Logical Consequence for Nominalists

Marcus ROSSBERG and Daniel COHNITZ

BIBLID [0495-4548 (2009) 24: 65; pp. 147-168]

ABSTRACT: It has repeatedly been argued that nominalistic programmes in the philosophy of mathematics fail, since they will at some point or other involve the notion of logical consequence which is unavailable to the nominalist. In this paper we will argue that this is not the case. Using an idea of Nelson Goodman and W.V. Quine's which they developed in Goodman and Quine (1947) and supplementing it with means that should be nominalistically acceptable, we present a way to explicate logical consequence in a nominalistically acceptable way.

Keywords: Philosophy of mathematics, nominalism, logical consequence, inferentialism, Nelson Goodman, W.V. Quine.

1. *The Argument from Logical Consequence*

We do not have any strong convictions concerning the question of the existence or non-existence of abstract objects. We do, however, believe that ontological fastidiousness is *prima facie* a good attitude to adopt. More precisely, ontological parsimony provides a *pro tanto* reason for theory choice. Nelson Goodman added to this fairly obvious principle the methodological observation that a more parsimonious theory can always easily be turned into a theory that is ontologically more lavish; and so a nominalistic theory can be turned into a platonistic theory without any problem. There is, however, no guarantee that the other way around is generally available Goodman (1977, p. L). So, hedging your bets, it is advisable to resist an extravagant ontology for as long as possible. The ontic extravaganza that Goodman zoomed in on was that of the calculus of classes, or set theory as it is commonly known today. Nominalism for Goodman meant the renunciation of classes, platonism their acceptance.

Goodman was certainly one of the most vocal defenders of nominalism in the last century. Nowadays, the attempts he had made together with W.V. Quine to achieve a nominalistic foundation of mathematics are often regarded as a complete failure. In a very influential synopsis of various nominalistic strategies, John P. Burgess and Gideon Rosen write:

> Goodman and Quine made it their priority to reconstrue the kind of science in which mathematics is applied, and especially the kind of mathematics applied in science. [...] After some modest initial progress, the project of Goodman and Quine reached an impasse. (Burgess and Rosen 1997, p. 5)
>
> [...] Quine, after the failure of his joint project with Goodman, soon came to reconsider, and eventually came to recant, his nominalism. (Burgess and Rosen 1997, p. 32)

It seems to us that this underestimates Goodman and Quine's achievements.[1] As we hope to show in this paper, the strategy that Goodman developed together with Quine

[1] Moreover, Burgess and Rosen's claim in the second part of the quotation does not strike us as historically correct regarding Quine's surrender to platonism.

in their paper "Steps Toward a Constructive Nominalism" Goodman and Quine (1947) can aid closing one of the most problematic open gaps in contemporary nominalism.

This open gap concerns the notion of logical consequence as employed by nominalists. It has been argued, on various grounds, that a nominalistically acceptable and at the same time adequate explication of logical consequence is unobtainable. This objection is usually raised against specific nominalistic programmes that aim to prove the dispensability of any ontological commitment to abstract objects for science.[2] Many nominalists have to rely in their projects on logical consequence at some point. In the next section we will briefly sketch two cases.

Prima facie, logical consequence should be unproblematic for the nominalist. Logic, of all things, should not carry any ontological commitments, let alone ontological commitments to any dubious entities such as abstract objects. In fact, many philosophers of logic believe that this is a mark of logicality. However, anti-nominalists have produced the following *Argument from Logical Consequence*:

(P1) Without using the notion of *logical consequence* the nominalist cannot explain mathematical practice and its contribution to science.

(P2) *Logical consequence* can only be satisfactorily explicated with appeal to abstract objects.

(P3) But *logical consequence* is a notion that stands in need of explication.

(C) The nominalist cannot explain mathematical practice and its contribution to science.[3]

As we said, the reasoning behind premise (P1) will be explained in more detail in the next section. (P2) and (P3) could be argued for as follows: One might begin by arguing that the most influential explication of logical consequence is the model-theoretic approach which goes back to the work of Alfred Tarski. This approach, however, makes use of set theory and is thus committed to the existence of abstract objects. Alternative, proof-theoretic explications also fail to be nominalistically acceptable, in that they employ sets or sequences of sentences, which are abstract objects too see, e.g., Hale and Wright (1992, p. 112). Indeed, sentences themselves are usually taken to be abstract objects: types, say. This much should support (P2).

(P3) blocks the move just to assume the semantic consequence relation to be primitive and therefore not in need of any explication. This move — which is in fact chosen by some contemporary nominalists — is in conflict with the epistemological motivation that many nominalists had in the first place. At least those nominalists who eschew abstract objects because of their "epistemology" should not accept a mysterious and inexplicable consequence relation either.

In this paper we aim to rebut this version of the Argument from Logical Consequence. In section 3., we argue that one of the main assumptions which this argument

[2] See, for instance, Hale and Wright (1992, 1994), Parsons (1990), Resnik (1983, 1985), Shapiro (1993, 1997).

[3] A similar argument, called "the argument from the philosophy of logic" is made in Wilholt (2006), the argument here is modeled after Wilholt's.

rests on is mistaken. We argue for an explication of logical consequence along inferentialist lines. We show how a consequence relation, so understood, can be explicated in a nominalistically acceptable way, using the techniques developed by Goodman and Quine in Goodman and Quine (1947). In the final section we discuss the prospects for using this explication of logical consequence in the two nominalistic programmes described in the following section.

2. *Two Nominalist Programmes*

Nominalists, as we understand them, renounce classes, and *a fortiori*, objects which are often considered to be constructed from them, such as numbers. Consequently, there are three positions that nominalists can adopt with respect to statements of mathematics: they can be regarded as true but not about numbers, sets or other mathematical objects; or as "strings of marks without meaning" Goodman and Quine (1947, p. 111); or as literally false Field (1980, p. 2 and *passim*). Nominalistic projects are usually concerned with the reconstruction of the success of mathematical *practice*. As Goodman and Quine expressed it, the challenge the nominalist has to face is to "account for the fact that mathematics can proceed with such remarkable agreement as to methods and results" Goodman and Quine (1947, p. 111). Moreover, mathematics apparently makes an important contribution to science, and the nominalist is challenged to account for this fact too. In what follows, we will sketch two ways in which contemporary nominalists have tried to give an account of mathematics, *eliminative structuralism* and *fictionalism*. We will see that both accounts make use of the notion of logical consequence at crucial places.

2.1. *Logical Consequence and Eliminative Structuralism*

Structuralism in the philosophy of mathematics holds that (pure) mathematics is the deductive study of structures as such. A nominalistic version of this view holds that mathematics does not, in fact, make statements about any abstract objects that constitute such structures (e.g., the natural numbers), but about whatever occupies the positions in a natural number system, a system of objects that happens to instantiate the relations that the realist assumes to be obtaining between the natural numbers. Accordingly, a mathematical statement like

(M) $3 + 9 = 12$

would be interpreted as

(M*) In any natural number system S, the object in the 3-place-of-S S-added to the object in the 9-place-of-S results in the object in the 12-place-of-S. Shapiro (1997, p. 85)

This way, the ontologically problematic statements of mathematics also come out nominalistically acceptable: 'numbers exist' comes to 'every natural number system has objects in its places':

> The programme of rephrasing mathematical statements as generalizations is a manifestation of structuralism, but it is one that does not countenance structures, or mathematical objects for that

> matter, as bona fide objects. Talk of numbers is convenient shorthand for talk about all systems that exemplify the structure. Talk of structures generally is convenient shorthand for talk about systems. (Shapiro 1997, p. 85)

Such an "eliminative structuralism" faces the *problem of vacuity*: in order to make sense of large parts of mathematics, one seems forced to accept a rich background ontology, and the nominalist just lacks the resources to provide one. To see how the problem arises, consider φ to be a sentence in the language of arithmetic.[4] Eliminative structuralism understands φ as something of the form

(φ^*) For any system S, if S exemplifies the natural number structure, then $\varphi[S]$.

