Recapture Results and Classical Logic

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Abstract

An old and well-known objection to non-classical logics is that they are too weak; in particular, they cannot prove a number of important mathematical results. A promising strategy to deal with this objection consists in proving so-called *recapture results*. Roughly, these results show that classical logic can be used in mathematics and other unproblematic contexts. However, the strategy faces some potential problems. First, typical recapture results are formulated in a purely logical language, and do not generalize nicely to languages containing the kind of vocabulary that usually motivates non-classical theories—for example, a language containing a naïve truth predicate. Second, proofs of recapture results typically employ classical principles that are not valid in the targeted non-classical system; hence, non-classical theorists do not seem entitled to those results. In this paper we analyze these problems and provide solutions on behalf of non-classical theorists. To address the first problem, we provide a novel kind of recapture result, which generalizes nicely to a truth-theoretic language. As for the second problem, we argue that it relies on an ambiguity and that, once the ambiguity is removed, there are no reasons to think that non-classical logicians are not entitled to their recapture results.

1   Introduction

There is an old and well-known objection to non-classical logics that is based on the indispensability of classical reasoning to mathematics. The idea is that many important mathematical results—including meta-logical results—rely on principles that are valid in classical logic but invalid in various prominent non-classical systems. In view of this, non-classical logicians appear to face an unpleasant dilemma. On the one hand, they can restrain themselves to reason only with principles that are valid in their favoured logic; if they do so, however, they must give up on some generally accepted mathematical results. On the other hand, they can retain these results but at the cost of employing principles that they themselves reject. Both horns are often seen as a serious embarrassment.

The so-called *classical recapture strategy* offers a promising route to escape this dilemma. Typically, revisions of logic are motivated by a specific collection of problematic statements. Thus,

non-classical logicians are not compelled to reject classical reasoning across the board. On the contrary, there seems to be nothing wrong if they use classical logic when the domain of discourse is safe, in the sense of not containing the problematic statements that led them to revise logic. The recapture strategy draws on these ideas, and it builds upon *recapture results*. Roughly, these results show that, if certain safety conditions obtain for a given set of statements (e.g. the statements in question satisfy this or that logical principle), then one can reason classically with those statements. If non-classical logicians manage to somehow apply this strategy to mathematical discourse, the dilemma can be avoided: they need not give up on classical mathematics, nor are they guilty of using principles that they reject. One can see, then, why the recapture strategy is of the utmost importance for non-classical logicians. If successful, it provides what they need to resist the indispensability objection.[1]

The strategy, however, faces some potential problems. First, typical recapture results for non-classical theories are formulated in a purely logical language, and they do not generalize nicely to languages containing the kind of vocabulary that usually motivates non-classical theories—for example, a language containing a naïve truth predicate. To illustrate, the recapture results in (Beall, 2013) assume that a theory is a set of statements, and this assumption is at odds with a naïve theory of truth based on a logic that has no theorems; however, if the results are reformulated so as not to assume that a theory is a set of statements, they face counterexamples in a truth-theoretic language. This problem is discussed by Nicolai (2022).

The second objection is that proofs of recapture results typically employ classical principles that are not valid in the targeted non-classical system. Hence, it is reasonable to conjecture that the results in question are not available unless classical logic is used. If that is the case, non-classical logicians do not seem to be epistemically entitled to those results. What they need to show, according to this line of thought, is that, by the standards of their own logic, recapture results are provable. This problem has been discussed by Woods (2019).[2]

In this paper we analyze these objections and provide solutions on behalf of non-classical theorists. To address the first one, we offer a novel kind of recapture result, which generalizes nicely to a language containing the truth predicate. In particular, the result does not assume that theories are just sets of statements and it avoids the truth-theoretic counterexamples alluded to earlier.

As for the second problem, we claim that it rests on a certain ambiguity. In a nutshell, the objection conflates two different requirements: on the one hand, the requirement that non-classical logicians ought not appeal to principles that are at odds with their *logic*; on the other hand, the

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[1]It is worth mentioning that there are non-classical logicians that do not try to avoid the dilemma at all. For example, some intuitionists and dialetheists fully take on the project of doing meta-theory for non-classical logic non-classically (cf. Dummett (2000) and Badia et al. (2016), respectively). This way of facing the dilemma eschews the recapture strategy (as we are understanding it) and embraces a revisionary attitude towards mathematics.

[2]To be sure, there are other objections to the idea of classical recapture, but they are not specifically about recapture *results*. For example, see Williamson (2018), Murzi and Rossi (2020) and Halbach and Nicolai (2018). Answers by non-classical logicians can be found in Rosenblatt (2021, 2022) and Field (2022).

requirement that they ought not appeal to principles that are at odds with their *theory*. We argue that, once the ambiguity is removed, there is no problem with the recapture strategy. The reason is that, while the proofs of recapture results typically violate the first constraint, this constraint is not reasonable anyway. In contrast, the second constraint seems to be in good standing, but—we argue—there are at least some non-classical theories that satisfy it.

The structure of the paper is simple. In §2 and §3, we present and then address the first problem. In §4 and §5, we present and then address the second problem. We finish with some concluding remarks in §6.

2   The weakness objection

What is a recapture result? There are various different things that may fall under this label. The thought underpinning many of them goes more or less like this. Let L be some sub-classical logic. Obviously, if some argument is valid in L, then it is valid in classical logic. Non-classical logicians seeking to uphold some version of classical recapture are interested in the conditions under which the converse of this conditional holds. That is, they want to identify what are the additional assumptions that must be in play for some argument to be valid in L if it is valid in classical logic.

Thus, recapture results typically adhere to the following template:

Recapture If the statements in Γ and the statement ϕ are unproblematic, then Γ classically entails ϕ only if Γ L-entails ϕ.

There are two place-holders in the template. First, one should make clear what it means to say that a statement is unproblematic. Secondly, one must specify what logic L is. Note that these two things are not independent of one another. The choice of logic might impinge on the characterization of the problematic statements. By way of example, if the logic L under discussion is paracomplete, then a statement might count as problematic if it fails to satisfy the law of excluded middle. But if L is a paraconsistent logic, then a statement may count as problematic if it fails to satisfy the rule of explosion.[3]

The point of a recapture result is to indicate to the non-classical theorist under what circumstances it is appropriate to reason classically. Thus, there is a sense in which these results are unlike other meta-theoretic results, like completeness, soundness, and so on. The availability of these other results depends on recapture. For example, to prove completeness for some non-classical logic, it seems that one needs to employ classically valid principles in the meta-theory, and the way to justify the employment of these principles is by providing a recapture result. Otherwise, it might well be

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[3] The template is not meant to cover every recapture result in the literature. For example, Neil Tennant's Core Logic (be it in its classical or its intuitionistic version) enjoys a very interesting recapture result that does not fit our template (see Tennant (2017)).

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that the principles that one is using in proving completeness are not justified from the non-classical logician's perspective.

One nice treatment of recapture results is offered by Beall (2013). The author shows how to formulate results of this kind for various logics that have been used to deal with the semantic paradoxes. We will focus on the paracomplete logic $K3$.[4] Since the objections to the recapture strategy that we will consider aim to be quite general, we take it that it is sufficient for our purposes to provide a single case where they can be resisted. Having said this, we think that the responses we outline below on behalf of the paracomplete logician can be adapted to theories based on other non-classical logics. Before stating Beall's recapture result for $K3$, we need to introduce a modicum of technical machinery. Let $\Gamma \cup \{\phi\}$ be a set of formulas belonging to a purely logical first-order language. Let $\psi_1, ..., \psi_n$ be the atomic formulas occurring as subformulas of ϕ. Given any formula ψ, we write $\overrightarrow{\forall x_\psi}$ to denote the sequence $\forall x_1...\forall x_m$ such that $x_1, ..., x_m$ are all the free variables in ψ. Also, $\vDash_{CL}$ stands for classical consequence and $\vDash_{K3}$ for $K3$-consequence. We can now state Beall's recapture result as follows:

($\star$) $\Gamma \vDash_{CL} \phi$ entails that $\Gamma, \overrightarrow{\forall x_{\psi_1}}(\psi_1 \vee \neg\psi_1), ..., \overrightarrow{\forall x_{\psi_n}}(\psi_n \vee \neg\psi_n) \vDash_{K3} \phi$.

