On the Logic of Theory Change: More Maps Between Different Kinds of Contraction Function

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1 INTRODUCTION

There are many ways to change a theory. The tasks of adding a sentence to a theory and of retracting a sentence from a theory are non-trivial because they are usually constrained by at least three requirements. The result of a revision or contraction of a theory should again be a theory, i.e., closed under logical consequence, it should be consistent whenever possible, and it should not change the original theory beyond necessity. In the course of the Alchourrón-Gärdenfors-Makinson research programme, at least three different methods for constructing contractions of theories have been proposed. Among these the "safe contraction functions" of Alchourrón and Makinson (1985, 1986) have played as it were the role of an outsider. Gärdenfors and Makinson (1988, p. 88) for instance state that 'another, *quite different*, way of doing this [contracting and revising theories] was described by Alchourrón and Makinson (1985).' (Italics mine.) The aim of the present paper is to show that this is a miscasting.

In any case, it seems that the intuitions behind *safe contractions* are fundamentally different from those behind its rivals, the *partial meet contractions* of Alchourrón, Gärdenfors and Makinson (1985) and the *epistemic entrenchment contractions* of Gärdenfors and Makinson (1988). Whereas the latter notions are tailored especially to handling *theories* (as opposed to sets of sentences which are not closed under a given consequence operation), safe contraction by its very idea focusses on *minimal sets of premises* sufficient to derive a certain sentence. Thus safe contraction has a certain "foundationalist" appearance, in contrast to the "coherentist" guise of its competitors. (The distinction between foundationalist and coherentist approaches to belief revision is due to Harman 1986 and elaborated in Gärdenfors 1990. Also see Doyle 1992.)

Safe contractions appear to possess some definite epistemological advantages over both partial meet and epistemic entrenchment contractions. Like epistemic entrenchment contractions, they are based on some kind of relation between *sentences* and not on a relation between *sets of sentences*, as it is the case with partial meet contractions. This constitutes an intuitive disadvantage of the latter. In addition, safe contractions rest on relatively weak requirements for the relation involved, which seem to give them the intuitive priority over epistemic entrenchment contractions.

This neat picture of a clear separability of different concepts of theory contraction, however, contrasts with some results on the intertranslatability of the different contraction functions. On the one hand, Alchourrón and Makinson (1986) revealed a far-reaching parallel between safe contractions and partial meet contractions of various strength for the finite case, while Rott (1991a) investigated close connections between partial meet contractions and epistemic entrenchment contractions. On the other hand, Alchourrón and Makinson (1985) showed that a distinguished kind of safe contraction conforms to the so-called *Gärdenfors postulates* for theory contraction, while Gärdenfors and Makinson (1988) proved that every such contraction is representable as an epistemic entrenchment contraction. (The situation is depicted in figure 1.) So it is clear that safe contractions can *somehow* be linked with epistemic entrenchment contractions. But for a serious epistemological comparison of safe contractions and epistemic entrenchment contractions it will not do just to graft one complicated construction found in the literature onto another. What we need is a natural, transparent and direct connection between safe contractions and epistemic entrenchment contractions, one of which we can gain an easy intuitive grasp. In particular, it would be nice to find an explicit map between the relations on which the respective contraction operations are built on. This is what I shall try to supply in this paper. The interest of the following constructions does not lie in the bare demonstration that safe contractions and epistemic entrenchment contractions *can* be related but in the fact that the relevant transitions are plain and that the relations involved can be linked directly to each other even in the infinite case.

We presuppose a language with the usual n-ary propositional operators $\bot$ and $\top$ ($n = 0$), $\neg$ (n=1), $\vee$, $\wedge$ and $\rightarrow$ (n=2), and a logic (consequence operation) Cn which includes classical propositional logic, is compact and satisfies the deduction theorem. By L, we denote the set of sentences of the language at hand, and we usually write $M \vdash \phi$ for $\phi \in Cn(M)$, for every set of sentences M and every sentence ϕ. By K, we denote an arbitrary theory, i.e., a subset of L that is closed under Cn.