$\varphi[S]$, here, is obtained from φ by interpreting the arithmetic terminology and the variables in terms of the objects and relations of S. Let us assume that there are only finitely many concrete objects. In this case, φ^* comes out vacuously true, no matter what φ is, since nothing exemplifies the natural number structure. Accordingly, an eliminative structuralist account of arithmetic will need to assume infinitely many objects in its background ontology, an eliminative structuralist account of Euclidean geometry a background ontology of the cardinality of the continuum, etc.

There seem to be two ways for the nominalist to meet this challenge. One way would be to assume that there are actual, concrete structures that can play the role of the background ontology. At least for all cases of applied mathematics the nominalist will need to hold that there is such an actual system that instantiates the mathematical structures applied. The problem here seems to be that mathematical truth and falsity will depend on contingent matters about the actual world, namely on whether there are actual systems of objects that instantiate the mathematical structures. But does the falsity of 'All even numbers are prime' depend on whether or not there exist infinite totalities of concrete entities that constitute ω-sequences?[5]

It is not entirely clear how strong this counter-argument is. After all, the nominalist typically feels forced to give an account for mathematics in the first place, because of the apparent usefulness of mathematics for science. But *that* mathematics is useful for science could be considered a contingent matter, accordingly, the nominalist need only explain why mathematics works, when it does. In a world with only a few objects, it might be possible to use mathematical structures that only require a comparatively small background ontology; in a finite universe, for instance, a mathematics of finite structures might be all that is require for science.

A similar reply, that would bring us closer to the second way for the nominalist to answer the problem of vacuity, could point out that also the platonist alternative will assume that there are *actually* enough abstract objects to play the role of the background ontology. Doesn't that make mathematical truth and falsity equally counterfactually dependent on the existence of abstract objects? The difference is that platonist are prepared to assume that their abstract objects exist *necessarily*, while the actual systems of concrete entities of the nominalist exist only contingently.

[4] The example is taken from Shapiro (1997, p. 85).
[5] For a related worry see Wilholt (2006).

But instead of assuming sufficiently rich actual systems that might or might not exist, depending on the way the world is, in order to instantiate the mathematical structures, one could rephrase eliminative structuralism as a claim about possible systems, the possibility of which depends on logic alone. Accordingly, our sentence of arithmetic φ would be interpreted as saying

(φ^{**}) For any *logically possible* system S, if S exemplifies the natural number structure, then $\varphi[S]$.

or as

(φ^{***}) Necessarily, for any system S, if S exemplifies the natural number structure, then $\varphi[S]$.

Geoffrey Hellman (1989) carries out such a programme of *modal eliminative structuralism*. Instead of assuming that there are actual systems that instantiate mathematical structures, the modal eliminative structuralist would only assume that there are possible systems that exemplify these structures — what could be ontologically less problematic? Merely assuming that some systems are possible should not commit one to anything, or so one might hope.

However, there is a problem: in order for his account to work, the modal eliminative structuralist either needs an account of logical possibility or must assume that the notion of logical possibility is primitive and not in need of further explication. The platonist can challenge the latter assumption and claim that our grasp of the modal terminology when applied to mathematics is too sophisticated to be considered primitive. It is rather mysterious how we manage to use these notions the way we do if it was not for the fact that they are themselves mathematically mediated:

> When beginning students are first told about logical possibility, logical consequence, etc., most of them seem to have *some* idea of what is meant, but consider how much their initial "intuitions" differ from our "refined" ones. The anti-realist owes us some account of how we plausibly could come to understand the notions in question (as applied here) *as we in fact do*, independent of our mathematics. Without this it is empty to use a word like "primitive" [...]. (Shapiro 1993, p. 475)

These worries concerning the involvement of modality might be silenced by pointing out that logical possibility is unproblematic since it can be defined with recourse to logical consequence or logical truth: φ is logically possible if and only if $\neg\varphi$ is not a logical truth. (We assume that logical truth is defined as a degenerate case of logical consequence — see section 3. for our concrete proposal.)

In this case, however, an account of logical consequence is needed that does not rely on abstract objects, in the way that the model-theoretic explication does.

2.2. *Logical Consequence and Fictionalism*

In order to see that this problem not only arises for the modal eliminative structuralist, we will briefly sketch the problem as it arises for Hartry Field's fictionalism.

Field tries to defend nominalism against the objection that mathematical objects must exist to the best of our knowledge, by trying to undercut the platonist's *Indispensability*

Argument. In particular, he tries to undercut the platonist's claim that mathematics is indispensable for science, instead he attempts to show how science can be done "without numbers" Field (1980, 1989).

The idea is the following: Field formulates part of physics — namely, Newtonian mechanics — in a way that does not involve reference to any abstract objects. He formulates this nominalistic theory N using a second-order mereology (at least in one version of the programme). He then proves metatheoretically that a platonistic extension of N, $N + S$, is *conservative* with respect to N: there are no nominalistically statable consequences of $N + S$ that are not consequences of N alone. In other words: in deriving nominalistically statable conclusions, mathematics basically does the same as logic. Logic might do it in a more long-winded fashion than mathematics, so the latter has a practical use in making derivations shorter and more elegant, but any consequence is also available without mathematics.

There are two places in which logical consequence seems to matter. First it seems to matter when spelling out the second-order logic the nominalist wants to use to accompany nominalized physics. Some have articulated the worry that the logic used here might already undermine the nominalistic enterprise (e.g. Resnik (1985, p. 163)). However, the more urgent problem seems to come with the fact that the conservativeness claim itself is formulated in terms of logical consequence:

> [T]he fictionalist thesis of conservativeness is stated in terms of logical consequence, and the two best historical explications of this are unavailable to the fictionalist. (Shapiro 1993, p. 461)

Again, Field assumes like Hellman that the notion of logical possibility can be taken as a primitive Field (1991). Of course he must then face the same epistemological challenges as the modal eliminative structuralist.

3. *Logical Consequence without Models*

3.1. *Inferentialism*

The most generally accepted explication of logical consequence is, no doubt, the model-theoretic construal which goes back to the work of Alfred Tarski. Logical consequence is, in the modern formulation, defined using set theory: a sentence φ is a logical consequence of a set of sentences Γ if and only if φ is satisfied by all models that satisfy all members of Γ. The models mentioned here are sets, and satisfaction is defined in set-theoretic terms.

That this notion is not available to the nominalist has been remarked above, and this point was raised by many authors, in particular in connection to Field's programme.[6] There is another approach to logical consequence, however, that at least *prima facie* does not involve nominalistically unpalatable notions, which goes back to the work of Gerhard Gentzen (1935) and puts the logical inference rules in the centre of attention. In recent times such approaches, which variably have been subsumed under the labels 'inferentialism' or 'proof-theoretic semantics', have received considerable attention; prominent

[6] In addition to Field himself, e.g. Field (1989, pp. 30–31), see for instance Resnik (1983, 1985, §2), Hale and Wright (1992, 1994), Shapiro (1983, 1993, §2), MacBride (1999, §4).

proponents include Michael Dummett, Ian Hacking and Dag Prawitz.[7] Inferentialism insists that the meaning of the logical constants is determined by their introduction- and elimination-rules, and that these rules (so far as they are the correct ones) are self-justifying. No further appeal to model-theoretic semantics, truth-tables or the like is needed in order to argue for the validity of the rules.