Informally, ($\star$) is the claim that the argument from Γ to ϕ is classically valid only if it is $K3$-valid provided all the atomic formulas in ϕ behave classically, i.e. satisfy the law of excluded middle.

From a technical point of view, ($\star$) is perfectly fine. Unfortunately, there are a few conceptual problems with it, as pointed out by Nicolai (2022). First, recapture results are supposed to play a role for languages that contain non-logical vocabulary. For example, if one favors a paracomplete theory of truth, the point of a recapture result is to show under what conditions classical reasoning can be retained in such a theory. However, it is not obvious that ($\star$) can be maintained for languages with non-logical vocabulary.

Second, ($\star$) seems to presuppose that a theory is a set of statements. One way to read it goes as follows: if Γ is a theory and ϕ is a classical consequence of that theory, then ϕ is also a $K3$-consequence of Γ provided excluded middle holds for the appropriate formulas. Although this way of understanding the workings of theories sits well with classical logic, it is problematic for $K3$ and other deviant logics that contain no theorems. A naïve theory of truth based on $K3$ should be formulated in terms of *rules*, not in terms of axioms; otherwise, the $K3$-theorist would not even be in a position to articulate her theory.[5]

Third, if one reformulates Beall's result in a way that doesn't presuppose that a theory is a set of statements, it is possible to find counterexamples to it. Here is one, taken from Nicolai (2022).

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[4]We assume that the reader is familiar with this and other logics that are compatible with naive truth in Kripke's (1975) sense. For details on $K3$-models, see Priest (2008).

[5]A theory of truth is said to be *naïve* if it validates a *transparency principle*, which states that a statement ϕ and its truth-predication are intersubstitutable. In $K3$ the rules for Tr yield intersubstitutability, but they are not enough to secure the T-Schema: the claim that 'ϕ' is true if and only if ϕ.

Let $PUTB$ (for positive uniform disquotation) be the theory characterized in terms of the sequents

$$\phi(x) \Rightarrow Tr\ulcorner\phi(\dot{x})\urcorner \qquad\qquad\qquad Tr\ulcorner\phi(\dot{x})\urcorner \Rightarrow \phi(x)$$

In these sequents $\phi(x)$ is a Tr-positive formula, viz. the truth predicate does not occur in the scope of an odd number of negations.[6] Let $PUTB_{K3}$ be the result of adding the sequents above as initial sequents to a sequent calculus for Peano arithmetic formulated over $K3$ and let $PUTB_{CL}$ be the result of adding the sequents above as initial sequents to the same sequent calculus for Peano arithmetic formulated over classical logic.[7] Let $Con(PUTB_{K3})$ be a consistency statement for $PUTB_{K3}$. It can be shown that $PUTB_{CL}$ proves the consistency of $PUTB_{K3}$, but due to Gödel's second incompleteness theorem, $PUTB_{K3}$ cannot prove its own consistency. In other words, $PUTB_{CL}$ proves $Con(PUTB_{K3})$ but $PUTB_{K3}$ does not. Given that $Con(PUTB_{K3})$ is a purely arithmetical statement and thus satisfies the law of excluded middle, the example shows that, if one does not assume that a theory is just a set of statements, Beall's result fails: it is not enough to assume that the law of excluded middle holds for the atomic statements occurring in the conclusion.

We shall bunch up these problems and call them the *weakness objection*. In order to overcome the objection, we will formulate a novel type of recapture result that does not presuppose that a theory is a set of statements, and that applies not only to logical languages but also to languages containing the truth predicate.

3   A Recapture Result

In what follows we will use the language of Peano arithmetic, $\mathcal{L}$, to which we will add a one place predicate $Tr(x)$ standing for truth. The resulting language is called $\mathcal{L}_{Tr}$. We will have $\neg$, $\vee$ and $\forall$ as logical expressions ($\wedge$ and $\exists$ can be defined in the usual way). The language will contain an identity predicate $=$, a constant $\overline{0}$ to denote the number 0, a one-place function symbol s standing for the successor function, and two two-place function symbols $+$ and $\times$ for addition and multiplication. We assume a fixed canonical Gödel numbering for $\mathcal{L}_{Tr}$-expressions and we follow the standard practice of using $\ulcorner\phi\urcorner$ as a name of (the Gödel code of) ϕ.

We proceed proof-theoretically, starting with a sequent calculus for the logic $K3$.

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[6]The function $\dot{x}$ maps each number n to its numeral $\overline{n}$ and $\ulcorner\phi(\dot{x})\urcorner$ stands for the name of the (Gödel code of the) sentence which results by substituting, within $\phi(y)$, the variable y for $\dot{x}$.

[7]There are some subtleties involved in setting up the rules of these calculi, but that need not detain us here. We refer the reader to Nicolai's paper (and the references therein) for the details.

- Structural rules:

$$Id \, \frac{}{\Gamma, \phi \Rightarrow \phi, \Delta} \qquad\qquad Cut \, \frac{\Gamma \Rightarrow \phi, \Delta \qquad \Gamma', \phi \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

- Operational rules

$$L\neg \, \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma, \neg\phi \Rightarrow \Delta}$$

$$L\neg\neg \, \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \neg\neg\phi \Rightarrow \Delta} \qquad\qquad R\neg\neg \, \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \neg\neg\phi, \Delta}$$

$$L\vee \, \frac{\Gamma, \phi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} \qquad\qquad R\vee \, \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}$$

$$L\neg\vee \, \frac{\Gamma, \neg\phi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\phi \vee \psi) \Rightarrow \Delta} \qquad\qquad R\neg\vee \, \frac{\Gamma \Rightarrow \neg\phi, \Delta \qquad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\phi \vee \psi), \Delta}$$

$$L\neg\forall \, \frac{\Gamma, \neg\phi(a) \Rightarrow \Delta}{\Gamma, \neg\forall x\phi \Rightarrow \Delta} \qquad\qquad R\forall \, \frac{\Gamma \Rightarrow \phi(a), \Delta}{\Gamma \Rightarrow \forall x\phi, \Delta}$$

$$L\forall \, \frac{\Gamma, \phi(t), \forall x\phi \Rightarrow \Delta}{\Gamma, \forall x\phi \Rightarrow \Delta} \qquad\qquad R\neg\forall \, \frac{\Gamma \Rightarrow \neg\phi(t), \neg\forall x\phi, \Delta}{\Gamma \Rightarrow \neg\forall x\phi, \Delta}$$

- Identity rules

$$SubL= \, \frac{\Gamma, \phi(s), \phi(t), s = t \Rightarrow \Delta}{\Gamma, \phi(t), s = t \Rightarrow \Delta}$$

$$Ref= \, \frac{\Gamma, s = s \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$SubR= \, \frac{\Gamma, s = t \Rightarrow \phi(s), \phi(t), \Delta}{\Gamma, s = t \Rightarrow \phi(t), \Delta}$$

In all the rules Γ and Δ are sets of formulas, in the quantifier rules a is an eigenvariable, i.e. a is a variable that does not occur in the conclusion-sequent of the rule, and in the identity rules s and t are any terms. The resulting system is the first-order version of the logic $K3$ with identity. If one adds the rule R¬ to this system, one obtains classical logic, CL:

$$R\neg \, \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\phi, \Delta}$$

Of course, the formulation of CL contains many redundant rules, but the point is to find presentations of $K3$ and CL wherein the only difference consists is the absence or presence of the rule R¬. This will come in handy in the formulation of the recapture result.

