



# HANS ROTT **Difference-Making Conditionals and Connexivity**

**Abstract.** Today there is a wealth of fascinating studies of connexive logical systems. But sometimes it looks as if connexive logic is still in search of a convincing interpretation that explains in intuitive terms *why* the connexive principles should be valid. In this paper I argue that difference-making conditionals as presented in Rott (*Review of Symbolic Logic* 15, 2022) offer one principled way of interpreting connexive principles. From a philosophical point of view, the idea of difference-making demands full, unrestricted connexivity, because neither logical truths nor contradictions or other absurdities can ever ‘make a difference’ (i.e., be relevantly connected) to anything. However, difference-making conditionals have so far been only partially connexive. I show how the existing analysis of difference-making conditionals can be reshaped to obtain full connexivity. The classical AGM belief revision model is replaced by a conceivability-limited revision model that serves as the semantic base for the analysis. The key point of the latter is that the agent should never accept any absurdities.

*Keywords:* Conditionals, Conditional logic, Connexive logic, Belief revision, Ramsey test, Difference-making, Relevance, Contingency, Aristotle, Boethius, Abelard.

## 1. Introduction: Connexive Logic, Conditional Logic and Difference-Making

Connexive logic has a distinguished history. The name ‘connexive logic’ seems to have been introduced in McCall’s Oxford Dissertation [22]. McCall in turn credits the term ‘connexive’ to Bocheński [5] who uses it in his reporting about Sextus Empiricus’ summary of the kinds of implication discussed by the Megarian-Stoic school,<sup>1</sup> However, the main historical sources of connexive logic are Aristotle and Boethius and, to a somewhat lesser extent, Abelard. Connexive logic is most frequently taken to be defined by the idea that certain ‘theses’ or ‘principles’ should be theorems of propositional

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<sup>1</sup> ‘*Konnexe*’ *Implikation* translated as ‘connexive’ implication, with Bocheński’s scare quotation marks.

Special Issue: Frontiers of Connexive Logic  
Edited by: **Hitoshi Omori and Heinrich Wansing.**

logic—even though they are non-theorems in the propositional logic that we today call classical.

I suggest to use somewhat non-committal and flexible formulations of three central principles of connexivity which are here formulated in a metalinguistic way. They all concern (non-nested) conditionals and negations:

(Arist1) Not  $A \rightarrow \neg A$ .

(Arist2) Not both  $A \rightarrow C$  and  $\neg A \rightarrow C$ .

(Boet-Abel) Not both  $A \rightarrow C$  and  $A \rightarrow \neg C$ .

Aristotle’s First Thesis (Arist1) also comes in a twin version ‘Not  $\neg A \rightarrow A$ ’ which I will neglect because it is equivalent to the above version in the context of the present paper. Aristotle’s Second Thesis<sup>2</sup> (Arist2) and the thesis (Boet-Abel) of Boethius and Abelard<sup>3</sup> are remarkably similar to each other and exhibit an appealing symmetry: the former concerns a negated antecedent, the latter a negated consequent. The principle of Boethius and Abelard is widely regarded as defining the very concept of connexive logic, while there is only little discussion about (Arist2) in the connexive logic community. This seems unjustified to me, both from a historical and from a systematic point of view. The historical origin of connexive logic is often located in Aristotle’s *Analytica Priora* 57b3–14 where he argues for (Arist2) (or something very similar) on the grounds of (Arist1) (or something very similar). Boethius and Abelard came nine and fifteen centuries later, respectively, but they had no intention of denying what Aristotle had said.<sup>4</sup> Systematically, the justification of (Boet-Abel) by means of (a version of)

<sup>2</sup>Since McCall [23, p. 416] these names have become fairly common. Martin [20, pp. 379–381], where the labels were introduced, had the numbering of the Aristotelian theses the other way round, and Martin has kept his numbering in later papers.

<sup>3</sup>According to their most common formalizations in modern connexive logic, (the strong) Boethius Thesis and Abelard’s (First) Principle (a.k.a. Strawson’s Thesis or Angell’s Principle of Subjunctive Contrariety) are similar in spirit, but differ in logical detail. Boethius’ Thesis  $(A \rightarrow C) \rightarrow \neg(A \rightarrow \neg C)$  (or its twin version  $(A \rightarrow \neg C) \rightarrow \neg(A \rightarrow C)$ ) features *nested conditionals* Abelard’s similar principle  $\neg((A \rightarrow C) \wedge \neg(A \rightarrow \neg C))$  does not. I do not think that this divergence can be historically substantiated. If it is granted, my formulation of the principle is closer to Abelard than to Boethius. In this paper, however, I want to avoid any embeddings of conditionals, even Boolean ones.

<sup>4</sup>Here is a telling passage of Abelard’s *Dialectica*: “Patet itaque ex inductione inconvenientium aristotelicam regulam constare, quae . . . negabat ad affirmationem et negationem eiusdem idem sequi; ostendamus quoque et quod nos adiecimus affirmationem et negationem eiusdem ad idem consequi non posse”. (Abaelardus [1, p. 291]) Or in English: “And so it is clear from the incidence of the embarrassments that the Aristotelian rule persists which . . . denied that from the affirmation and negation of the same thing the

(Arist1) runs exactly parallel to the justification of (Arist2) by means of (the twin version of) (Arist1).<sup>5</sup> So perhaps the preference that contemporary connexive logicians give to (Boet-Abel) over (Arist2) is only a matter of historical contingency. In any case, one of the fruits of the present paper is to have a perfectly equal treatment of (Arist2) and (Boet-Abel).

Today there is a great wealth of fascinating studies of connexive logical systems. However, connexive logic is only defined by the postulated validity of a few characteristic principles for conditionals and negations. Sometimes it looks as if it is still in search of a convincing interpretation that explains in intuitive terms *why* these connexive principles should be valid. The present paper offers such an interpretation, and in fact one that treats Aristotle's Second Thesis and the Thesis of Boethius and Abelard in a parallel way. I do not claim, of course, that it is the only reasonable interpretation of connexive logic, or that it is the interpretation that matches the intentions of its classical advocates. But it is one consistent interpretation.

The vantage point of this paper is conditional logic rather than connexive logic, and we'll mark this by the use of different symbols for the conditional connective. One of the standard readings of a conditional 'If  $A$  then  $C$ ' is the *suppositional* one that is captured by the *Ramsey Test*:

(RT) Accept a (suppositional) conditional 'If  $A$ , then  $C$ ' ( $A > C$ ) just in case you believe  $C$  under the supposition of  $A$ .

The Ramsey Test with 'Believe a conditional' substituted for 'Accept a conditional' was pioneered by Stalnaker [33, pp. 43–45] and Gärdenfors [8]. We will not interpret conditionals as objects of belief and thus talk of the *acceptance* of conditionals. The main idea of *difference-making* conditionals is expressed by the *Relevant Ramsey Test* (Rott [28,31]):

(RRT) Accept a (difference-making) conditional 'If  $A$ , then  $C$ ' ( $A \gg C$ ) just in case

- (i) you believe  $C$  under the supposition of  $A$ , but
- (ii) you don't believe  $C$  under the supposition of  $\neg A$ .

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same follows. Let us also show what we added: that it is not possible that the affirmation and the negation of the same thing follows from the same." (Translation HR)

<sup>5</sup>The equal status of (Arist2) and (Boet-Abel) in Abelard [1, pp. 290–292] is nicely highlighted by Martin [21, p. 190], who shows that Abelard's justification of (Boet-Abel) is entirely parallel to Aristotle's justification of (Arist2) and presupposes contraposition and transitivity of  $\rightarrow$ . There is also a direct, immediate link between (Arist2) and (Boet-Abel), if one allows free use of double negation and contraposition (as pointed out by McCall [22, p. 21], and again by Martin [19, p. 568]).

Clause (i) is just the ordinary Ramsey Test for suppositional conditionals, clause (ii) compares the result of the supposition of the negated antecedent  $\neg A$  with that of the unnegated  $A$ . This is a way of encoding the idea that the antecedent of a conditional should *make a difference* for the consequence, or that the former is (*positively*) *relevant* to the latter. ‘If ... then’ is thereby taken as a *contrastive* connective, much like the factive connective ‘because’.

I want to emphasize a difference in interpretation between conditionals in connexive logic and difference-making conditionals. In historical and systematic studies of connexive logic, conditionals of the form  $A \rightarrow C$  are frequently glossed as ‘ $A$  entails  $C$ ’, ‘ $A$  implies  $C$ ’ or ‘ $C$  follows from  $A$ ’. It is hard to avoid reading this as ‘ $A$  *logically* entails  $C$ ’, ‘ $A$  *logically* implies  $C$ ’ or ‘ $C$  *logically* follows from  $A$ ’. It is intended that conditionals reflect some meta-logical relationship between the antecedent and the consequent (only this, I believe, would justify calling conditionals ‘implications’), and this seems to be a natural view if the task of logic is taken to be the identification of *logical truths*.

Natural-language conditionals like ‘If Lisa comes, Max will be happy’ (indicative) or ‘If Lisa had come, Max would have been happy’ (counterfactual) are statements expressing the speaker’s view about the world. They do not state logical relationships between antecedent and consequent—as long as one understands logic in the sense of formal, context-free logic. One may say, ‘Given a certain set of facts, it follows from Lisa’s coming that Max will be happy’, or ‘Given my current beliefs, it follows from Lisa’s coming that Max will be happy’, but this involves a non-standard, extended concept of ‘following’ (‘entailment’, ‘implication’), something like following-in-the-context-of-certain-cotenable-propositions.<sup>6</sup>

Since I won’t interpret conditionals as reflecting logical implication, but at best only some sort of context-dependent implication, it is often difficult to import arguments pertaining to connexive logic into the context of difference-making conditionals. In particular, it is problematic to make free use of inference schemes such as contraposition and transitivity, as it has often been done by Aristotle, Boethius, Abelard, as well as in discussions of connexive logic. Difference-making conditionals are close in kind to the

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<sup>6</sup>We are then close to the so-called metalinguistic theory of conditionals and counterfactuals associated with philosophers like Roderick Chisholm, Nelson Goodman, John L. Mackie and Nicholas Rescher. Of course this extended concept of ‘implication’ is in much need of explication. Three explications are reviewed by Lewis [18, sections 3.1–3.3] Belief revision theories may be understood as offering an alternative way of explicating ‘cotenable propositions’, viz., as the beliefs (and expectations) that are still held after a doxastic state has been revised.

conditionals studied in conditional logic, and for the founders of this field, contraposition and transitivity are paradigmatically *invalid* for indicative conditionals and counterfactuals<sup>7</sup>

Like Rott ([31], ‘DMC’ from now on), the present paper does not aim at presenting a conditional logic in the sense of delineating a set of logical truths or theorems. It rather endorses the view that the task of logic lies in identifying what can be validly *inferred* from what. And this concept will be explicated semantically, not with the help of possible worlds and truth conditions, but with the help of rational belief states and conditions of belief or acceptance. Something can be validly inferred from something else if and only if the former is believed or accepted whenever the latter is believed or accepted, in *every* rational belief state. That the inference from  $A \gg B$  and  $A \gg C$  to  $A \gg (B \wedge C)$  is valid, for instance, means that in every rational belief state in which the former conditionals are accepted, the latter conditional is accepted, too. It will turn out that the logic developed in the present paper has no ‘logical truths’, i.e., no conditionals that are accepted in all rational belief states. None of the typical candidates (like  $A \gg A$ ,  $(A \wedge B) \gg A$ ,  $A \gg (A \vee B)$ ,  $\top \gg \top$ ,  $\perp \gg \perp$  or  $\perp \gg \top$ ) is universally acceptable.<sup>8</sup> There is, however, a conditional that is true *a priori* and depends on the particular belief state the agent is in: if  $X_{bel}$  it the agent’s strongest belief, then  $X_{bel} \gg X_{bel}$  is always accepted.

Implementing the idea of relevance into conditionals by means of the Relevant Ramsey Test is *a very principled way of obtaining connexivity*.<sup>9</sup> As noted in ‘DMC’ (pp. 145, 148), difference-making conditionals have many distinctively connexive features. But they fail to be fully connexive in that they validate (Boet-Abel) only in a restricted form.<sup>10</sup> In the light of the last

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<sup>7</sup>See Adams [2, pp. 166–168, 177–178, 188–192] Stalnaker [33, pp. 48–49] and Lewis [18, pp. 32–36]. Also cf. the counterexample to contraposition given at the end of section 5.1 of ‘DMC’.—I take it that both connexive logic and the logic of difference-making conditionals are projects of expressing ‘reasonable condition[s] for the behaviour of conditionals’ (Lenzen [16, p. 25]). I do not presuppose that there is just one kind of conditional that is used in ordinary or scientific language, nor do I think that any deviation from classical propositional logic or (normal) modal logic is a mistake.

<sup>8</sup>This is different from ‘DMC’, which has  $\perp \gg \perp$  as a basic principle.

<sup>9</sup>Ramsey’s famous footnote indeed contains a formulation which is close to the Abelard-Boethius Thesis. This was first pointed out by McCall [23, p. 420].

<sup>10</sup>I am using the term ‘full connexivity’ in order to mark the unrestricted validity of the three connexive principles (Arist1), (Arist2) and (Boet-Abel), not in a sense that presupposes embeddings or nestings of conditionals. More specifically, full connexivity concerns the lifting of the restriction of the principle (Boet-Abel) that is present in ‘DMC’.—An important feature of difference-making conditionals as presented in ‘DMC’ is that they

paragraphs, the reasons for this connexivity are likely to be different from the reasons that drove the historical development of connexive logic. The questions I want to address in this paper are these: Is it possible to make difference-making conditionals “more connexive”, i.e., to make them satisfy all the properties of connexive logic? This is an important question if, as I have claimed above, the idea of difference-making conditionals offers a very principled route to connexivity. I will answer this question in the positive.

In terms of general ideas and structure, the remainder of this paper is quite similar to ‘DMC’. But the reader should not be led into thinking that this is just a trifling, if somewhat laborious technical exercise. The change carries considerable philosophical weight. I now think that if one takes the idea—or rather: my interpretation of the idea—of connexivity seriously, then one needs to go all the way and endorse *unrestricted* connexivity. An important point of ‘DMC’ has been this one:

(Irrelevant  $\top$ ) Tautologies are never relevant to anything, nor is anything ever relevant to them.

Tautologies have no role to play in difference-making conditionals because they have no content. They are always true. But the status of contradictions is very different in ‘DMC’. They are (or more precisely, the falsity constant ‘ $\perp$ ’ is) very frequently used, and indeed they serve the purpose of defining absurdities and belief in a very convenient way. But intuitively, this is strange. Because they have ‘much too much content’—so much they are never true—, I now think that contradictions (and in fact other absurdities as well) should receive equal treatment. My new thesis thus is:

(Irrelevant  $\perp$ ) Contradictions are never relevant to anything, nor is anything ever relevant to them.

Contradictions should have no role to play in difference-making conditionals because they have too much content, as it were—so much that they are never true. Clearly, this is a reflection of the basically classical background logic

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Footnote 10 continued

validate neither ‘(Conjunctive) Simplification’,  $(A \wedge B) \rightarrow A$ , nor ‘(Disjunctive) Addition’,  $A \rightarrow (A \vee B)$ . Aside from some limiting cases, a difference-making conditional of the form  $(A \wedge B) \gg A$  is accepted iff  $A$  is not believed under the supposition of  $\neg(A \wedge B)$ ; a difference-making conditional of the form  $A \gg (A \vee B)$  is accepted iff  $B$  is not believed under the supposition  $\neg A$ . Thus both conditionals make substantive, non-trivial claims and are far from being universally acceptable. Routley, Meyer, Plumwood and Brady [32, pp. 3–4, 47, 79–85] consider the violation of Simplification (and, to a lesser degree, the violation of Addition) as a defining feature of connexive logics.

on which the conditional logic to be developed is based. And yet the re-interpreted difference-making conditionals studied in this paper are fully connexive.<sup>11</sup>

## 2. Two Decisions in the Theory of Belief Revision: What Are Absurdities, and What Is One to Do With Them?

The analysis of difference-making conditionals in ‘DMC’ is conducted in terms of a belief revision semantics. The classical theory of belief revision is due to Alchourrón, Gärdenfors and Makinson—hence the widely used acronym ‘AGM’ (Alchourrón, Gärdenfors and Makinson [3], Gärdenfors [9]).

The analysis of difference-making conditionals is sensitive to certain choices made on the level of belief revision theory. We need to distinguish between the genuine revision of the agent’s beliefs that is prompted by the acceptance of new information on the one hand, and a merely hypothetical revision prompted by the making of a supposition on the other. Our starting point is the view that AGM-style belief revision is capable of modelling suppositional belief change, too.<sup>12</sup>

We will now consider two important decisions for the conceptualization of belief revision. These design decisions concern the way that ‘absurdities’ are defined and dealt with. By an *absurdity* (or *doxastic impossibility*) I informally mean a proposition that is extremely hard—or even impossible—to believe or suppose as true.

We may safely take it that all contradictions (logical falsehoods) are absurdities. The decisions mentioned above can be characterised by the answers they give to the following two questions:

(1) *Are all absurdities contradictions?* Classical AGM theory says ‘yes’. This is a consequence of AGM’s postulate of ‘consistency preservation’: Unless the input is a contradiction, the agent’s posterior belief state is supposed to be consistent. But it is quite easy to say ‘no’ and liberalize AGM here, and

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<sup>11</sup>I apologize to the advocates of *relevance logic* or *relevant logic* for using the same term ‘relevance’ with a very different meaning. There is simply no better word for (positive) relevance in my sense than ‘relevance’. The term ‘difference-making conditionals’ has been chosen in order to avoid confusion with ‘relevance conditionals’ in the sense of Iatridou [13, p. 50] and other linguists taking up her usage of the term. Finally, let me add that my project is fundamentally different from the projects of *paraconsistent* and *hyperintensional logic*.

<sup>12</sup>Levi [17, pp. 5–8] argues that it is even better at modelling suppositional belief change than at modelling genuine change in belief.

this is precisely what was done in ‘DMC’, by using weaker axioms (\*5a) and (\*5b) rather than AGM’s stronger consistency postulate (\*5).<sup>13</sup> I think this latter answer is philosophically desirable for the genuine revision of beliefs: If an agent were forced to accept ‘for real’ that she is the empress of China, her beliefs might ‘explode’ and she might be ready to accept anything. But the liberalization is less compelling for hypothetical reasoning: She will not land in ‘epistemic hell’ (Gärdenfors [9, p. 51]) if she only *supposes* that she were the empress of China. She can make sense of more propositions as suppositions than she can sensibly digest as genuine beliefs. But even conceivability appears to have its limits. Can an agent suppose that she is a fried egg?<sup>14</sup> Can she, a woman of the 21st century, suppose that she is Napoleon?

Absurdities may alternatively be called doxastic impossibilities. I have not yet said how to formally define an absurdity. This will be done later, but we may introduce already here a bit of further terminology. A sentence  $A$  is called *conceivable* iff it is not an absurdity. It is a *doxastic necessity* iff its negation  $\neg A$  is an absurdity. It is *contingent* iff it is neither an absurdity nor a doxastic necessity, i.e., iff both  $A$  and  $\neg A$  are conceivable.<sup>15</sup> More on the notions of absurdity, conceivability, doxastic necessity and contingency later.

(2) *Should agents accept absurdities?* Classical AGM theory says ‘yes’. This is a consequence of AGM’s unrestricted postulate of ‘success’: Each input for a revision is to be ‘successfully’ accepted, even if the input is absurd. The resulting belief state is epistemic hell, but due to AGM’s design decision regarding (1), only contradictions lead us into epistemic hell. Philosophically, however, it seems a good idea so say ‘no’ here when the genuine revision of beliefs is concerned. Agents should avoid epistemic hell—this seems to be *the* motivating motivation behind the whole project of belief revision theories. One rather principled way of dealing with the problem of absurdities is to just refuse to change anything and keep one’s doxastic state when presented with an absurd input. So a good option seems to be to restrict the success postulate to conceivable inputs and universally require

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<sup>13</sup>(\*5) says that a belief set revised by  $A$  is inconsistent only if  $A$  is inconsistent. For (\*5a) and (\*5b), see the list in Section 4 below.

<sup>14</sup>This question is inspired by Priest [25]. Poets have particularly strong powers of imagination: “y si yo fuera sal/ tú serías una lechuga/ una palta o al menos un huevo frito/ y si tú fueras un huevo frito/ yo sería un pedazo de pan” (Bertoni [4], translation: “And if I were salt/ you’d be lettuce/ an avocado or at least a fried egg/ and if you were a fried egg/ I’d be a piece of bread.”)

<sup>15</sup>The latter deviates from the rather nonchalant usage of ‘contingent’ in ‘DMC’ which now seems infelicitous to me.



the consistency of an agent's belief states,<sup>16</sup> Again the situation is somewhat less clear for hypothetical reasoning. It seems quite possible to reason on the basis of a supposition that strikes one as absurd (at least if it is logically consistent). In 'DMC', AGM's design decision is followed, but there is a hint at the alternative as a potential way of connexifying difference-making conditionals.<sup>17</sup> The present paper expands these rather casual side remarks there into a more proper treatment.

So where AGM say 'yes' twice, we say 'no' in reply to both questions in this paper. While our first decision appears to be desirable from a philosophical point of view, it is not particularly relevant to the problem of connexivity. However, our second decision will turn out to have important consequences for the connexivity of difference-making conditionals, both from a philosophical and a formal point of view.

### 3. Technical Preliminaries

We denote a belief state by the letter ' $\mathfrak{B}$ '. The nature of  $\mathfrak{B}$  is left completely open, except that we assume that one can always determine, with the help of a uniform method *Bel*, the agent's set of beliefs in belief state  $\mathfrak{B}$ . The beliefs thus determined are expressed in a certain language. Our object language in this paper features the logical constants  $\top$  (*verum*) and  $\perp$  (*falsum*), the usual truth-functional propositional operators  $\neg$ ,  $\wedge$ ,  $\vee$  and  $\supset$ , as well as a conditional connective  $\gg$ . Sentences without any occurrences of  $\gg$  are called *factual* sentences. In this paper, I follow AGM in assuming that the factual language is governed by some reflexive, monotonic and idempotent consequence operation ('background logic')  $Cn$  which is supraclassical and compact and satisfies the deduction theorem:  $\Gamma \subseteq Cn(\Gamma)$ ; if  $\Gamma \subseteq \Delta$  then  $Cn(\Gamma) \subseteq Cn(\Delta)$ ;  $Cn(Cn(\Gamma)) = Cn(\Gamma)$ ; if  $Cn_0$  is classical tautological implication then  $Cn_0(\Gamma) \subseteq Cn(\Gamma)$ ; if  $A \in Cn(\Gamma)$  then  $A \in Cn(\Gamma_0)$  for some finite subset  $\Gamma_0$  of  $\Gamma$ ; and finally,  $B \in Cn(\Gamma \cup \{A\})$  iff  $A \supset B \in Cn(\Gamma)$ . I use the notation  $A \vdash B$  for  $B \in Cn(\{A\})$ .

