{cases}$$

Again, we have to show that $\star(y_i)$ holds for all i. The induction step is similar as for the z_i. We only treat the case $i = 0$. Therefore, assume $\Delta_2 \vdash \bigvee_{h=1}^m A_h$. Whence there are $k, l \in \omega$, $(B_i \to D_i) \in \Delta_2$ such that

$$\vdash \bigwedge_{i=1}^{k} (B_i \to D_i) \to \bigvee_{h=1}^{m} A_h.$$

Let $B = \bigwedge_{i=1}^k (B_i \to D_i)$. As V_{km} is admissible for L by Remark 1.1, an application of V_{km} (with C empty) gives

$$\vdash \bigvee_{i=1}^{k} (B \to B_i) \vee \bigvee_{h=1}^{m} (B \to A_h).$$

By (2) it follows that $x \cap x_0$ contains $(B \to B_i)$ for some $i \leq k$, or $(B \to A_h)$ for some $h \leq m$. Since $B \in x_0 \cap x$ and $B_i \notin x_0 \cap x$, we have to be in the latter case. Because $B \in x \cap x_0$, this implies that $A_h \in x \cap x_0$ for some h, which is what we had to show.

Let Let $x' = \cup_i y_i$. The proof that x' is a tight predecessor of x, x_0 is analogues to the proof for x above, and therefore omitted. This proves the second part of the lemma.

For the proof of the first part of the lemma, the case that only the restricted Visser's rules are admissible, take $\Delta = \Delta_1$, then the construction of the tight predecessor is completely similar to the construction of x above. $\square$

Theorem 18 For any intermediate logic L, Visser's rules are admissible for L if and only if L has the offspring property.

Proof In the proof we omit reference to L, i.e. saturated means L-saturated, $\vdash$ denote $\vdash_{\mathsf{L}}$ etc. First the direction from right to left. Let U be a class of models with the offspring property with respect to which L is sound and complete. Let

$$A = \bigwedge_{i=1}^{n} (A_i \to B_i), \quad A' = A_{n+1} \vee A_{n+2}, \quad B = \bigvee_{j=1}^{n+2} (A \to A_j),$$

and suppose $\mathsf{L} \vdash (A \to A') \vee C$. We have to show that $\mathsf{L} \vdash B \vee C$. Arguing by contradiction, assume this is not the case. Then there is a model $K \in U$ with root k_0 such that $K \not\models B \vee C$. We show that there is a model K'' such that

$K'' \not\models (A \to A') \vee C$. Note that $k_0 \nVdash B$ and $k_0 \nVdash C$. Thus there are $k_i \in K$ such that $k_i \Vdash A$ and $k_i \nVdash A_i$, for all $i \leq n+2$. First suppose all k_i are distinct from the root of K. Then by assumption, there is a variant M_1 of $(\sum_{i=1}^{n+2} K_{k_i})'$ such that a bounded morphic image M of some variant M_0 of $(M_1 + K)'$ is contained in U. Recall that M and M_0 validate the same formulas (Section 2.4). We leave it to the reader to verify that the root of M_1 forces A but not A'. This gives $M_1 \not\models (A \to A')$. Whence M_0 does not force $(A \to A')$. As is clearly does not force C either, this gives $M_0 \not\models (A \to A') \vee C$. Thus $M \not\models (A \to A') \vee C$. Since $M \in U$, this implies $\mathsf{L} \nvdash (A \to A') \vee C$. If one of the k_i is the root of K, say k_j, this implies that A is forced at the root, but as none of the A_i are forced at the root, A' is not forced there either. Thus $k_j \not\models (A \to A')$. Since also $k_j \nVdash C$, $K \not\models (A \to A') \vee C$ follows. This also implies $\mathsf{L} \nvdash (A \to A') \vee C$.