2 HIERARCHIES AND SAFE CONTRACTION FUNCTIONS

Some ten years ago, Peter Gärdenfors put up a set of eight postulates for theory contraction that have become widely known as the *Gärdenfors postulates*. We repeat

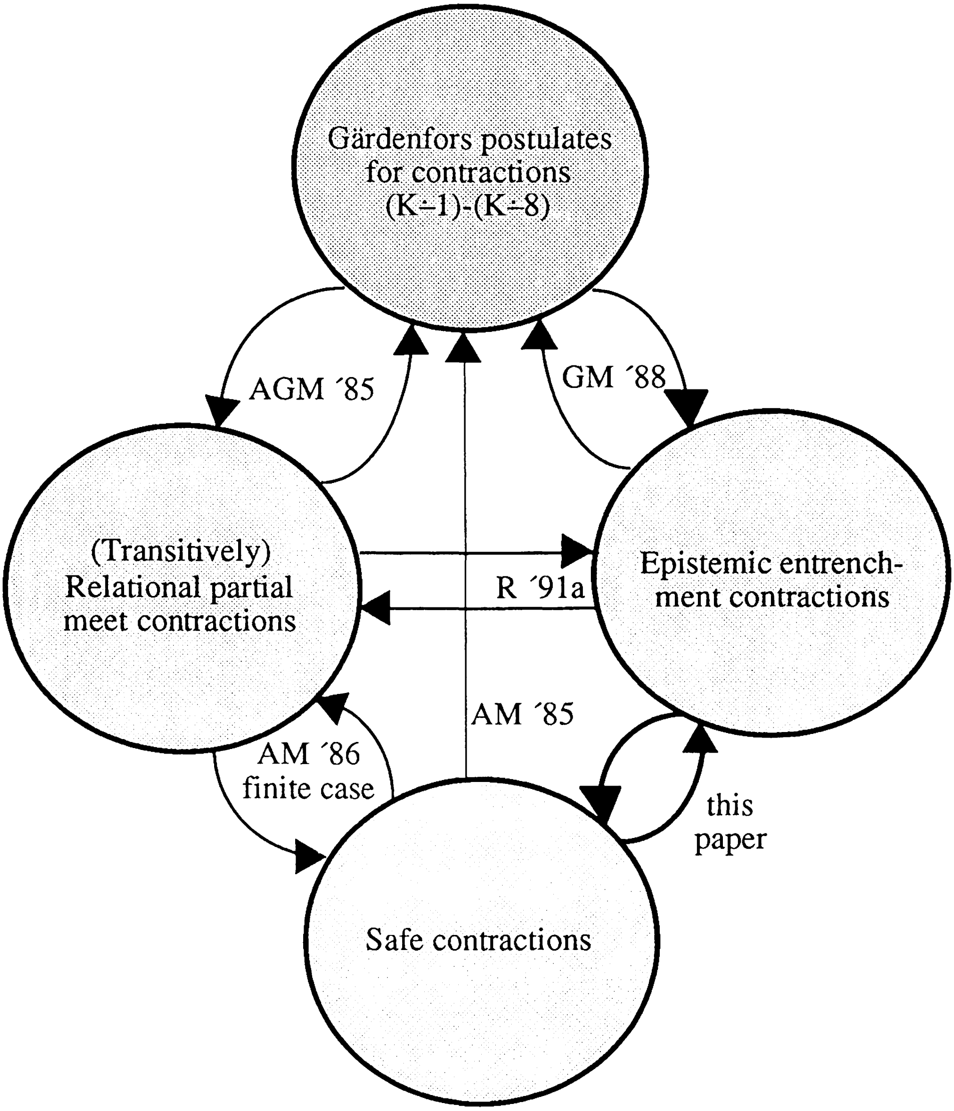

Figure 1: Three methods of explicit construction of Gärdenfors contractions

them in order to make this paper self-contained. $K \dot{-} \phi$ is to be read as 'the theory K contracted with respect to the sentence ϕ'.