As is well known, conditional, negation, and universal quantification suffice to characterise classical logic,[8] so we will restrict our attention to these connectives. According to the moderate inferentialism recommended here, logical consequence is characterised by the following rules which can be added to any formal language that contains '$\supset$', '$\neg$', and '$\forall$', and has appropriate syntactical categories for these rules to operate on:[9]

$$\frac{\begin{array}{c}[\varphi]\\ \vdots\\ \psi\end{array}}{\varphi \supset \psi}\supset\mathrm{I} \qquad \frac{\begin{array}{cc}[\varphi] & [\varphi]\\ \vdots & \vdots\\ \psi & \neg\psi\end{array}}{\neg\varphi}\neg\,\mathrm{I} \qquad \frac{\Phi(t)}{\forall x\Phi(x)}\forall\,\mathrm{I}$$

$$\frac{\varphi \quad \varphi \supset \psi}{\psi}\supset\mathrm{E} \qquad \frac{\neg\neg\varphi}{\varphi}\neg\,\mathrm{E} \qquad \frac{\forall x\Phi(x)}{\Phi(t)}\forall\,\mathrm{E}\ .$$

The $\forall$I-rule has the standard proviso that 't' does not occur free in Φ or in any of the relevant assumptions. The square parentheses in $\supset$I and $\neg$I indicate that φ is an assumption that is discharged by the application of these rules (thus, strictly speaking, we have also the rule of assumption).[10]

Logical consequence is then explicated using these rules:

> A sentence φ is a *logical consequence* of some premises Γ if and only if there is a derivation of φ from Γ whose single lines are either sentences of Γ, result by applications of the above rules from previous lines, or are assumptions that are discharged by applications of $\neg$I or $\supset$I.
>
> φ is a *logical truth* if and only it is thus derived using no undischarged premises.

The trouble with this definition is that it mentions notions that do not appear to be readily available to the nominalist: sentence, line and derivation, which are usually taken to be abstract types. If we want to spell out everything in detail, we will also have to mention variables, terms, logical symbols, and more. Also, 'Γ' looks suspiciously like the

---

[7] See, e.g., Dummett (1973), Hacking (1979), Prawitz (1965, 1971, 1991), but also Milne (1994), Read (2000), Wright (2007).

[8] There is, of course, an issue concerning whether the logic should indeed be classical. This is, however, assumed by Goodman and most contemporary nominalists and will not be discussed here. Needless to say, inferentialism is not committed to classical rules of inference.

[9] For simplicities sake, we will assume that no other logical axioms or rules are present for the language in question prior to the introduction of these rules.

[10] For more details see, for example, Prawitz (1965).

name of a set. The definition has, hence, as of yet some significant gaps that are still to be filled in — in a nominalistically acceptable way. This problem will be addressed in section 4.

3.2. Second-Order Logic

With this conception of logical consequence at hand (*modulo* the gaps that will be filled in §4.), logical means more powerful than those provided by first-order logic become available. We can supplement our rules by a pair of rules for second-order quantifiers. Second-order logic allows generalisation into predicate-position in much the same way that first-order logic allows generalisation into name-position: with quantifiers binding variables that take the place of these expressions.[11] As we will see in the final section, the adoption of second-order logic is crucial to several nominalistic projects. We will see in §4., however, that the second-order quantifier also comes in handy in our explication of logical consequence, albeit without being strictly required.

The rule for the second-order quantifiers we add are:

$$\frac{\Phi(T)}{\forall X^n \Phi(X^n)}\,\forall^2\,\mathrm{I} \qquad\qquad \frac{\forall X^n \Phi(X^n)}{\Phi(\Xi)}\,\forall^2\,\mathrm{E}\ .$$

In $\forall^2$ I, T is a n-place predicate letter or free variable that must not occur free in Φ or any of the relevant assumptions. In $\forall^2$ E, Ξ is an open sentence with n argument places;[12] no variables in Ξ are to be bound in $\Phi(\Xi)$ that are not already bound in Ξ.

For sure, second-order logic has attracted a profusion of criticism. Most prominent amongst the complaints are incompleteness allegations, and Quine's famous claim that second-order logic is nothing but "set theory in sheep's clothing" Quine (1970, pp. 66–68). The former complaint is usually framed thus: the non-axiomatisable consequence relation of second-order logic on the standard model-theoretic conception is intractable and does, hence, not qualify as *logical* consequence. We will content ourselves here with the observation that the second-order consequence relation we are after is proof-theoretic, and that therefore this intractability objection does not arise. This paper is not the place for a more detailed discussion.[13]

---

[11] For more details compare the current "bible" of second-order logic: Shapiro (1991). Shapiro favours the model-theoretic conception of logic. For a proof-theoretic charactersation see, e.g., Prawitz (1965).

[12] We allow instantiation with open sentences in this rule, rather than just predicate letters, in order to gain the proof-theoretic strength of what, in axiomatic systems of second-order logic, is know as the *comprehension schema*: $\exists X^n \forall \langle x \rangle_n (X^n \langle x \rangle_n \equiv \Phi \langle x \rangle_n)$ (where '$\langle x \rangle_n$' abbreviates a string of variables, '$x_1 x_2 \ldots x_n$'). We bracket a discussion of comprehension here, and also dodge the issue of quantification over functions which is usually included in a formulation of second-order logic. For details see Shapiro (1991, §3.2). Although it is irrelevant here, the reader might be interested in noting that n-place functions can be simulated by using $(n+1)$-place predicates: the clause $\ulcorner \forall \langle x \rangle_n \exists! y\, F^{n+1} \langle x \rangle_n y \urcorner$ indicates that $\ulcorner F^{n+1} \urcorner$ is in effect an n-place function (where '$\exists!$' stands for the first-order definable 'there is exactly one').

[13] Such a discussion can, however, be found in Rossberg (2004). The for the inferentialist more pressing problem of a proof-theoretic notion of incompleteness is discussed in Rossberg (2006) and Wright (2007).

Quine's complaint about the set-theoretic commitment that second-order quantification allegedly brings about has been contested in various places, on various grounds. To name but a few: George Boolos famously provided a plural interpretation of the monadic second-order quantifiers that dispenses with any commitment to sets and which at least David Lewis found nominalistically acceptable.[14] Crispin Wright argued that second-order quantification cannot bring about new ontological commitment: if predicates, as Quine contends for instance in Quine (1948), do not themselves carry any ontological commitment to sets (or properties), then this commitment cannot suddenly arise when one generalises into predicate position; much like first-order quantification does not suddenly bring about commitment to new objects when applied in a language that contains non-referential terms.[15] Even assuming that Quine is correct about the ontological commitment of a theory being exhibited by the first-order quantifiers, there does not appear to be any way of arguing from there that the second-order quantifiers bring about a commitment to sets.

As mentioned above, the involvement of sets is obvious if a model-theoretic approach to logical consequence is chosen. This, however, is nothing peculiar to second-order logic on this conception, but is the case for ordinary first-order logic as well. But since we here attempt to manage without model theory altogether, this problem does not arise either. We thus leave the discussion at this stage in order to return to the question of the significant gaps that still remain in our explication of logical consequence.

4. *Proofs and Tokens*

4.1. *Concatenation Theory*

The problem of providing a nominalistically acceptable theory of syntax and proof-theory for a formal language was tackled by Goodman and Quine in their joint paper "Steps Toward a Constructive Nominalism" using the Calculus of Individuals, developed by Henry S. Leonard together with Goodman, and a theory of token-concatenation.[16] Their effort has been found wanting due to a couple of limitations, which we aim to overcome here.

---

[14] Boolos (1984, 1985), Lewis (1991). The plural interpretation has subsequently been further developed, see e.g. Rayo (2002), Rayo and Uzquiano (1999). Agustín Rayo and Stephen Yablo Rayo and Yablo (2001) attempt to provide an interpretation of polyadic second-order quantification, roughly along Boolos's lines and inspired by Arthur Prior's work Prior (1971, chapter 2). Peter Simons also draws conclusions with respect to the debate about higher-order logic from Prior; see Simons (1993, 1997). The plural interpretation itself has been found wanting in various respects — see, e.g., Linnebo (2003), Resnik (1988) or Shapiro (1991, §9.1.1) — and is not further pursued for the purpose of the present paper.

[15] Wright (1983, pp. 132–133), more recently again and in more detail in Wright (2007). Rayo and Yablo (2001) suggest a similar principle.

[16] Goodman and Quine (1947); see also Martin (1958) for a detailed study of token-concatenation theories in this context. The Calculus of Individuals was published in Leonard and Goodman (1940); for a study of its development see Rossberg (2009). An investigation into second-order versions of calculi of individuals can be found in Niebergall (2009).