The direction from left to right. We show that the class of all models of L has the offspring property. This will prove that L has the offspring property. Consider a model K of L and nodes $k_1, \ldots, k_n$ in K that are distinct from the root. Let k_0 be the root of K, let K_i be K_{k_i} and $x_i = \{A \mid k_i \Vdash A\}$. Note that the x_i are saturated sets such that $x_0 \subseteq x_1 \cap \ldots \cap x_n$. By Lemma 17 there exists saturated sets x, x' such that x is a tight predeccessor of $x_1, \ldots, x_n$ and x' is a tight predeccessor of x, x_0. This means that $x \subseteq x_1 \cap \ldots \cap x_n$ and $x' \subseteq x \cap x_0$, and that for all saturated sets y

$$\big(x \subset y \Rightarrow \exists i \leq n (x_i \subseteq y)\big) \;\wedge\; \big(x' \subset y \Rightarrow (x_0 \subseteq y \vee x \subseteq y)\big). \tag{3}$$

We first define a variant K' of $(\sum K_i)'$ by defining for the root k' of $(\sum K_i)'$, $k' \Vdash p$ iff $p \in x$, for atoms p. Then we define a variant K'' of $(K' + K)'$ by defining for the root k'' of K'', $k'' \Vdash p$ iff $p \in x'$. Note that the fact that $x' \subseteq x \cap x_0$ and $x \subseteq x_1 \cap \ldots \cap x_n$ guarantees the upward persistency in the model. To show that this is a model of L it suffices to show that for all formulas A

$$k' \Vdash A \text{ iff } A \in x \qquad k'' \Vdash A \text{ iff } A \in x'. \tag{4}$$

We use formula induction, and only treat implication, for the case k', x. Consider $A = (B \to C)$. If $(B \to C) \in x$ then $k' \Vdash (C \to D)$ follows easily. For the other direction, assume $(B \to C) \notin x$. This implies that there is a saturated set $y \supseteq x$ such that $B \in y$ and $C \notin y$. By (3), $x = y$ or $x_i \subseteq y$ for some $i = 1, \ldots, n$. In the first case the induction hypothesis gives $k' \Vdash B$ and $k' \nVdash C$, thus $k' \nVdash (B \to C)$. In the latter case $(B \to C) \notin x_i$, and thus $k_i \nVdash (B \to C)$. Hence $k' \nVdash (B \to C)$. This proves (4), and thereby the theorem. ☐

Theorem 19 For any intermediate logic L, the restricted Visser rules are admissible for L if and only if L has the weak extension property.

Proof Similar as the proof above, using the first part of Lemma 17 instead of the second part. ☐

From the Theorem 18 and 19 we also derive:

Corollary 20 For any intermediate logic L, the (restricted) Visser rules are admissible for L if and only if the class of all models of L has the offspring (weak extension) property. Whence L has the offspring (weak extension) property iff and only the class of all models of L has the offspring (weak extension) property.

Proof If a logic has the offspring property, then Visser's rules are admissible by Theorem 18. As the proof of this theorem shows, this again implies that the class of all models of L has the offspring property. Similar reasoning applies to the weak extension property. $\square$

5 Intermediate logics

In this section we apply the results of the previous theorems to the following specific intermediate logics. When we say "axiomatized by ..." we mean "axiomatized over IPC by ...". For a class of frames F, L is called the *logic of the frames F* when L is sound and complete with respect to F.

$\mathsf{Bd_n}$ The logic of frames of depth at most n. $\mathsf{Bd_1}$ is axiomatized by $bd_1 = p_1 \vee \neg p_1$, and $\mathsf{Bd_{n+1}}$ by $bd_{n+1} = (p_{n+1} \vee (p_{n+1} \to bd_n))$ [3].

$\mathsf{D_n}$ The Gabbay-de Jongh logics [5], axiomatized by the following scheme: $\bigwedge_{i=0}^{n+1}((A_i \to \bigvee_{j\neq i} A_j) \to \bigvee_{j\neq i} A_j) \to \bigvee_{i=0}^{n+1} A_i$. $\mathsf{D_n}$ is complete with respect to the class of finite trees in which every point has at most $(n+1)$ immediate successors.

$\mathsf{G_k}$ The Gödel logics, first introduced by Gödel [8]. They are extensions of LC axiomatized by $A_1 \vee (A_1 \to A_2) \vee \ldots \vee (A_1 \wedge \ldots \wedge A_{k-1} \to A_k)$. $\mathsf{G_k}$ is the logic of the linearly ordered Kripke frames with at most $k-1$ nodes [1].

KC De Morgan logic (also called Jankov logic), axiomatized by $\neg A \vee \neg\neg A$. The logic of the frames with one maximal node.