Because I want to remain non-committal as to whether or not conditionals express propositions, conditionals formed with the help of the connective  $\gg$  are not embedded in more complex sentences. For the same reason I will say that we *accept* or *reject* conditionals, whereas we *believe* or *disbelieve*

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<sup>16</sup>The earliest discussion in print of unrestricted consistency and restricted success is probably due to Hansson [11, pp. 343–344] who refers to an unpublished manuscript of mine that became part of Rott [29, cf. pp. 111 and 128–138].

<sup>17</sup>In footnotes 17 and 27 of that paper.

or *suspend judgment on* factual sentences (or on the propositions expressed by them). Only factual sentences are in the belief set  $Bel$ . The ‘logic’ of conditionals as studied in this paper takes place in the meta-language, just as the logic of belief revision does within the AGM approach.

In the following,  $Bel$  is short for  $Bel(\mathfrak{B})$ , and  $Bel * A$  is short for  $Bel(\mathfrak{B} * A)$ , which denotes the belief set obtained after revising one’s belief state  $\mathfrak{B}$  by a new piece of information or by a hypothetical assumption  $A$ . Throughout this paper, we will make *three presuppositions* about the initial, unrevised belief set  $Bel$ : (i) it is consistent; (ii) it can be characterised by a strongest belief  $X_{bel} \in Bel$  and the condition  $A \in Bel$  just in case  $X_{bel}$  logically implies  $A$ <sup>18</sup>; and (iii) there is at least one belief  $A$  such that  $\neg A$  is conceivable for the agent (in other words, her strongest belief  $X_{bel}$  is not a doxastic necessity). Taken together, this means that our beliefs are—or more precisely, that  $X_{bel}$  is—assumed to be contingent.

Since there are no embeddings of conditionals governed by the Ramsey Tests,<sup>19</sup> many of the following considerations will be formulated in the metalanguage. We will use ‘ $A \not\gg C$ ’ as an abbreviation of the metalinguistic statement ‘not  $A \gg C$ ’.

#### 4. Classical Belief Revision

For ease of reference, let us repeat the belief revision axioms endorsed in ‘DMC’ which are a slight variation of AGM’s. In the following,  $Bel$  is the agent’s belief set and  $*$  is the revision function that complies with the decisions made in ‘DMC’: *no*, not all absurdities are contradictions, but *yes*, agents who are presented with absurdities should accept them, even though this will lead them into a state in which everything is accepted.

- (\*0)  $Bel$  is consistent. (Initial Consistency)
- (\*1)  $Bel * A = Cn(Bel * A)$ . (Closure)
- (\*2)  $A \in Bel * A$ . (Success)
- (\*3)  $Bel * A \subseteq Cn(Bel \cup \{A\})$ . (Inclusion)
- (\*4) If  $\neg A \notin Bel$ , then  $Bel \subseteq Bel * A$ . (Preservation)

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<sup>18</sup>In the finite case, we may think of  $X_{bel}$  as a very long conjunction. Note that (ii) implies that  $Bel$  is closed under logical consequences.

<sup>19</sup>This applies to the present paper. But it is very well possible to admit right-nested Ramsey Test conditionals if methods for iterated belief revision are available. See Rott [30].

- (\*5a) If  $\perp \in Bel * A$ , then  $\perp \in Bel * A \wedge B$ . (Absurdity 1)
- (\*5b) If  $\perp \in Bel * A$  and  $\perp \in Bel * B$ , then  $\perp \in Bel * A \vee B$ . (Absurdity 2)
- (\*6) If  $A \dashv\vdash B$ , then  $Bel * A = Bel * B$ . (Intensionality)
- (\*7c) If  $B \in Bel * A$ , then  $Bel * (A \wedge B) \subseteq Bel * A$ . (Cut)
- (\*7)  $Bel * (A \wedge B) \subseteq Cn((Bel * A) \cup \{B\})$ . (Conditionalisation)
- (\*8c) If  $B \in Bel * A$ , then  $Bel * A \subseteq Bel * (A \wedge B)$ . (Cautious Monotonicity)
- (\*8) If  $\neg B \notin Bel * A$ , then  $Bel * A \subseteq Bel * (A \wedge B)$ . (Rational Monotonicity)

Some comments on the (minor) deviations from the original AGM axioms are given in ‘DMC’. For the present paper, it is important to emphasize that both (\*4) and (\*8) are weakened versions of AGM’s original fourth and eighth axioms which have the consequents  $Cn(Bel \cup \{A\}) \subseteq Bel * A$  and  $Cn((Bel * A) \cup \{B\}) \subseteq Bel * (A \wedge B)$ , respectively.<sup>20</sup> Given Closure (\*1) and Success (\*2), the weakenings don’t matter, because the stronger versions can easily be recovered. However, in the next section, when we won’t have unrestricted Success any more, we will see that the differences matter. The postulates (\*5a) and (\*5b) are weakenings of AGM’s fifth postulate. This change reflects the fact that in ‘DMC’ a sentence  $A$  is an *absurdity* of a belief state  $\mathfrak{B}$  if and only if  $Bel * A$  is inconsistent, i.e., iff  $\perp \in Bel * A$ .<sup>21</sup> (\*1) and (\*2) together imply that contradictions are absurdities. While AGM’s postulate (\*5) answers the first question of Section 2 with ‘yes’, postulates (\*5a) and (\*5b) allow absurdities that aren’t contradictions. A sentence  $A$  is a *doxastic necessity* if  $\perp \in Bel * \neg A$ , and it is a *contingency* if neither  $\perp \in Bel * A$  nor  $\perp \in Bel * \neg A$ . Postulates (\*7c) and (\*8c) are two prominent weakenings of AGM’s ‘supplementary’ postulates (\*7) and (\*8) that are well known in the literature of defeasible reasoning as characterising ‘Cumulative Reasoning’.<sup>22</sup>

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<sup>20</sup>If we endorse the extra postulate that  $Bel = Bel * \top$ , (\*3) follows from (\*7), and (\*4) follows from (\*8).

<sup>21</sup>Thus, in line with the terminology introduced above, a sentence  $A$  is *conceivable* iff  $\perp \notin Bel * A$ ;  $A$  is a *doxastic necessity* iff  $\perp \in Bel * \neg A$ ; and  $A$  is *contingent* iff neither  $\perp \in Bel * A$  nor  $\perp \in Bel * \neg A$ .

<sup>22</sup>More precisely, (\*8c) is only a weakening of (\*8) for non-absurd  $A$ , i.e., only if  $\perp \notin Bel * A$ .

The most striking fact about difference-making conditionals, i.e., conditionals governed by the Relevant Ramsey Test, is that they do not validate Right Weakening which has long seemed entirely innocuous to conditional logicians. That is, for difference-making conditionals,  $A \gg C$  and  $C \vdash B$  together do not imply  $A \gg B$ . In ‘DMC’, the invalidity of Right Weakening is called *the hallmark of difference-making conditionals* and indeed *of the relevance relation*. Another important property of difference-making conditionals is that  $A \vdash C$  does not imply that  $A \gg C$  is accepted: neither ‘(Conjunctive) Simplification’  $A \wedge B \gg A$  nor ‘(Disjunctive) Addition’  $A \gg A \vee B$  is universally acceptable. If  $C$  is accepted *anyway* (like for instance a logical truth  $C$  is), then  $A$  cannot be relevant to  $C$ , even if it logically implies  $C$ .

Supposing that conditionals are meant to express that the antecedent is positively relevant for the consequent, the symmetry between (Arist2) and (Boet-Abel) appears to be plausible:  $A$  cannot be positively relevant simultaneously to both  $C$  and  $\neg C$ . And  $A$  and  $\neg A$  cannot both be positively relevant to  $C$ .<sup>23</sup>

We now consider the behaviour of difference-making conditionals regarding these principles (the formal proofs are very short and given in ‘DMC’, p. 155).

- (i) We have unrestricted (Arist1): Not  $A \gg \neg A$ .<sup>24</sup> The reason for the unacceptability of  $A \gg \neg A$  here is that we always have  $\neg A \in Bel * \neg A$  (and  $A \in Bel * \neg \neg A$ ), due to the unrestricted success condition.
- (ii) We have unrestricted (Arist2): Not both  $A \gg C$  and  $\neg A \gg C$ . This is a direct consequence of the definition of difference-making conditionals.  $A \gg C$  says that  $C \in Bel * A$  and  $C \notin Bel * \neg A$ , and  $\neg A \gg C$  essentially says that it is just the other way round.
- (iii) We only have a *substantially restricted version of (Boet-Abel)*: Not both  $A \gg C$  and  $A \gg \neg C$ , unless  $A$  is an absurdity (i.e.,  $\perp \in Bel * A$ ) and neither  $C$  nor  $\neg C$  is a belief. If, however,  $A$  is an absurdity and neither  $C$  nor  $\neg C$  is a belief, then both  $A \gg C$  and  $A \gg \neg C$  are accepted.<sup>25</sup>

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<sup>23</sup>This is true as well if relevance is captured by probabilities. From  $\Pr(C|A) > \Pr(C)$ , both  $\Pr(\neg C|A) < \Pr(\neg C)$  and  $\Pr(C|\neg A) < \Pr(C)$  follow easily (provided that  $0 < \Pr(A) < 1$ ).

<sup>24</sup>And similarly, not  $\neg A \gg A$ . In the AGM framework employed, which is based on a supraclassical background logic  $Cn$ , there is no real difference between these two versions. Thus we will neglect this and all similar twin theses in the following.

<sup>25</sup>It should be mentioned that there is a straightforward, philosophically very different way of making difference-making conditionals fully connexive. As explained in the final section of ‘DMC’, it is not necessary to implement the idea of difference-making in terms of

Summing up: Given the design decisions for belief revision taken in ‘DMC’, the two connexive theses associated with Aristotle are valid in unrestricted form. The connexive thesis associated with Boethius and Abelard, however, is valid only in a restricted, ‘humble’ (Kapsner [14]) form.

## 5. Conceivability-Limited Belief Revision

Now we embark on the project of making difference-making conditionals “fully connexive”. The central idea is that agents should simply refuse to accept absurdities.<sup>26</sup> The crucial step is to adapt the AGM postulates (or the postulates used in ‘DMC’) in order to represent our new design decisions: *no*, not all absurdities are contradictions, and *no*, agents who are presented with absurdities should not accept them. In the following axioms, ‘ $\star$ ’ is the symbol for the revision function that complies with these decisions.

( $\star$ 0)  $Bel$  and  $Bel\star A$  are consistent. (Unrestricted Consistency)

( $\star$ 0b)  $\neg X_{bel} \in Bel\star \neg X_{bel}$ . (Contingent Belief)

( $\star$ 1)  $Bel\star A = Cn(Bel\star A)$ . (Closure)

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a belief revision semantics. One might opt for a possible-worlds semantics for suppositional conditionals  $A > C$  in the style of Stalnaker [33] and Lewis [18], endorse Lewis’s Centering condition, and define the difference-making conditional  $A \gg C$  as the conjunction of  $A > C$  and  $\neg(\neg A > C)$ . Many results obtained in the belief revision framework then carry over to the possible worlds framework (see Raidl [26]). But the two kinds of semantics diverge on Boethius/Abelard’s Principle. The unless clause encountered in (iii) includes the condition that neither  $C$  nor  $\neg C$  is a belief. This condition is met whenever the agent suspends judgment on  $C$ . But truth is different from belief. Possible worlds are points of evaluation where every (factual) sentence is either true or false. Possible worlds do not offer the option of “suspending truth” on  $C$ . If  $A$  is absurd from the perspective of a possible world  $w$ , which entails that both  $A > C$  and  $A > \neg C$  are true at  $w$  for arbitrary  $C$ , then  $\neg A$  is true at  $w$ . Because either  $C$  or  $\neg C$  is true at  $w$ , one of the conditionals  $\neg A > C$  or  $\neg A > \neg C$  is true at  $w$ . But then either  $A \gg C$  or  $A \gg \neg C$  is false at  $w$ . Thus the unless clause (with truth substituted for belief) in the restricted Boethius/Abelard’s Principle mentioned in (iii) cannot be satisfied, and thus the possible-worlds interpretation of difference-making conditionals supports the unrestricted version of the principle of (Boet-Abel). As it also validates (Arist1) and (Arist2), it is fully connexive.

<sup>26</sup>This approach has some similarity to the view of *negation as cancellation* (Strawson 1952, Routley 1978, Routley et al 1982, Routley and Routley 1985, Priest 1999) maintaining that in  $A \wedge \neg A$ , the negation of  $A$  cancels the information provided by  $A$ . The contradiction thus provides no information at all. Also compare Wittgenstein’s [34, p. 209] famous dictum: “Well then, don’t draw any conclusions from a contradiction; make that a rule.” In the present approach, a contradiction produces no change in the agent’s belief state at all. The same holds for absurdities of other kinds.

- ( $\star 2$ )  $A \in Bel\star A$  or  $Bel\star A = Bel$ . (Restricted Success)
- ( $\star 3$ )  $Bel\star A \subseteq Cn(Bel \cup \{A\})$ . (Inclusion)
- ( $\star 4$ ) If  $\neg A \notin Bel$ , then  $Cn(Bel \cup \{A\}) \subseteq Bel\star A$ . (Preservation+)
- ( $\star 5a$ ) If  $A \wedge B \in Bel\star (A \wedge B)$ , then  $A \in Bel\star A$ . (Absurdity 1)
- ( $\star 5b$ ) If  $A \vee B \in Bel\star (A \vee B)$ , then  $A \in Bel\star A$  or  $B \in Bel\star B$ .  
(Absurdity 2)
- ( $\star 6$ ) If  $A \dashv\vdash B$ , then  $Bel\star A = Bel\star B$ . (Intensionality)
- ( $\star 7c$ ) If  $B \in Bel\star A$ , then  $Bel\star (A \wedge B) \subseteq Bel\star A$ . (Cut)
- ( $\star 7$ )  $Bel\star (A \wedge B) \subseteq Cn((Bel\star A) \cup \{B\})$ . (Conditionalsation)
- ( $\star 8c$ ) If  $B \in Bel\star A$ , then  $Bel\star A \subseteq Bel\star (A \wedge B)$ .  
(Cautious Monotonicity)
- ( $\star 8$ ) If  $\neg B \notin Bel\star A$ , then  $Bel\star A \subseteq Bel\star (A \wedge B)$ .  
(Rational Monotonicity)

Like in classical belief revision, we call ( $\star 0$ )–( $\star 6$ ) the *basic* postulates and ( $\star 7c$ )–( $\star 8$ ) the *supplementary* postulates. Postulate ( $\star 0$ ) is new and strengthens AGM’s original fifth postulate which is only a restricted consistency postulate (it is restricted to consistent inputs  $A$ ). Obviously, an absurdity cannot be defined with reference to inconsistent belief sets in this adapted context, because ( $\star 0$ ) rules out such sets. Instead we now say that a sentence  $A$  is an *absurdity* (or a *doxastic impossibility*) of a belief state  $\mathfrak{B}$  iff the agent refuses to accept  $A$  when  $A$  is presented to her as an input to the set of beliefs or assumptions, i.e., iff  $A \notin Bel\star A$ . ( $\star 0$ ) implies that contradictions are absurdities. Thus, in line with the terminology introduced above, a sentence  $A$  is *conceivable* iff  $A \in Bel\star A$ ;  $A$  is a *doxastic necessity* iff  $\neg A \notin Bel\star \neg A$ ; and  $A$  is *contingent* iff both  $A \in Bel\star A$  and  $\neg A \in Bel\star \neg A$ .

Postulate ( $\star 0b$ ) is one half of the claim that the agent’s beliefs are, or more precisely: that her strongest belief is contingent. The other half, namely that  $X_{bel} \in Bel\star X_{bel}$ , follows from ( $\star 2$ ), or alternatively from ( $\star 0$ ) and ( $\star 4$ ).

Restricted Success ( $\star 2$ ) says that the agent should not change her beliefs at all if the input  $A$  is absurd. If a new piece of information or a hypothetical assumption is absurd, then it seems a good strategy not just to refuse to accept it, but simply to ignore it and change nothing. The postulates of *Unrestricted Consistency* and *Restricted Success* are the key deviations from AGM.

Postulate ( $\star 4$ ) is the original fourth AGM postulate, not the weaker postulate of Preservation that is used in ‘DMC’. It is important to have this

stronger version here because it entails a restricted success axiom: no sentence  $A$  that is compatible with the agent's beliefs is absurd.<sup>27</sup>

Postulates (\*5a) and (\*5b) are counterparts of postulates (\*5a) and (\*5b) of 'DMC'. They put exactly the same constraints on the set of absurdities, but rely on our new notion of 'absurdity'.

The supplementary postulates (\*7c)–(\*8) are identical to those of the previous section. Rational Monotonicity (\*8) is a sound principle, but the stronger original eighth AGM postulate does not hold for absurd inputs  $A$  in conceivability-limited revision, since  $B$ 's being consistent with  $Bel$  does not entail that  $B$  is an element of  $Bel$ .<sup>28</sup>

Given the background postulates (\*0) and (\*1), it is immediate that (\*7) implies (\*7c) and (\*8) implies (\*8c). Apart from these simple implications, I think this set of postulates is free from redundancies.

LEMMA 1. [*Derived properties of  $\star$* ] *The following conditions are derivable from the principles (\*0)–(\*8) of conceivability-limited belief revision.*

$$(r1) \quad Bel = Bel \star \top = Bel \star X_{bel}.$$

$$(r2) \quad A \in Bel \text{ iff } X_{bel} \wedge \neg A \notin Bel \star (X_{bel} \wedge \neg A).$$

$$(r3) \quad \text{Not } Bel \star A \subsetneq Bel.$$

$$(r4) \quad \text{If } Bel \subsetneq Bel \star A, \text{ then } \neg A \notin Bel.$$

$$(r5) \quad \text{If } A \notin Bel \star A, \text{ then } \neg A \in Bel.$$

$$(r6) \quad \text{If } A \notin Bel \star A, \text{ then } A \notin Bel.$$

$$(r7) \quad \text{If } A \notin Bel \star A, \text{ then } \neg A \in Bel \star A.$$

$$(r8) \quad Bel \star A = Bel \text{ iff either } A \in Bel \text{ or } A \notin Bel \star A.$$

$$(r9) \quad \text{If } A \text{ is not contingent, then } Bel \star A = Bel = Bel \star \neg A.$$

$$(r10) \quad \text{If } C \in Bel \star A \text{ and } C \notin Bel \star \neg A, \text{ then } A \text{ is contingent.}$$

$$(r11) \quad \text{If } Bel \not\subseteq Bel \star A, \text{ then } Bel \star A \text{ is inconsistent with } Bel.$$

*(Consistent Expansion)*

$$(r12) \quad \text{If } A \in Bel \star B, \text{ then } A \in Bel \star A.$$

*(Regularity)*

<sup>27</sup>As already mentioned, in the presence of unrestricted Success, the two postulates are equivalent. Here, with Restricted Success only, we need (\*4) in order to derive the conditions (r5)–(r13) listed in Lemma 1.

<sup>28</sup>If we endorse the extra postulate that  $Bel = Bel \star \top$ , (\*3) follows from (\*7). But (\*4) does not quite follow from (\*8). For this we need an additional success postulate saying that if  $\neg A \notin Bel$ , then  $A \in Bel \star A$ . This is condition (r5) of Lemma 1, which we will derive from (\*4). On the other hand, (\*8c) follows from (\*8) without restrictions for conceivability-limited belief revision (cf. footnote 22).

- (r13) If  $A \notin \text{Bel}\star A$ , then  $\neg A \in \text{Bel}\star B$ . (Strong Regularity)
- (r14)  $A \in \text{Bel}\star A$  or  $\neg A \in \text{Bel}\star \neg A$ . (Disjunctive Success)
- (r15) If  $C \in \text{Bel}\star A$  and  $C \notin \text{Bel}\star \neg A$ , then  $C$  is contingent.
- (r16) If  $A \in \text{Bel}\star A$  and  $\neg B \notin \text{Bel}\star A$ , then  
 $\text{Cn}((\text{Bel}\star A) \cup \{B\}) \subseteq \text{Bel}\star (A \wedge B)$ . (Guarded Subexpansion)
- (r17) If  $A \in \text{Bel}\star B$  and  $B \in \text{Bel}\star A$ , then  $\text{Bel}\star A = \text{Bel}\star B$ . (Reciprocity)
- (r18) If  $\text{Bel}\star (A \vee B) = \text{Bel}\star A$  or  $\text{Bel}\star (A \vee B) = \text{Bel}\star B$  or  
 $\text{Bel}\star (A \vee B) = (\text{Bel}\star B) \cap (\text{Bel}\star A)$ . (Disjunctive Factoring)

The derivations of (r1)–(r11) need only the basic postulates  $(\star 0)$ – $(\star 6)$ , those of (r12)–(r18) also need at least one of the supplementary principles  $(\star 7c)$ – $(\star 8)$ .

The longer names of the conditions mentioned in Lemma 1 are taken from Hansson, Fermé, Cantwell and Falappa [12].

It is easy to see that the above collection of axioms is consistent. Use a system-of-spheres modelling in the style of Grove [10], but do *not* require these systems to exhaust the space of all possible worlds, i.e., to be universal in the sense of Lewis [18, p. 16]—first design decision! Then let the set of models of  $\text{Bel}\star A$  be intersection of the  $A$ -models with the smallest sphere that contains any  $A$ -models if such a sphere exists, and let it be the smallest sphere otherwise—second design decision!