To build a syntax for a formal language, we have to be able to say what a well-formed formula — or (open) sentence — of this language is. To do this in a nominalistically acceptable way, not only mention of sets has to be avoided, the sentences themselves also have to turn out not to be abstract entities. Goodman and Quine suggest to make sense of what a sentence is by identifying it with its concrete inscriptions. Marks on paper, in instance, can be said to fall under a predicate 'Fmla' (for well-formed formula). In order to give a definition of this predicate, they start out with primitive predicates that are true of concrete inscriptions if these have the familiar shape of the logical symbols, variables, etc., and then build up the language in a way analogous to the common recursive definition of a language that the platonist uses. To do so, they use a primitive three-place predicate 'C' which applies to token inscriptions. '$\mathrm{C}(x, y, z)$' expresses that x is a token inscription that is the concatenation of y and z. For convenience, we can define a four-place predicate '$\mathrm{C}(x, y, z, w)$' as '$\exists t(\mathrm{C}(x, y, t) \wedge \mathrm{C}(t, z, w))$', and analogously for five- and six-place predicates for concatenation (the number of terms following the 'C' will disambiguate which predicate it is).

But first things first. The primitive unary predicates we will be using are 'Vee', 'UVee', 'Ac', 'LPar', 'RPar', 'Neg', 'Cond', 'UpsA', which are true of physical objects if they have the shape of a lower case 'v', and upper case 'V' (for use as first- and second-order variables, respectively), an accent '$'$', a left parenthesis '(', a right parenthesis, ')', a negation sign, '$\neg$', a conditional sign, '$\supset$', and an upside-down A, '$\forall$', respectively.[17]

Let a character, 'Char', be any of the things that the predicates above are true of Goodman and Quine (1947, p. 113):

$$\mathrm{Char}(x) =_{df} \mathrm{Vee}(x) \vee \mathrm{UVee}(x) \vee \mathrm{Ac}(x) \vee \mathrm{LPar}(x) \vee \mathrm{RPar}(x) \vee \mathrm{Neg}(x) \vee \mathrm{Cond}(x) \vee \mathrm{UpsA}(x).$$

And let an inscription, 'Insc', be either a character, or a concatenation (note that concatenation was introduced as applying only to inscriptions — fully explicit, a recursive definition would be in order):

$$\mathrm{Insc}(x) =_{df} \mathrm{Char}(x) \vee \exists y \exists z \mathrm{C}(x, y, z).$$

The first thing we obviously need for the syntax is a sufficient supply of distinct variables. These can be formed out of lower- and upper-case vees, concatenated with strings of accents. For this, we can define a string of accents as

$$\mathrm{AcString}(x) =_{df} \mathrm{Insc}(x) \wedge \forall y((\mathrm{Part}(x, y) \wedge \mathrm{Char}(y)) \supset \mathrm{Ac}(y)).$$

[17] Goodman and Quine are more economic in their choice: the replace '$\neg$' and '$\supset$' by the Sheffer stroke, '|', for alternative denial, in terms of which the former two are definable; they also form the universal quantifier using parenthesis, '(v)', in lieu of '$\forall v$', and do not have upper case variables. Since they aim to provide a syntax for first-order set theory, they have the additional 'ϵ' for membership. There might be concerns about left and right parentheses having the same shape, albeit rotated by 180°. To dissolve such worries, one could either appeal to the orientation of the inscription, or use a different shape for one of the parenthesis, say, ']' instead of ')'. See Goodman and Quine (1947, p. 112).

(Again, a recursive definition would have been possible that takes one accent as the base case, and defines in the recursion step an accent-string to be any concatenation of accent strings.) Note that 'Part' is the the two-place predicate that is introduced and axiomatised in the class-free subsystem of the Calculus of Individuals Leonard and Goodman (1940).

First- and second-order variable, 'FVbl' and 'SVbl', respectively, can thus be defined:

$$\mathrm{FVbl}(x) =_{df} \mathrm{Vee}(x) \vee \exists y \exists z(\mathrm{Vee}(y) \wedge \mathrm{AcString}(z) \wedge \mathrm{C}(x, y, z))$$
$$\mathrm{SVbl}(x) =_{df} \mathrm{UVee}(x) \vee \exists y \exists z(\mathrm{UVee}(y) \wedge \mathrm{AcString}(z) \wedge \mathrm{C}(x, y, z)).$$

That is, any lower-case vee, possibly followed by a string of accents, is a first-order variable, and any upper-case vee, possibly followed by a string of accents, is a second-order variable.

Goodman and Quine go on to develop the syntax like this in a painstaking detail which we will not go into here. They define quantifiers, (in our case an upside-down A followed by a variable, orders distinguished by the order of the variable), atomic formulae, and formulae. They then inscriptionally set up some logical axioms, which we, of course, dispense with here. It follows the definition of a substitution, immediate consequence (a formula that can be arrive at by one application of a rule), that of a line (of a proof), and lastly a proof itself (as a list of lines all of which are immediate consequences of previous lines or axioms). Since we allow for assumptions, our construction first defines a derivation as a list of lines all of which are immediate consequences of previous lines or assumptions; a proof (of φ) will then be a derivation whose last line (φ) does not depend on any undischarged assumption. A theorem, finally, is the last line of a proof.

These sketchy remarks on the construction must here suffice as a hint on the actual construction. Goodman and Quine give their construction in full detail,[18] and this is easily amended to suit our proposal here if our hints above are followed. Note that we have not included any constants in the language, neither names nor predicate constants. Thus, all our formulae so far contain only variables and logical constants ('$\forall$', '$\neg$', '$\supset$'). Identity is standardly defined in second-order logic (by Leibniz' Law), and other constants can easily be introduced into the construction of the language.[19]

4.2. *The Proof is Out There*

If the notion of proof thus defined only encompassed discernible marks on paper, the consequence relation defined with its help would be very restrictive. Goodman and Quine, indeed, suggest that instead we take inscriptions to be any appropriately formed portion of matter, whether it is against a contrasting background or not.

[18] See Goodman and Quine (1947). Note that Goodman and Quine's definition D10 is defective, but easily mended, as noted by Henkin in Henkin (1962, p. 192, fn. 3). See also Martin (1958).

[19] Goodman and Quine mention in various passages of Goodman and Quine (1947) the problem they have in defining the ancestral. Leon Henkin Henkin (1962) provides a solution; Goodman later states Goodman (1972, p. 153) that the technique he developed himself in Goodman (1977, chapters IX and X) will serve the purpose. Since we assume that the version of second-order logic presented here is nominalistically acceptable, we can simply rely on Frege's original defintion of the ancestral in Frege (1879, §26).

> Then the only syntactical description that will fail to have inscriptions answering to them will be those that describe inscriptions too long to fit into the whole spatio-temporally extended universe. This limitation is hardly likely to prove embarassing. (If we ever should be handicapped by gaps in the proof of an inscription wanted as a theorem, however, we can strengthen our rules of inference to bridge such gaps; for, the number of steps required in a proof depends on the rules, and the rules we have adopted can be altered or supplemented considerably without violation of nominalistic standards.) (Goodman and Quine 1947, p. 121)

We suggest the amendment to allow to count as inscriptions any appropriately formed space-time region, whether it is occupied by matter or not. Let us also note that even employing more and more abbreviations, other useful definitions and rules other our primitive ones, in a finite universe, we will *eventually* run out of actual concrete inscriptions, no matter how generously construed. We thus suggest to side with Field Field (1980) and take it for granted that our universe is infinite, and, in fact, contains a continuum of space-time points (i.e., $2^{\aleph_0}$-many); and we do not consider this as a violation of nominalism[20] — until further notice. We discuss the infinity of the universe in this and other respects in the next section.

This generous conception of an inscription might seems objectionable to some, at first glance, since it means that all proofs are already out there — and quite literally so. We literally discover proofs, that is, the space-time regions that are proofs, e.g. by outlining the proof-shaped regions with a pencil (note, however, that this is not the only way to learn that a proof exists). The initial feeling of offence will in most cases subside when it is pointed out that the situation is exactly analogous for the platonist who takes proofs to be abstract objects: types, for instance. These are also commonly assumed to exist independently of anyone finding them (e.g. by tokening them). The only difference is that the nominalist's proofs are concrete objects. In principle, it should thus also be possible to use nominalistic analogues of any way of demonstrating the existence of a proof that the platonist uses. Next to transcribing it, there is, for example, proving that the inscription must exist. (This proof will be an inscription again, but it need not be the proof whose existence is thus demonstrated.)