Since we are interested in theories that contain Peano arithmetic, we add the Peano axioms in rule form.[8] That is, if ϕ is an axiom of Peano arithmetic, we have the following rule:

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

We also add induction in rule form:

$$Ind \; \frac{\Gamma, \phi(x) \Rightarrow \phi(s(x))\Delta}{\Gamma, \phi(\overline{0}) \Rightarrow \phi(t), \Delta}$$

In Ind, t is an arbitrary term and ϕ may contain the truth predicate. The variable x cannot occur freely in $\phi(\overline{0})$, Γ or Δ.

The system containing all these rules except for R¬ will be called $\mathcal{S}_{K3}$. The system that results from adding the rule R¬ to $\mathcal{S}_{K3}$ will be called $\mathcal{S}_{CL}$. $\mathcal{S}_{CL}$ is a system for classical Peano arithmetic and $\mathcal{S}_{K3}$ is a system for Peano arithmetic with $K3$ as the underlying logic. A few remarks about both systems are in order. First, the only initial sequents are instances of Id. Hence, every derivation must start from sequents of the form $\Gamma, \phi \Rightarrow \phi, \Delta$. Second, it is not hard to verify that Weakening is an admissible rule in both systems. That is, if there is a derivation of $\Gamma \Rightarrow \Delta$, there are also derivations of $\Gamma, \phi \Rightarrow \Delta$ and $\Gamma \Rightarrow \phi, \Delta$ for any formula ϕ. Third, rules L∀, R¬∀, SubL= and SubR= are formulated as they are, viz. with the principal formula(s) repeated in the premise-sequent, to secure Lemma 1 below. Fourth, even though there are no specific rules for the truth predicate, it can feature in any of the logical rules and in Ind.

Eventually, we want to use the systems $\mathcal{S}_{K3}$ and $\mathcal{S}_{CL}$ to reason under assumptions. That is, we want to see how they differ upon the addition of extra premises involving the truth predicate. Because these systems are sequent calculi, assumptions will themselves be sequents. Let's say that a conclusion-sequent $\Sigma \Rightarrow \Pi$ *follows from* the premise-sequents $\Gamma_1 \Rightarrow \Delta_1,...,\Gamma_n \Rightarrow \Delta_n$ in $\mathcal{S}_{K3}$ ($\mathcal{S}_{CL}$) if there is a derivation of $\Sigma \Rightarrow \Pi$ from $\Gamma_1 \Rightarrow \Delta_1,...,\Gamma_n \Rightarrow \Delta_n$ using the rules (and possibly initial sequents) of $\mathcal{S}_{K3}$ ($\mathcal{S}_{CL}$). Let's say that a formula *occurs in a derivation* if it is a subformula of a formula that occurs in a sequent that is part of that derivation. Also, for any atomic formula $\phi^{at}(t_1, ..., t_k)$, let $\phi^{at}(t_1', ..., t_k')$ be an atomic formula that is like $\phi^{at}(t_1, ..., t_k)$ except perhaps in that it replaces some variables with (open or closed) terms. For the recapture result below it will be crucial that any derivation has the following property:

Lemma 1 *Let $\mathcal{D}$ be any derivation of $\mathcal{S}_{K3}$ or $\mathcal{S}_{CL}$. If an atomic formula $\phi^{at}(t_1', ..., t_k')$ occurs in $\mathcal{D}$, then $\phi^{at}(t_1, ..., t_k)$ occurs in one of the premise-sequents of $\mathcal{D}$ or in one of the initial sequents employed in $\mathcal{D}$.*

Proof By induction on the height of the derivation. We leave the details to the reader.

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[8] We could instead add as initial sequents $\Gamma \Rightarrow \phi, \Delta$ for each axiom ϕ of Peano arithmetic, but the formulation in terms of rules simplifies the proofs to be given below.

A related fact is that if the law of excluded middle holds for a formula $\phi^{at}(t_1, ..., t_k)$, it must also hold for $\phi^{at}(t'_1, ..., t'_k)$. That is:

Lemma 2 *If there there is a derivation of $\Rightarrow \phi^{at}(t_1, ..., t_k) \vee \neg\phi^{at}(t_1, ..., t_k)$ in $\mathcal{S}_{K3}$, then there is a derivation of $\Rightarrow \phi^{at}(t'_1, ..., t'_k) \vee \neg\phi^{at}(t'_1, ..., t'_k)$ in $\mathcal{S}_{K3}$.*

Proof Let the universal closure of the formula ϕ, denoted by $UC(\phi)$, be the sentence $\overrightarrow{\forall x_\phi}(\phi)$. Then from $\Rightarrow \phi^{at}(t_1, ..., t_k) \vee \neg\phi^{at}(t_1, ..., t_k)$ one can infer $\Rightarrow UC(\phi^{at}(t_1, ..., t_k) \vee \neg\phi^{at}(t_1, ..., t_k))$ by (possibly several applications of) R$\forall$. Since $UC(\phi^{at}(t_1, ..., t_k) \vee \neg\phi^{at}(t_1, ..., t_k)) \Rightarrow \phi^{at}(t'_1, ..., t'_k) \vee \neg\phi^{at}(t'_1, ..., t'_k)$ is derivable by Id and L$\forall$, one can apply Cut to obtain $\Rightarrow \phi^{at}(t'_1, ..., t'_k) \vee \neg\phi^{at}(t'_1, ..., t'_k)$.

Another important fact about $\mathcal{S}_{K3}$ is that for any formula, if all its subformulas satisfy excluded middle, then the formula satisfies excluded middle too.

Lemma 3 *If there is a derivation of $\Rightarrow \phi^{at} \vee \neg\phi^{at}$ in $\mathcal{S}_{K3}$ for all the atomic subformulas of a formula ϕ, then there is also a derivation in $\mathcal{S}_{K3}$ of $\Rightarrow \phi \vee \neg\phi$.*

Proof By induction on the complexity of ϕ. We leave the details to the reader.

Before formulating the recapture result, we need one last lemma about $\mathcal{S}_{K3}$ stating that to guarantee classical reasoning for a formula ϕ it is sufficient to assume excluded middle for ϕ. Since the only difference between $\mathcal{S}_{K3}$ and $\mathcal{S}_{CL}$ is that the former lacks but the latter has the rule R$\neg$, we can state this fact more formally as follows:

Lemma 4 *If there is a derivation of $\Rightarrow \phi \vee \neg\phi$ in $\mathcal{S}_{K3}$, then R$\neg$ holds for ϕ.*

Proof If there is a derivation of $\Rightarrow \phi \vee \neg\phi$, there is also a derivation of $\Rightarrow \phi, \neg\phi$, courtesy of Id, L$\vee$ and Cut. Then, we can apply Cut again to $\Rightarrow \phi, \neg\phi$ and $\Gamma, \phi \Rightarrow \Delta$ to obtain $\Gamma \Rightarrow \neg\phi, \Delta$.

Now we have all we need to present the recapture result.