OBSERVATION 2. (a) The axioms  $(\star 0)$ – $(\star 8)$  are sound with respect to this simple sphere-based modelling, provided that the system of spheres  $\$$  contains at least two spheres (which means that  $\bigcap \$ \neq \bigcup \$$ ).

(b) Every revision function satisfying  $(\star 0)$ – $(\star 8)$  can be represented as generated from a system of spheres  $\$$  by the recipe mentioned above.

The proviso mentioned in part (a) of the observation is necessary for  $(\star 0b)$ . Apart from this addition, the modelling is formally identical to what Hansson, Fermé, Cantwell and Falappa [12] call *non-prioritized* or *credibility limited sphere-based revision*. But there is a clear difference in interpretation here. For Hansson et al. the propositions that initiate successful revisions are *credible* propositions, while in this paper they only need to be *conceivable*, i.e., non-absurd. It is true that no credible proposition is absurd, but not every proposition that isn't absurd is credible. So we call the model using  $\star$  *conceivability-limited revision*.<sup>29</sup>

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<sup>29</sup>Also compare Garapa [7] on a refined model of credibility-limited belief revision.



We will now summarize what consequences or our deviations from classical belief revision are for the connexivity of difference-making conditionals. A formal development will be given in Section 6.

- (i) We again have unrestricted (Arist1):  $\text{Not } A \gg \neg A$ ; and also not  $\neg A \gg A$  (there is still no real difference between these two versions). The reason for its validity is that the acceptance of  $A \gg \neg A$  would require that  $\neg A \in \text{Bel}\star A$  and  $\neg A \notin \text{Bel}\star \neg A$ , i.e., that both  $A$  and  $\neg A$  are absurd, which is impossible, by (r14).
- (ii) We again have unrestricted (Arist2): Not both  $A \gg C$  and  $\neg A \gg C$ . The reason is exactly the same as in Section 4.
- (iii) In contrast to the context of ‘DMC’, now (Boet-Abel) is valid *in its unrestricted form*: Not both  $A \gg C$  and  $A \gg \neg C$ . This is because we cannot simultaneously have  $C \in \text{Bel}\star A$  and  $\neg C \in \text{Bel}\star A$ , due to the axiom of Unrestricted Consistency.

Summing up: Given the alternative design decision regarding the axiomatization of belief revision, difference-making conditionals validate all connexive theses in unrestricted (‘unhumble’, ‘ambitious’) form. Difference-making conditionals thus can be fully connexified.

## 6. The Logic of Difference-Making Conditionals Based on Conceivability-Limited Belief Revision

Remember that we suppose throughout this paper that the belief set  $\text{Bel}$  is consistent (and that  $\text{Bel}\star A$  is consistent, too). This is the Relevant Ramsey Test based on  $\star$ :

$$A \gg C @ \mathfrak{B} \quad \text{iff} \quad C \in \text{Bel}\star A \quad \text{and} \quad C \notin \text{Bel}\star \neg A. \quad (\text{RRT})$$

We will see later that  $A \gg C @ \mathfrak{B}$  implies that both  $A$  and  $C$  are contingent. This is in accord with the intuitions captured by (Irrelevant  $\top$ ) and (Irrelevant  $\perp$ ): Only contingent sentences can be relevant to one another.

Notice that RRT has recourse only to revised belief sets. It does not refer to any property of the original belief set  $\text{Bel}$ .

### 6.1. What Can Be Expressed by Difference-Making Conditionals?

First of all, we specify the ‘meanings’ of some basic difference-making conditionals:

Table 1. The acceptance conditions of some (combinations of) simple difference-making conditionals

$A \gg A$	$A$ is contingent.
$A \gg A \wedge C$	$C \in \text{Bel}\star A$ and $A$ is contingent.
$A \gg A \vee C$	$C \notin \text{Bel}\star \neg A$ and $A$ is contingent.
$A \wedge C \gg C$	$C \notin \text{Bel}\star \neg(A \wedge C)$ and $A \wedge C$ is contingent. <sup>30</sup>
$A \vee C \gg C$	$C \in \text{Bel}\star(A \vee C)$ and $A \vee C$ is contingent.
$A \wedge X_{bel} \gg A \wedge X_{bel}$	$A$ is consistent with $\text{Bel}$ .
$X_{bel} \gg X_{bel} \wedge A$	$A \in \text{Bel}$ ( $A$ is a belief).
$X_{bel} \gg X_{bel} \wedge A$ and $A \not\gg A$	$\neg A \notin \text{Bel}\star \neg A$ ( $A$ is doxastic necessity).
$X_{bel} \gg X_{bel} \wedge \neg A$ and $A \not\gg A$	$A \notin \text{Bel}\star A$ ( $A$ is absurd).

LEMMA 3. *Let the basic conditions  $(\star 0)$ – $(\star 6)$  for conceivability-limited belief revision be given. Then the acceptance conditions of some (combinations of) simple difference-making conditionals are as detailed in Table 1.*

Difference-making conditionals lose their contrastive character when the antecedent is logically stronger or weaker than, or equally strong as, the consequent. A conditional of the form  $A \gg A \wedge C$  is a de-relevantised conditional; it expresses a suppositional conditional ‘If  $A$  then  $C$ ’ (plus the contingency of  $A$  as a side condition) using the difference-making conditional  $\gg$ . It is almost equivalent to the standard Ramsey conditional  $A > C$  to which it only adds that  $A$  is a contingent sentence.  $A \gg A \wedge C$  is strictly weaker than  $A \gg C$ . This should come as no surprise, because the acceptance of a difference-making conditional  $A \gg C$ , in contrast to that of its de-relevantised counterpart, also depends on what happens when its antecedent  $A$  is supposed to be false.

There are various things we would like to express in the language of difference-making conditionals. It is more difficult to express some things with difference-making conditionals based on the alternative revision functions  $\star$  than with those based on  $*$  as in ‘DMC’. For instance, it is not directly possible to express that  $A$  is an absurdity (with respect to  $\mathfrak{B}$ ), since  $A \not\gg A$  only says that  $A$  is an absurdity *or*  $\neg A$  is an absurdity. Similarly,

<sup>30</sup> $C \notin \text{Bel}\star \neg(A \wedge C)$  means that  $C$  is not more entrenched than  $A$ . An explanation of this claim can be found in Rott [29, Ch. 8]

expressing that  $A$  is a doxastic necessity (with respect to  $\mathfrak{B}$ ) is slightly complicated, since  $A \gg A$  only says that either  $A$  is a doxastic necessity or  $\neg A$  is a doxastic necessity.

## 6.2. Basic Principles

If a conditional of the form ‘If  $A$  then  $C$ ’ is used in everyday discourse and meant to convey that  $A$  is relevant for  $C$ , then  $A$  and  $C$  are hardly ever logically related. What is common usage in a seminar on propositional logic is apt to cause bewilderment in practical contexts. In the sense of ‘relevance’ intended here, it sounds odd to say that a sentence is relevant for some of its subsentences or for some Boolean compounds containing it. Still odder does it sound to say that a sentence is relevant for itself. Yet, if one wants to present a conditional logic, it is just one’s principal task to deal with such statements. We should not expect that the principles valid for difference-making conditionals are generally intuitively appealing. The most important thing to bear in mind is that the Relevant Ramsey Test RRT provides a clear and simple doxastic semantics that is applicable to arbitrary compounds of factual sentences as antecedents and consequents. We are now going to explore the logic of conditionals governed by RRT.

We are now in a position to list the *basic principles of difference-making conditionals*. All these principles are to be read as quantified over all belief states  $\mathfrak{B}$ , but the clause ‘@  $\mathfrak{B}$ ’ after each conditional is left implicit throughout: ‘ $A \gg C$ ’ is short for ‘ $A \gg C @ \mathfrak{B}$ ’ and ‘ $A \not\gg C$ ’ is short for ‘not  $A \gg C @ \mathfrak{B}$ ’. I trust that this somewhat sloppy notation will not cause any confusion. A principle of the form ‘If  $\Phi$ , then  $\Psi$ ’ formulates a *validity* in the sense that *for every belief state*  $\mathfrak{B}$ , if the conditionals mentioned in  $\Phi$  are all accepted/rejected in  $\mathfrak{B}$ , then the conditionals mentioned in  $\Psi$  are accepted/rejected in  $\mathfrak{B}$ . The variables  $A$ ,  $B$  and  $C$  range over propositional sentences without any occurrences of the conditional connective.

$$(\gg 0) X_{bel} \gg X_{bel}.$$

$$(\gg 1) \text{ If } A \gg B \wedge C, \text{ then } A \gg B \text{ or } A \gg C.$$

$$(\gg 2a) A \gg C \text{ iff } (A \gg A \wedge C \text{ and } A \gg A \vee C).$$

$$(\gg 2b) A \gg A \wedge C \text{ iff } (\neg A \not\gg \neg A \vee C \text{ and } A \gg A).$$

$$(\gg 3) \text{ If } A \gg A \wedge C, \text{ then } A \wedge \neg C \wedge X_{bel} \not\gg A \wedge \neg C \wedge X_{bel}.$$

$$(\gg 4) \text{ If } A \gg A \vee C \text{ and } \neg A \wedge X_{bel} \gg \neg A \wedge X_{bel}, \text{ then } \neg C \wedge X_{bel} \gg \neg C \wedge X_{bel}.$$

$$(\gg 5a) A \gg A \text{ iff } \neg A \gg \neg A.$$

( $\gg 5b$ ) If  $A \wedge B \gg A \wedge B$ , then  $A \gg A$  or  $B \gg B$ .

( $\gg 5c$ ) If  $A \wedge B \gg A \wedge B$  and  $A \vee C \gg A \vee C$ , then  $A \gg A$ .

( $\gg 6$ ) If  $A \dashv\vdash B$  and  $C \dashv\vdash D$ , then:  $A \gg C$  iff  $B \gg D$ .

Let us emphasize again that in the set-up of both ‘DMC’ and the present paper, Reflexivity is not in general valid. In ‘DMC’, the conditional  $A \gg A$  is accepted if and only if  $\neg A$  is not absurd. In our new set-up,  $A \gg A$  is accepted if and only if neither  $A$  nor  $\neg A$  is absurd, i.e.,  $A$  is contingent.<sup>30</sup> Condition ( $\gg 0$ ) says that the strongest belief is contingent. This implies not only that it is consistent, but also that there are non-absurd non-beliefs.

Condition ( $\gg 1$ ) is a weaker replacement of the condition of ‘Right Weakening’ which is valid for suppositional conditionals. It says that when  $A$  is a relevant antecedent to  $B \wedge C$ , then it must also be a relevant antecedent to at least one of  $B$  and  $C$ .

Condition ( $\gg 2a$ ) is valid but close to trivial for reasoning with suppositional conditionals. It says that if one accepts ‘If  $A$  then relevantly  $C$ ’, i.e., if  $A$  is a relevant antecedent for  $C$ , then  $A$  is also a relevant antecedent for both the conjunction  $A \wedge C$  and the disjunction  $A \vee C$ —and vice versa. As we can gather from Table 1, the two conjuncts correspond rather neatly to the two parts of the Relevant Ramsey Test.

Condition ( $\gg 2b$ ) says that if  $A$  is a relevant antecedent for  $A \wedge C$ , then  $A$  is also a relevant antecedent for itself, but  $\neg A$  is not a relevant antecedent for  $\neg A \vee C$ —and vice versa. This means that  $\neg A \gg \neg A \vee C$  is essentially the negation of  $A \gg A \wedge C$ : The one is accepted if and only if the other isn’t (neglecting the contingency condition for  $A$ ). By chaining ( $\gg 2a$ ) and ( $\gg 2b$ ), we find that only contingent sentences can be relevant antecedents: If  $A \gg C$ , then  $A \gg A$ . (We will later derive a similar claim for relevant consequents: If  $A \gg C$ , then  $C \gg C$ ; see Lemma 11.)

Conditions ( $\gg 3$ ) and ( $\gg 4$ ) correspond to the Inclusion and Preservation+ postulates for revisions. Like in the numbering of belief revision principles, the third and fourth conditions are the only ones that refer to the agent’s initial beliefs. In view of Lemma 3, we can see that contingencies play a crucial role here.

Conditions ( $\gg 5a$ )–( $\gg 5c$ ) deal with contingencies. The first two say that the set of non-contingencies is closed under negation and conjunction (and thus under disjunction, too, as we will shortly verify). The third one is

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<sup>30</sup>See Lemma 3.—It is known that some connexive logics do not satisfy Idempotence in the form of  $(A \wedge A) \rightarrow A$  or  $A \rightarrow (A \wedge A)$  (Estrada-González and Ramírez-Cámara [6]). Given our essentially classical background logic, this is equivalent to a failure of Reflexivity.

a kind of interpolation property for contingencies. The sentence  $A \gg A$  is contingent whenever both a stronger sentence  $A \wedge B$  and a weaker sentence  $A \vee C$  are contingent. These three conditions correspond to axiom schemes A1–A3 of the ‘minimal non-contingency logic’ of Kuhn [15].

( $\gg 6$ ) is an intensionality principle that corresponds to AGM’s sixth and part of their first postulates. Relevant antecedents and consequents may be replaced by logically equivalent sentences.

Now we need to show that the basic principles for difference-making conditionals are valid.

OBSERVATION 4. [Basic principles for  $\gg$  are valid] Let  $\star$  be a revision function satisfying the basic postulates mentioned in Section 5, and let  $\gg$  be obtained from  $\star$  by RRT. Then  $\gg$  satisfies the basic principles ( $\gg 0$ ), ( $\gg 1$ ), ( $\gg 2a$ ), ( $\gg 2b$ ), ( $\gg 3$ ), ( $\gg 4$ ), ( $\gg 5a$ ), ( $\gg 5b$ ), ( $\gg 5c$ ) and ( $\gg 6$ ).

The following lemma offers a rather long list of derived principles. Some of them are interesting in their own right, some will be useful in later proofs. (All proofs are collected in the Appendix.)

LEMMA 5. [Derived properties of  $\gg$ ] Let the principles Let the principles ( $\gg 0$ )–( $\gg 6$ ) for difference-making conditionals be given. Then

- (c1) If  $A \gg C$ , then  $A \gg A$ .
- (c2)  $A \not\gg \perp$ .
- (c3)  $A \not\gg \top$ .
- (c4)  $\top \not\gg C$ .
- (c5)  $\perp \not\gg C$ .
- (c6)  $A \gg A$  iff ( $A \gg A \vee C$  or  $A \gg A \vee \neg C$ ).
- (c7) If  $A \wedge B \vdash \perp$ , then  $A \not\gg B$ .
- (c8) Not both  $A \gg A \wedge C$  and  $A \gg A \wedge \neg C$ .
- (c9) If  $A \vee B \gg A \vee B$  then  $A \gg A$  or  $B \gg B$ .
- (c10) If  $\top \vdash A \vee B$ , then  $A \not\gg B$ .
- (c11) If  $A \gg A \wedge B \wedge C$ , then  $A \gg A \wedge C$ .
- (c12) If  $A \gg A \vee B \vee C$ , then  $A \gg A \vee C$ .
- (c13) If  $A \gg B$  and  $A \gg C$ , then  $A \gg B \wedge C$ .
- (c14)  $A \gg A \vee C$  iff ( $\neg A \not\gg \neg A \wedge C$  and  $A \gg A$ ).
- (c15) If  $A \not\gg A$  and  $A \wedge B \gg A \wedge B$ , then  $\neg A \wedge B \not\gg \neg A \wedge B$ .

- (c16) If  $X_{bel} \gg X_{bel} \wedge A$ , then  $X_{bel} \wedge A \gg X_{bel} \wedge A$ .
- (c17) If  $A \not\gg A$  and  $A \wedge B \gg A \wedge B$ , then  $X_{bel} \gg X_{bel} \wedge A$ .
- (c18) If  $A \not\gg A$  and  $A \vee B \gg A \vee B$ , then  $X_{bel} \gg X_{bel} \wedge \neg A$ .
- (c19) If  $A \not\gg A$ , then  $X_{bel} \gg X_{bel} \wedge A$  or  $X_{bel} \gg X_{bel} \wedge \neg A$ .
- (c20) If  $X_{bel} \gg X_{bel} \wedge A$  and  $A \gg A \wedge C$ , then  $X_{bel} \gg X_{bel} \wedge C$ .
- (c21) If  $A \gg A \wedge C$ , then  $X_{bel} \gg X_{bel} \wedge (A \supset C)$ .
- (c22) If  $A \gg A \vee C$  and  $X_{bel} \gg X_{bel} \wedge C$ , then  $X_{bel} \gg X_{bel} \wedge A$ .
- (c23) If  $A \gg A$ ,  $X_{bel} \gg X_{bel} \wedge C$  and  $X_{bel} \not\gg X_{bel} \wedge \neg A$ , then  $A \gg A \wedge C$ .

A few brief comments on some of these conditions are in order. It turns out that it does not make sense to place tautologies or contradictions to the left or to the right of difference-making conditionals based on  $\star$  (parts (c2)–(c5) of the Lemma). *They can never be relevant antecedents or relevant consequents!* And this is indeed what one would expect intuitively. So this is an important advantage over the setup of ‘DMC’ where contradictions are not only taken to make sense, but are very heavily made use of both as antecedents and as consequents. Given our new belief revision semantics, contradictions have no interesting role to play. The reason that contradictions are not acceptable antecedents is that  $Bel\star\top = Bel\star\perp = Bel$ , and the reason that they are not acceptable consequents is that  $\top$  is in every and  $\perp$  is in no belief set.

Condition (c10) is a qualitative counterpart of the findings of Popper and Miller [24, Theorems 2 and 5] that there can be no positive probabilistic support between  $A$  and  $A \supset B$  in either direction (notice that  $\top \vdash A \vee (A \supset B)$ ).

As we pointed out above, Right Weakening is in general *invalid* for difference-making conditionals. Like ( $\gg 1$ ), the principle (c11) is a weakened form of Right Weakening. Condition (c13) says that from ‘If  $A$  then relevantly  $B$ ’ and ‘If  $A$  then relevantly  $C$ ’, one can infer ‘If  $A$  then relevantly  $B$  and  $C$ ’. This important principle is called ‘Conjunction in the Consequent’ or simply ‘And’ in the literature, and it is the only inference pattern of suppositional conditionals that remains valid for the difference-making conditional  $\gg$ !

Conditions (c16)–(c19) say that beliefs are consistent with  $Bel$ , that doxastic necessities are beliefs, that absurdities are disbeliefs and that non-contingencies are either beliefs or disbeliefs.

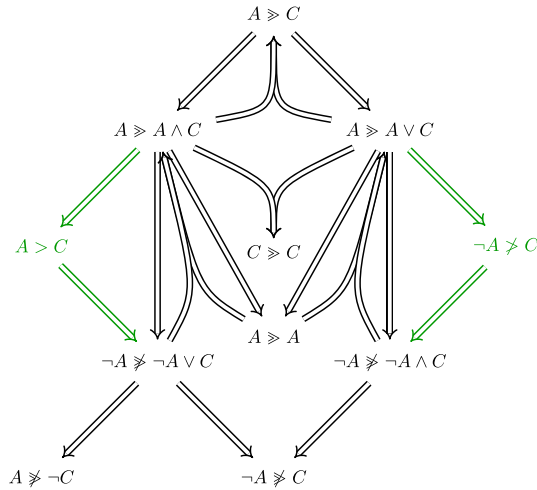


Figure 1. Some useful implications. Here ‘>’ is the suppositional conditional according to (RT) and ‘ $\gg$ ’ is the difference-making conditional according to (RRT), both based on conceivability-limited belief revision

Conditions (c21) and (c22) look like simpler versions of ( $\gg$ 3) and ( $\gg$ 4) (see Lemma 7(i) below), but as far as I can see, they are not strong enough to replace the latter.

A number of important elementary implications are represented in Figures 1 and 2.

The following observation is our most important result from the point of view of connexivity.

**OBSERVATION 6.** The difference-making conditional  $\gg$  satisfies the following connexivity principles:

( $\gg$ Arist1) Not  $A \gg \neg A$ .

( $\gg$ Arist2) Not both  $A \gg C$  and  $\neg A \gg C$ .

( $\gg$ Boet-Abel) Not both  $A \gg C$  and  $A \gg \neg C$ .

( $\gg$ Arist1) are ( $\gg$ Arist2) Aristotle’s First and Second Thesis, now in the particular specialization for difference-making conditionals. ( $\gg$ Boet-Abel) is the principle of Boethius and Abelard. Thus the difference-making conditional  $\gg$  based on  $\star$  is fully connexive. In particular, it is excluded that ‘If  $A$  then relevantly  $C$ ’ and ‘If  $\neg A$  then relevantly  $C$ ’ are accepted simultaneously, and also that ‘If  $A$  then relevantly  $C$ ’ and ‘If  $A$  then relevantly  $\neg C$ ’ are accepted simultaneously.

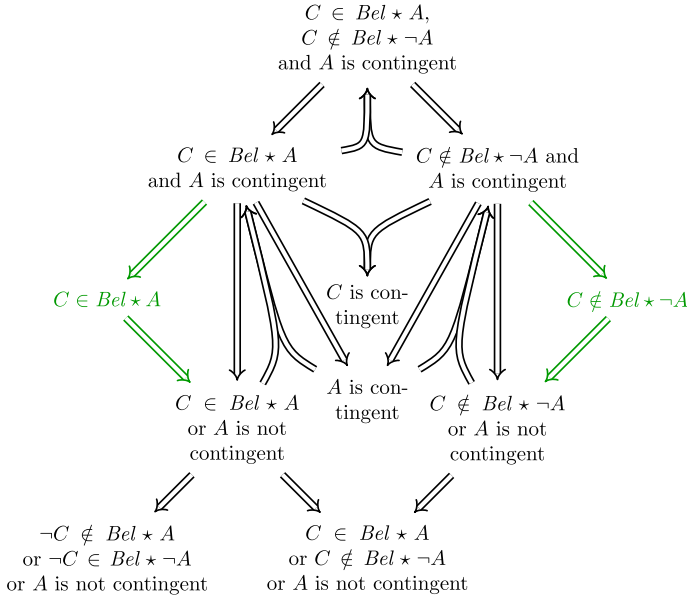


Figure 2. The meanings of the conditions of Figure 1

The following lemma uses a few facts about the relation between belief, doxastic necessity and absurdity. First,  $A$  is a doxastic necessity if and only if it is a non-contingent belief. Second,  $A$  is an absurdity if and only if  $\neg A$  is a doxastic necessity, i.e.,  $\neg A$  is a non-contingent belief, or, again equivalently,  $A$  is a non-contingent non-belief.