Rejecting the generous conception of an inscription, and thus not only denying an actual infinity of proofs, but also the existence of proofs that have not been written down, would mean to adopt a position even more radical than Goodman's nominalism. It seems that Stanisław Leśniewski embraced this very restrictive conception of proof see Simons (2002), which not only entails that proofs come into existence when they are first written down, but also that they cease to exist when the last inscription (narrowly construed) is destroyed. Irrespective of how appealing this position is, it does not appear that a criticism along these lines could coherently be put forward by the platonist.

In our infinite universe there are thus infinitely many concrete proof-inscriptions (understood in the generous way specified above). Moreover, there are enough in the sense that there are *all* the proofs that a platonistic version of inferentialism accounts for. Our explication of logical consequence is thus co-extensive with the platonistic inferentialist account of logical consequence.

---

[20] This, to be sure, has been contested; see, e.g., MacBride (1999).

5. *Is this Enough? (Or is it too much?)*

With the help of concatenation theory, an inferentialist conception of logical consequence, and speculations about the size of the actual universe, we arrived at an explication of logical consequence that might seem nominalistically acceptable. Whether it indeed *is* nominalistically acceptable is a question that does not allow for a straightforward answer. First of all, nominalists might disagree with one another about what resources in fact count as nominalistically acceptable.

Further, they might disagree on what an explication is supposed to do, and, in particular, disagree about the relation in which explicatum and explicandum must stand to each other, in order for an explication to be adequate.

Finally, they might disagree about the use they want to make of the explicatum in their theories. We mentioned two nominalist programmes above and said what role the notion of logical consequence plays in their projects. Below, we will address the question to what extend the explication of logical consequence suggested here can be used in these projects. Space constraints will not allow us to pursue all these questions in sufficient detail.

5.1. *A Ballet Dancing Brick Layer*

Let us first turn to the assumptions we made about the size of the universe. To be on the safe side, we assumed that our universe is large enough to contain the inscriptions of all proofs the platonist assumes to exist. But does the assumed size of the universe not sin against the nominalist's standards and put everything in jeopardy that we have achieved so far? Some anti-nominalists have questioned our assumption that space-time points could be considered nominalistically acceptable, since they believe that the presumed mark of the concrete (having causal powers) is merely metaphorically instantiated by space-time points e.g. Resnik (1985), MacBride (1999). We will not go into this discussion here, and simply assume that *if* a nominalist (like Hartry Field) finds space-times points ultimately acceptable, then so be it. But we want to claim that our explication of logical consequence would also be acceptable for Goodman. Would he have accepted an assumption of uncountably many points of physical space-time?

Remembering Goodman's early paper with Quine, one might think that he would not have accepted such an assumption. The project by Goodman and Quine was not only nominalist, it was also finitist:

> We decline to assume that there are infinitely many objects. Not only is our own experience finite, but there is no general agreement among physicists that there are more than finitely many objects in all space-time. If in fact the concrete world is finite, acceptance of any theory that presupposes infinity would require us to assume that in addition to the concrete objects, finite in number, there are also abstract entities. (Goodman and Quine 1947, p. 106)

Goodman and Quine's project descended from their joint efforts with Alfred Tarski and Rudolf Carnap in the early 1940s to develop the foundations of arithmetic in a way that respects finitism (there is only a finite number of individuals), physicalism/reism (there are only physical things), and nominalism (there are only variables for individuals, not for universals).[21] The motivations for this project were already at that time rather heterogeneous. Tarski claimed not to understand languages that do not satisfy these conditions, thus citing an epistemic reason for these constraints. However, he apparently also thought that finitism is just a consequence of there being only finitely many objects in the world.

Carnap, on the other hand, shared Tarski's insistence on finitism only to a certain degree. Insofar as he was motivated to require finitism for the foundations of arithmetic, his motivation was empiricist: since each confirmation is based on finite observations, our knowledge is limited to the finite (independent of how many objects the world contains). But Carnap also claimed to understand infinite conceptions of arithmetic, making sense of them in terms of what we characterized above as modal eliminative structuralism:

> It seems to me that I actually understand, in a certain degree, infinite arithmetic [...]. To the question of Tarski and Quine, how I interpret this, when the number of things is perhaps finite: I do not know exactly but perhaps through mere positions instead of things [...]. A position is an ordering possibility for a thing. I do not have the intuitive rejection of the concept of possibility as Tarski and Quine do. (Carnap's notes on the discussions with Tarski and Quine, RC-090-16-25, Carnap Archives Pittsburgh, as cited in (Mancosu 2005, p. 344))

As Goodman and Quine report in the beginning of their 1947 paper, the discussion in the early 40s did not lead to a final solution. Goodman and Quine had a new idea though how to address the problem. Instead of trying to formulate arithmetic with finitely many objects, platonist mathematics was simply treated as a meaningless language that did not require any interpretation in terms of acceptable objects. Instead, they went meta-mathematical: they devised a nominalistically acceptable way to speak about the way that platonist mathematics, considered as a mere "apparatus", can work. Since this way of doing meta-mathematics did not involve notions of arithmetic, the problem of interpreting the numbers as concrete objects, which had bothered Carnap and Tarski, did not reoccur for their proposal.

Finitism is however not fully unproblematic for an inscriptionist account. Above we assumed — to be on the safe side — that the universe should comprise an uncountable number of space-time points in order to allow for inscriptions of proofs of arbitrary length. Goodman and Quine believed that such an assumption is not needed, as quoted above, since stronger rules and auxiliary definitions could be introduced.

As we said above, it is not clear that this move can help in all cases. Many authors have pointed out, however, that the nominalist is free to chose other ressources. Michael Resnik straightforwardly suggests that in order to develop a nominalistically acceptable meta-mathematics that is workable, the nominalist has to assume an infinite universe. A Fieldian conception of space-time would help providing one:

> If we followed Field we would find it much easier to develop a nominalistic syntax than did Quine and Goodman, because we find an infinitude of inscriptions in his already posited continuous space-time. (Resnik 1983, p. 518)

[21] Cf. Mancosu (2005).

Also Henkin (1962) admits that Goodman and Quine need not make any assumptions regarding finitism for their *nominalistic* project. Goodman, in particular, was prepared to divorce nominalism from finitism. In "A World of Individuals" (Goodman (1956), reprinted in Goodman (1972)) Goodman pointed out that nominalism is not logically connected to finitism:

> The nominalist is unlikely to be a nonfinitist only in much the same way a bricklayer is unlikely to be a ballet dancer. The two things are at most incongruous, not incompatible. Obviously by the stated criterion for nominalism [essentially, the rejection of classes], some systems with infinite ontologies are nominalistic, and some systems with finite ontologies are platonistic. (Goodman 1972, p. 166)

And later, in *Problems and Projects* Goodman (1972) he clearly seems to be ready to give up finitism for the sake of nominalism, in response to Alonzo Church's challenges to Goodman and Quine's finitistic syntax:

> In the first place, I should point out that this letter [by Alonzo Church, in which Church lists tasks that he thinks the finitist still has to accomplish] predated "A World of Individuals", where nominalism is carefully distinguished from finitism. Our position in "Steps" was indeed fintistic as well as nominalistic; but finitism, although a friendly companion of nominalism, is neither identical with nor necessary to it. (Goodman 1972, p. 154)

We thus content ourselves with the fact that Goodman would have found the assumption of an infinite universe nominalistically acceptable.[22]

However, there remains a problem: the assumption made about the size of the universe was introduced as an empirical hypothesis about the actual world. We usually assume such hypotheses to be *contingently* true, if true at all. This, however, appears to conflict with the very nature of logic. Logical consequence is usually assumed to be a matter of necessity. How can a contingent assumption serve as its foundation?

5.2. The Size of the Universe and Logical Consequence

Three problems need to be distinguished here. The first concerns a vague feeling, the second and third can be put forward in a precise way.