(K$\dot{-}$1) $K \dot{-} \phi$ is a theory

(K$\dot{-}$2) $K \dot{-} \phi \subseteq K$

(K$\dot{-}$3) if $\phi \notin K$ then $K \dot{-} \phi = K$

(K$\dot{-}$4) if $\nvdash \phi$, then $\phi \notin K \dot{-} \phi$

(K$\dot{-}$5) $K \subseteq Cn((K \dot{-} \phi) \cup \{\phi\})$

(K$\dot{-}$6) if $\vdash \phi \leftrightarrow \psi$ then $K \dot{-} \phi = K \dot{-} \psi$

(K$\dot{-}$7) $K \dot{-} \phi \cap K \dot{-} \psi \subseteq K \dot{-} \phi \wedge \psi$

(K$\dot{-}$8) if $\phi \notin K \dot{-} \phi \wedge \psi$ then $K \dot{-} \phi \wedge \psi \subseteq K \dot{-} \phi$

A brief account of the motivation of these postulates is given in Gärdenfors's (1992) introduction to this volume. Contraction functions $\dot{-}$ satisfying (K$\dot{-}$1) – (K$\dot{-}$8) are called *Gärdenfors contractions* in what follows. In this paper we are mainly interested in contractions that meet exactly the Gärdenfors postulates.

We take over the appropriate set of conditions for the hierarchies used in safe contractions from Alchourrón and Makinson (1985):

Definition 1. Let K be a set of sentences and $<_H$ be a relation over K. We call $<_H$ a *hierarchy over* K iff $<_H$ is *acyclic* over K, in the sense that for every $\phi_1, \ldots, \phi_n$ in K,

(H1) if $\phi_1 <_H \ldots <_H \phi_n$ then not $\phi_n <_H \phi_1$

A hierarchy $<_H$ is said to *continue up* $\vdash$ over K, or *continue down* $\vdash$ over K, iff for every ϕ, ψ, χ in K,

(H2$^\uparrow$) if $\phi <_H \psi$ and $\psi \vdash \chi$ then $\phi <_H \chi$, or respectively,

(H2$^\downarrow$) if $\phi <_H \psi$ and $\chi \vdash \phi$ then $\chi <_H \psi$

A hierarchy $<_H$ is said to be *regular* over K iff it continues up and down $\vdash$ over K. Finally, a hierarchy $<_H$ is said to be *virtually connected* over K iff for every ϕ, ψ, χ in K,

(H3) if $\phi <_H \psi$ then $\phi <_H \chi$ or $\chi <_H \psi$.

In the presence of (H1) and (H3), conditions (H2$^\downarrow$) and (H2$^\uparrow$) are equivalent. Virtual connectivity of $<_H$, which is sometimes called *negative transitivity* (the name derives from a contrapositive reading of (H3)), guarantees that the symmetric complement $\sim_H$ of a hierarchy $<_H$, defined by $\phi \sim_H \psi$ iff neither $\phi <_H \psi$ nor $\psi <_H \phi$, is an equivalence relation. According to Alchourrón and Makinson (1985, p. 411), the relation $\phi <_H \psi$ is to reflect the idea that ϕ is less "secure or reliable or plausible", or more "exposed" or "vulnerable" than ψ. The authors go on and define:

Definition 2. If K is a theory and $<_H$ is a hierarchy over K then the associated *safe contraction function* $\dot{-} = C(<_H)$ is given by $K \dot{-} \phi = Cn(K/\phi)$, where K/ϕ is the set of sentences in K that are *$<_H$-safe with respect to* ϕ in the sense that they are not $<_H$-minimal in any $\subseteq$-minimal subset M of K such that $M \vdash \phi$ (i.e., if a sentence ψ which is $<_H$-safe with respect to ϕ is in such an M, then there is a χ in M such that $\chi <_H \psi$).

To facilitate the notation below, we generalize hierarchies to relations between sets of sentences.

Definition 3. Let K be a set of sentences and $<_H$ be a hierarchy over K. Then the associated *generalized hierarchy* $<_{GH} = GH(<_H)$ *over* 2^L is given by

$$M <_{GH} N \text{ iff } M \neq \emptyset \text{ and for every } \psi \text{ in } N \text{ there is a } \phi \text{ in } M \text{ such that } \phi <_H \psi.$$

That is, a set of sentences N is safer (in terms of $<_{GH}$) than a set of sentences M if and only if M is non-empty and for every element ψ in N there is an element ϕ in M which is less safe (in terms of $<_H$) than ψ. Obviously, $\phi <_H \psi$ iff $\{\phi\} <_{GH} \{\psi\}$, for every ϕ and ψ in K. The relation $<_{GH}$ is not particularly well-behaved. It is not even irreflexive in general, as we may have $M <_{GH} M$ for an infinite set M. We can take down, however, some nice properties which will be useful later.