LEMMA 7. [Alternative ways of expressing beliefs, doxastic necessities and absurdities] Let the principles  $(\geq 0)$ – $(\geq 6)$  for difference-making conditionals be given.

(i) The following conditions are equivalent:

- (i.0)  $A$  is a belief;
- (i.1)  $X_{bel} \geq X_{bel} \wedge A$ ;
- (i.2)  $\neg A \wedge X_{bel} \not\geq \neg A \wedge X_{bel}$ ;
- (i.3) for all  $B$ ,  $\neg A \wedge B \wedge X_{bel} \not\geq \neg A \wedge B \wedge X_{bel}$ .

(ii) The following conditions are equivalent:

- (ii.0)  $A$  is a doxastic necessity;
- (ii.1)  $X_{bel} \geq X_{bel} \wedge A$  and  $A \not\geq A$ ;
- (ii.2)  $A \wedge X_{bel} \geq A \wedge X_{bel}$  and  $A \not\geq A$ ;



(ii.3) there is a  $B$  such that  $A \wedge B \gg A \wedge B$  and  $A \not\gg A$ .

(ii.4) for all  $B$ ,  $A \vee B \not\gg A \vee B$ .

(iii) The following conditions are equivalent:

(iii.0)  $A$  is an absurdity;

(iii.1)  $X_{bel} \gg X_{bel} \wedge \neg A$  and  $A \not\gg A$ ;

(iii.2)  $A \wedge X_{bel} \not\gg A \wedge X_{bel}$  and  $A \not\gg A$ ;

(iii.3)  $X_{bel} \not\gg X_{bel} \wedge A$  and  $A \not\gg A$ ;

(iii.4)  $\neg A \wedge X_{bel} \gg \neg A \wedge X_{bel}$  and  $A \not\gg A$ ;

(iii.5) there is a  $B$  such that  $A \vee B \gg A \vee B$  and  $A \not\gg A$ .

(iii.6) for all  $B$ ,  $A \wedge B \not\gg A \wedge B$ .

Comments. For the ‘canonical’ ways of expressing beliefs, doxastic necessities and absurdities, see Lemma 3.

On (i).  $A$  is a belief if and only if  $X_{bel} \wedge \neg A$  is an absurdity, i.e., if and only if  $X_{bel} \supset A$  is a doxastic necessity. In this case, since  $X_{bel}$  is contingent,  $X_{bel} \wedge \neg A$  is even a contradiction and  $X_{bel} \supset A$  is even a logical truth.

It is easy to check that the representation in (i.3) reveals that the set of beliefs is closed under singleton entailment and conjunction.

Concerning (i.1) and (i.2), both ideas are attractive. The former is clearly simpler and thus seems preferable. I will endorse it as the “official” definition of beliefs in terms of conditionals. The latter reduces questions about beliefs to questions about contingencies, and there are particularly perspicuous rules for reasoning about contingencies, even in the absence of the supplementary AGM postulates.

On (ii) and (iii). Weakening a doxastic necessity results in another doxastic necessity, and strengthening an absurdity results in another absurdity, but not the other way round. A comparison of (ii.1) and (i.1) shows immediately that every doxastic necessity is a belief.

(iii.1) and (iii.2) express that absurdities are non-contingent sentences the negations of which are believed. (iii.3) and (iii.4) express that absurdities are non-contingent non-beliefs. In order for the idea of (iii.6) to work, we have to presuppose that there is a contingency. (Otherwise any proper strengthening of a doxastic necessity will result in an absurdity.) Since we have posited in ( $\gg 0$ b) that the strongest belief  $X_{bel}$  is a contingency, this presupposition is met.

The following principles are reminiscent of inference patterns studied intensively in the psychology of reasoning. They concern the interrelations



making extensive use of de-relevantised conditionals of the form ‘ $A \gg A \wedge C$ ’. These are *supplementary principles of difference-making conditionals*:

- ( $\gg 7c$ ) If  $A \gg A \wedge B$  and  $A \wedge B \gg A \wedge B \wedge C$ , then  $A \gg A \wedge C$ .
- ( $\gg 7$ ) If  $A \wedge B \gg A \wedge B \wedge C$  and  $A \gg A$ , then  $A \gg A \wedge (B \supset C)$ .
- ( $\gg 7'$ ) If  $A \gg A \wedge C$  and  $B \gg B \wedge C$  and  $A \vee B \gg A \vee B$ , then  $A \vee B \gg (A \vee B) \wedge C$ .
- ( $\gg 8c$ ) If  $A \gg A \wedge B$  and  $A \gg A \wedge C$ , then  $A \wedge B \gg A \wedge B \wedge C$ .
- ( $\gg 8$ ) If not  $A \gg A \wedge \neg B$  and  $A \gg A \wedge C$ , then  $A \wedge B \gg A \wedge B \wedge C$ .

Now we show that the supplementary principles for difference-making conditionals are valid, given the supplementary postulates for conceivability-limited belief revision.

OBSERVATION 9. [Supplementary principles for  $\gg$  are valid] Let  $\star$  be a basic revision function in the sense of Section 5, and let  $\gg$  be obtained from  $\star$  by RRT. If  $\star$  satisfies ( $\star 7c$ ), ( $\star 7$ ), ( $\star 7'$ ), ( $\star 8c$ ) or ( $\star 8$ ), then  $\gg$  satisfies ( $\gg 7c$ ), ( $\gg 7$ ), ( $\gg 7'$ ), ( $\gg 8c$ ) or ( $\gg 8$ ), respectively.

LEMMA 10. [*Derivabilities within compound-antecedent principles*] Given the basic principles, ( $\gg 7$ ) is inter-derivable with ( $\gg 7'$ ), and ( $\gg 7c$ ) is implied by either of them. Given the basic principles, ( $\gg 8c$ ) is implied by ( $\gg 8$ ).

Here are more derived principles that depend on the compound-antecedent principles just introduced.

LEMMA 11. [*Further derived properties of  $\gg$* ] Let the basic principles ( $\gg 0$ )–( $\gg 6$ ) plus the supplementary principles ( $\gg 7c$ )–( $\gg 8$ ) for difference-making conditionals be given. Then

- (c24) If  $A \gg C$ , then  $A \gg A$ .
- (c25) If  $A \gg C$ , then  $A \wedge C$  and  $A \vee C$  are contingent.<sup>32</sup>
- (c26) If  $A \gg A$  and  $A \wedge B \not\gg A \wedge B$ , then  $A \gg A \wedge \neg B$ .
- (c27) If  $A \not\gg A$  and  $A \wedge B \gg A \wedge B$ , then  $\neg(A \wedge B) \gg \neg B$ .
- (c28) If  $B \not\gg B$  and  $A \wedge B \gg A \wedge B \wedge C$ , then  $A \gg A \wedge C$ .
- (c29) If  $B \not\gg B$  and  $A \vee B \gg (A \vee B) \wedge C$ , then  $A \gg A \wedge C$ .
- (c30) If  $A \not\gg A$  and  $A \vee B \gg A \vee B$ , then  $A \vee B \gg B$ .

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<sup>32</sup>One cannot infer from  $A \gg C$  that either of  $A \wedge \neg C$  and  $\neg A \wedge C$  is contingent.

(c31) If  $A \gg A$  and  $A \vee B \not\gg A \vee B$ , then  $\neg A \gg \neg A \wedge B$ .

I have been unable to find a proof of condition (c24) that does not require the strong principle ( $\gg 8$ ). The simple, but important condition (c24) says that if there is any relevant antecedent to  $C$ , then  $C$  is a relevant antecedent to itself, i.e., that  $C$  is contingent. From this and condition (c1) it transpires that *only contingent sentences can be relevant antecedents or consequents*.

#### 6.4. Constructing Revisions from Difference-Making Conditionals

So far we have taken a revision function  $\star$  in the alternative sense as given, and analysed the conditional connective  $\gg$  as obtained from  $\star$  by RRT. Now we also take the converse perspective. Given the set of difference-making conditionals accepted by the agent in her belief state  $\mathfrak{B}$ , can we determine the result of the revision of her beliefs by a new sentence  $A$ ?

We need to find a way to express, in terms of accepted and rejected conditionals, the inclusion of a sentence  $C$  in a revised belief set  $Bel \star A$ . Here is the solution we will use, reminding the reader that both sides of the definition are universally quantified statements about the acceptance or non-acceptance of sentences in belief states:

$$C \in Bel \star A \quad \text{iff} \quad A \gg A \wedge C \quad \text{or} \quad (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge C). \quad (\text{Def}\star)$$

One can read (Def $\star$ ) as saying that  $C \in Bel \star A$  iff either (i)  $A$  is contingent and an appropriate conditional, viz.,  $A \gg A \wedge C$ , is accepted, or (ii)  $A$  is not contingent and  $C \in Bel$ . (i) is expressed by the first disjunct and (ii) by the second disjunct of the defining condition in (Def $\star$ ).<sup>33</sup>

According to (Def $\star$ ), a revision by  $A$  is successful, i.e.,  $A \in Bel \star A$ , if and only if  $(A \gg A \text{ or } X_{bel} \gg X_{bel} \wedge A)$ , i.e., if and only if  $A$  is contingent or a belief.

If we apply the identity (r1),  $Bel = Bel \star \top$ , to (Def $\star$ ), we can characterise the agent's belief set in terms of the difference-making conditionals accepted by her. Since  $\top \gg \top \wedge A$  and  $\top \gg \top$  are always unacceptable, by (c4), the definition of  $A \in Bel \star \top$  reduces to

$$A \in Bel \quad \text{iff} \quad X_{bel} \gg X_{bel} \wedge A. \quad (\text{DefBel})$$

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<sup>33</sup>Notice that the non-contingency of  $A$  in (ii) implies that  $A$  is an absurdity or a belief, so either way  $Bel \star A = Bel$ , by (r9). (Def $\star$ ) is surely a roundabout way of expressing that  $C$  is in the revision of  $Bel$  by  $A$ , but it is not much worse than the condition (Def $\star$ ) that is used in 'DMC':  $C \in Bel \star A$  iff  $A \gg A \wedge C$  or  $(\neg A \gg \perp \text{ and } \perp \not\gg C)$ .

This is exactly the characterization of belief we have endorsed in our discussion after Lemma 7.

LEMMA 12. (i) Using the definition (*DefBel*), beliefs are closed under conjunction and singleton entailment.

(ii) Using the definitions (*DefBel*) and (*Def★*), revisions by input that is consistent with the prior beliefs are always successful: If  $\neg A \notin Bel$ , then  $A \in Bel \star A$ .

The property mentioned in part (ii) of Lemma 12 is exactly what has to be added to Preservation in order to get Preservation+, i.e., ( $\star 4$ ).

Importantly, the reconstruction (*Def★*) of a revision function is a kind of inverse to the acceptance conditions of conditionals given by the Relevant Ramsey Test.

OBSERVATION 13. [RRT and (*Def★*) fit together] (i) Let  $\star$  satisfy ( $\star 0$ )–( $\star 6$ ) and let  $\gg$  be the difference-making conditional defined from  $\star$  by RRT. Let  $\star'$  be the revision function defined from  $\gg$  by (*Def★*). Then  $\star'$  is identical to  $\star$ .

(ii) Let  $\gg$  satisfy ( $\gg 0$ )–( $\gg 6$ ) and let  $\star$  be the revision function defined from  $\gg$  by (*Def★*). Let  $\gg'$  be the difference-making conditional defined from  $\star$  by RRT. Then  $\gg'$  is identical to  $\gg$ .

In a converse to Observations 4 and 9, it can be shown that any set of conditionals that satisfies the basic principles characterising difference-making conditionals can be represented as generated by the Relevant Ramsey Test based on a revision function  $\star$  in the sense of Section 5.

OBSERVATION 14. [Representation theorem] Let  $\gg$  be a conditional satisfying the principles ( $\gg 0$ ), ( $\gg 1$ ), ( $\gg 2a$ ), ( $\gg 2b$ ), ( $\gg 3$ ), ( $\gg 4$ ), ( $\gg 5a$ )–( $\gg 5c$ ) and ( $\gg 6$ ), and let  $\star$  be obtained from  $\gg$  by (*Def★*). Then  $\star$  is a basic AGM revision function in the sense of Section 5, and RRT is satisfied.

If  $\gg$  in addition satisfies ( $\gg 8c$ ) as well as one of ( $\gg 7c$ ), ( $\gg 7$ ) and ( $\gg 7'$ ), then  $\star$  satisfies ( $\star 7c$ ), ( $\star 7$ ) or ( $\star 7'$ ), respectively.

If  $\gg$  in addition satisfies ( $\gg 8c$ ) or ( $\gg 8$ ), then  $\star$  satisfies ( $\star 8c$ ) or ( $\star 8$ ), respectively.

## 7. Conclusion

I submit that the qualitative positive relevance idea embodied in the Relevant Ramsey Test offers a very clear motivation for connexivity. Some authors (Kapsner [14], Lenzen [16]) have argued that one should restrict

(Arist1) and (Boet-Abel) to consistent antecedents. The idea of difference-making conditionals, however, requires *unrestricted, full connexivity*, and not only one that is restricted to consistent or conceivable antecedents. We have achieved this aim by basing difference-making conditionals on an alternative, ‘conceivability-limited’ conception of belief revision.

The interpretation of difference-making conditionals in the spirit of connexive logic is already present in ‘DMC’. But the account of conditionals presented there is not fully connexive, because it restricts the Principle of Abelard and Boethius to non-absurd antecedents. Sometimes an absurdity is positively relevant to a proposition and its negation at the same time, viz., if the agent does not have any belief about this proposition. But the account is not entirely ‘humble’ either (in Kapsner’s sense), because it endorses unrestricted versions of Aristotle’s two theses. No proposition is positively relevant for its own negation.

The new set-up based on conceivability-limited revision has a number of advantages over the more conservative treatment of difference-making conditionals in ‘DMC’. Obtaining full connexivity is the first one, the connexive principles are validated without restrictions. A second advantage is that the second Aristotelian thesis (Arist2) is as valid and as important as the thesis associated with Boethius and Abelard (Boet-Abel), just as it had been in Aristotle’s and Abelard’s writings. The third advantage is of a more immediate intuitive nature. Absurdities (and in particular contradictions) should not be understood as relevantly promoting anything, nor should anything be relevant for them. This is exactly the result that the new analysis of difference-making conditionals delivers. We also obtain a perfect symmetry in the roles of logical truths and contradictions and respect both principles (Irrelevant  $\top$ ) and (Irrelevant  $\perp$ ). And fourthly, these symmetries are reflected in the symmetry displayed by Figures 1 and 2.

But there is a price to pay. The new approach offers less expressive power. While in the context of ‘DMC’ it is very easy to express that  $A$  is a non-belief ( $\perp \gg A$ ) or that  $A$  is an absurdity ( $A \gg \perp$ ), this simple expressibility gets lost. In the new set-up, neither  $\perp \gg A$  nor  $A \gg \perp$  is ever acceptable for any  $A$ , and from a philosophical point of view, this is as it should be. In ‘DMC’,  $\perp$  plays an important theoretical role, both as an antecedent and as a consequent. On the one hand, employing  $\perp$  in difference-making conditionals is attractive, precisely because we can conveniently avail ourselves of a lot of expressive power. On the other hand, making use of the above-mentioned expressions looks a bit like an *abuse* of  $\perp$ , since intuitively—as we have emphasized in the introduction—contradictions are not relevant

for anything, nor is anything relevant for them. In a way, using contradictions as antecedents and consequents of conditionals appears to be sinning against the very idea of difference-making conditionals. The new set-up has contingency rather than (in-)consistency as a key concept.

We have seen that there are ways of expressing in the new setting of difference-making conditionals that  $A$  is a belief (Lemma 7(i)). These ways, however, depend on having a sentential representation  $X_{bel}$  of one's strongest belief, something that is impractical in natural language.<sup>34</sup> In addition, we had to assume that this strongest belief is contingent. This assumption amounts to a kind of enlightened fallibilism: the agent shouldn't regard it as absurd that at least one of her beliefs is false. The fact that agents cannot easily rephrase their beliefs as conditionals is a limitation of expressive power, but it is not worrying at all. Beliefs are propositional, and agents have a perfect way of expressing them: by uttering them straight away.

There are also several ways of expressing that  $A$  is a doxastic necessity or an absurdity (Lemma 7(ii) and (iii)). But most of them are certainly rather unnatural ways of speaking.

There is another question of expressiveness. With the original Ramsey test, it is easy to encode the fact that  $C$  is a belief after revising  $Bel$  by  $A$ : this is indeed exactly what is expressed by a simple 'suppositional' conditional  $A > C$ . In the set-up of 'DMC', there are two ways of expressing this, one less intuitive but rather elegant ( $\neg A \not\approx \neg A \vee C$ ), and one more intuitive but less elegant ( $A \approx A \wedge C$  or ( $\neg A \approx \perp$  and  $\perp \not\approx A$ )). The only way to express the same fact in the new approach, (Def $\star$ ), is even somewhat less elegant than the less elegant way of 'DMC'.

Absurdities are always somewhat embarrassing, both as pieces of new information and as hypothetical assumptions. In section 2 we saw that there are two ways of making a design decision regarding absurdities. On the first, absurdities lead into the inconsistent 'epistemic hell' (AGM and 'DMC'), on the second, absurdities are simply not processed at all (this paper). Which way of treating absurdities is better, the first or the second?

One way of accounting for the difference between genuine belief revision—'You are the emperor of China. Believe me!'—and hypothetical reasoning—'Suppose you were the emperor of China!'—might be the claim that they require different design decisions. For genuine belief revision, the new design

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<sup>34</sup>A different solution to a similar expressibility problem is offered by Raidl [27] who introduces an additional primitive belief modality into the formal language and connects it axiomatically with conditionals.

decision seems preferable; absurd testimony, for instance, is simply too incredible to be taken seriously. For hypothetical reasoning, however, allowing suppositions that strike one as absurd appears to make more sense. And it is hypothetical reasoning which is more suitable for the interpretation of conditionals.

But there is another way of dealing with the difference between genuine belief revision and hypothetical reasoning. One might simply hold that for belief revision, more things are absurd than for hypothetical reasoning. In hypothetical reasoning, many things are conceivable. It is not easy to find hypotheses so absurd that rational agents refuse to entertain them at all. Logical contradictions are good candidates, but Priest's [25] 'If I were a fried egg . . .' is perhaps just as good. It is not implausible to assume that an agent simply refuses to modify her belief state at all in order to accommodate the supposition that she is a fried egg. Considering the philosophical arguments in favour of the new set-up mentioned above, I endorse this view and advocate full, unrestricted connexivity for difference-making conditionals.

**Acknowledgements.** I would like to thank Andreas Kapsner, Wolfgang Lenzen, Hitoshi Omori, Eric Raidl, Heinrich Wansing, the participants of *Trends in Logic XXI* in Bochum and an anonymous referee of *Studia Logica* for helpful and stimulating comments on earlier versions of this paper.

**Funding** Open Access funding enabled and organized by Projekt DEAL.

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## Appendix: Proofs

*Proof of Lemma 1.* Let  $\star$  satisfy the principles  $(\star 0)$ – $(\star 8)$ .



(r1) Both  $\top$  and  $X_{bel}$  are in  $Bel$ , and  $Bel$  is consistent, by  $(\star 0)$ . So by  $(\star 3)$  and  $(\star 4)$ ,  $Bel \star \top = Cn(Bel \cup \{\top\}) = Bel$  and  $Bel \star X_{bel} = Cn(Bel \cup \{X_{bel}\}) = Bel$ .

(r2) If  $A \in Bel$ , then  $X_{bel} \wedge \neg A \vdash \perp$ , so  $X_{bel} \wedge \neg A \notin Bel \star (X_{bel} \wedge \neg A)$  by  $(\star 0)$ . For the converse, suppose that  $A \notin Bel$ . Then, by  $(\star 1)$ ,  $\neg(X_{bel} \wedge \neg A) \notin Bel$ , and so, by  $(\star 4)$ ,  $X_{bel} \wedge \neg A \in Bel \star (X_{bel} \wedge \neg A)$ .

(r3) Suppose for reductio that  $Bel \star A \subseteq Bel$  and  $Bel \star A \neq Bel$ . From the latter we get, by  $(\star 2)$ ,  $A \in Bel \star A$ . So  $A \in Bel$  and  $\neg A \notin Bel$ , by  $(\star 0)$ . From the latter we get  $Bel \subseteq Bel \star A$ , by  $(\star 4)$ , and thus  $Bel \star A = Bel$  after all. We have a contradiction.

(r4) Suppose that  $Bel \subseteq Bel \star A$  and  $\neg A \in Bel$ . Then  $\neg A \in Bel \star A$  and, by  $(\star 0)$ ,  $A \notin Bel \star A$ . Thus, by  $(\star 2)$ ,  $Bel \star A = Bel$ .

(r5) We show the contraposition. Suppose  $\neg A \notin Bel$ . So by  $(\star 4)$ ,  $Cn(Bel \cup \{A\}) \subseteq Bel \star A$  and thus  $A \in Bel \star A$ .

(r6) This follows from (r5) and  $(\star 0)$ .

(r7) Suppose  $A \notin Bel \star A$ . Then by  $(\star 2)$ ,  $Bel \star A = Bel$ . Since by (r5)  $\neg A \in Bel$ , we get  $\neg A \in Bel \star A$ .

(r8) From right to left. If  $A \in Bel$ , then  $Bel \star A = Bel$  by  $(\star 0)$ ,  $(\star 3)$  and  $(\star 4)$ . If  $A \notin Bel \star A$ , then  $Bel \star A = Bel$  by  $(\star 2)$ .

From left to right. Suppose that  $A \notin Bel$  and  $A \in Bel \star A$ . This immediately implies that  $Bel \star A \neq Bel$ .

(r9) Let  $A$  be non-contingent, i.e., let either  $A$  or  $\neg A$  be absurd. Suppose without loss of generality that  $A$  is absurd (the other case is similar). Then  $Bel \star A = Bel$ , by  $(\star 2)$ , and  $Bel \star \neg A = Bel$ , by (r5),  $(\star 3)$  and  $(\star 4)$ .

(r10) This follows immediately from (r9).