There might be a vague and uncomfortable feeling arising, given our explication of logical consequence, that the size of the universe, or the existence of some peculiarly shaped space-time regions, just should have nothing to do, generally speaking, with what follows *logically* from what. Vague worries are difficult to address, but here are some remarks which might help, at least, to get into the spirit.

First, a semi-technical point: the explication of logical consequence only indirectly depends on the size of the universe. The consequence relation is pinned down by the inferentialist proposal: what follows from what is determined by the inference rules. The trouble only arises in the *metatheory* when an explicit definition of logical consequence is asked for. There, logical consequence is defined as a certain relation. A relation requires relata, and for the nominalist nothing but concrete things can serve as such. The explication of a sentence, or, more generally, the provision of a nominalistically acceptable *syntax*, primarily involves the worrisome sentence tokens. But if infinitely

[22] Compare also Goodman's brief remark on Field in Goodman (1984, p. 53).

many sentences are indeed needed, and sentences are concrete objects, then there will have to be infinitely many concrete things.

Further, the nominalist might ask back what the opponent expected a nominalistically acceptable definition to look like. Obviously, it will only mention concrete entities, what else could it do? These are the only entities that nominalists allow themselves, after all. Note, however, that no cunning coding tricks are used in our proposal here which are employed elsewhere, in order to achieve arguably nominalistic reconstructions of mathematics. Let us emphasise that, rather than assuming a large enough ontology to allow for the interpretation of some mathematical notion,[23] the proposal here is in exact correspondence with the actual practice of proof (or, rather, the idealised version of it that is commonly assumed in the discussion of logical consequence). The ontology here assumed, concrete inscriptions of proofs, is precisely the gold standard of proof in logical and mathematical practice: the provision of a written down version of a proof to demonstrate that the inference holds. The explication of logical consequence presented here thus comes with an epistemology already attached. This was one of the motivations for the project in the first place.

The first precisely formulated problem concerns the counterfactual dependence of the extension of logical consequence given our proposal on the size of the universe. Let us here take for granted that the actual universe is infinite and thus big enough for the nominalist definition of logical consequence to be extensionally equivalent to the platonist inferentialist conception of logical consequence. The dependence on actual inscriptions makes logical consequence nevertheless counterfactually dependent on the size of the universe: suppose there is a possible finite universe and that there is some proof inscription that "uses up" all space-time in that universe. Say the last line of this proof, the consequence of the argument, is ψ, and one of the premises it rests on is φ, then it seems that the nominalist would be forced to say, that $\ulcorner \varphi \supset \psi \urcorner$ is *not* a logical consequence of the rest of the premises that ψ was originally derived from. The rule for $\supset$-introduction, also know as *conditional proof*, would of course licence the inference, but since we ran out of space-time, this inference cannot be drawn.[24] We would be forced to say that in *this* universe, $\ulcorner \varphi \supset \psi \urcorner$ is *not* a logical consequence of the premises, while in our (infinite) universe it is. But, surely, the size of the universe should not matter for the question, what follows from what.[25]

Goodman, we think, would not have considered this objection to be seriously damaging. His meta-philosophical conception of explications would have counted the explication we arrived at as adequate, since extensional equivalence is sufficient for this purpose. Indeed, Goodman is famous for insisting that even co-extensionality is too strong a requirement for adequacy of explications Goodman (1977, pp. 3–22). Goodman's weak requirements for adequate explications are certainly met by the account

---

[23] As, e.g., in Lewis (1991) or Niebergall (2005).

[24] This, in effect, amounts to Alonzo Church's demand, that the inscriptional account suggested by Goodman and Quine would have to be able, *inter alia*, to sustain the deduction theorem. The letter in which Church raises this criticism is published in Goodman (1972, pp. 153–154), alongside with Goodman's dismissive response.

[25] See Wilholt (2006, pp. 122–123) for a related worry.

proposed here. Thus we have at least an account of logical consequence before us that would have satisfied Goodman.

But there is the second problem which questions whether the the actual universe is, in fact, infinite and thus big enough for the nominalist definition of logical consequence to be extensionally equivalent to the platonist inferentialist conception of logical consequence.[26] Suppose the above derivation described again. If the universe turns out to be finite so that there is not enough space-time for an actual concrete derivation of $\ulcorner\varphi \supset \psi\urcorner$. In this case, $\ulcorner\varphi \supset \psi\urcorner$ is actually not a logical consequence of the relevant premises, despite the fact that the rule of conditional proof would licences the inference if only the universe was bigger.

Besides biting the bullet and admitting that $\ulcorner\varphi \supset \psi\urcorner$ does, in this case, not actually follow from the relevant premises, we can see three strategies that nominalists of different temperaments might adopt to avoid this problem.

If we are happy to allow ourselves modal notions, we could amend the proposal to let logical consequence be a relation between *possible* concrete sentence tokens. The nominalist could point out that the modality required for this treatment does not fall foul of the common obscurity objections with which modal notions are traditionally attributed. No vague speculation or guesswork is required to work out how the relation will extend beyond the actual universe. Our schematic rules determine this extension precisely. A slightly bigger universe would contain no surprises as to what follows from what. We merely have more inscriptions there that look exactly the same, only that some are longer, and behave in exactly the same way as the actual inscriptions do.

The case would, indeed, be much like the following. Imagine a mathematician who runs out of paper while scribbling down a certain proof just before she can write down the conclusion of the proof (which is an application of conditional proof). It would be madness to claim that, owing to the lack of paper and the resulting lack of the last line, there is no telling how the concrete inscription of the proof would go on, or that it is obscure to say that it is *possible* to extend the proof. It is perfectly obvious how the proof would continue if she had another sheet of paper. It does not matter for this purpose whether she *actually* bothers to get another sheet. It would also make no difference if there happened to be no more paper in the universe at all, or, indeed, anything else to write on. Obviously, the conclusion follows, whether our mathematician finds another sheet of paper or not, whether there exists any more paper in the universe or nor, or whether there is enough space-time for the last line or not. But this unproblematic notion of possibility is all that is needed in the "modal" version of the explication of logical consequence.

Further, this strategy has a variant: instead of quantifying over possible sentence tokens in the definition of logical consequence, one could employ a *constructibility* operator akin to that which Charles Chihara introduced in his nominalistic programme Chihara (1990). Logical consequence would not be a relation between possible sentence tokens, but between *constructible* sentence tokens, where 'constructible' is obviously not to be

[26] Note that it would be sufficient if *any* aspect of the (concrete) physical universe were infinite for a similar construction to go through. Thus, the case described here involves that *nothing* physical is infinite or infinitely divisible: not just space, but also time, electromagnetic force, wave-length, gravity, etc.

analysed as 'possible to construct' — otherwise this variant would be no more than a complication of the modal strategy above. What is constructible, again, would have to take us beyond what actually can be constructed given the finite cardinality of the universe that we here assume.

Lastly, a *counterfactual* approach could be taken. Counterfactuals should again best not be taken as to be reduced to possibility in order to not merely complicate the modal proposal, and there are reasons to refrain from such attempts in any case.[27] If, for instance, a Stalnaker-Lewis-style analysis of counterfactuals was given, one would end up not only with a yet to be explained notion of possibility (or possible world), but also in want of an explication of the similarity (or closeness) relations which analyses of this ilk require in addition. The definition of logical consequence on this account would thus still be in terms of a relation between concrete sentence tokens; it would, however, state something like: a sentence φ is a *logical consequence* of some premises Γ if and only if, if the universe were large enough to contain the required inscriptions, there would be a derivation of φ from Γ whose single lines are either sentences of Γ, etc.

These three proposals contain notions — possibility, constructiblity, counterfactual — which themselves are in want of explanation. If any of the three is adopted, one would have to let go of the hope to define it in terms of logical consequence, on pain of vicious circularity. This might seem a high price to pay. On the other hand, perhaps the techniques suggested here could be utilised for such explications. Attempting this is beyond the scope of this paper, but the beginning of an explication of possibility along the lines suggested here might be made using the inferentialist account of modality.[28]

In what follows, we discuss these proposals in relation to the above mentioned nominalistic projects.