Theorem 5 *If there is a derivation $\mathcal{D}$ of the conclusion-sequent $\Sigma \Rightarrow \Pi$ from the premise-sequents $\Gamma_1 \Rightarrow \Delta_1, ..., \Gamma_n \Rightarrow \Delta_n$ in $\mathcal{S}_{CL}$, then there is also a derivation of $\Sigma \Rightarrow \Pi$ from $\Gamma_1 \Rightarrow \Delta_1, ..., \Gamma_n \Rightarrow \Delta_n$ in the system that can be obtained from $\mathcal{S}_{K3}$ by the addition of the initial sequent $\Rightarrow \phi^{at} \vee \neg\phi^{at}$ for each atomic subformula of formulas occurring in $\Gamma_i \Rightarrow \Delta_i$ with $1 \leq i \leq n$ or occurring in instances of Id employed in $\mathcal{D}$.*[9]

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[9]It would be routine to adapt this theorem to other non-classical logics in the $K3$ family, such as Priest's logic of paradox LP and Belnap-Dunn logic FDE (cf. Priest (1979) and Belnap (1977), respectively). It is also possible, though slightly less straightforward, to adapt it to substructural logics, such as the non-transitive logic ST and the non-reflexive logic TS (cf. Ripley (2013) and French (2016), respectively).

Proof Let's assume that there is a derivation $\mathcal{D}$ of $\Sigma \Rightarrow \Pi$ from $\Gamma_1 \Rightarrow \Delta_1,...,\Gamma_n \Rightarrow \Delta_n$ in $\mathcal{S}_{CL}$. Either $\mathcal{D}$ contains an application of R¬ or it doesn't. If it doesn't, then $\mathcal{D}$ is also a derivation of $\mathcal{S}_{K3}$ and we are done. If it does, let ϕ be a formula to which R¬ applies. If $\phi^{at}(t'_1, ..., t'_k)$ is an atomic subformula of ϕ, it follows by Lemma 1 that the formula $\phi^{at}(t_1, ..., t_k)$ must occur either in some sequent $\Gamma_i \Rightarrow \Delta_i$ for $1 \leq i \leq n$, or in some instance of Id employed in $\mathcal{D}$. By assumption, we have $\Rightarrow \phi^{at}(t_1, ..., t_k) \vee \neg\phi^{at}(t_1, ..., t_k)$ as an initial sequent for any of these atomic formulas. Thus, by Lemma 2 there is a derivation of $\Rightarrow \phi^{at}(t'_1, ..., t'_k) \vee \neg\phi^{at}(t'_1, ..., t'_k)$ and by Lemma 3 there is also a derivation of $\Rightarrow \phi \vee \neg\phi$. Lastly, using Lemma 4 we infer that R¬ holds for ϕ. Thus, there is a derivation of $\Sigma \Rightarrow \Pi$ from $\Gamma_1 \Rightarrow \Delta_1,...,\Gamma_n \Rightarrow \Delta_n$ in the enriched $\mathcal{S}_{K3}$ system.

We will give some examples to illustrate how the result can be applied. Let λ be a liar sentence, i.e. a sentence such that $\lambda \Rightarrow \neg Tr\ulcorner\lambda\urcorner$ and $\neg Tr\ulcorner\lambda\urcorner \Rightarrow \lambda$ are provable. In $\mathcal{S}_{CL}$ there is a derivation of the conclusion-sequent $\Rightarrow Tr\ulcorner\lambda\urcorner \wedge \neg Tr\ulcorner\lambda\urcorner$ from the premise-sequents $Tr\ulcorner\lambda\urcorner \Rightarrow \lambda$ and $\lambda \Rightarrow Tr\ulcorner\lambda\urcorner$. Of course, this derivation is not available in $\mathcal{S}_{K3}$. The point of $\mathcal{S}_{K3}$ is precisely to retain consistency even in the presence of a naïve truth predicate. But Theorem 5 guarantees that the $K3$-theorist can recover the classical logician's contradiction if she wishes to, for there will be a derivation of $\Rightarrow Tr\ulcorner\lambda\urcorner \wedge \neg Tr\ulcorner\lambda\urcorner$ if we add to $\mathcal{S}_{K3} \Rightarrow Tr\ulcorner\lambda\urcorner \vee \neg Tr\ulcorner\lambda\urcorner$ as an initial sequent.

A second example is given by the theories $PUTB_{K3}$ and $PUTB_{CL}$ from above. As Nicolai pointed out, the consistency statement for $PUTB_{K3}$ can be used to act as a counterexample of a plausible reformulation of Beall's result. However, it leaves Theorem 5 untouched. If there is a derivation of $\Rightarrow Con(PUTB_{K3})$ from the sequents $\phi(x) \Rightarrow Tr\ulcorner\phi(\dot{x})\urcorner$ and $Tr\ulcorner\phi(\dot{x})\urcorner \Rightarrow \phi(x)$ in $\mathcal{S}_{CL}$, there will be a corresponding derivation in $\mathcal{S}_{K3}$ from these sequents and the appropriate instances of the law of excluded middle for statements containing the truth predicate.

A third and final example goes as follows. It can be shown that $\mathcal{S}_{K3}$ and $\mathcal{S}_{CL}$ have the same arithmetical content, but $\mathcal{S}_{CL}$ proves more formulas containing the truth predicate. More specifically, for $\mathcal{L}$-formulas both $\mathcal{S}_{CL}$ and $\mathcal{S}_{K3}$ prove the schema of transfinite induction up to any ordinal below ϵ_0. Yet, for $\mathcal{L}_{Tr}$-formulas $\mathcal{S}_{CL}$ proves transfinite induction up to ϵ_0, whereas $\mathcal{S}_{K3}$ only reaches ω^ω.[10] Crucially, the instances of induction that are only provable in $\mathcal{S}_{CL}$ all involve the truth predicate, but note that no specific truth-theoretic assumptions are required for this. So the example shows that it is not sufficient to assume that the law of excluded middle holds for the atomic formulas occurring in the premise-sequents. In this case there are no premise-sequents, only initial sequents (i.e. instances of Id). Be that as it may, our recapture result is not in danger because, remember, it also assumes excluded middle for the atomic formulas in the instances of Id employed in the derivation. Since the conclusion-sequent contains a formula $Tr(t')$ that has an occurrence of the truth predicate, the formula $Tr(t)$ must occur in some instance of Id in the derivation, by Lemma 2. It is enough, then, to assume the law of excluded middle for $Tr(t)$ to recover the extra

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[10]For the details, see Halbach (2014) or Halbach and Nicolai (2018).

truth-theoretic content given by $\mathcal{S}_{CL}$.

At this point, a few worries might arise.[11] To begin with, one may have the impression that our recapture result is not very informative, for it seems to say that the $K3$-theorist can mimic classical reasoning provided all the predicates involved behave classically. But if so, the recapture result is of no help to her, for it requires abandoning paracompleteness!

However, the impression is misleading. The recapture result we have presented is local, in the sense that it applies to specific derivations. It does not assume excluded middle for all the predicates involved in a given derivation. Rather, what it assumes is that excluded middle holds for all the atomic subformulas occurring in the premise-sequents and initial sequents of the derivation. In many cases, the latter will not imply the former. For a simple example, consider the theories $PUTB_{CL}$ and $PUTB_{K3}$ again. $PUTB_{CL}$ proves the sequent $\Rightarrow \neg Tr \ulcorner 0 = 1 \urcorner$, which cannot be proved in $PUTB_{K3}$. Now, one derivation of this sequent in $PUTB_{CL}$ runs as follows:

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\frac{\vdots \qquad \frac{\dfrac{Tr \ulcorner 0 = 1 \urcorner \Rightarrow 0 = 1}{\dfrac{\neg 0 = 1, Tr \ulcorner 0 = 1 \urcorner \Rightarrow}{\neg 0 = 1 \Rightarrow \neg Tr \ulcorner 0 = 1 \urcorner} \text{R}\neg} \text{L}\neg}{\Rightarrow \neg 0 = 1}}{\Rightarrow \neg Tr \ulcorner 0 = 1 \urcorner} Cut
\tag{1}
$$

According to Theorem 5, to recapture this theorem the $K3$-theorist does not need to assume excluded middle for the truth predicate in general, but only for the statement $Tr \ulcorner 0 = 1 \urcorner$. Thus, it is inaccurate to say, without further qualifications, that our recapture result assumes excluded middle for all the predicates in a given derivation.