(r11) (Consistent Expansion) Let  $Bel \not\subseteq Bel \star A$ . Then by  $(\star 2)$ ,  $A \in Bel \star A$ , and by  $(\star 4)$ ,  $\neg A \in Bel$ . This proves the claim. (Notice that given Inclusion  $(\star 3)$ , Consistent Expansion implies Preservation  $(\star 4)$ : If  $A$  is consistent with  $Bel$ , then, by  $(\star 3)$ ,  $Bel \star A$  is consistent with  $Bel$ , and so by (r11)  $Bel \subseteq Bel \star A$ .)

(r12) (Regularity) Let  $A \in Bel \star B$ . Suppose for reductio that  $A \notin Bel \star A$ . From the former we get, by  $(\star 8c)$ ,  $Bel \star B \subseteq Bel \star (A \wedge B)$ . Since  $A \notin Bel \star A$ , also  $A \wedge B \notin Bel \star (A \wedge B)$ , by  $(\star 5a)$ . So by  $(\star 2)$ ,  $Bel \star (A \wedge B) = Bel$ . Thus  $Bel \star B \subseteq Bel$  and further, by (r3),  $Bel \star B = Bel$ . But we know from (r5) that  $\neg A \in Bel$ . So  $\neg A \in Bel \star B$ , and we have a contradiction with  $(\star 0)$ .

(r13) (Strong Regularity) Let  $A \notin Bel \star A$ . Suppose for reductio that  $\neg A \notin Bel \star B$  for some  $B$ . Then by  $(\star 8)$ ,  $Bel \star B \subseteq Bel \star (A \wedge B)$ . Since  $A \notin Bel \star A$ , also  $A \wedge B \notin Bel \star (A \wedge B)$ , by  $(\star 5a)$ . So by  $(\star 2)$ ,  $Bel \star (A \wedge B) = Bel$ . Thus  $Bel \star B \subseteq Bel$  and further, by (r3),  $Bel \star B = Bel$ . But we know from (r5) that  $\neg A \in Bel$ . So  $\neg A \in Bel \star B$ , and we have a contradiction.

(r14) (Disjunctive Success) This follows directly from (r13).

(r15) Suppose that  $C \in Bel\star A$  and  $C \notin Bel\star \neg A$ . From  $C \in Bel\star A$ , we can infer that  $C$  is not absurd, by (r12). And from  $C \notin Bel\star \neg A$ , we can infer that  $\neg C$  is not absurd, by (r13).

(r16) (Guarded Subexpansion) Let  $A \in Bel\star A$  and  $\neg B \notin Bel\star A$ . From the latter we get, by ( $\star 8$ ), that  $Bel\star A \subseteq Bel\star (A \wedge B)$ .

We now show that  $B \in Bel\star (A \wedge B)$ . Suppose this is not the case. Then, by ( $\star 1$ ),  $A \wedge B \notin Bel\star (A \wedge B)$ , and by ( $\star 2$ ),  $Bel\star (A \wedge B) = Bel$ . So  $Bel\star A \subseteq Bel$ . Thus, by (r3),  $Bel\star A = Bel$ . So we have  $A \in Bel$  and  $\neg B \notin Bel$ . It follows from ( $\star 1$ ) that  $\neg A \vee \neg B \notin Bel$ . Thus, by ( $\star 4$ ),  $Cn(Bel \cup \{A \wedge B\}) \subseteq Bel\star (A \wedge B)$ . So  $B \in Bel\star (A \wedge B)$ .

From  $Bel\star A \subseteq Bel\star (A \wedge B)$  and  $B \in Bel\star (A \wedge B)$  we get, with the help of ( $\star 1$ ), that  $Cn((Bel\star A) \cup \{B\}) \subseteq Bel\star (A \wedge B)$ , as desired.

(r17) (Reciprocity) Let  $A \in Bel\star B$  and  $B \in Bel\star A$ . By ( $\star 0$ ), we get  $\neg A \notin Bel\star B$  and  $\neg B \notin Bel\star A$ , so by ( $\star 7$ ) and ( $\star 8$ ), we get  $Bel\star A = Bel\star (A \wedge B) = Bel\star B$ .

(r18) (Disjunctive Factoring) If  $A \vee B \notin Bel\star (A \vee B)$ , or if  $A \notin Bel\star A$  and  $B \notin Bel\star B$ , then  $Bel\star (A \vee B) = Bel = Bel\star A = Bel\star B$ , by ( $\star 2$ ), ( $\star 5a$ ) and ( $\star 5b$ ).

So suppose without loss of generality that  $A \in Bel\star A$ . Then  $A \vee B \in Bel\star (A \vee B)$ , by ( $\star 5a$ ) and ( $\star 6$ ). By ( $\star 0$ ), we cannot have both  $\neg A \in Bel\star (A \vee B)$  and  $\neg B \in Bel\star (A \vee B)$ .

Case 1. Suppose  $\neg A \in Bel\star (A \vee B)$ . Then also  $B \in Bel\star (A \vee B)$ , by ( $\star 1$ ), and  $\neg B \notin Bel\star (A \vee B)$ , by ( $\star 0$ ). Thus, by ( $\star 6$ ) and ( $\star 7$ ),  $Bel\star B = Bel\star((A \vee B) \wedge (\neg A \vee B)) \subseteq Cn((Bel\star(A \vee B)) \cup \{\neg A \vee B\}) = Bel\star(A \vee B)$ . On the other hand, we have  $\neg(\neg A \vee B) \notin Bel\star(A \vee B)$ , since  $\neg B \notin Bel\star(A \vee B)$ . So by ( $\star 8$ ),  $Bel\star(A \vee B) \subseteq Bel\star((A \vee B) \wedge (\neg A \vee B)) = Bel\star B$ . So  $Bel\star(A \vee B) = Bel\star B$ .

Case 2. Suppose  $\neg B \in Bel\star(A \vee B)$ . Then by exactly the same reasoning as in Case 1,  $Bel\star(A \vee B) = Bel\star A$ .

Case 3. Suppose  $\neg A, \neg B \notin Bel\star(A \vee B)$ . Then  $Bel\star(A \vee B) \subseteq Bel\star A$  and  $Bel\star(A \vee B) \subseteq Bel\star B$ , by ( $\star 8$ ) and ( $\star 6$ ). On the other hand, using ( $\star 6$ ) and ( $\star 7$ ), we can show  $Bel\star A \subseteq Cn((Bel\star(A \vee B)) \cup \{A \vee \neg B\})$  and  $Bel\star B \subseteq Cn((Bel\star(A \vee B)) \cup \{\neg A \vee B\})$ . Hence  $(Bel\star A) \cap (Bel\star B) \subseteq Cn((Bel\star(A \vee B)) \cup \{A \vee \neg B\}) \cap Cn((Bel\star(A \vee B)) \cup \{\neg A \vee B\}) = Cn((Bel\star(A \vee B)) \cup \{(A \vee \neg B) \vee (\neg A \vee B)\}) = Bel\star(A \vee B)$ , by ( $\star 1$ ). In sum,  $Bel\star(A \vee B) = (Bel\star A) \cap (Bel\star B)$ . ■

*Proof of Observation 2.* (a) This is left as an exercise to the reader.

(b) Suppose that the revision function  $\star$  satisfies  $(\star 0)$ – $(\star 8)$ . Then it satisfies the eleven conditions listed in Theorem 13 of Hansson, Fermé, Cantwell and Falappa [12].

Hansson et al. list eight “core postulates”: Closure, Relative Success, Inclusion, Strong Consistency, Extensionality and Disjunctive Distribution are all included (in almost literally the same form) as postulates in the postulate set here. Strict Improvement is equivalent to  $(\star 5a)$ . Consistent Expansion is (r11).

Hansson et al. have three additional postulates: Vacuity is equivalent to the conjunction of  $(\star 3)$  and  $(\star 4)$ . Strong Regularity and Disjunctive Factoring are (r13) and (r18), respectively.<sup>35</sup>

So their representation result (Theorem 13) applies, and the revision function  $\star$  can be represented as a conceivability-limited spheres-based revision. ■

*Proof of Lemma 3.* Let  $\star$  satisfy the basic conditions  $(\star 0)$ – $(\star 6)$  for conceivability-limited belief revision.

$A \gg A$ : We need to show that

$$\begin{array}{l} A \in Bel\star A \text{ and} \\ A \notin Bel\star \neg A \end{array} \quad \text{iff} \quad \begin{array}{l} A \in Bel\star A \text{ and} \\ \neg A \in Bel\star \neg A \end{array} .$$

But  $A \notin Bel\star \neg A$  is equivalent to  $\neg A \in Bel\star \neg A$ , by  $(\star 0)$  and (r7). So  $A \gg A$  is equivalent to the contingency of  $A$ .

$A \gg A \wedge C$ : We need to show that

$$\begin{array}{l} A \wedge C \in Bel\star A \text{ and} \\ A \wedge C \notin Bel\star \neg A \end{array} \quad \text{iff} \quad \begin{array}{l} C \in Bel\star A \text{ and} \\ A \in Bel\star A \text{ and } \neg A \in Bel\star \neg A \end{array} .$$

The LHS implies that  $A$  is contingent, by (r10), and that  $C \in Bel\star A$ , by  $(\star 1)$ . The RHS implies that  $\neg A$  is not absurd, so  $A \wedge C \notin Bel\star \neg A$ , by  $(\star 0)$ , and  $A \wedge C \in Bel\star A$  follows by  $(\star 1)$ .

$A \gg A \vee C$ : We need to show that

$$\begin{array}{l} A \vee C \in Bel\star A \text{ and} \\ A \vee C \notin Bel\star \neg A \end{array} \quad \text{iff} \quad \begin{array}{l} C \notin Bel\star \neg A \text{ and} \\ A \in Bel\star A \text{ and } \neg A \in Bel\star \neg A \end{array} .$$

The LHS implies that  $A$  is contingent, by (r10). The RHS implies that  $A$  is not absurd, so  $A \vee C \in Bel\star A$ , by  $(\star 1)$ . Since  $\neg A \in Bel\star \neg A$  and  $C \notin Bel\star \neg A$ , it follows that  $A \vee C \notin Bel\star \neg A$ , by  $(\star 1)$ .

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<sup>35</sup>Disjunctive Factoring is in fact equivalent to the conjunction of  $(\star 7)$  and  $(\star 8)$ , similarly to the situation in AGM belief revision. But proving this here would lead us too far afield.

$A \wedge C \gg C$ : We need to show that

$$\begin{aligned} C \in Bel\star A \wedge C \text{ and } & \text{iff} \\ C \notin Bel\star \neg(A \wedge C) & \\ & C \notin Bel\star \neg(A \wedge C) \text{ and} \\ & A \wedge C \in Bel\star A \wedge C \text{ and } \neg(A \wedge C) \in Bel\star \neg(A \wedge C) \end{aligned}$$

The LHS implies that  $A \wedge C$  is contingent, by (r10). The RHS implies that  $A \wedge C \in Bel\star A \wedge C$ , so  $C \in Bel\star A \wedge C$  follows, by ( $\star$ 1).

$A \vee C \gg C$ : We need to show that

$$\begin{aligned} C \in Bel\star A \vee C \text{ and } & \text{iff} \\ C \notin Bel\star \neg(A \vee C) & \\ & C \in Bel\star (A \vee C) \text{ and} \\ & A \vee C \in Bel\star A \vee C \text{ and } \neg(A \vee C) \in Bel\star \neg(A \vee C) \end{aligned}$$

The LHS implies that  $A \vee C$  is contingent, by (r10). The RHS implies that  $\neg(A \vee C)$  is not absurd, so  $C \notin Bel\star \neg(A \vee C)$ , by ( $\star$ 0).

$A \wedge X_{bel} \gg A \wedge X_{bel}$ : We need to show that

$$\begin{aligned} A \wedge X_{bel} \in Bel\star (A \wedge X_{bel}) \text{ and } & \text{iff} \quad \neg A \notin Bel. \\ A \wedge X_{bel} \notin Bel\star \neg(A \wedge X_{bel}) & \end{aligned}$$

From left to right. Suppose that  $\neg A \in Bel$ . Then  $A \wedge X_{bel}$  is inconsistent, by the definition of  $X_{bel}$ , so  $A \wedge X_{bel} \notin Bel\star (A \wedge X_{bel})$ , by ( $\star$ 0). From right to left. Assume  $\neg A \notin Bel$ . Then, since  $X_{bel} \in Bel$ ,  $\neg(A \wedge X_{bel}) \notin Bel$ , by ( $\star$ 1). So by ( $\star$ 3) and ( $\star$ 4),  $Bel\star (A \wedge X_{bel}) = Cn(Bel \cup \{A \wedge X_{bel}\})$ . So  $A \wedge X_{bel} \in Bel\star (A \wedge X_{bel})$ . For the second disjunct, we use  $\neg X_{bel} \in Bel\star \neg X_{bel}$  and get that  $\neg(A \wedge X_{bel}) \in Bel\star \neg(A \wedge X_{bel})$ , by ( $\star$ 5a). So  $A \wedge X_{bel} \notin Bel\star \neg(A \wedge X_{bel})$ , by ( $\star$ 0).

$X_{bel} \gg X_{bel} \wedge A$ : We need to show that

$$\begin{aligned} A \wedge X_{bel} \in Bel\star X_{bel} \text{ and } & \text{iff} \quad A \in Bel. \\ A \wedge X_{bel} \notin Bel\star \neg X_{bel} & \end{aligned}$$

From left to right. Assume the LHS. We know that  $Bel\star X_{bel} = Bel$ , by (r1), so  $A \wedge X_{bel} \in Bel$  and  $A \in Bel$  by ( $\star$ 1). From right to left. Assume  $A \in Bel$ . Then  $A \wedge X_{bel} \in Bel$ . Since  $X_{bel} \in Bel$ ,  $A \wedge X_{bel} \in Bel\star X_{bel}$ , by ( $\star$ 0), ( $\star$ 3) and ( $\star$ 4). Moreover, since  $\neg X_{bel} \in Bel\star \neg X_{bel}$ , by ( $\star$ 0b), we get  $A \wedge X_{bel} \notin Bel\star \neg X_{bel}$ , by ( $\star$ 0).

$X_{bel} \gg A \wedge X_{bel}$  and  $A \not\gg A$ : We need to show that

$$X_{bel} \gg A \wedge X_{bel} \text{ and } A \not\gg A \quad \text{iff} \quad \neg A \notin Bel\star \neg A.$$

From left to right. Suppose that  $X_{bel} \gg A \wedge X_{bel}$  and  $A \not\gg A$ . We already know that  $X_{bel} \gg A \wedge X_{bel}$  is equivalent to  $A \in Bel$ . So  $\neg A \notin Bel$ , by  $(\star 0)$ , and thus  $A \in Bel \star A$ , by  $(\star 4)$ . But  $A \not\gg A$  means that either  $A \notin Bel \star A$  or  $A \in Bel \star \neg A$ . We have just refuted the former. So  $A \in Bel \star \neg A$  and thus, by  $(\star 0)$ ,  $\neg A \notin Bel \star \neg A$ . From right to left. Suppose that  $\neg A \notin Bel \star \neg A$ . Thus, by  $(r5)$  and  $(\star 1)$ ,  $A \in Bel$ , which we already know to be equivalent to  $X_{bel} \gg A \wedge X_{bel}$ . And by  $(r7)$  and  $(\star 1)$ ,  $A \in Bel \star \neg A$  which is sufficient to give us  $A \not\gg A$ .

$X_{bel} \gg \neg A \wedge X_{bel}$  and  $A \not\gg A$ : We need to show that

$$X_{bel} \gg \neg A \wedge X_{bel} \text{ and } A \not\gg A \quad \text{iff} \quad A \notin Bel \star A.$$

From left to right. Suppose that  $X_{bel} \gg \neg A \wedge X_{bel}$  and  $A \not\gg A$ . We already know that  $X_{bel} \gg \neg A \wedge X_{bel}$  is equivalent to  $\neg A \in Bel$ . So  $A \notin Bel$ , by  $(\star 0)$ , and thus  $\neg A \in Bel \star \neg A$ , by  $(\star 4)$ . So  $A \notin Bel \star \neg A$ , by  $(\star 0)$ . But  $A \not\gg A$  means that either  $A \notin Bel \star A$  or  $A \in Bel \star \neg A$ . We have just refuted the latter. So  $A \notin Bel \star A$ . From right to left. Suppose that  $A \notin Bel \star A$ . Thus, by  $(r5)$  and  $(\star 1)$ ,  $\neg A \in Bel$ , which we already know to be equivalent to  $X_{bel} \gg \neg A \wedge X_{bel}$ . And by  $(r7)$  and  $(\star 1)$ ,  $\neg A \in Bel \star A$ , so  $A \notin Bel \star A$ , by  $(\star 0)$ , which is sufficient to give us  $A \not\gg A$ . ■

*Proof of Observation 4.* Let  $\star$  be a basic revision function in the sense of Section 5, and let  $\gg$  be obtained from  $\star$  by RRT.

$(\gg 0)$  We know that  $X_{bel} \gg X_{bel}$  expresses that (our strongest belief)  $X_{bel}$  is contingent, and this follows from  $(\star 0b)$  and  $(\star 2)$ .

$(\gg 1)$  We show that  $A \gg B \wedge C$  implies that  $A \gg B$  or  $A \gg C$ . Suppose that  $B \wedge C \in Bel \star A$  and  $B \wedge C \notin Bel \star \neg A$ . But then, with the help of  $(\star 1)$ , it follows immediately that either  $B \in Bel \star A$  and  $B \notin Bel \star \neg A$  or that  $C \in Bel \star A$  and  $C \notin Bel \star \neg A$ .

$(\gg 2a)$  We show that  $A \gg C$  if and only if  $A \gg A \wedge C$  and  $A \gg A \vee C$ . By RRT, this means that

$$\begin{aligned} & C \in Bel \star A \text{ and } \\ & C \notin Bel \star \neg A \end{aligned} \quad \text{iff} \quad \left( \begin{array}{l} A \wedge C \in Bel \star A \text{ and } \\ A \wedge C \notin Bel \star \neg A \end{array} \text{ and } \begin{array}{l} A \vee C \in Bel \star A \text{ and } \\ A \vee C \notin Bel \star \neg A \end{array} \right).$$

For the left-to-right direction, we first infer from the LHS that  $A$  is contingent, by  $(r10)$ . So we get from  $C \in Bel \star A$  that  $A \wedge C \in Bel \star A$  and  $A \vee C \in Bel \star A$ , by  $(\star 1)$  and  $(\star 2)$ . Since  $C \notin Bel \star \neg A$ ,  $A \wedge C \notin Bel \star \neg A$ , by  $(\star 1)$ . Suppose  $A \vee C \in Bel \star \neg A$ . Then since  $\neg A \in Bel \star \neg A$ ,  $C \in Bel \star \neg A$ , by  $(\star 1)$  and  $(\star 2)$ , contradicting the supposition.

For the right-to-left direction, we get  $C \in Bel \star A$  from  $A \wedge C \in Bel \star A$ , and  $C \notin Bel \star \neg A$  from  $A \vee C \notin Bel \star \neg A$ , both by  $(\star 1)$ .

( $\geq 2b$ ) We show that  $A \geq A \wedge C$  if and only if not  $\neg A \geq \neg A \vee C$  and  $A \geq A$ . By RRT, this means that

$$\begin{aligned} & A \wedge C \in Bel \star A \text{ and} \\ & A \wedge C \notin Bel \star \neg A \quad \text{iff} \\ & \left( \begin{array}{l} \neg A \vee C \notin Bel \star \neg A \text{ or} \\ \neg A \vee C \in Bel \star A \end{array} \text{ and } \begin{array}{l} A \in Bel \star A \text{ and} \\ A \notin Bel \star \neg A \end{array} \right). \end{aligned}$$

For the left-to-right direction, we first infer from the LHS that  $A$  is contingent, by (r10). So  $A \in Bel \star A$ , and also  $A \notin Bel \star \neg A$ , by  $(\star 0)$ . Since  $A \wedge C \in Bel \star A$ ,  $\neg A \vee C \in Bel \star A$ , by  $(\star 1)$ .

For the right-to-left direction, we first infer from the RHS that  $A$  is contingent, by (r10). Thus  $\neg A \vee C \notin Bel \star \neg A$  is impossible, by  $(\star 1)$  and  $(\star 2)$ . So  $\neg A \vee C \in Bel \star A$ , from which we get  $A \wedge C \in Bel \star A$ , by  $(\star 1)$  and  $(\star 2)$ . Finally, we get  $A \wedge C \notin Bel \star \neg A$  from  $A \notin Bel \star \neg A$ , by  $(\star 1)$ .

( $\geq 3$ ) We show that  $A \wedge \neg C \wedge X_{bel} \geq A \wedge \neg C \wedge X_{bel}$  implies  $A \not\geq A \wedge C$ . By RRT, this means that

$$\text{If } A \wedge \neg C \wedge X_{bel} \text{ is contingent, then } \begin{array}{l} A \wedge C \notin Bel \star A \text{ or} \\ A \wedge C \in Bel \star \neg A \end{array}.$$

Assume the antecedent. Then, by  $(\star 0)$ ,  $A \wedge \neg C \wedge X_{bel}$  is consistent, so  $\neg(A \wedge \neg C) \notin Bel$ . By the logical closure of  $Bel$ , then  $C \notin Cn(Bel \cup \{A\})$ , and by  $(\star 3)$ , this implies that  $C \notin Bel \star A$  which proves the consequent, using  $(\star 1)$ .

( $\geq 4$ ) We show that  $\neg A \wedge X_{bel} \geq \neg A \wedge X_{bel}$  and  $A \geq A \vee C$  together imply  $\neg C \wedge X_{bel} \geq \neg C \wedge X_{bel}$ . By RRT, this means that

$$\text{If } \neg A \wedge X_{bel} \text{ is contingent and } \begin{array}{l} A \vee C \in Bel \star A \text{ and} \\ A \vee C \notin Bel \star \neg A \end{array}, \text{ then } \neg C \wedge X_{bel} \text{ is contingent.}$$

Assume the antecedent. Then, by  $(\star 0)$ ,  $\neg A \wedge X_{bel}$  is consistent, so  $A \notin Bel$ , by the logical closure of  $Bel$ . Now we can conclude with  $(\star 4)$  from  $A \vee C \notin Bel \star \neg A$ , that  $A \vee C \notin Bel$ . So  $C \notin Bel$ , by the logical closure of  $Bel$ , and also, since  $X_{bel} \in Bel$ ,  $\neg(\neg C \wedge X_{bel}) \notin Bel$ . Now we use  $(\star 4)$  once more and conclude  $\neg C \wedge X_{bel} \in Bel \star (\neg C \wedge X_{bel})$ . On the other hand, since  $X_{bel}$  is contingent,  $\neg X_{bel}$  is not absurd, thus  $\neg(\neg C \wedge X_{bel})$  is not absurd either and so  $\neg(\neg C \wedge X_{bel}) \in Bel \star \neg(\neg C \wedge X_{bel})$ , by  $(\star 5a)$ . This establishes that  $\neg C \wedge X_{bel}$  is contingent.