5.3. *Two Nominalist Programmes Revisited*

It seems that, although the notion of logical consequence is available to the nominalist, there will still remain open problems in the nominalistic programmes that we have introduced above. Let us first discuss the problem as it arises for fictionalism, and then turn to the problem that seems to remain for modal eliminative structuralism.

As we have said, Field wants to prove the conservativeness of mathematics, in order to show that mathematics is — despite being a useful tool for shortening derivations — dispensable for the derivation of physical consequences from physical theory. It seems obvious that Field cannot establish the relevant conservativeness result using the explication of logical consequence provided here. As Stewart Shapiro Shapiro (1983) has shown, one can formulate a Gödel sentence, G, in the nominalistic language, such that G is not provable in the nominalistic theory N, but provable (via Field's bridge principles) in $N + S$, where S is, e.g., Zermelo-Fraenkel set theory. According to our analysis, mathematics is thus not conservative over nominalistic physics: there are logical consequences of $N + S$, formulated in the nominalistic language, that are not logical consequences of N, in the explicated sense of logical consequence.

[27] In fact, Timothy Williamson has recently suggest to do it the other way around: to analyse modal notions in terms of counterfactuals; see Williamson (2005).

[28] See for instance Prawitz (1965) and Read (2008).

For Hellman's project, things do not seem to be much better. First of all, explicating the notion of possibility that modal eliminative structuralism assumes in terms of logical consequence won't help if modal notions are essential for the explication of logical consequence (cf. the previous section). By this variant of the explication we would be led back in a full circle.

Taking either the counterfactual option or the proposal involving the notion of constructibility, the circularity worry does not arise (at least not initially), but what we have achieved is to patch up an account that was found wanting because it used an unexplained notion of logical possibility by using a further unexplained notion. This is little progress, if progress at all.

As we indicated above, if the nominalist accepts Goodmanian standards for the adequacy of explications, this problem might not arise, since in that case no notion of possibility needs to be assumed for the explication of logical consequence. However, the rich actual system of space-time could directly be put to use as an instantiation of all structures that a structuralist should[29] be interested in. If there is no reason to be a modal eliminativist in the case of logical consequence, there surely does not seem to be such a reason in the case of mathematics. Thus, while logical consequence on this variant of the explication might help the structuralist nominalist, it at the same time seems wholly superfluous.

A platonist might also sense a problem if — as in the case of Hellman's original proposal — the logic is assumed to be second-order. In this case, there will be a sentence, G, formulated exclusively in (second-order) logical vocabulary, such that G is not a logical truth of second-order logic, but a logical truth of third-order logic.[30] 'Logical truth' is here used in the inferentialist sense, such that something is a logical truth if it is a logical consequence of the empty set of premises, i.e. (in nominalistic terms) if there is a derivation-inscription of an inscription of that sentence that accords to the rules and contains no undischarged assumptions. We can call this the *non-conservativeness* of third-over second-order logic.[31]

The objection might be formulated thus: although G is not a logical truth (of second-order logic), it is nevertheless not the case that $\Diamond\neg G$, since $\neg G$ is ruled out by third-order logic. G is a theorem of third-order logic, and thus $\Box G$, i.e. $\neg\Diamond\neg G$.

It seems, however, that this problem might be surmountable with an amendment in the notion of possibility. There are two options that come to mind. First, one could be a relativist about logical possibility, second, one could be a minimalist about this notion.

According to relativism, $\neg G$ is not logically impossible *simpliciter*, but logically possible relative to second-order logic and logically impossible relative to third-order logic,

---

[29] If we assume that the structuralist is interested in making sense of mathematics as it is put to use in the sciences.

[30] Third-order logic is the next step up in an infinite hierarchy of n-th order logics, that is in many ways similar to the hierarchy of simple type theory. Second-level predicates are introduced to apply to the "ordinary" predicates of first- and second-order logic which are now called first-level predicates. Third-order variables stand in the place of second-level predicates in the same way that second-order variables stand in the place of first-level, i.e. ordinary predicates. Third-order quantifiers bind these.

[31] For a proof and discussion of the non-conservativeness of third- over second-order logic see Rossberg (2006); the proof is based on a result of Leivant (1994, §3.7).

because G is a theorem of third-order logic, but no theorem of second-order logic. Since G contains only second-order vocabulary, this move seems only promising if we assume that logical consequence and logical truth (and *a fortiori* the "meaning" of a sentence containing logical vocabulary), also in case of sentences that use only a restricted selection of logical vocabulary, is always determined "holistically" by all introduction and elimination rules of a given logic. Since G is a sentence of two different logics (second-order and third-order logic), it thus can be a logical truth in one, while failing to be a logical truth in the other.

One might also attempt to surmount the problem by being a minimalist about logical possibility, such that some sentence S counts as a logical possibility *simpliciter* only if there is no logic in the hierarchy of (n-th)-order logics (or maybe an even wider range of logics), such that $\neg S$ would be a logical truth in that logic (and, accordingly, some sentence T is a logical truth *simpliciter* if it is a logical truth of some logic). This paper is not the place to discuss these options in any detail, but note that Crispin Wright suggests, for related reasons, introducing the quantifiers of all orders up to ω (and possibly beyond) at once.[32]

That we have no better news for nominalistic programmes hardly is the fault of the notion of logical consequence. If logical consequence in a nominalistically acceptable conception is less "powerful" than the platonist notion, then this merely means that there is work yet to be done for the nominalist, but not that they should resign themselves to relying on platonistic smoke-screens.

If logical consequence turns out to be insufficient to explicate some notion that a particular nominalist project requires, then the task will have to be done with the assistance of other nominalistically acceptable means. Using, on the other hand, a notion of logical possibility as an inexplicable primitive seems to have the same advantages as theft over honest toil.[33]

REFERENCES

Boolos, G., 1984. To Be is to Be the Value of a Variable (or to Be Some Values of Some Variables). *Journal of Philosophy*, 81: 430–450.

Boolos, G., 1985. Nominalist Platonism. *Philosophical Review*, 94: 327–344.

Burgess, J.P. and Rosen, G., 1997. *A Subject with No Object: Strategies for Nominalistic Interpretation of Mathematics*. Oxford: Clarendon Press.

Chihara, C., 1990. *Constructibility and Mathematical Existence*. Oxford: Oxford University Press.

Dummett, M., 1973. The Philosophical Basis of Intuitionistic Logic. In *Truth and Other Enigmas*, Dummett 1978, pp. 215–247.

Dummett, M., 1978. *Truth and Other Enigmas*. London: Duckworth.

Dummett, M., 1991. *The Logical Basis of Metraphysics*. Cambridge, MA: Harvard University Press.

Ernst, G., Scholz, O. and Steinbrenner, J. (eds), 2009. *Nelson Goodman: From Logic to Art*. Frankfurt: Ontos.

---

[32] Wright (2007), but see also Rossberg (2006) for a problem with this proposal.

[33] This paper was presented at the workshop "Abstract Objects in Semantics and the Philosophy of Mathematics" in Paris, February 28 – March 1, 2008. We would like to thank the audience for the helpful discussion. We are also indebted to Catarina Dutilh Novaes and Michael Gabbay for comments and criticisms of an earlier version of this paper.