A more justified worry is that, while not always, in *many* cases our result makes such an assumption. For instance, if we wanted the statement $\neg Tr \ulcorner 0 = 1 \urcorner$ to witness the claim $\neg \forall x Tr(x)$, then our derivation in $PUTB_{CL}$ would now be:

$$
\frac{\vdots \qquad \frac{\dfrac{\dfrac{Tr \ulcorner 0 = 1 \urcorner \Rightarrow 0 = 1 \qquad 0 = 1 \Rightarrow 0 = 1, \neg \forall x Tr(x)}{Tr \ulcorner 0 = 1 \urcorner \Rightarrow 0 = 1, \neg \forall x Tr(x)} Cut}{\dfrac{\neg 0 = 1, Tr \ulcorner 0 = 1 \urcorner \Rightarrow \neg \forall x Tr(x)}{\neg 0 = 1 \Rightarrow \neg Tr \ulcorner 0 = 1 \urcorner, \neg \forall x Tr(x)} \text{R}\neg} \text{L}\neg}{\Rightarrow \neg 0 = 1}}{\dfrac{\Rightarrow \neg Tr \ulcorner 0 = 1 \urcorner, \neg \forall x Tr(x)}{\Rightarrow \neg \forall x Tr(x)} \text{R}\neg\forall} Cut
\tag{2}
$$

According to Theorem 5, to recapture this derivation the $K3$-theorist needs to assume excluded middle in general: $\Rightarrow Tr(x) \vee \neg Tr(x)$. Indeed, she needs to make this assumption whenever the formula $Tr(x)$ occurs in the classical derivation to be recaptured.

The objection is accurate, but we think that it does not pose a serious threat to the recapture

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[11] We are grateful to an anonymous reviewer for raising these worries. We think that some of them are anticipated in the last paragraphs of Nicolai's paper.

strategy as such, for it heavily relies on the way in which Theorem 5 is formulated. But Theorem 5 is not, and is not *meant* to be, the only recapture result available to the paracomplete theorist. In fact, it is easy to come up with results which do not require excluded middle for Tr in general to recover derivation (2). For instance, as an immediate consequence of Lemma 4 we have:

Fact 6 *If there is a derivation $\mathcal{D}$ of the conclusion-sequent $\Sigma \Rightarrow \Pi$ from the premise-sequents $\Gamma_1 \Rightarrow \Delta_1,...,\Gamma_n \Rightarrow \Delta_n$ in $\mathcal{S}_{CL}$, then there is also a derivation of $\Sigma \Rightarrow \Pi$ from $\Gamma_1 \Rightarrow \Delta_1,...,\Gamma_n \Rightarrow \Delta_n$ in the system that can be obtained from $\mathcal{S}_{K3}$ by the addition of the initial sequent $\Rightarrow \phi \vee \neg\phi$ for each formula ϕ that is an active premise in an application of $R\neg$ in $\mathcal{D}$.*

Fact 6 looks at the internal structure of the derivation to be recovered, rather than at its premises. The paracomplete theorist might well prefer this or some other result to Theorem 5. However, the question of which is the most satisfactory result available to her is a difficult one and it is not among our present aims to provide an answer to it. We only claim to have provided a reasonable improvement of the recapture results present in the literature—which, as already said, either assume that a theory is a set of statements or fail for non-logical languages.

The last and most substantial objection we shall consider is the following. For all we know, it may well happen that, even under the strongest recapture result available to the $K3$-theorist, the proof of some important mathematical theorems requires excluded middle across the board. For instance, it may well be that any derivation in $PUTB_{CL}$ of the consistency statement for $PUTB_{K3}$ requires excluded middle in general. But if this were the case, the recapture result would be useless to the $K3$-theorist, for the simple reason that she does not accept the general law of excluded middle—quite the contrary, the result would be music to the ears of the classical theorist, who claims that, without excluded middle, some important theorems cannot be obtained.

However, the scope of this objection remains to be determined. The $K3$-theorist might have principled reasons to accept specific collections of instances of excluded middle that go beyond the language of pure mathematics. For example, she may want to accept excluded middle for (a recursively axiomatizable subset of) the statements that are grounded in the sense of Kripke (1975) (more on this in §5). If the collection in question suffices to recover some or many of the theorems under discussion, the objection would lose part of it force.

But even if not—that is, even if recovering some of those theorems requires excluded middle in general—we think that the objection can be resisted. First, the $K3$-theorist might argue that, in spite of lacking those theorems, her theory is not *arithmetically* weak. To be sure, if one endorses an instrumentalist conception of truth and thus conceives of the truth predicate as an expressive resource whose main purpose is to bolster the deductive power of theories, then one will very likely think of the unavailability of these theorems as a cost. However, the motivation underpinning a paracomplete theory of truth, as we are understanding it, has little to do with the search for new mathematical theorems. Rather, its aim is to provide a plausible characterization (of some aspects)

of how the truth predicate is used in natural language and, perhaps, in philosophical discourse.[12] So the fact that these theories fail to prove some mathematical results that go beyond Peano arithmetic does little harm to them. After all, they prove everything that Peano arithmetic proves, and having these additional theorems was not a part of their initial motivation.

Secondly, from the fact that these theorems are not available in a theory of truth over Peano arithmetic, it does not follow that they are not available tout court.[13] In the truth-theoretic literature, Peano arithmetic is typically taken as a base theory due to its simplicity and familiarity. But if one is seeking to offer a theory of mathematical truth, then the real-deal base theory is arguably set theory, say ZFC. The theorems that where unattainable in $\mathcal{S}_{K3}$ can be derived in ZFC (or a $K3$-friendly version thereof) without the need to endorse instances of excluded middle involving the truth predicate. Of course, assuming as before that the relevant axiom-schemata are unrestricted, there may still be a disparity between a theory of truth (or satisfaction) formulated over classical ZFC and a theory of truth (or satisfaction) formulated over a $K3$ version of ZFC, in the sense that there may be purely mathematical consequences available in the classical theory that are not available in the non-classical theory. But it is moot whether this surplus mathematical strength is of any philosophical significance. If one grants that ZFC is sufficient as a foundation of mathematics, then these additional mathematical consequences are inessential.[14]

In sum, we provided a recapture result for $\mathcal{S}_{K3}$ that improves on Beall's result on two counts. On the one hand, it can be applied to languages that go beyond logic. On the other hand, it is not susceptible to potential counterexamples like those of Nicolai (2022). By formulating the result in terms of sequents, we make sure that the truth-rules can be appropriately expressed, and by requiring that the atomic formulas in the premise-sequents (in addition to the atomic formulas that occur in instances of Id) satisfy excluded middle, we make sure that every classically valid step can be recovered. Also, we considered various objections and putative limitations of the result we provided. The upshot of our discussion was that none of them undercuts the recapture strategy as we understand it.

4   The faithfulness objection

The second objection against classical recapture that we address in this paper is due to Woods (2019), although in the context of a different discussion. His main goal is to challenge the theoretical stance known as *anti-exceptionalism about logic*. This is the claim that logic is, in some relevant respects, unexceptional compared to the other sciences (cf. Hjortland, 2017; Williamson, 2017).