KP The logic axiomatized by $(\neg A \to B \vee C) \to \big((\neg A \to B) \vee (\neg A \to C)\big)$, called Kreisel-Putnam logic. It constituted the first counterexample to Łukasiewicz' conjecture that IPC is the greatest intermediate logic with the disjunction property [12].

LC Gödel-Dummett logic [4], the logic of the linear frames. It is axiomatized by the scheme $(A \to B) \vee (B \to A)$.

$\mathsf{M_n}$ The logic of frames with at most n maximal nodes. Note that $\mathsf{M_1} = \mathsf{KC}$.

Sm The greatest intermediate logic properly included in classical logic. It is axiomatized by $\big((A \to B) \vee (B \to A)\big) \wedge (A \vee (A \to B \vee \neg B))$ and it is complete with respect to frames of at most 2 nodes [3].

V The logic axiomatized by $V_1^{\to}$, that is by the implication corresponding to the rule V_1: $\big((A_1 \to B) \to A_2 \vee A_3\big) \to \bigvee_{i=1}^{3}\big((A_1 \to B) \to A_i\big)$.

Theorem 21 Visser's rules form a basis for the admissible rules of the logics KC and $\mathsf{M_n}$. Visser's rules are not derivable in these logics.

Proof The first part is proved by showing that the classes of models based on the frames for these logics as mentioned in the list above, have the offspring property. Then apply Theorem 18. To see that the logics have the offspring property, the following claim suffices.

Claim Let W be a class of frames, and let U be the class of models based on frames in W. U has the offspring property if for every $F \in W$, for all $k_1, \ldots, k_n$ in F distinct from the root k_0, the frame that consists of attaching a new node l_1 below $k_1, \ldots, k_n$ and a new node l_0 below l_1, k_0, also belongs to W.

Proof of Claim Left to the reader.

The second part of the theorem can be shown by constructing appropriate countermodels to the formulas $V_n^{\rightarrow}$, which we leave to the reader. □ Note that all the logics in the previous theorem are also examples of logics which have the weak extension property, but not the extension property, as they do not have the disjunction property (see Fact 4). That they do not have the disjunction property follows from the fact that the only logic with the disjunction property for which all Visser's rules are admissible is IPC, recall Corollary 16.

Theorem 22 Visser's rules are derivable in $\mathsf{Bd_1}$, $\mathsf{G_k}$, LC, Sm and V. Whence these logics do not have nonderivable admissible rules.

Proof The first four logics are complete w.r.t. to classes that contain only models with linear frames. It is easy to see that in any linear model the implications $V_n^{\rightarrow}$ are valid (for the notation $R^{\rightarrow}$ see Section 2.1). In fact, even $(A \rightarrow B \vee C) \rightarrow (A \rightarrow B) \vee (A \rightarrow C)$ holds in every linear model. Then apply Corollary 10. For V one uses the fact that all $V_n^{\rightarrow}$ are derivable from $V_1^{\rightarrow}$, which was first observed in [13]. □

Theorem 23 For $\mathsf{Bd_n}$, $n \geq 2$, the restricted Visser rules are admissible but not derivable.

Proof Use Theorem 19 by showing that the class of models of depth n has the weak extension property. To see that the restricted Visser rules are not derivable it suffices to construct a countermodel of depth 2 to $V_1^{\rightarrow}$, which is left to the reader. □

Theorem 24 V_1 is not admissible for KP. For the logics $\mathsf{D_n}$ ($n \geq 1$), V_{n+1} is admissible, while V_{n+2} is not.

Proof The part about the $\mathsf{D_n}$'s is proved in [9]. For KP, let $X = (\neg p \rightarrow q \vee r)$ and $Y = (\neg p \rightarrow q) \vee (\neg p \rightarrow r)$. Then $(X \rightarrow Y)$ is derivable in KP. If V_1 would be admissible, then KP would derive

$$\big(X \rightarrow (\neg p \rightarrow q)\big) \vee \big(X \rightarrow (\neg p \rightarrow r)\big) \vee (X \rightarrow \neg p).$$

Since KP has the disjunction property, this would imply that at least one of $\big(X \to (\neg p \to q)\big)$, $\big(X \to (\neg p \to r)\big)$ or $(X \to \neg p)$ is derivable in KP. However, these formulas are note even derivable in classical logic. $\Box$

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