Field, H., 1980. *Science Without Numbers*. Princeton: Princeton University Press.
Field, H., 1989. *Realism, Mathematics and Modality*. Oxford: Blackwell.
Field, H., 1991. Metalogic and Modality. *Philosophical Studies*, 62: 1–22.
Frege, G., 1879. *Begriffsschrift: Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Verlag L. Nebert, Halle a.d. Saale. English translation by S. Bauer-Mengelberg in van Heijenoort 1967, pp. 1–82.
Gentzen, G., 1935. Untersuchungen über das Logische Schliessen. *Mathematische Zeitschrift*, 39: 176–210, 405–431.
Goodman, N., 1956. A World of Individuals. In I. M. Bochenski, A. Church and N. Goodman (eds), *The Problem of Universals*. Notre Dame, Indiana: University of Notre Dame Press, pp. 13–31.
Goodman, N., 1972. *Problems and Projects*. Indianapolis and New York: The Bobbs-Merrill Company, Inc.
Goodman, N., 1977. *The Structure of Appearance*. Dordrecht: Reidel.
Goodman, N., 1984. *Of Mind and Other Matters*. Cambridge, MA: Harvard University Press.
Goodman, N. and Quine, W. V., 1947. Steps Toward a Constructive Nominalism. *The Journal of Symbolic Logic*, 12: 105–122.
Hacking, I., 1979. What is Logic?. *Journal of Philosophy*, 76: 285–319.
Hale, B. and Wright, C., 1992. Nominalism and the Contingency of Abstract Objects. *Journal of Philosophy*, 89: 111–135.
Hale, B. and Wright, C., 1994. A *Reductio Ad Surdum*? Field on the Contingency of Mathematical Objects. *Mind*, 103: 169–184.
Hellman, G., 1989. *Mathematics Without Numbers*. Oxford: Oxford University Press.
Henkin, L., 1962. Nominalistic Analysis of Mathematical Language. In E. Nagel, P. Suppes and A. Tarski (eds), *Logic, Methodology and Philosophy of Science: Proceedings of the 1960 International Congress*. Stanford: Stanford University Press, pp. 187–193.
Leivant, D., 1994. Higher-Order Logic. In D. M. Gabbay, C. J. Hogger, J. A. Robinson and J. Siekmann (eds), *Handbook of Logic in Artificial Intelligence and Logic Programming. Volume 2: Deduction Methodologies*. Oxford: Oxford University Press, pp. 230–321.
Leonard, H.S. and Goodman, N., 1940. The Calculus of Individuals and its Uses. *Journal of Symbolic Logic*, 5: 45–55.
Lewis, D., 1991. *Parts of Classes*. Oxford: Basil Blackwell.
Linnebo, Ø., 2003. Plural Quantification Exposed. *Noûs*, 37: 71–92.
MacBride, F., 1999. Listening to Fictions: A Study of Fieldian Nominalism. *British Journal for the Philosophy of Science*, 50: 431–455.
Mancosu, P., 2005. Harvard 1940–1941: Tarski, Carnap and Quine on a Finitistic Language of Mathematics for Science. *History and Philosophy of Logic*, 26: 327–357.
Martin, R.M., 1958. *Truth and Denotation: A Study in Semantical Theory*. Chicago: University of Chicago Press.
Milne, P., 1994. Classical Harmony: Rules of Inference and the Meaning of the Logical Constants. *Synthese*, 100: 49–94.
Niebergall, K.-G., 2005. Zur nominalistischen Behandlung der Mathematik. In J. Steinbrenner, O. Scholz and G. Ernst (eds), *Symbole, Systeme, Welten: Studien zur Philosophie Nelson Goodmans*. Heidelberg: Synchron Wissenschaftsverlag der Autoren, pp. 235–260.
Niebergall, K.-G., 2009. On Second Order Variants of Calculi of Individuals. *Theoria* in this issue.
Parsons, C., 1990. The Structuralist View of Mathematical Objects. *Synthese*, 84, pp. 303–346.
Prawitz, D., 1965. *Natural Deduction: A Proof-Theoretical Study*. Stockholm: Almqvist and Wiksell.
Prawitz, D., 1971. Ideas and Results in Proof Theory. In J. E. Fenstad (ed), *Proceedings of the 2nd Scandinavian Logic Symposium*. Amsterdam: North-Holland, pp. 235–308.
Prior, A.N., 1971. *Objects of Thought*. Oxford: Oxford University Press.
Quine, W.V., 1948. On What There Is. *Review of Metaphysics*, 2: 21–38.
Quine, W. V., 1970. *Philosophy of Logic*. Cambridge, MA: Harvard University Press.
Rayo, A., 2002. Word and Objects. *Noûs*, 36: 436–464.

Rayo, A. and Uzquiano, G., 1999. Toward a Theory of Second-Order Consequence. *Notre Dame Journal of Formal Logic*, 40: 315–325.

Rayo, A. and Yablo, S., 2001. Nominalism Through De-Nominalization. *Noûs*, 35: 74–92.

Read, S., 2000. Harmony and Autonomy in Classical Logic. *Journal of Philosophical Logic*, 29: 123–154.

Read, S., 2008. Harmony and Modality. In C. Dégremont, L. Keiff and H. Rückert (eds), *Dialogues, Logics and other Strange Things: Essays in Honour of Shahid Rahman*. London: College Publications, pp. 285–303.

Resnik, M.D., 1983. Review of Field 1980. *Noûs*, 17: 514–519.

Resnik, M.D., 1985. How Nominalist is Hartry Fields Nominalism?. *Philosophical Studies*, 47: 163–181.

Resnik, M.D., 1988. Second-Order Logic Still Wild. *Journal of Philosophy*, 85: 75–87.

Rossberg, M., 2004. First-Order Logic, Second-Order Logic, and Completeness. In V. Hendricks, F. Neuhaus, S. A. Pedersen, U. Scheffler and H. Wansing (eds), *First-Order Logic Revisited*. Berlin: Logos, pp. 303–321.

Rossberg, M., 2006. Inferentialism and Conservativeness. ms.

Rossberg, M., 2009. Leonard, Goodman, and the Development of the *Calculus of Individuals*. In Ernst et al. 2009, pp. 51–69.

Shapiro, S., 1983. Conservativeness and Incompleteness. *Journal of Philosophy*, 80: 521–531.

Shapiro, S., 1991. *Foundations Without Foundationalism: A Case For Second-Order Logic*. Oxford: Oxford University Press.

Shapiro, S., 1993. Modality and Ontology. *Mind*, 102: 455–481.

Shapiro, S., 1997. *Philosophy of Mathematics: Structure and Ontology*. Oxford: Oxford University Press.

Simons, P., 1993. Who's Afraid of Higher-Order Logic? *Grazer Philosophische Studien* 44: 253–264.

Simons, P., 1997. Higher-Order Quantification and Ontological Commitment. *Dialectica*, 51: 255–271.

Simons, P., 2002. Reasoning on a Tight Budget: Lesniewski's Nominalistic Metalogic. *Erkenntnis*, 56: 99–122.

van Heijenoort, J. (ed), 1967. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*. Cambridge, MA: Harvard University Press.

Wilholt, T., 2006. Think about the Consequences! Nominalism and the Argument from the Philosophy of Logic. *dialectica*, 60: 115–133.

Williamson, T., 2005. Armchair Philosophy, Metaphysical Modality and Counterfactual Thinking. *Proceedings of the Aristotelian Society*, 105: 1–23.

Wright, C., 1983. *Frege's Conception of Numbers as Objects*. Aberdeen: Aberdeen University Press.

Wright, C., 2007. On Quantifying into Predicate Position: Steps towards a New(tralist) Position. In M. Leng, A. Paseau and M. Potter (eds), *Mathematical Knowledge*. Oxford: Oxford University Press.

Marcus ROSSBERG is Assistant Professor at the University of Connecticut and Associate Fellow of the Northern Institute of Philosophy at the University of Aberdeen. Together with Daniel Cohnitz he co-authored a book on Nelson Goodman, and published a number of articles in the philosophy of mathematics and logic. His further areas of research include the philosophy of language, metaphysics and aesthetics.

ADDRESS: Department of Philosophy, 344 Mansfield Road, University of Connecticut, Storrs, CT 06269-2054, USA. E-mail: marcus.rossberg@uconn.edu

Daniel COHNITZ is Professor of Theoretical Philosophy at the University of Tartu, Estonia. Together with Marcus Rossberg he co-authored *Nelson Goodman* (Acumen, 2006). He is also the author of *Gedanken-experimente in der Philosophie* (Mentis, 2006), and (with Manuel Bremer) *Information and Information Flow* (Ontos, 2004). He published articles on metaphilosophy, philosophy of language, epistemology, and philosophy of science.

ADDRESS: Department of Philosophy, University of Tartu, Ülikooli 18, 50090 Tartu, Estonia.
E-mail: cohnitz@ut.ee Web: http://daniel.cohnitz.de