( $\geq 5a$ ) We show that  $A \geq A$  iff  $\neg A \geq \neg A$ . But as we have already seen, these conditionals just say that  $A$  and  $\neg A$ , respectively, are contingent, and this is the same thing.

( $\geq 5b$ ) We show that  $A \wedge B \geq A \wedge B$  implies  $A \geq A$  or  $B \geq B$ . That is to say, if  $A \wedge B$  is contingent, so is either  $A$  or  $B$ . That is, we have to show that

$$\begin{array}{l} \text{If } A \wedge B \in \text{Bel}\star A \wedge B \text{ and} \\ \quad \neg(A \wedge B) \in \text{Bel}\star \neg(A \wedge B), \\ \text{then } \left( A \in \text{Bel}\star A \text{ and } \neg A \in \text{Bel}\star \neg A \text{ or } B \in \text{Bel}\star B \text{ and } \neg B \in \text{Bel}\star \neg B \right). \end{array}$$

This follows immediately from ( $\star 5a$ ) and ( $\star 5b$ ).

( $\geq 5c$ ) We show that  $A \wedge B \geq A \wedge B$  and  $A \vee C \geq A \vee C$  together imply  $A \geq A$ . That is to say, if both  $A \wedge B$  and  $A \vee C$  are contingent, so is  $A$ . That is, we have to show that

$$\begin{array}{l} \text{If } A \wedge B \in \text{Bel}\star A \wedge B \text{ and } \neg(A \wedge B) \in \text{Bel}\star \neg(A \wedge B) \text{ and } \\ \quad A \vee C \in \text{Bel}\star A \vee C \text{ and } \neg(A \vee C) \in \text{Bel}\star \neg(A \vee C), \\ \text{then } A \in \text{Bel}\star A \text{ and } \neg A \in \text{Bel}\star \neg A. \end{array}$$

But by ( $\star 5a$ ),  $A \wedge B \in \text{Bel}\star A \wedge B$  implies  $A \in \text{Bel}\star A$ , and  $\neg(A \vee C) \in \text{Bel}\star \neg(A \vee C)$  implies  $\neg A \in \text{Bel}\star \neg A$ .

( $\geq 6$ ) follows straightforwardly from ( $\star 1$ ) and ( $\star 6$ ). ■

*Proof of Lemma 5.* Let the principles ( $\geq 0$ ), ( $\geq 1$ ), ( $\geq 2a$ ), ( $\geq 2b$ ), ( $\geq 3$ ), ( $\geq 4$ ), ( $\geq 5a$ ), ( $\geq 5b$ ), ( $\geq 5c$ ) and ( $\geq 6$ ) for difference-making conditionals be given.

(c1) Suppose that  $A \geq C$ . Then, by ( $\geq 2a$ ),  $A \geq A \wedge C$ . Then, by ( $\geq 2a$ ) again,  $A \geq A \vee (A \wedge C)$ . So  $A \geq A$ , by ( $\geq 6$ ).

(c2) Suppose for reductio that  $A \geq \perp$ . Then, by ( $\geq 6$ ),  $A \geq A \wedge \neg A$ . Using ( $\geq 2b$ ), we get that  $\neg A \not\geq \neg A \vee \neg A$  and  $A \geq A$ . Using ( $\geq 6$ ) again, this means that  $\neg A \not\geq \neg A$ , but  $A \geq A$ . This contradicts (5a).

(c3) Suppose for reductio that  $A \geq \top$ . Then  $A \geq A$ , by (c1). By ( $\geq 5a$ ) and ( $\geq 6$ ), we get  $\neg A \geq \neg A \wedge \neg A$ . By ( $\geq 2b$ ) and ( $\geq 6$ ), we can infer that  $A \not\geq A \vee \neg A$ , and thus, by ( $\geq 6$ ) again,  $A \not\geq \top$ , contradicting our supposition.

(c4) Suppose for reductio that  $\top \geq C$ . By ( $\geq 6$ ),  $\top \geq \top \wedge A$ . So by ( $\geq 2b$ ),  $\top \geq \top$ . But this is impossible, according to (c3).

(c5) Suppose for reductio that  $\perp \geq C$ . Then  $\perp \geq \perp \wedge C$ , by ( $\geq 2a$ ). Thus  $\perp \geq \perp$ , by ( $\geq 6$ ). But this is impossible, according to (c2).

(c6) For the left-to-right direction, note that  $A$  is equivalent to  $(A \vee C) \wedge (A \vee \neg C)$  and apply ( $\geq 1$ ). For the right-to-left direction, note that  $A \geq A \vee C$

implies  $A \gg A \wedge (A \vee C)$ , by ( $\gg 2a$ ), which is equivalent to  $A \gg A$ , by ( $\gg 6$ ). Similarly for  $A \gg A \vee \neg C$ .

(c7) Let  $A \wedge B \vdash \perp$  and suppose for reductio that  $A \gg B$ . Then, by ( $\gg 2a$ ),  $A \gg A \wedge B$ , and thus, by ( $\gg 6$ ),  $A \gg \perp$ , contradicting (c2).

(c8) follows from ( $\gg 2b$ ) and (c6).

(c9) Let  $A \vee B \gg A \vee B$ . Then, by ( $\gg 5a$ ),  $\neg(A \vee B) \gg \neg(A \vee B)$ , and by ( $\gg 6$ ),  $\neg A \wedge \neg B \gg \neg A \wedge \neg B$ . So by ( $\gg 5b$ ), either  $\neg A \gg \neg A$  or  $\neg B \gg \neg B$ . Finally, by ( $\gg 5a$ ) again, either  $A \gg A$  or  $B \gg B$ .

(c10) follows from ( $\gg 2a$ ), ( $\gg 6$ ) and (c3).

(c11) Suppose  $A \gg A \wedge B \wedge C$ . Then, by ( $\gg 6$ ),  $A \gg A \wedge C \wedge (A \supset B)$ . So by ( $\gg 1$ ), either  $A \gg A \wedge C$  or  $A \gg A \supset B$ . But the latter is impossible, by (c10) and ( $\gg 6$ ).

(c12) Suppose that  $A \gg A \vee B \vee C$ . Then  $A \gg A$ , by (c1), and  $\neg A \gg \neg A$ , by ( $\gg 5a$ ). By ( $\gg 2b$ ),  $\neg A \not\gg \neg A \wedge (B \vee C)$ . Hence, by (c11),  $\neg A \not\gg \neg A \wedge (B \vee C) \wedge (\neg B \vee C)$ , and, by ( $\gg 6$ ),  $\neg A \not\gg \neg A \wedge C$ . So, by ( $\gg 2b$ ) again,  $A \gg A \vee C$ .

(c13) Suppose that  $A \gg B$  and  $A \gg C$ . By the left-to-right directions of ( $\gg 2a$ ) and ( $\gg 2b$ ), we get  $A \gg A \vee B$  and  $A \gg A \vee C$ , and  $A \gg A$ ,  $\neg A \not\gg \neg A \vee B$  and  $\neg A \not\gg \neg A \vee C$ . From the latter two, we can deduce with ( $\gg 1$ ) that  $\neg A \not\gg (\neg A \vee B) \wedge (\neg A \vee C)$ . Thus by ( $\gg 6$ ),  $\neg A \not\gg \neg A \vee (B \wedge C)$ . Since  $A \gg A$ , we can apply ( $\gg 2b$ ) in the right-to-left direction and get  $A \gg A \wedge (B \wedge C)$ . On the other hand, we know that  $A \gg A \vee B$ , which is equivalent to  $A \gg A \vee (B \wedge C) \vee (B \wedge \neg C)$ , by ( $\gg 6$ ). So, by (c12),  $A \gg A \vee (B \wedge C)$ . Putting this and  $A \gg A \wedge (B \wedge C)$  together with the help of the right-to-left direction of ( $\gg 2a$ ), we get  $A \gg B \wedge C$ .

(c14) From left to right. Let  $A \gg A \vee C$ . Then by ( $\gg 2b$ )  $\neg A \not\gg \neg A \wedge C$ . Also, by (c1)  $A \gg A$ . From right to left. Let  $\neg A \not\gg \neg A \wedge C$  and  $A \gg A$ . From the latter we get by ( $\gg 5a$ )  $\neg A \gg \neg A$ . From this and  $\neg A \not\gg \neg A \wedge C$  we get  $A \gg A \vee C$ , by ( $\gg 2b$ ) and ( $\gg 6$ ).

(c15) By ( $\gg 5a$ ) and ( $\gg 5c$ ).

(c16) Suppose that  $X_{bel} \gg X_{bel} \wedge A$ . From ( $\gg 0$ ) and ( $\gg 6$ ), we get  $(X_{bel} \wedge A) \vee (X_{bel} \wedge \neg A) \gg (X_{bel} \wedge A) \vee (X_{bel} \wedge \neg A)$ . So by (c9) either  $X_{bel} \wedge A \gg X_{bel} \wedge A$  or  $X_{bel} \wedge \neg A \gg X_{bel} \wedge \neg A$ . But from  $X_{bel} \gg X_{bel} \wedge A$  it follows by ( $\gg 3$ ) and ( $\gg 6$ ) that  $X_{bel} \wedge \neg A \not\gg X_{bel} \wedge \neg A$ . So  $X_{bel} \wedge A \gg X_{bel} \wedge A$ , as desired.

(c17) Suppose that  $A \wedge B \gg A \wedge B$  and  $X_{bel} \not\gg X_{bel} \wedge A$ . We show that  $A \gg A$ . From  $X_{bel} \not\gg X_{bel} \wedge A$  we infer, by ( $\gg 0$ ) and ( $\gg 2b$ ), that  $\neg X_{bel} \gg \neg X_{bel} \vee A$ . From this we get  $\neg A \wedge X_{bel} \gg \neg A \wedge X_{bel}$ , by ( $\gg 0$ ) and ( $\gg 4$ ). That is, by ( $\gg 5a$ ),  $A \vee \neg X_{bel} \gg A \vee \neg X_{bel}$ . Finally, from this and  $A \wedge B \gg A \wedge B$ , we get  $A \gg A$ , by ( $\gg 5c$ ).



(c18) Suppose that  $A \not\approx A$  and  $A \vee B \approx A \vee B$ . Thus  $X_{bel} \wedge A \not\approx X_{bel} \wedge A$ , by ( $\approx 5c$ ). By ( $\approx 0$ ) and ( $\approx 4$ ), we get  $\neg X_{bel} \not\approx \neg X_{bel} \vee \neg A$ . By ( $\approx 0$ ) and ( $\approx 2b$ ), this gives us  $X_{bel} \approx X_{bel} \wedge \neg A$ .

(c19) Suppose that  $A \not\approx A$  and  $X_{bel} \not\approx X_{bel} \wedge \neg A$ . By ( $\approx 0$ ) and ( $\approx 2b$ ),  $\neg X_{bel} \approx \neg X_{bel} \vee \neg A$ . Since by ( $\approx 0$ )  $X_{bel} \wedge X_{bel} \approx X_{bel} \wedge X_{bel}$ , we can apply ( $\approx 4$ ) and get  $A \wedge X_{bel} \approx A \wedge X_{bel}$ . From this and  $A \not\approx A$ , we get that  $\neg X_{bel} \vee A \not\approx \neg X_{bel} \vee A$ , by ( $\approx 5c$ ). So by ( $\approx 5a$ ),  $X_{bel} \wedge \neg A \not\approx X_{bel} \wedge \neg A$ . Now we use  $X_{bel} \wedge X_{bel} \not\approx X_{bel} \wedge X_{bel}$  and ( $\approx 4$ ) again and infer that  $\neg X_{bel} \not\approx \neg X_{bel} \vee A$ . So by ( $\approx 0$ ) and ( $\approx 2b$ )  $X_{bel} \approx X_{bel} \wedge A$ , as desired.

(c20) Suppose that  $X_{bel} \approx X_{bel} \wedge A$  and  $A \approx A \wedge C$ . From the latter we get, by ( $\approx 3$ ), that  $X_{bel} \wedge A \wedge \neg C \not\approx X_{bel} \wedge A \wedge \neg C$ . So, by Lemma 7, (i.2) $\Rightarrow$ (i.1), we get that  $X_{bel} \approx X_{bel} \wedge (A \supset C)$ . From this and  $X_{bel} \approx X_{bel} \wedge A$ , we infer with (c13) and (c11) that  $X_{bel} \approx X_{bel} \wedge C$ , as desired.

(c21) Suppose that  $A \approx A \wedge C$ . Then, by ( $\approx 3$ ),  $X_{bel} \wedge A \wedge \neg C \not\approx X_{bel} \wedge A \wedge \neg C$ . By ( $\approx 0$ ) and ( $\approx 6$ ), we have  $X_{bel} \wedge X_{bel} \approx X_{bel} \wedge X_{bel}$ . Now we can apply ( $\approx 4$ ) and get  $\neg X_{bel} \not\approx \neg X_{bel} \vee \neg(A \wedge \neg C)$ . From this and  $X_{bel} \approx X_{bel}$ , we get  $X_{bel} \approx X_{bel} \wedge (A \supset C)$ , by ( $\approx 2b$ ) and ( $\approx 6$ ).

(c22) Suppose that  $A \approx A \vee C$  and  $X_{bel} \approx X_{bel} \wedge C$ . From the latter we get, by ( $\approx 3$ ) and ( $\approx 6$ ),  $X_{bel} \wedge \neg C \not\approx X_{bel} \wedge \neg C$ . Now we can apply ( $\approx 4$ ) and get  $X_{bel} \wedge \neg A \not\approx X_{bel} \wedge \neg A$ . From this and  $X_{bel} \wedge X_{bel} \approx X_{bel} \wedge X_{bel}$ , we get, by ( $\approx 4$ ) again,  $\neg X_{bel} \not\approx \neg X_{bel} \vee \neg A$ , and thus, by ( $\approx 2b$ ) and ( $\approx 6$ ),  $X_{bel} \approx X_{bel} \wedge A$ .

(c23) Suppose that  $A \approx A$ ,  $X_{bel} \approx X_{bel} \wedge C$  and  $X_{bel} \not\approx X_{bel} \wedge \neg A$ . From this we get, by (c22), that  $\neg A \not\approx \neg A \vee C$ . So, by ( $\approx 2b$ ),  $A \approx A \wedge C$  or  $A \not\approx A$ . Since the latter is excluded by supposition, we get  $A \approx A \wedge C$ .  $\square$

*Proof of Observation 6.* For ( $\approx$ Arist1), suppose that  $A \approx \neg A$ . By ( $\approx 2a$ ), we get  $A \approx A \wedge \neg A$ . But this is impossible, by ( $\star 6$ ) and (c2).

For ( $\approx$ Arist2), suppose that  $A \approx C$ . By ( $\approx 2a$ ), we get that  $A \approx A \wedge C$ . Thus, by ( $\approx 2b$ ),  $\neg A \not\approx \neg A \vee C$ . So, by ( $\approx 2a$ ) again,  $\neg A \not\approx C$ .

For ( $\approx$ Boet-Abel), suppose that both  $A \approx C$  and  $A \approx \neg C$ . By (c13), we get that  $A \approx C \wedge \neg C$ . But this is impossible, by ( $\star 6$ ) and (c2).  $\blacksquare$

*Proof of Lemma 7.* Let the principles ( $\approx 0$ )–( $\approx 6$ ) for difference-making conditionals be given.

(i) *Beliefs.* (i.0) is equivalent to (i.1). See Lemma 3.

(i.1) implies (i.2). This follows immediately from ( $\approx 3$ ) and ( $\approx 6$ ).

(i.2) implies (i.3). Suppose for reductio that  $\neg A \wedge X_{bel} \not\approx \neg A \wedge X_{bel}$  and  $\neg A \wedge B \wedge X_{bel} \approx \neg A \wedge B \wedge X_{bel}$ . Then on the one hand, by ( $\approx 4$ ),  $\neg(\neg A \wedge B) \not\approx \neg(\neg A \wedge B) \vee \neg \neg A$ , i.e., by ( $\approx 6$ ),  $A \vee \neg B \not\approx A \vee \neg B$ . On the

other hand, the supposition gives us, by ( $\geq 6$ ) and ( $\geq 5b$ ),  $\neg A \wedge B \geq \neg A \wedge B$ . This contradicts ( $\geq 5a$ ).

(i.3) implies (i.1). First choose  $B = \neg A$  and get (i.2),  $\neg A \wedge X_{bel} \not\geq \neg A \wedge X_{bel}$ . By ( $\geq 0$ ) and ( $\geq 6$ ), we have  $X_{bel} \wedge X_{bel} \geq X_{bel} \wedge X_{bel}$ . Now we can apply ( $\geq 4$ ) and get  $\neg X_{bel} \not\geq \neg X_{bel} \vee A$ . From this and  $X_{bel} \geq X_{bel}$ , we get  $X_{bel} \geq A \wedge X_{bel}$ , by ( $\geq 2b$ ).

(ii) *Doxastic necessities*. (ii.0) is equivalent to (ii.1). See Lemma 3.

(ii.1) implies (ii.2). Let  $X_{bel} \geq A \wedge X_{bel}$  and  $A \not\geq A$ . We have  $(A \wedge X_{bel}) \vee (\neg A \wedge X_{bel}) \geq (A \wedge X_{bel}) \vee (\neg A \wedge X_{bel})$ , by ( $\geq 0$ ) and ( $\geq 6$ ). So by (c9), either  $A \wedge X_{bel} \geq A \wedge X_{bel}$  or  $\neg A \wedge X_{bel} \geq \neg A \wedge X_{bel}$ . But from  $X_{bel} \geq A \wedge X_{bel}$ , we get that  $\neg A \wedge X_{bel} \not\geq \neg A \wedge X_{bel}$ , by the equivalence of (i.1) and (i.2). So  $A \wedge X_{bel} \geq A \wedge X_{bel}$ , which gives us (ii.2).

(ii.2) implies (ii.3). This is trivial with  $B = X_{bel}$ .

(ii.3) implies (ii.4). Let  $A \wedge B \geq A \wedge B$  and  $A \not\geq A$ . Suppose for reductio that there is a  $B'$  such that  $A \vee B' \geq A \vee B'$ . But then, by ( $\geq 5c$ ),  $A \geq A$ , and we have a contradiction.

(ii.4) implies (ii.1). Let  $A \vee B \not\geq A \vee B$  for all  $B$ .  $A \not\geq A$  follows immediately if we choose  $B = A$ . We also get  $A \vee \neg X_{bel} \not\geq A \vee \neg X_{bel}$  if we choose  $B = \neg X_{bel}$ . Thus  $\neg A \wedge X_{bel} \not\geq \neg A \wedge X_{bel}$ , by ( $\geq 5a$ ). This implies  $X_{bel} \geq A \wedge X_{bel}$ , by the equivalence of (i.1) and (i.2), and so we have (ii.1).

(iii) *Absurdities*. (iii.0) is equivalent to (iii.1). See Lemma 3.

(iii.1) implies (iii.2), by the equivalence of (i.1) and (i.2) established above.

(iii.2) implies (iii.3), by exactly the same argument that we used for the inference from (ii.1) to (ii.2).

(iii.3) implies (iii.4), by the equivalence of (i.1) and (i.2).

(iii.4) implies (iii.5). Let  $\neg A \wedge X_{bel} \geq \neg A \wedge X_{bel}$  and  $A \not\geq A$ . From the former we get, by ( $\geq 5a$ ) and ( $\geq 6$ ), that  $A \vee \neg X_{bel} \geq A \vee \neg X_{bel}$ . So we have (iii.5) with  $B = \neg X_{bel}$ .

(iii.5) implies (iii.6). Let  $A \vee B \geq A \vee B$  and  $A \not\geq A$ . Suppose for reductio that there is a  $B'$  such that  $A \wedge B' \geq A \wedge B'$ . But then, by ( $\geq 5c$ ),  $A \geq A$ , and we have a contradiction.

(iii.6) implies (iii.1). That (iii.6) implies (iii.2) follows immediately if we first choose  $B = A$  and second choose  $B = X_{bel}$ . And (iii.2) implies (iii.1), by the equivalence of (i.1) and (i.2). ■

*Proof of Observation 8.* For ( $\geq MP$ ), suppose that  $A \geq C$  and  $X_{bel} \geq X_{bel} \wedge A$ . From the former we get, by ( $\geq 2a$ ),  $A \geq A \wedge C$ . Thus, by (c20),  $X_{bel} \geq X_{bel} \wedge C$ .

For ( $\gg$ MT), suppose that  $A \gg C$  and  $X_{bel} \gg X_{bel} \wedge \neg C$ . From the former we get, by ( $\gg$ 2a),  $A \gg A \wedge C$ . Thus, by (c8),  $A \not\gg A \wedge \neg C$ . Since  $A \gg C$  also implies  $A \gg A$ , by (c1), we can apply ( $\gg$ 2b) and get  $\neg A \gg \neg A \vee \neg C$ . By (c22), this taken together with  $X_{bel} \gg X_{bel} \wedge \neg C$  gives us  $X_{bel} \gg X_{bel} \wedge \neg A$ , as desired.

For ( $\gg$ AC), suppose that  $A \gg C$  and  $X_{bel} \gg X_{bel} \wedge C$ . From the former we get, by ( $\gg$ 2a),  $A \gg A \vee C$ . So by (c22),  $X_{bel} \gg X_{bel} \wedge A$ .