Lemma 1. Let K be a set of sentences and $<_H$ be a hierarchy over K. Then the generalized hierarchy $<_{GH} = GH(<_H)$ over 2^L satisfies

(GH1) if $M <_{GH} N$, $M \subseteq M'$ and $N' \subseteq N$, then $M' <_{GH} N'$

(GH2) if $M <_{GH} N_i$ for every i in an index set I, then $M <_{GH} \bigcup\{N_i : i \in I\}$

(GH3) if $M_1, \ldots, M_n$ are finite and $M_1 <_{GH} M_2$, $M_2 <_{GH} M_3$, ..., $M_{n-1} <_{GH} M_n$, then $M_n \not<_{GH} M_1$

(GH4) if M and N are finite, $M \cup N \neq \emptyset$ and $M \cup N <_{GH} M$, then $M \cup N \not<_{GH} N$

(GH5) $N <_{GH} M$ for every $N \neq \emptyset$ iff $M = \emptyset$.

If $<_H$ is transitive then $<_{GH}$ satisfies

(GH6) if $M <_{GH} N$ and $N <_{GH} P$ then $M <_{GH} P$

(GH7) if M is finite and $M \cup N <_{GH} M$ then $N <_{GH} M$

If $<_H$ is virtually connected then $<_{GH}$ satisfies

(GH8) if $M <_{GH} N$ then $M <_{GH} P$ or $P <_{GH} N$.

Proof. (GH1) and (GH2) are immediate from the definition of $<_{GH}$. — For (GH3), suppose for reductio that $M_1, \ldots, M_n$ are all finite and that $M_1 <_{GH} M_2 <_{GH} \cdots <_{GH} M_n <_{GH} M_1$. Letting $+$ denote addition modulo n, we have by definition for any $i = 1, \ldots, n$: $M_i \neq \emptyset$, and for every $\phi \in M_{i+1}$ there is a $\psi \in M_i$ such that $\psi <_H \phi$. Now take any $\phi \in M_1$. Then we find a $\psi \in M_n$ such that $\psi <_H \phi$. Again, we find a $\chi \in M_{n-1}$ such that $\chi <_H \psi$, and a $\rho \in M_{n-2}$ such that $\rho <_H \chi$, and so on, and so on. Since the M_i's form a cycle under $<_{GH}$, we can continue this process infinitely many times. But $\bigcup\{M_i : i = 1, \ldots, n\}$ is finite. So at least one element of $\bigcup\{M_i\}$ must be mentioned twice in the infinite descending chain $\ldots <_H \rho <_H \chi <_H \psi <_H \phi$, which therefore must contain a cycle. This contradicts (H1). — For (GH4), let M and N be finite, $M \cup N \neq \emptyset$ and suppose that $M \cup N <_{GH} M$ and $M \cup N <_{GH} N$. By definition, this means that for every ϕ in M and every ψ in N there is a χ in $M \cup N$ such that $\chi <_H \phi$ or $\chi <_H \psi$ respectively. This again generates an infinite descending $<_H$-chain in the finite set $M \cup N$, which is ruled out by (H1). — The direction from right to left in (GH5) is obvious. From left to right, suppose that $M \neq \emptyset$. Take some finite non-empty subset $M_0 \subseteq M$. By (GH3), $M_0 \not<_{GH} M_0$. Hence, by (GH1), $M_0 \not<_{GH} M$, so there is a non-empty set N such that $N \not<_{GH} M$. — Now let $<_H$ be transitive. Then (GH6), the transitivity of $<_{GH}$, is immediate by the definition of $<_{GH}$. — For (GH7), let M be finite and $M \cup N <_{GH} M$. By definition, this means that for every ϕ in M there is a ψ in $M \cup N$ such that $\psi <_H \phi$. Now take any $\phi \in M$. We have to show that there is a ψ in N such that $\psi <_H \phi$. By hypothesis, we know that there is such a ψ in $M \cup N$. If ψ is in N, we are ready. If ψ is in M then we know that there is a $\chi \in M \cup N$ such that $\chi <_H \psi$. We are ready, if $\chi \in N$, because, by the transitivity of $<_H$, $\chi <_H \phi$, as desired. If $\chi \in M$, then there is a $\rho \in M \cup N$ such that $\rho <_H \chi$, and so on, and so on. This process must lead us to some sentence in N, because there can be no infinite descending chain $\ldots <_H \rho <_H \chi <_H \psi <_H \phi$ in M, since M is finite and $<_H$ is acyclic. Using the transitivity of $<_H$, we conclude that this sentence in N is indeed smaller under $<_H$ than ϕ, as desired. — Now let $<_H$ be virtually connected. Suppose for reductio that, first, $M <_{GH} N$, secondly $M \not<_{GH} P$, and thirdly $P \not<_{GH} N$. That is, first, $M \neq \emptyset$ and for every $\phi \in N$ there is a $\psi \in M$ such that $\psi <_H \phi$. By $M \neq \emptyset$, the second supposition yields that there is a $\chi_1 \in P$ such that $\rho_1 \not<_H \chi_1$ for every $\rho_1 \in M$. Hence $P \neq \emptyset$. So the third supposition yields that there is a $\chi_2 \in N$ such that $\rho_2 \not<_H \chi_2$ for every $\rho_2 \in P$. Combining these two facts with the help of (H3), we get that there is a $\chi_2 \in N$ such that $\rho_1 \not<_H \chi_2$ for every $\rho_1 \in M$. But this just means that $M \not<_{GH} N$, contradicting the first supposition. □