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[12] In this regard, we are more sympathetic to the way in which Kripke motivates such theories in the Outline than to more deflationary approaches such as Field's (2008).

[13] Field (2022) briefly suggests this idea without endorsing it.

[14] More can be said about the philosophical issues raised by this last objection. Our hope is that our observations are enough to convince the reader that the non-classical theorist has tools at her disposal to meet the objection. We plan to take this up in future work.

Woods argues that anti-exceptionalists should commit to a methodological principle he calls 'Logical Partisanhood':

> Unless the output of weighing the merits of my background logic against an alternative—on one hand by the lights of my own background logic and on the other hand by the lights of the proposed alternative—agree that moving to the alternative is no worse than staying with our current background logic, we ought to hold fast to our background logic. (2019, p. 1205)

That is, a change of one's background logic to some alternative logic is warranted only if the change can be justified both in the background logic and in the alternative. The reason why anti-exceptionalists should commit to Logical Partisanhood is that, unless a maxim of this sort is in force, straightforward applications of the abductive methodology (which is considered to be the correct methodology if logic is unexceptional) can lead to revision cycles. Very roughly, the idea is that in a dispute between two logical theories, there may be situations where the methodology instructs one to change one's theory and then change back to the original theory.

Woods appeals to Logical Partisanhood to argue that non-classical logicians seeking to use a recapture result to show that classical reasoning can be recovered in certain contexts—and thus bolster the case for the suitability of their favored logic—should be in a position to prove this result without availing themselves of principles that are not valid according to their own background logic. However—Woods observes—typical proofs of recapture results appear to employ classical principles that are *not* valid in the target sub-classical logic.

It is not our aim here to discuss whether logical anti-exceptionalism is tenable or to take a stand on whether Logical Partisanhood is a reasonable principle. The requirement that one ought to prove a recapture result for one's non-classical logic without bringing into play resources that go beyond that logic could stand on its own. That is, the requirement may be seen as reasonable—which doesn't mean that it is so—even if one doesn't endorse logical anti-exceptionalism and Logical Partisanhood. In fact, we think that Wood's objection rests on a simpler maxim that the author extracts from Logical Partisanhood. The simpler maxim is more relevant to our current purposes, and it has the added benefit of being neutral, in that it can be endorsed by logical exceptionalists and anti-exceptionalists alike. Let's call it, for lack of a better name, 'Logical Faithfulness'.

Logical Faithfulness In proving a result about some theory T that one endorses one cannot employ principles that are not sanctioned by T's underlying logic L.

For the purposes of this paper, then, the issue raised by Woods can be put as follows: if L is the non-classical theorist's favored logic, she typically provides a classically valid but L-invalid argument for classical recapture, and thereby infringes Logical Faithfulness. We will refer to this as the *faithfulness objection*.[15]

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[15]In (2018, p. 417) Williamson echoes the worry.

Woods analyzes, as a test case, Hartry Field's claim that if one takes excluded middle as a non-logical fact about a particular domain of discourse, then his consequence relation 'collapses' to classical consequence.[16] The problem, according to Woods, is that Field proves this claim using principles that are not available in his own logic. We will follow Woods in considering the case of a paracomplete logic, but our focus will be on $K3$-based theories lacking an intensional conditional, since that would bring additional complications that we would like to ignore.[17]

Now, if one inspects the proof of the recapture result for $K3$ in §3 (Theorem 5), it is fairly easy to see that it relies on the law of excluded middle, which is not available in $K3$. Recall that we are using the fact that in the derivation of $\Sigma \Rightarrow \Pi$ from $\Gamma_1 \Rightarrow \Delta_1,...,\Gamma_n \Rightarrow \Delta_n$ in $\mathcal{S}_{CL}$, either there is an application of the rule $R\neg$ or there is not. This is an instance of the schema $\phi \lor \neg\phi$.

Of course, there may be a different way of proving Theorem 5 that doesn't rely on the law of excluded middle. But even if there is, the proof will be carried out in the meta-theory, which is presumably classical; it will assume, among other things, that the notion of derivability can be defined in classical set theory. Also, it will unavoidably use the principle of mathematical induction, which is needed for the proofs of lemmata 1 and 3. Now, the *law* form of this principle is not unrestrictedly valid in $K3$. In its strong variant, it is the claim that if for every number x, it holds that every number y less than x has the property ϕ only if x has the property ϕ, then every number has the property ϕ:

$$\forall x(\forall y(y < x \to \phi(y)) \to \phi(x)) \to \forall x\phi(x).$$

In a context where semantic paradoxes are lurking around, the $K3$-theorist cannot endorse this claim in general. The reason is that this form of induction fails when semantic predicates are available for substitution in ϕ. Let $\phi(x)$ be the statement $\lambda \land x = x$. The $K3$-conditional, $\phi \to \psi$, can be defined in terms of disjunction and negation in the usual way, $\neg\phi \lor \psi$. Then $\forall x(\forall y(y < x \to \phi(y)) \to \phi(x)) \to \forall x\phi(x)$ is neither true nor false. In contrast, the *rule* form of mathematical induction, given e.g. by Ind, is consistent with a paracomplete theory of truth. However, it is not obvious whether lemmata 1 and 3 require the rule or the law form of induction. If what is needed is the law, we have another violation of Logical Faithfulness.

So it appears that the faithfulness objection was successful, in the sense that Woods was right when he claimed that in establishing recapture results non-classical theorists make essential use of classically valid principles that they themselves would not be willing to admit. End of story?

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[16]For the result under discussion, see Field (2003).

[17]Woods also considers the case of Neil Tennant's Core Logic, but for obvious reasons having to do with the narrative of this paper we will not discuss Tennant-style classical recapture (cf. Footnote 3).

5   Theoretical Faithfulness

We submit that Logical Faithfulness is not a correct methodological principle as it stands. To be sure, there is something appealing about it, but we think that its intuitive strength comes from a much simpler thought: that in proving some result about a theory that one endorses, one should not employ principles that one rejects. Logical Faithfulness, however, claims something stronger: that in proving some result about a theory that one endorses, one should not employ principles that are not sanctioned by the theory's underlying logic. This latter requirement, we take it, is not reasonable. If one appeals to a certain principle of reasoning, one has of course to make sure that the principle is sound. But it should not matter whether the principle is logical or not—or, in other words, whether the reasons for endorsing it have a logical or a non-logical character. Thus, we claim that the simple thought that underpins the idea of faithfulness requires a different formulation.

Unfortunately, it is not obvious what a correct formulation would look like. To begin with, one could take Logical Faithfulness and modify it as follows:

(*) In proving a result about a theory T that one endorses, one cannot employ principles that are not sanctioned by T.

This requirement no longer imposes the condition that one is only allowed to employ principles that are sanctioned by the logic of one's theory. Instead, it merely asks one to employ principles that are sanctioned by one's theory, regardless of whether those principles are logical or not.

But (*) is still too strong, and thus unreasonable. By Gödel, we know that, sometimes, in order to prove a result about a theory T one needs a theory stronger than (or different from) T. The use of a stronger theory is not in general ruled out by our driving intuition. For example, if T is Peano arithmetic and Con_{PA} its consistency statement, we know that Con_{PA} cannot be a part of T. Yet, there appears to be nothing wrong if one defends T by appealing to a theory T' that proves Con_{PA}—provided one already has independent reasons to accept T'. But (*) claims otherwise; hence, it does not fit the bill.