For ( $\gg$ DA<sup>w</sup>), suppose that  $A \gg C$  and  $X_{bel} \gg X_{bel} \wedge \neg A$ . Suppose further for reductio that  $X_{bel} \gg X_{bel} \wedge C$ . Then we can apply ( $\gg$ AC) and get  $X_{bel} \gg X_{bel} \wedge A$ . But by (c8), this contradicts  $X_{bel} \gg X_{bel} \wedge \neg A$ . So  $X_{bel} \not\gg X_{bel} \wedge C$ . ■

*Proof of Observation 9.* For ( $\gg$ 7c), we show that  $A \gg A \wedge B$  and  $A \wedge B \gg A \wedge B \wedge C$  imply  $A \gg A \wedge C$ . By RRT, this means that

$$\begin{array}{l} \text{If } \begin{array}{l} A \wedge B \in Bel \star A \text{ and} \\ A \wedge B \notin Bel \star \neg A \end{array} \text{ and } \begin{array}{l} A \wedge B \wedge C \in Bel \star A \wedge B \text{ and} \\ A \wedge B \wedge C \notin Bel \star \neg A \vee \neg B \end{array} , \\ \text{then } \begin{array}{l} A \wedge C \in Bel \star A \text{ and} \\ A \wedge C \notin Bel \star \neg A \end{array} . \end{array}$$

It follows from  $B \in Bel \star A$  and  $C \in Bel \star A \wedge B$  by ( $\ast$ 7c) that  $C \in Bel \star A$ , which gives us the upper line. For the lower line, suppose that  $A \wedge C \in Bel \star \neg A$ . Then, by ( $\ast$ 2),  $\neg A$  is absurd, and  $A$  is a belief, by (r5). But then  $Bel \star A = Bel = Bel \star \neg A$ , by ( $\ast$ 2)–( $\ast$ 4), which contradicts the premises  $A \wedge B \in Bel \star A$  and  $A \wedge B \notin Bel \star \neg A$ .

For ( $\gg$ 7), we show that  $A \wedge B \gg A \wedge B \wedge C$  and  $A \gg A$  imply  $A \gg A \wedge (B \supset C)$ . By RRT, this means that

$$\begin{array}{l} \text{If } \begin{array}{l} A \wedge B \wedge C \in Bel \star A \wedge B \text{ and} \\ A \wedge B \wedge C \notin Bel \star \neg A \vee \neg B \end{array} \text{ and } A \text{ is contingent,} \\ \text{then } \begin{array}{l} A \wedge (B \supset C) \in Bel \star A \text{ and} \\ A \wedge (B \supset C) \notin Bel \star \neg A \end{array} . \end{array}$$

Since  $A$  is contingent,  $A \in Bel \star A$ . And it follows from  $C \in Bel \star A \wedge B$  by ( $\ast$ 7) that  $C \in Cn((Bel \star A) \cup \{B\})$ . By the deduction theorem and ( $\ast$ 1), this entails  $B \supset C \in Bel \star A$ . Since  $A$  is contingent,  $\neg A \in Bel \star \neg A$ , so  $A \wedge (B \supset C) \notin Bel \star \neg A$ , by ( $\ast$ 0).

For ( $\gg$ 7'), we show that  $A \gg A \wedge C$ ,  $B \gg B \wedge C$  and  $A \vee B \gg A \vee B$  imply  $A \vee B \gg (A \vee B) \wedge C$ . By RRT, this means, after a few routine simplifications

using  $(\star 1)$  and  $(\star 2)$ , that

$$\begin{array}{l} \text{If } A \wedge C \in \text{Bel}\star A \text{ and } B \wedge C \in \text{Bel}\star B \text{ and} \\ A \wedge C \notin \text{Bel}\star \neg A \quad \text{and} \quad B \wedge C \notin \text{Bel}\star \neg B \\ \text{and } A \vee B \in \text{Bel}\star A \vee B \text{ and } (A \vee B) \wedge C \in \text{Bel}\star A \vee B \text{ and} \\ A \vee B \notin \text{Bel}\star \neg A \wedge \neg B, \text{ then } (A \vee B) \wedge C \notin \text{Bel}\star \neg A \wedge \neg B. \end{array}$$

It follows from  $C \in \text{Bel}\star A$  and  $C \in \text{Bel}\star B$  by  $(\star 7')$  that  $C \in \text{Bel}\star A \vee B$ . Since the first two premises say that neither  $A$  nor  $B$  is absurd,  $A \vee B$  isn't absurd either, by  $(\star 5b)$ , and we have the upper line. For the lower line, the third premise is sufficient.

For  $(\geq 8c)$ , we show that  $A \geq A \wedge B$  and  $A \geq A \wedge C$  imply  $A \wedge B \geq A \wedge B \wedge C$ . By RRT, this means that

$$\begin{array}{l} \text{If } A \wedge B \in \text{Bel}\star A \text{ and } A \wedge C \in \text{Bel}\star A \text{ and} \\ A \wedge B \notin \text{Bel}\star \neg A \quad \text{and} \quad A \wedge C \notin \text{Bel}\star \neg A, \\ \text{then } A \wedge B \wedge C \in \text{Bel}\star A \wedge B \text{ and} \\ A \wedge B \wedge C \notin \text{Bel}\star \neg A \vee \neg B. \end{array}$$

It follows from  $B \in \text{Bel}\star A$  and  $C \in \text{Bel}\star A$  by  $(\star 8c)$  that  $C \in \text{Bel}\star A \wedge B$ , for arbitrary  $C$ . We can also take  $C = A \wedge B$  and get  $A \wedge B \in \text{Bel}\star A \wedge B$ . Thus we have the upper line. For the lower line, suppose that  $A \wedge B \wedge C \in \text{Bel}\star \neg A \vee \neg B$ . Then, by  $(\star 2)$ ,  $\neg A \vee \neg B$  is absurd. But then, by  $(\star 5a)$ ,  $\neg A$  is absurd, too, and  $A$  is a belief, by  $(r5)$ . But then  $\text{Bel}\star A = \text{Bel} = \text{Bel}\star \neg A$ , by  $(\star 2)$ – $(\star 4)$ , which contradicts the premises  $A \wedge B \in \text{Bel}\star A$  and  $A \wedge B \notin \text{Bel}\star \neg A$ .

For  $(\geq 8)$ , we show that  $A \geq A \wedge \neg B$  and  $A \geq A \wedge C$  imply  $A \wedge B \geq A \wedge B \wedge C$ . By RRT, this means that

$$\begin{array}{l} \text{If } A \wedge \neg B \notin \text{Bel}\star A \text{ or } A \wedge C \in \text{Bel}\star A \text{ and} \\ A \wedge \neg B \in \text{Bel}\star \neg A \quad \text{and} \quad A \wedge C \notin \text{Bel}\star \neg A, \\ \text{then } A \wedge B \wedge C \in \text{Bel}\star A \wedge B \text{ and} \\ A \wedge B \wedge C \notin \text{Bel}\star \neg A \vee \neg B. \end{array}$$

Case 1:  $A \wedge \neg B \notin \text{Bel}\star A$ . Then, by  $A \wedge C \in \text{Bel}\star A$ ,  $\neg B \notin \text{Bel}\star A$ . Thus, by  $(\star 8)$ ,  $\text{Bel}\star A \subseteq \text{Bel}\star A \wedge B$ . Now suppose for reductio that  $A \wedge B$  is absurd. Then, by  $(r13)$ ,  $\neg(A \wedge B) \in \text{Bel}\star A$ . Since  $A \wedge C \in \text{Bel}\star A$ ,  $\neg B \in \text{Bel}\star A$ , and we have a contradiction. So  $A \wedge B \in \text{Bel}\star A \wedge B$ . But since also  $C \in \text{Bel}\star A \subseteq \text{Bel}\star A \wedge B$ , we get  $A \wedge B \wedge C \in \text{Bel}\star A \wedge B$ , as desired. For the second part, suppose for reductio that  $A \wedge B \wedge C \in \text{Bel}\star \neg A \vee \neg B$ . Then  $\neg A \vee \neg B$  is absurd, by  $(\star 0)$ . But then  $\neg A$  is also absurd, by  $(\star 6)$  and  $(\star 5a)$ . So  $\text{Bel}\star \neg A = \text{Bel}\star \neg A \vee \neg B = \text{Bel}$ , by  $(\star 2)$ , and  $A \wedge C \notin \text{Bel}\star \neg A$

implies  $A \wedge B \wedge C \notin Bel\star \neg A \vee \neg B$ , by  $(\star 1)$ . We have a contradiction which proves  $A \wedge B \wedge C \notin Bel\star \neg A \vee \neg B$ .

Case 2:  $A \wedge \neg B \in Bel\star \neg A$ . Then  $\neg A$  is absurd, by  $(\star 0)$ . So  $Bel\star \neg A = Bel$  and  $A \wedge \neg B \in Bel$ . But then  $Bel\star \neg A \vee \neg B = Bel$ , by  $(\star 3)$  and  $(\star 4)$ , and we get  $A \wedge B \wedge C \notin Bel\star \neg A \vee \neg B$ , by  $(\star 0)$ . ■

*Proof of Lemma 10.* In the following proofs, we assume the basic conditions  $(\geq 0)$ – $(\geq 6)$  as given.

$(\geq 7)$  implies  $(\geq 7c)$ . Let  $A \geq A \wedge B$  and  $A \wedge B \geq A \wedge B$ . We want to show that  $A \geq A \wedge C$ . From  $A \geq A \wedge B$  we get  $A \geq A$ , by (c1), and this taken together with  $A \wedge B \geq A \wedge B$  gives us  $A \geq A \wedge (B \supset C)$ , by  $(\geq 7)$ . Now we take the latter and  $A \geq A \wedge B$ , use (c13) to get  $A \geq (A \wedge B) \wedge (A \wedge (B \supset C))$ , which is equivalent to  $A \geq A \wedge B \wedge C$ . By (c11), we get  $A \geq A \wedge C$ .

$(\geq 7)$  implies  $(\geq 7')$ . Let  $A \geq A \wedge C$ ,  $B \geq B \wedge C$  and  $A \vee B \geq A \vee B$ . We want to show that  $A \vee B \geq (A \vee B) \wedge C$ . From  $A \geq A \wedge C$ , which is equivalent to  $(A \vee B) \wedge (A \vee \neg B) \geq (A \vee B) \wedge (A \vee \neg B) \wedge C$ , together with  $A \vee B \geq A \vee B$ , we derive  $A \vee B \geq (A \vee B) \wedge ((A \vee \neg B) \supset C)$ ,  $(\geq 7)$ . By an analogous argument, we get  $A \vee B \geq (A \vee B) \wedge ((\neg A \vee B) \supset C)$ . Using (c13) and  $(\geq 6)$ , we get  $A \vee B \geq (A \vee B) \wedge C$ .

$(\geq 7')$  implies  $(\geq 7)$ . Let  $A \wedge B \geq A \wedge B \wedge C$  and  $A \geq A$ . We want to show that  $A \geq A \wedge (B \supset C)$ .

Case 1: Suppose that  $A \wedge \neg B \geq A \wedge \neg B$ . This is equivalent to  $A \wedge \neg B \geq (A \wedge \neg B) \wedge (B \supset C)$ , by  $(\geq 6)$ . On the other hand,  $A \wedge B \geq A \wedge B \wedge C$  is equivalent to  $A \wedge B \geq (A \wedge B) \wedge (B \supset C)$ , also by  $(\geq 6)$ . From the last two facts and  $A \geq A$  we get, using  $(\geq 7')$ , that  $(A \wedge B) \vee (A \wedge \neg B) \geq ((A \wedge B) \vee (A \wedge \neg B)) \wedge (B \supset C)$  which reduces to  $A \geq A \wedge (B \supset C)$ .

Case 2: Suppose that  $A \wedge \neg B \not\geq A \wedge \neg B$ . From this and  $A \geq A$ , we conclude, using  $(\geq 5a)$ , that  $\neg A \vee B \not\geq \neg A \vee B$ . On the other hand,  $A \wedge B \geq A \wedge B \wedge C$  is equivalent to  $A \wedge (\neg A \vee B) \geq A \wedge (\neg A \vee B) \wedge (B \supset C)$ , by  $(\geq 6)$ . Now we can apply (c28) and get  $A \geq A \wedge (B \supset C)$ .

$(\geq 8)$  implies  $(\geq 8c)$ . Let  $A \geq A \wedge B$  and  $A \geq A \wedge C$ . We want to show that  $A \wedge B \geq A \wedge B$ . But  $A \geq A \wedge B$  entails  $A \not\geq A \wedge \neg B$ , by (c8), so we can apply  $(\geq 8)$  and get the desired claim immediately. ■

*Proof of Lemma 11.* (c24) Let  $A \geq C$ . By  $(\geq 2a)$ , we get  $A \geq A \wedge C$  and  $A \geq A \vee C$ , and by  $(\geq 2b)$ , we get  $A \geq A$ . From  $A \geq A \wedge C$ , we get  $A \wedge C \geq A \wedge C$ , by  $(\geq 8c)$ . From  $A \geq A \vee C$  and  $A \geq A$ , we get  $\neg A \not\geq \neg A \wedge C$ , by  $(\geq 2b)$ . By  $(\geq 5a)$  and  $(\geq 6)$ , we get  $\neg A \geq \neg A \wedge \neg A$ . Now we can use  $(\geq 8)$  and get that  $\neg A \wedge \neg C \geq \neg A \wedge \neg C \wedge \neg A$ , and thus  $\neg A \wedge \neg C \geq \neg A \wedge \neg C$ , by  $(\geq 6)$ . We can now use  $(\geq 5a)$  to get  $A \vee C \geq A \vee C$ . From this and  $A \wedge C \geq A \wedge C$ , we finally get  $A \geq A$ , by  $(\geq 5c)$ .

(c25) Suppose that  $A \succcurlyeq C$ . Then by ( $\succcurlyeq$ 2a),  $A \succcurlyeq A \wedge C$  and  $A \succcurlyeq A \vee C$ . So by (c24),  $A \wedge C \succcurlyeq A \wedge C$  and  $A \vee C \succcurlyeq A \vee C$ .

(c26) Suppose that  $A \succcurlyeq A$  and  $A \wedge B \not\succeq A \wedge B$ . From the latter, we get  $A \not\succeq A \wedge B$ , by ( $\succcurlyeq$ 8c). So  $\neg A \not\succeq \neg A \vee \neg B$ , by ( $\succcurlyeq$ 5a). From this and  $A \succcurlyeq A$ , we finally get  $A \succcurlyeq A \wedge \neg B$ , by ( $\succcurlyeq$ 2b).

(c27) Suppose that  $A \not\succeq A$  and  $A \wedge B \succcurlyeq A \wedge B$ . From the former, we get  $A \wedge B \not\succeq A$ , by (c24), and this means  $A \wedge B \not\succeq (A \wedge B) \vee A$ , by ( $\succcurlyeq$ 6). Since we also have  $\neg A \not\succeq \neg A$ , by ( $\succcurlyeq$ 5a), we get  $\neg(A \wedge B) \succcurlyeq \neg(A \wedge B) \wedge A$ , by ( $\succcurlyeq$ 2b), or equivalently,  $\neg(A \wedge B) \succcurlyeq \neg(A \wedge B) \wedge A \wedge \neg B$ , by ( $\succcurlyeq$ 6). So  $\neg(A \wedge B) \succcurlyeq \neg(A \wedge B) \wedge \neg B$ , by (c11), and  $\neg(A \wedge B) \succcurlyeq \neg B$ , by ( $\succcurlyeq$ 6).

(c28) Let  $B \not\succeq B$  and  $A \wedge B \succcurlyeq A \wedge B \wedge C$ . From the latter, we get  $A \wedge B \succcurlyeq A \wedge B$ , by (c1). From this and  $B \not\succeq B$ , we get two things: First,  $A \succcurlyeq A$ , by ( $\succcurlyeq$ 5b), and second,  $A \wedge \neg B \not\succeq A \wedge \neg B$ , by ( $\succcurlyeq$ 5a) and ( $\succcurlyeq$ 5c). From the last two, we get  $A \succcurlyeq A \wedge B$ , by (c26). This, taken together with  $A \wedge B \succcurlyeq A \wedge B \wedge C$ , gives us  $A \succcurlyeq A \wedge C$ , by ( $\succcurlyeq$ 7c).

(c29) Let  $B \not\succeq B$  and  $A \vee B \succcurlyeq (A \vee B) \wedge C$ . From the latter, we get  $A \vee B \succcurlyeq A \vee B$ , by (c1). From this and  $B \not\succeq B$ , we get two things: First,  $A \succcurlyeq A$ , by (c9), and second,  $A \vee B \succcurlyeq (A \vee B) \wedge \neg(A \vee B)$ , by (c26). By ( $\succcurlyeq$ 6), the latter reduces to  $A \vee B \succcurlyeq (A \vee B) \wedge A \wedge \neg B$ . Using (c11), we infer  $A \vee B \succcurlyeq (A \vee B) \wedge A$ , which reduces to  $A \vee B \succcurlyeq A$ . This, taken together with  $A \vee B \succcurlyeq (A \vee B) \wedge C$ , gives us  $A \succcurlyeq A \wedge C$ , by ( $\succcurlyeq$ 8c).

(c30) Suppose that  $A \not\succeq A$  and  $A \vee B \succcurlyeq A \vee B$ . From the former, we get  $(A \vee B) \wedge (A \wedge \neg B) \not\succeq (A \vee B) \wedge (A \wedge \neg B)$ , by ( $\succcurlyeq$ 6), and thus  $A \vee B \not\succeq (A \vee B) \wedge (A \wedge \neg B)$ , by ( $\succcurlyeq$ 8c). By, ( $\succcurlyeq$ 5a),  $\neg(A \vee B) \not\succeq \neg(A \vee B) \vee \neg(A \wedge \neg B)$ . From this and  $A \vee B \succcurlyeq A \vee B$ , ( $\succcurlyeq$ 2b) gives us  $A \vee B \succcurlyeq (A \vee B) \wedge \neg(A \vee \neg B)$  or, equivalently,  $A \vee B \succcurlyeq (A \vee B) \wedge \neg A \wedge B$ . So by (c11) and ( $\succcurlyeq$ 6),  $A \vee B \succcurlyeq B$ .

(c31) Suppose that  $A \succcurlyeq A$  and  $A \vee B \not\succeq A \vee B$ . From the latter, we get  $A \not\succeq A \vee B$ , by (c24), and this means  $A \not\succeq A \vee (A \vee B)$ , by ( $\succcurlyeq$ 6). Since we also have  $\neg A \not\succeq \neg A$ , by ( $\succcurlyeq$ 5a), we get  $\neg A \succcurlyeq \neg A \wedge (A \vee B)$ , by ( $\succcurlyeq$ 2b), or equivalently,  $\neg A \succcurlyeq \neg A \wedge B$ , by ( $\succcurlyeq$ 6). ■

*Proof of Lemma 12.* Suppose that the definitions (DefBel) and (Def $\star$ ) are used.

(i) *Closure of Bel under conjunction.* Suppose that  $A$  and  $B$  are beliefs, i.e., both  $X_{bel} \succcurlyeq X_{bel} \wedge A$  and  $X_{bel} \succcurlyeq X_{bel} \wedge B$ . Then, by (c13),  $X_{bel} \succcurlyeq (X_{bel} \wedge A) \wedge (X_{bel} \wedge B)$ , so also  $X_{bel} \succcurlyeq X_{bel} \wedge (A \wedge B)$ , by ( $\succcurlyeq$ 6), which means that  $A \wedge B$  is a belief.

*Closure of Bel under singleton entailment.* We show that if  $A \wedge B$  is a belief, so is  $A$ . Suppose that  $A \wedge B$  is a belief, i.e.,  $X_{bel} \succcurlyeq X_{bel} \wedge A \wedge B$ . Hence, by (c11),  $X_{bel} \succcurlyeq X_{bel} \wedge A$ , which means that  $A$  is a belief.

(ii) We need to show that  $X_{bel} \not\approx X_{bel} \wedge \neg A$  implies that  $A \gg A$  or  $X_{bel} \gg A \wedge X_{bel}$ . Suppose that  $X_{bel} \not\approx X_{bel} \wedge A$  and  $A \not\approx A$ . From the former, we get  $A \wedge X_{bel} \gg A \wedge X_{bel}$ , by Lemma 7, part (i.2) $\Rightarrow$ (i.1). Now we apply (c15) and get  $\neg A \wedge X_{bel} \not\approx \neg A \wedge X_{bel}$ . Finally, we use Lemma 7, part (i.2) $\Rightarrow$ (i.1), again and get  $X_{bel} \gg A \wedge X_{bel}$ , as desired. ■

*Proof of Observation 13.* (i) Let  $\star$  satisfy  $(\star 0)$ – $(\star 6)$ . We want to show that  $\star' = \star$ . We have

$$C \in Bel\star'A \quad \text{iff (by Def}\star\text{)}$$

$$A \gg A \wedge C \text{ or } (A \not\approx A \text{ and } X_{bel} \gg X_{bel} \wedge C) \quad \text{iff (by RRT)}$$

$$A \wedge C \in Bel\star A \text{ and } \\ A \wedge C \notin Bel\star\neg A \quad \text{or}$$

$$\left( A \notin Bel\star A \text{ or } A \in Bel\star\neg A \text{ and } \begin{array}{l} X_{bel} \wedge C \in Bel\star X_{bel} \text{ and} \\ X_{bel} \wedge C \notin Bel\star\neg X_{bel} \end{array} \right). \quad (\dagger)$$

$C \in Bel\star A$  implies  $C \in Bel\star'A$ . Let  $C \in Bel\star A$ .

Case 1. Suppose  $A \in Bel\star A$ . Then  $A \wedge C \in Bel\star A$ , by  $(\star 1)$ .

Case 1a: Suppose that  $A \notin Bel\star\neg A$ , then by  $(\star 1)$   $A \wedge C \notin Bel\star\neg A$ , and we have shown the first disjunct of  $(\dagger)$ .

Case 1b: Suppose that  $A \in Bel\star\neg A$ , then by  $(\star 0)$   $\neg A \notin Bel\star\neg A$ , so by  $(\star 2)$   $Bel\star\neg A = Bel$ . So  $A \in Bel$ , and  $Bel\star A = Bel$ , by  $(\star 3)$  and  $(\star 4)$ . So  $C \in Bel$ , and thus  $X_{bel} \wedge C \in Bel = Bel\star X_{bel}$ , by  $(\star 1)$  and (r1). On the other hand,  $X_{bel} \wedge C \notin Bel\star\neg X_{bel}$ , by  $(\star 0)$  and  $(\star 0b)$ . So we have shown the second disjunct of  $(\dagger)$ .