Corollary. (GH9) If M is finite, then $M \not<_{GH} M$

(GH10) if M and N are finite and $M <_{GH} N$, then $N \not<_{GH} M$

(GH11) if M is finite and $N \subseteq M$, then $N \not<_{GH} M$

(GH12) $M \neq \emptyset$ iff $M <_{GH} \emptyset$.

We see that $<_{GH}$ is well-behaved as long as we restrict our attention to finite sets of sentences. It should be noted that the proofs of lemma 1 and its corollary do not use the regularity conditions ($H2^{\uparrow}$) and ($H2^{\downarrow}$).

3 RELATIONS OF EPISTEMIC ENTRENCHMENT AND THEIR ASSOCIATED CONTRACTION FUNCTIONS

Relations of *epistemic entrenchment*, or simply *E-relations*, were introduced by Gärdenfors in 1984, but applied systematically only in Gärdenfors (1988) and Gärdenfors and Makinson (1988).

Definition 4. Let K be a theory and $\leq_E$ be a relation over L. We call $\leq_E$ an *E-relation with respect to* K iff for all sentences ϕ, ψ, χ,

(E1) if $\phi \leq_E \psi$ and $\psi \leq_E \chi$ then $\phi \leq_E \chi$

(E2) if $\phi \vdash \psi$ then $\phi \leq_E \psi$

(E3) $\phi \leq_E \phi \wedge \psi$ or $\psi \leq_E \phi \wedge \psi$

(E4) if $\perp \notin K$, then $\phi \leq_E \rho$ for every ρ iff $\phi \notin K$

(E5) if $\rho \leq_E \phi$ for every ρ then $\vdash \phi$.

A brief account of the motivation of these postulates is given in Gärdenfors's (1992) introduction to this volume. The correct interpretation of E-relations is very similar to that of hierarchies. I think that a good paraphrase of $\phi \leq_E \psi$ is 'Giving up ψ is not easier than giving up ϕ'.

E-relations are employed in the construction of contraction functions as follows (Gärdenfors and Makinson 1988):

Definition 5. If K is a theory and $\leq_E$ is an E-relation with respect to K then the associated *epistemic entrenchment contraction function* $\dot{-} = C(\leq_E)$ is given by $K \dot{-} \phi = K \cap \{\psi : \phi <_E \phi \vee \psi\}$ for sentences ϕ such that $\nvdash \phi$, and $K \dot{-} \phi = K$ for sentences ϕ such that $\vdash \phi$.