A tempting alternative stems from the thought that theories come with so-called *implicit commitments*. Sometimes, in endorsing a theory T one implicitly commits oneself to (or is at least entitled to accept) principles that are not (and cannot be) a part of T.[18] For instance, it seems clear that if one endorses Peano arithmetic, one is implicitly committed to the claim that it is consistent. The general thought is that, in defending a theory T, faithfulness concerns do not rule out the use of such implicit commitments:

(**) In proving a result about a theory T that one endorses, one cannot employ principles that are not among the (implicit or explicit) commitments of T.

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[18]For the idea that theories come with a set of implicit commitments and/or entitlements, see Wright (2004), Dean (2015), Cieśliński (2017), Nicolai and Piazza (2019), Horsten (2021) and Łełyk and Nicolai (2022).

Thus, the intuitive counterexample given above against (*) would not apply to (**). However, there are at least two reasons to cast doubts on the adequacy of (**). First, the notion of 'implicit commitment' is relatively recent in the literature, and its precise scope remains to be determined. Thus, by settling on (**) we would be relying on a notion that many would find controversial. Secondly, we think that (**) is too strict anyway. For example, many meta-*logical* results require induction, a principle that can hardly be seen as an implicit commitment of a logical theory. Also, many results about formal theories of truth formulated over an arithmetical base theory require set-theoretic principles that are beyond arithmetic. The general point is that to prove results about some theory T one often needs claims that are not a part of T's implicit commitments. Hence, we should look for another option.

The strongest candidate that we find plausible is what we shall call *theoretical faithfulness*. It relinquishes the idea that one should only use principles that one is somehow already committed to; instead, it just requires one's principles to be compatible with one's theory:

Theoretical Faithfulness In proving a result about some theory T that one endorses, one ought not employ principles that are inconsistent with T.[19]

We think Theoretical Faithfulness constitutes a plausible rendering of the simple thought that underpins Wood's objection.

With this in the background we are now in a position to flesh out our claim that Woods' objection rests on an ambiguity. The intuitive idea behind it was, remember, that one should not use principles that one rejects in proving results about a theory that one endorses. However, this idea admits of at least two readings. On the one hand, it can be understood as saying that one cannot use principles that are somehow at odds with one's logic—this yields Logical Faithfulness. On the other hand, it can be understood as saying that one cannot use principles that are somehow at odds with one's theory—this yields Theoretical Faithfulness. Woods seems to embrace the first reading. We claim that this reading is misguided, for Logical Faithfulness is not a reasonable requirement. While we think that the intuitive idea that one should remain faithful to one's commitments is correct, we claim that Theoretical Faithfulness is a more appropriate way of rendering that idea. So, in a nutshell, what we are suggesting is that the discussion ought to shift its focus from logics to theories. Faithfulness does not require that in proving a recapture result for one's theory one cannot avail oneself of resources that are beyond one's *logic*, but that one ought not bring into play resources that are inconsistent with one's *theory*.[20]

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[19]For obvious reasons, in the case of paraconsistent theories the requirement should not be formulated in terms of principles that are inconsistent with T but in terms of principles that would *trivialize T*.

[20]There may be another reason to move from Logical to Theoretical Faithfulness. The former—but not the latter—seems to presuppose that it is possible to draw a sharp distinction between logical and non-logical principles. However, there are results such as Diaconescu's theorem, stating that the law of excluded middle is derivable from the axiom of choice, that cast doubts on whether this distinction can be feasibly made. Thanks to Lorenzo Rossi for bringing this result to our attention.

How is it that the shift from Logical Faithfulness to Theoretical Faithfulness can help with the faithfulness objection? One may think that the paracomplete logician is infringing not only Logical Faithfulness but also Theoretical Faithfulness, since in the proof of Theorem 5 she seems to be availing herself of the law of excluded middle when she claims that either there is an application of the rule R¬ or there isn't. The worry is that the law is inconsistent with the theory she is endorsing. Therefore, Theorem 5 would not be available to her.

However, note that the proof of this theorem only requires a specific instance of the law of excluded middle. Crucially, the instance in question—unlike the full law—is consistent with a paracomplete theory of truth. Thus, the paracomplete logician appealing to Theorem 5 abides by Theoretical Faithfulness—provided, of course, she does not appeal to the full law of excluded middle in her justification of the relevant instance. Moreover, she has independent reasons to accept that instance. The property *there is an application of R¬ in the derivation x* can be defined in classical set theory. The $K3$-theorist may argue that this predicate satisfies excluded middle precisely because it stands for a mathematically definable property, so the proof is after all available to her. More generally, rejecting classical logic does not prevent the $K3$-theorist from accepting, *on non-logical grounds*, certain instances of classical principles that are invalid in $K3$. The instance of excluded middle needed for the proof of Theorem 5 does not feature any of the vocabulary that, according to the $K3$-theorist, motivates a revision of logic. Thus, she can coherently reject classical logic and accept this instance.[21]

What about induction? As we saw, $K3$-based mathematical theories can express induction as the rule *Ind*, rather than as a law. Given that the deduction theorem is not $K3$-valid, the paradoxical instances that affect the law form of induction do not pose a threat to *Ind*. This is relevant because we think that it is the rule form of induction that is needed for the proof of Theorem 5 in particular and for meta-theoretic proofs in general.

But even if we are wrong, i.e. even if it is the law form of induction that is needed, this should not be a problem. The reason is that, again, only certain instances of induction are needed for the proof, and these instances are not inconsistent with a paracomplete theory of truth. To illustrate, one of the claims that are being proved by induction is that if there is a derivation in $\mathcal{S}_{K3}$ of $\Rightarrow \phi^{at} \vee \neg\phi^{at}$ for all the atomic subformulas of a formula ϕ, then there is also a derivation in $\mathcal{S}_{K3}$ of $\Rightarrow \phi \vee \neg\phi$. One first shows that this holds if ϕ is itself an atomic formula and, second, that if it holds for formulas of complexity less than n, then it holds for formulas of complexity n. The induction is performed on the conditional property *if there is a derivation of $\Rightarrow x^{at} \vee \neg x^{at}$ in $\mathcal{S}_{K3}$ for each atomic subformula x^{at} of the formula x, then there is a derivation of $\Rightarrow x \vee \neg x$ in $\mathcal{S}_{K3}$*. Clearly, this is a purely set-theoretic property. Therefore, there are independent reasons for thinking that it is available to the $K3$-theorist. Put differently, the $K3$-theorist is only availing herself of an instance of induction which is not inconsistent with her theory, which means that there is no

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[21]In connection to this, cf. Field (2008, p. 15).

violation of Theoretical Faithfulness. Analogous considerations apply to the other uses of induction in the proof.

There might be a worry about the line of argument we have deployed. In attaining the recapture result, the $K3$-theorist *assumes* that the law of excluded middle can be retained for a certain class of statements. But then one may ask: isn't that precisely what a recapture result is supposed to deliver? The question might be seen as revealing a kind of circularity in the proof for recapture. However, a little thought shows that the worry is misplaced. The point of a recapture result for a paracomplete theory is *not* to prove that excluded middle holds for some set of statements. Rather, the point of a recapture result is to show that, *if* there is some set of statements all of which obey the law of excluded middle, *then* classical reasoning will be available for those statements. The objection conflates these two ideas.

Let us stress that we are not claiming that Theoretical Faithfulness is the whole story about which principles one is entitled to use in one's meta-theory. It only expresses a negative constraint: it points to some principles that one is not licensed to use. Perhaps classical logicians will be tempted to complain that this is not enough. The thought would be that the non-classical logician ought to provide a positive criterion, since if she is silent on which principles it is appropriate to use in meta-theoretic proofs, then there is an explanatory gap in her position. But classical logicians should resist this temptation, for the same complaint can be made against them. The question of which principles one is allowed to use in proving a result about a theory that one endorses is pressing for the non-classical logician only if it is also pressing for her classical rivals.