Case 2. Suppose  $A \notin Bel\star A$ . Then  $Bel\star A = Bel$ , by  $(\star 2)$ . So  $C \in Bel$ , and the rest is as in case 1b.

$C \in Bel\star'A$  implies  $C \in Bel\star A$ . Let  $C \in Bel\star'A$ , i.e., suppose that  $(\dagger)$  is true.

Case 1. Suppose that  $A \wedge C \in Bel\star A$ . Then  $C \in Bel\star A$ , by  $(\star 1)$ , and we are done.

Case 2. Suppose that  $A \wedge C \notin Bel\star A$ . Thus the second disjunct of  $(\dagger)$  must be true, so we have that  $X_{bel} \wedge C \in Bel\star X_{bel}$ . Hence  $C \in Bel$ , by  $(\star 1)$  and (r1). The second disjunct of  $(\dagger)$  says that either  $A \notin Bel\star A$  or  $A \in Bel\star\neg A$ . If the former, we have  $C \in Bel\star A$ , by  $(\star 2)$ . If the latter, then  $\neg A \notin Bel\star\neg A$ , by  $(\gg 0)$ , and  $A \in Bel$ , by (r5). But this means that  $Bel\star A = Bel$ , by (r8), and so we have  $C \in Bel\star A$  again.

(ii) Let  $\gg$  satisfy  $(\gg 0)$ – $(\gg 6)$ . We want to show that  $\gg' = \gg$ . We have

$$A \gg' C \quad \text{iff (by RRT)}$$

$C \in Bel\star A$  and  $C \notin Bel\star \neg A$  iff (by Def $\star$ )

$A \gg A \wedge C$  or  
 $(A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge C)$  and  $\neg A \not\gg \neg A \wedge C$  and  
 $(\neg A \gg \neg A \text{ or } X_{bel} \not\gg X_{bel} \wedge C)$ . ( $\ddagger$ )

$A \gg C$  implies  $A \gg' C$ . Let  $A \gg C$ . Then  $A \gg A \wedge C$ , by ( $\gg 2a$ ). Also by ( $\gg 2a$ )  $A \gg A \vee C$  and by ( $\gg 2b$ )  $\neg A \not\gg \neg A \wedge C$ . From (c1), we get  $A \gg A$ , so  $\neg A \gg \neg A$ , by ( $\gg 5a$ ).

$A \gg' C$  implies  $A \gg C$ . Let  $A \gg' C$ , i.e., suppose that ( $\ddagger$ ) is true.

Case 1. Let  $A \gg A \wedge C$ . So by (c1)  $A \gg A$  and by ( $\gg 5a$ )  $\neg A \gg \neg A$ . Using  $\neg A \not\gg \neg A \wedge C$  of ( $\ddagger$ ), we get that  $A \gg A \vee C$ , by ( $\gg 2b$ ). Since also  $A \gg A \wedge C$ , we get that  $A \gg C$ , by ( $\gg 2a$ ).

Case 2. Let  $A \not\gg A$  and  $X_{bel} \gg X_{bel} \wedge C$ . But by ( $\gg 5a$ ), this contradicts  $\neg A \gg \neg A$  or  $X_{bel} \not\gg X_{bel} \wedge C$ , so the case is impossible. ■

*Proof of Observation 14.* Let  $\gg$  satisfy ( $\gg 0$ ), ( $\gg 1$ ), ( $\gg 2a$ ), ( $\gg 2b$ ), ( $\gg 3$ ), ( $\gg 4$ ), ( $\gg 5a$ )–( $\gg 5c$ ) and ( $\gg 6$ ), and let  $\star$  be obtained from  $\gg$  by (Def $\star$ ).

( $\star 1a$ ) Closure under conjunction. We show that  $B \in Bel\star A$  and  $C \in Bel\star A$  imply  $B \wedge C \in Bel\star A$ . By (Def $\star$ ), this means

If  $A \gg A \wedge B$  or  
 $(A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge B)$  and  $A \gg A \wedge C$  or  
 $(A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge C)$ ,  
 then  $A \gg A \wedge B \wedge C$  or  
 $(A \not\gg A \text{ and } X_{bel} \gg (B \wedge C) \wedge X_{bel})$ .

Case 1:  $A \gg A \wedge B$  and  $A \gg A \wedge C$  imply  $A \gg A \wedge B \wedge C$ , by (c13).

Case 2:  $A \gg A \wedge B$  and  $(A \not\gg A \text{ and } X_{bel} \gg C \wedge X_{bel})$  is impossible, by (c1).

Case 3:  $A \gg A \wedge C$  and  $(A \not\gg A \text{ and } X_{bel} \gg C \wedge X_{bel})$  is similar to case 2.

Case 4:  $A \not\gg A$  and  $X_{bel} \gg X_{bel} \wedge B$  and  $X_{bel} \gg X_{bel} \wedge C$  imply  $X_{bel} \gg X_{bel} \wedge (B \wedge C)$ , by (c13) and ( $\gg 6$ ).

( $\star 1b$ ) Closure under singleton entailment. We show that  $B \wedge C \in Bel\star A$  implies  $C \in Bel\star A$ . By (Def $\star$ ), this means

If  $A \gg A \wedge B \wedge C$  or  
 $(A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge (B \wedge C))$  then  $A \gg A \wedge C$  or  
 $(A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge C)$

Case 1:  $A \gg A \wedge B \wedge C$  implies  $A \gg A \wedge C$ , by (c11).

Case 2:  $X_{bel} \gg X_{bel} \wedge (B \wedge C)$  implies  $X_{bel} \gg X_{bel} \wedge C$ , by (c11).



(★2) Restricted success. We show that either  $A \in Bel\star A$  or  $Bel\star A = Bel$ . By (Def★) and (DefBel), this means

$$\left( \begin{array}{l} A \gg A \wedge A \text{ or} \\ (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge A) \end{array} \right) \quad \text{or} \\ \left( \begin{array}{l} A \gg A \wedge C \text{ or} \\ (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge C) \end{array} \text{ iff } X_{bel} \gg X_{bel} \wedge C \right).$$

Suppose the the first disjunct is false, i.e.,  $A \not\gg A$  and  $X_{bel} \not\gg X_{bel} \wedge A$ . We need to show that the second disjunct is true. From  $A \not\gg A$ , we get  $A \not\gg A \wedge C$ , by (c1). But given this and  $A \not\gg A$ , the LHS of the second disjunct reduces to the RHS of the second disjunct.

(★3) We showed by proving (★1a), (★1b) and (r1) that  $Bel$  is logically closed. Thus it suffices to show that  $C \in Bel\star A$  implies  $A \supset C \in Bel$ . By (Def★) and (DefBel), this means

$$\text{If } \begin{array}{l} A \gg A \wedge C \text{ or} \\ (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge C) \end{array}, \text{ then } X_{bel} \gg X_{bel} \wedge (A \supset C).$$

Case 1:  $A \gg A \wedge C$  implies  $A \wedge \neg C \wedge X_{bel} \not\gg A \wedge \neg C \wedge X_{bel}$ , by ( $\gg$ 3). By Lemma 7, (i.2) $\Rightarrow$ (i1), we get  $X_{bel} \gg X_{bel} \wedge (A \supset C)$ .

Case 2:  $X_{bel} \gg X_{bel} \wedge C$  is equivalent to  $X_{bel} \gg X_{bel} \wedge C \wedge (A \supset C)$  which implies  $X_{bel} \gg X_{bel} \wedge (A \supset C)$ , by (c11).

(★4) Given that  $Bel$  is logically closed (which we have already shown), it suffices to show that  $\neg A \notin Bel$  and  $A \supset C \in Bel$  together imply  $C \in Bel\star A$ . By (Def★) and (DefBel), this means

$$\text{If } X_{bel} \not\gg X_{bel} \wedge \neg A \text{ and } X_{bel} \gg X_{bel} \wedge (A \supset C) \\ \text{then } \begin{array}{l} A \gg A \wedge C \text{ or} \\ (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge C) \end{array}.$$

From the antecedent we get, by Lemma 7 (i), that  $X_{bel} \wedge A \gg X_{bel} \wedge A$  and  $X_{bel} \wedge (A \wedge \neg C) \not\gg X_{bel} \wedge (A \wedge \neg C)$ . It follows with ( $\gg$ 4) that  $\neg A \not\gg \neg A \vee \neg(A \wedge \neg C)$ , or equivalently, by ( $\gg$ 6),  $\neg A \not\gg \neg A \vee C$ .

Case 1: Suppose that  $A \gg A$ . Then we get  $A \gg A \wedge C$ , by ( $\gg$ 2b).

Case 2: Suppose that  $A \not\gg A$ . From this and  $X_{bel} \not\gg X_{bel} \wedge \neg A$  we get, by Lemma 12 (ii), that  $X_{bel} \gg X_{bel} \wedge A$ . From this and  $X_{bel} \gg X_{bel} \wedge (A \supset C)$ , we get  $X_{bel} \gg X_{bel} \wedge A \wedge C$ , by (c13) and ( $\gg$ 6). So  $X_{bel} \gg X_{bel} \wedge C$ , by (c11).

(★5a) We show that  $(A \not\gg A \text{ and } X_{bel} \not\gg X_{bel} \wedge A)$  implies  $(A \wedge B \not\gg A \wedge B \text{ and } X_{bel} \not\gg X_{bel} \wedge (A \wedge B))$ . From  $A \not\gg A$  we get  $\neg A \not\gg \neg A$ , by ( $\gg$ 5a), and from  $X_{bel} \not\gg X_{bel} \wedge A$  we get  $X_{bel} \wedge \neg A \gg X_{bel} \wedge \neg A$ , by Lemma 7, (i.2) $\Rightarrow$ (i.1). Thus, by ( $\gg$ 5c),  $(\neg A \vee \neg B) \not\gg (\neg A \vee \neg B)$ . So by ( $\gg$ 5a) again,

$A \wedge B \not\approx A \wedge B$ . Moreover,  $X_{bel} \not\approx X_{bel} \wedge A$  implies  $X_{bel} \not\approx X_{bel} \wedge (A \wedge B)$ , by (c11).

(★5b) We show that  $A \vee B \approx A \vee B$  or  $X_{bel} \approx X_{bel} \wedge (A \vee B)$  implies  $A \approx A$  or  $X_{bel} \approx X_{bel} \wedge A$  or  $B \approx B$  or  $X_{bel} \approx X_{bel} \wedge B$ . First, let that  $A \vee B \approx A \vee B$ . Then the claim follows from (c9). Second, let  $X_{bel} \approx X_{bel} \wedge (A \vee B)$ . Suppose for reductio that the claim is false. Then it follows from  $A \not\approx A$  and  $X_{bel} \not\approx X_{bel} \wedge A$  that  $X_{bel} \approx X_{bel} \wedge \neg A$ , by Lemma 7, (iii.3) $\Rightarrow$ (iii.1). In a similar way, we get  $X_{bel} \approx X_{bel} \wedge \neg B$ . By two-fold application of (c13), then, we get  $X_{bel} \approx X_{bel} \wedge (A \vee B) \wedge \neg A \wedge \neg B$ . So by ( $\approx$ 6),  $X_{bel} \approx \perp$ . But this contradicts (c2).

(★6) follows from ( $\approx$ 6).

(★7c) We show that  $B \in Bel\star A$  and  $C \in Bel\star(A \wedge B)$  implies  $C \in Bel\star A$ . Suppose the hypothesis, i.e., that

$A \approx A \wedge B$  or  $A \wedge B \approx A \wedge B \wedge C$   
 $(A \not\approx A \text{ and } X_{bel} \approx X_{bel} \wedge B)$  and  $(A \wedge B \not\approx A \wedge B \text{ and } X_{bel} \approx X_{bel} \wedge C)$

We need to show that  $A \approx A \wedge C$  or  $(A \not\approx A \text{ and } X_{bel} \approx X_{bel} \wedge C)$ .

Case 1. Suppose the two upper rows are true. Then we get  $A \approx A \wedge C$ , by ( $\approx$ 7c) straight away.

Case 2. Suppose the upper left and the lower right rows are true. Since  $A \approx A \wedge B$  entails  $A \wedge B \approx A \wedge B$ , by ( $\approx$ 8c), this is impossible.

Case 3. Suppose the lower left and the upper right rows are true. From  $A \wedge B \approx A \wedge B \wedge C$ , we get  $A \wedge B \approx A \wedge B$ , by (c1). From this and  $A \not\approx A$ , it follows by (c17) that  $X_{bel} \approx X_{bel} \wedge A$ . With  $X_{bel} \approx X_{bel} \wedge B$ , we conclude that  $X_{bel} \approx X_{bel} \wedge A \wedge B$ , using (c13). So by (c20),  $X_{bel} \approx X_{bel} \wedge C$ , as desired.

Case 4. Suppose the two lower rows are true. Then we have  $A \not\approx A$  and  $X_{bel} \approx X_{bel} \wedge C$ .

(★7) We show that  $C \in Bel\star(A \wedge B)$  implies  $B \supset C \in Bel\star A$ . Suppose the former, i.e., that  $A \wedge B \approx A \wedge B \wedge C$  or  $(A \wedge B \not\approx A \wedge B \text{ and } X_{bel} \approx X_{bel} \wedge C)$ . We need to show that  $A \approx A \wedge (B \supset C)$  or  $(A \not\approx A \text{ and } X_{bel} \approx X_{bel} \wedge (B \supset C))$ .

Case 1. Suppose that  $A \wedge B \approx A \wedge B \wedge C$  and  $A \approx A$ . Then we get  $A \approx A \wedge (B \supset C)$ , by ( $\approx$ 7) straight away.

Case 2. Suppose that  $A \wedge B \approx A \wedge B \wedge C$  and  $A \not\approx A$ . We will show that  $X_{bel} \approx X_{bel} \wedge (B \supset C)$ . On the one hand,  $A \wedge B \approx A \wedge B \wedge C$  implies  $A \wedge B \approx A \wedge B$ , by (c1). From this and  $A \not\approx A$ , it follows by (c17) that  $X_{bel} \approx X_{bel} \wedge A$ . On the other hand, we can infer from  $A \wedge B \approx A \wedge B \wedge C$  that  $A \wedge B \wedge \neg C \wedge X_{bel} \not\approx A \wedge B \wedge \neg C \wedge X_{bel}$ , by ( $\approx$ 3). So by Lemma 7, (i.2) $\Rightarrow$ (i.1), we get  $X_{bel} \approx X_{bel} \wedge ((A \wedge B) \supset C)$ . So by (c13),  $X_{bel} \approx X_{bel} \wedge A \wedge ((A \wedge B) \supset C)$ . From ( $\approx$ 6) and (c11), we finally get  $X_{bel} \approx X_{bel} \wedge (B \supset C)$ , as desired.

Case 3. Suppose that  $A \wedge B \not\approx A \wedge B$  and  $X_{bel} \approx X_{bel} \wedge C$ . Case 3a. Suppose that in addition  $A \not\approx A$ . Now  $X_{bel} \approx X_{bel} \wedge C$  gives us  $X_{bel} \approx X_{bel} \wedge (B \supset C)$ , by ( $\approx 6$ ) and (c11), so we are done. Case 3b. Suppose that in addition  $A \approx A$ . From this and  $A \wedge B \not\approx A \wedge B$ , we get  $A \approx A \wedge \neg B$ , by (c26). With the help of ( $\approx 6$ ), this can be rewritten as  $A \approx A \wedge \neg B \wedge (B \supset C)$ . Thus, by (c11),  $A \approx A \wedge (B \supset C)$ , as desired.

( $\star 7'$ ). We show that  $C \in Bel\star A$  and  $C \in Bel\star B$  implies  $C \in Bel\star(A \vee B)$ . Suppose the hypothesis, i.e., that

$$\begin{array}{l} A \approx A \wedge C \text{ or} \\ (A \not\approx A \text{ and } X_{bel} \approx C \wedge X_{bel}) \end{array} \quad \text{and} \quad \begin{array}{l} B \approx B \wedge C \text{ or} \\ (B \not\approx B \text{ and } X_{bel} \approx X_{bel} \wedge C) \end{array}.$$

We need to show that  $A \vee B \approx (A \vee B) \wedge C$  or  $(A \vee B \not\approx A \vee B \text{ and } X_{bel} \approx X_{bel} \wedge C)$ .

Case 1. Suppose the two upper rows are true. Case 1a. Suppose further that  $A \vee B \approx A \vee B$ . Then we can use ( $\approx 7'$ ) and get  $A \vee B \approx (A \vee B) \wedge C$  right away. Case 1b. Suppose further that  $A \vee B \not\approx A \vee B$ . We will show that  $X_{bel} \approx X_{bel} \wedge C$ . On the one hand, we infer from  $A \approx A \wedge C$  that  $A \wedge \neg C \wedge X_{bel} \not\approx A \wedge \neg C \wedge X_{bel}$ , by ( $\approx 3$ ), and use Lemma 7, (i.2) $\Rightarrow$ (i.1), to get  $X_{bel} \approx X_{bel} \wedge (A \supset C)$ . Similarly, we get that  $X_{bel} \approx X_{bel} \wedge (B \supset C)$ . On the other hand, we infer from  $A \approx A \wedge C$  that  $A \approx A$ , by (c1), which together with  $A \vee B \not\approx A \vee B$  gives us  $X_{bel} \approx X_{bel} \wedge (A \vee B)$ , by (c17). Finally, we apply (c1) twice and get  $X_{bel} \approx X_{bel} \wedge (A \vee B) \wedge (A \supset C) \wedge (B \supset C)$ , which is equivalent to  $X_{bel} \approx X_{bel} \wedge (A \vee B) \wedge C$ , by ( $\approx 6$ ). By (c11), this implies that  $X_{bel} \approx C$ , as desired.

Case 2. Suppose the upper left and the lower right rows are true. Then we have  $A \approx A \wedge C$ ,  $B \not\approx B$  and  $X_{bel} \approx X_{bel} \wedge C$ . If  $A \vee B \not\approx A \vee B$ , then we are done. So suppose further that  $A \vee B \approx A \vee B$ . We show that  $A \vee B \approx (A \vee B) \wedge C$ . But from  $B \not\approx B$  and  $A \vee B \approx A \vee B$ , we get  $A \vee B \approx A$ , by (c30), i.e.,  $A \vee B \approx (A \vee B) \wedge (A \vee \neg B)$ . From this and  $A \approx A \wedge C$ , we get  $A \vee B \approx (A \vee B) \wedge C$ , by ( $\approx 6$ ) and ( $\approx 7c$ ).

Case 3. Suppose the lower left and the upper right rows are true. This is similar to Case 2.

Case 4. Suppose both of the lower rows are true. From  $A \not\approx A$  and  $B \not\approx B$ , we get  $A \vee B \not\approx A \vee B$ , by (c9). But since we also have  $X_{bel} \approx X_{bel} \wedge C$ , we are done.

( $\star 8c$ ). We show that  $B \in Bel\star A$  and  $C \in Bel\star A$  implies  $C \in Bel\star(A \wedge B)$ . Suppose the hypothesis, i.e., that

$$\begin{array}{l} A \approx A \wedge B \text{ or} \\ (A \not\approx A \text{ and } X_{bel} \approx X_{bel} \wedge B) \end{array} \quad \text{and} \quad \begin{array}{l} A \approx A \wedge C \text{ or} \\ (A \not\approx A \text{ and } X_{bel} \approx X_{bel} \wedge C) \end{array}.$$

We need to show that  $A \wedge B \gg A \wedge B \wedge C$  or  $(A \wedge B \not\gg A \wedge B$  and  $X_{bel} \gg X_{bel} \wedge C)$ .

Case 1. Suppose the two upper rows are true. Then we get  $A \wedge B \gg A \wedge B \wedge C$ , by ( $\gg 8c$ ), straight away.

Cases 2–3. Suppose one of the upper and one of the lower rows are true. But this is impossible, since each of  $A \gg A \wedge B$  and  $A \gg A \wedge C$  entails that not  $A \not\gg A$ , by (c1).

Case 4. Suppose the two lower rows are true, i.e.,  $A \not\gg A$ ,  $X_{bel} \gg X_{bel} \wedge B$  and  $X_{bel} \gg X_{bel} \wedge C$ . Case 1: Suppose that in addition  $A \wedge B \not\gg A \wedge B$ . Then we are ready immediately. Case 2: Suppose that in addition  $A \wedge B \gg A \wedge B$ . From this last supposition and  $A \not\gg A$  it follows by (c17) that  $X_{bel} \gg X_{bel} \wedge A$ . So by (c13)  $X_{bel} \gg X_{bel} \wedge A \wedge B$  and by (c8)  $X_{bel} \not\gg X_{bel} \wedge \neg(A \wedge B)$ . We can finally apply (c23) and get that  $A \wedge B \gg (A \wedge B) \wedge C$ .

( $\star 8$ ). We show that  $\neg B \notin Bel \star A$  and  $C \in Bel \star A$  together imply  $C \in Bel \star (A \wedge B)$ . Suppose the hypothesis, i.e., that

$$A \not\gg A \wedge \neg B \text{ and } (A \gg A \text{ or } X_{bel} \not\gg X_{bel} \wedge \neg B) \quad \text{and} \quad A \gg A \wedge C \text{ or } (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \wedge C).$$

We need to show that  $A \wedge B \gg A \wedge B \wedge C$  or  $(A \wedge B \not\gg A \wedge B$  and  $X_{bel} \gg X_{bel} \wedge C)$ .

Case 1. Suppose the upper right row is true. Then we use  $A \not\gg A \wedge \neg B$  and get  $A \wedge B \gg A \wedge B \wedge C$ , by ( $\gg 8$ ), straight away.

Case 2. Suppose the lower right row is true. Case 2a. Suppose further that  $A \wedge B \not\gg A \wedge B$ . Then the second disjunct of what we need to show is true. Case 2b. Suppose further that  $A \wedge B \gg A \wedge B$ . From this last supposition and  $A \not\gg A$  it follows by (c17) that  $X_{bel} \gg X_{bel} \wedge A$ . So by (c13) and (c11)  $X_{bel} \not\gg X_{bel} \wedge \neg(A \wedge B)$ , because otherwise  $X_{bel} \gg X_{bel} \wedge \neg B$  which is excluded by the lower left row. So we can finally apply (c23) and get that  $A \wedge B \gg (A \wedge B) \wedge C$ . ■

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