Still, the classical logician could insist by claiming that we should impose a demand which is slightly stronger than Theoretical Faithfulness; namely, that one ought not use principles that are inconsistent with one's theory and, *if using instances of such principles, one ought to give some reason to think they are true or some justification for their use besides.* Without such a justification it could be said that the $K3$-theorist is, so to speak, cherry-picking instances of excluded middle as she goes along— e.g. the instances required for the proof of the recapture result. And if so, whether or not a piece of mathematical reasoning if available to her must be determined on a case-by-case basis, and that seems to have an important abductive cost.[22]

However, we think that the $K3$-theorist is not forced to cherry-pick. In fact, we submit that she is in a position to provide a general philosophical explanation to justify her decision to retain some instances of excluded middle and to reject others. We will not attempt to provide a full defense of this idea here, for one of us has already done this elsewhere (cf. Rosenblatt, 2022), but we will offer a sketch of how that defense would go.

To begin with, it is useful to make a distinction between two different (though related) questions

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[22]Williamson (2018) develops this line of objection in depth. In fact, this also seems to be what Woods has in mind at some points in his paper. He acknowledges that non-classical theorists might be in a position to justify their use of instances of principles that they do not endorse in general. He merely complains that they have not attempted to supply such justifications in a legitimate way.

that one might pose in connection to the idea of classical recapture. First, one can ask what are the safety conditions needed to reason classically with a given set of statements. Second, one may ask whether the statements in question satisfy those safety conditions. In this paper we were mostly concerned with the first question, viz. the challenge of providing a (faithful) recapture result. But it is clear that, without an answer to the second question, the recapture strategy is incomplete. Thus, in the case at study, in addition to providing a (faithful) proof of Theorem 5, the $K3$-theorist needs to argue that excluded middle holds for some interesting fragment of the truth-theoretic language.

There are various different criteria that the one may adopt to separate the legitimate instances of excluded middle from the illegitimate ones. However, not all of them are on an equal footing. For example, it could be suggested that one ought to retain excluded middle for a statement ϕ just in case ϕ contains no occurrences of the truth predicate. This arguably allows one to reason classically in pure mathematics. However, we do not find this position convincing. For one thing, there are many instances of excluded middle containing occurrences of the truth predicate that the $K3$-theorist can and thus should accept (for instance, 'It is or isn't true that $0 = 1$'). For another, if the non-classical theorist is interested in applied mathematics, she will admit the induction schema in rule form to be instantiated with formulas containing the truth predicate. She may even want to admit *some* instances of the law of induction with occurrences of the truth predicate. But accepting such instances without endorsing any instance of excluded middle with the truth predicate would arguably lead her to an unstable position. There is no obvious justification for allowing these instances in one case but not in the other.

A more robust alternative would be to retain excluded middle for ϕ just in case ϕ is not paradoxical in Kripke's sense.[23] This position is certainly not weak; alas, it is untenable. The set composed by all non-paradoxical instances of excluded middle is inconsistent in $K3$. There are pairs of instances of excluded middle that are consistent on their own but that are inconsistent when taken together. So there is no unique maximal consistent set of instances of excluded middle that the $K3$-theorist could endorse. There are many such sets, all of which are incompatible with one another, and there is no principled way of choosing one of these sets over the other ones.[24]

The option that we find most plausible, and to which we subscribe, is that one should endorse excluded middle for ϕ just in case ϕ is a grounded statement, in Kripke's sense again.[25] There are a couple of things to say in favor of this view. First, if one believes that 'ϕ or not ϕ' ought to be endorsed whenever there is a non-semantic fact of the matter as to whether ϕ is the case or not, then it seems that one ought to accept $\phi \lor \neg \phi$ just in case ϕ is grounded. Second, if one accepts

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[23]A statement is paradoxical in this sense if, and only if, it lacks a classical truth-value at every interpretation. See Kripke (1975, p. 708) for the definition.

[24]A proof of this fact can be found in Rosenblatt (2020). For the cognoscenti, the fact can be understood as a non-classical variant of McGee's (1992) well-known result that there is no unique maximal consistent set of instances of Tarski's schema.

[25]A statement is grounded just in case it has a classical truth value at every interpretation. See Kripke (1975, p. 706).

excluded middle for statements without the truth predicate and, moreover, one endorses that ϕ and $Tr\ulcorner\phi\urcorner$ are intersubstitutable for any ϕ, then there is a sense in which one is *already* committed to excluded middle for grounded statements; the reason is that, when intersubstitutability is assumed, the truth or falsity of a grounded statement is ultimately determined by the truth or falsity of a statement or a set of statements that does not contain the truth predicate at all.[26]

To be sure, these rough-and-ready remarks just scratch the surface and are only intended to illustrate one way in which the $K3$-theorist may justify her idea of retaining some instances of excluded middle without having to cherry-pick. At bottom, our view is that the recapture strategy only bares fruit when the answers to the two questions we mentioned above are harmonically assembled together. If we may use a Kantian dictum, we can say that without a recapture result indicating which instances of excluded middle one ought to endorse, the recapture strategy is blind; and without a justification for those instances, the strategy is empty.

6 Conclusion

Non-classical logics are sometimes dismissed on the grounds that any case in favor of them must ultimately rely on classical logic. For example, it is often suggested that in order to prove results such as completeness, compactness, undecidability, and so on, the non-classical theorist needs the full power classical logic because the proofs of these results implicitly rely on classical set theory. What we have suggested is that there is an important ambiguity in this line of argument that is more often than not overlooked. There is a sense in which it is true that the non-classical theorist needs classical logic in the meta-language, since in proving meta-theoretic and other mathematical results she occasionally appeals to classically valid principles that she herself rejects. Yet, the crucial point is that it would be erroneous to conclude from this that she endorses those principles in full generality—in that sense, the claim is false. All that can be inferred is that she uses specific instances of those principles, and this is perfectly compatible with her rejection of other instances of those very same principles. In other words, it is simply inaccurate to say, without qualification, that the non-classical theorist 'employs classical logic in the meta-theory', at least if by that one means that she accepts those principles unrestrictedly. Once the ambiguity is removed, there is a clear sense in which she can do classical meta-theory while remaining faithful to her favored theory. With a recapture result at her disposal, she can accept classical set theory and reject classical logic.

What should the classical logician make of this? For example, Williamson has suggested that even if it is admitted that a proof of a recapture result is available to the non-classical theorist

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[26]It is well-known that the set of grounded statements is very complex from a computational point of view. A consequence of this is that it is not axiomatizable (cf. Burgess, 1986). Thus, if one takes a purely axiomatic approach to classical recapture, all one can say is that one can reason classically with some recursively enumerable subset of the set of grounded statements (cf. Burgess (2014) for an axiomatic theory of grounded truth). However, one should not be discouraged by this. In this context completeness was never an attainable goal to begin with.

and it is admitted that the non-classical theorist has a general criterion to identify the statements that satisfy excluded middle, the recapture strategy will involve the postulation of additional metalinguistic premises to guarantee that the relevant statements or expressions are well-behaved, and this has an important explanatory cost. In Williamson's words (2018, p. 418), '(...) such metalinguistic premises seem (...) out of place in an ordinary natural scientific explanation (...) Thus the recovery strategy has a tendency to degrade ordinary explanations in natural science'.

But we think that at this point in the discussion, the non-classical theorist can happily concede that the strategy has a cost. The non-classical theorist can have her cake (i.e. she can attain naivety for semantic concepts) and eat it too (i.e. she can retain every mathematical theorem). If the cake is a bit pricey, so be it. The cost can be paid.[27]

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