Notice that this definition makes reference to the asymmetric part $<_E = \leq_E - (\leq_E)^{-1}$ of an E-relation $\leq_E$. Conditions (E1) - (E3) imply the connectivity of $\leq_E$, i.e. that for every pair of sentences ϕ and ψ either $\phi \leq_E \psi$ or $\psi \leq_E \phi$. Hence we may identify $<_E$ with the converse complement of $\leq_E$: $\phi <_E \psi$ iff $\psi \not\leq_E \phi$. As we wish to work with strict relations later on, we restate, in a 1-1-fashion, the conditions (E1) - (E5) as conditions for the converse complement $<_E$ of $\leq_E$:

Definition 6. Let K be a theory and $<_E$ be a binary relation over L. We call $<_E$ a *(strict) E-relation with respect to* K iff for all sentences ϕ, ψ, χ,

(E1) if $\phi <_E \psi$ then $\phi <_E \chi$ or $\chi <_E \psi$

(E2) if $\psi \vdash \phi$ then $\phi \not<_E \psi$

(E3) if $\phi \wedge \psi <_E \phi$ then $\phi \wedge \psi \not<_E \psi$

(E4) if $\perp \notin K$, then $\rho <_E \phi$ for some ρ iff $\phi \in K$

(E5) if $\not\vdash \phi$ then $\phi <_E \rho$ for some ρ.

A suitable reading of $\phi <_E \psi$ is 'Giving up ψ is harder than giving up ϕ'. For the rest of this paper, we will always refer to the strict versions when we speak of "E-relations" and when we mention (E1) - (E5).

Two further conditions of considerable interest are

($E3^{\uparrow}$) if $\phi <_E \psi$ and $\phi <_E \chi$ then $\phi <_E \psi \wedge \chi$

($E3^{\downarrow}$) if $\phi \wedge \psi <_E \psi$ then $\phi <_E \psi$

Like (E3), ($E3^{\uparrow}$) and ($E3^{\downarrow}$) are conditions concerning conjunctions — conjunctions which appear in ($E3^{\uparrow}$) on the right-hand side of $<_E$ and in ($E3^{\downarrow}$) on the left-hand side of $<_E$. It is easy to show that (E1) and (E3) jointly imply ($E3^{\uparrow}$) and ($E3^{\downarrow}$). On the other hand, ($E3^{\uparrow}$) implies (E3), provided that $<_E$ is irreflexive (which follows from (E2)). And ($E3^{\downarrow}$) implies (E3), provided that $<_E$ is asymmetric and we may substitute logical equivalents on the left of $<_E$. ($E3^{\uparrow}$) and ($E3^{\downarrow}$) are useful if one wants to get along without virtual connectivity, (E1). A generalized concept of epistemic entrenchment can be axiomatized by (H1), ($H2^{\uparrow}$), ($H2^{\downarrow}$), ($E3^{\uparrow}$) and ($E3^{\downarrow}$) (see Rott 1991c, where in fact (H1) is replaced by the weaker axiom $\top \not< \top$).

4 CONNECTING SAFE AND EPISTEMIC ENTRENCHMENT CONTRACTIONS

Clearly, constructing contractions with the help of relations of epistemic entrenchment is easier than with the help of hierarchies (contrast definition 2 with definition 5).

On the other hand, we shall presently verify that the requirements for hierarchies are weaker than those for E-relations. It would therefore be desirable to combine the "cheap" method of definition 5 with "cheap" relations of definition 1. However, it turns out that contractions thus constructed fail to satisfy the most basic rationality postulates $(K\dot{-}1)$ and $(K\dot{-}4)$, even if $<_H$ is regular and virtually connected:

Example 1. Consider the propositional language L with two atoms p and q, let Cn be classical propositional logic and $K = Cn(\{p\})$. Let $<_H$ be the regular and virtually connected hierarchy over K which is characterized by $p <_H p \vee q \sim_H p \vee \neg q <_H \top$ (where $\phi \sim_H \psi$ iff neither $\phi <_H \psi$ nor $\psi <_H \phi$). See figure 2. As every element

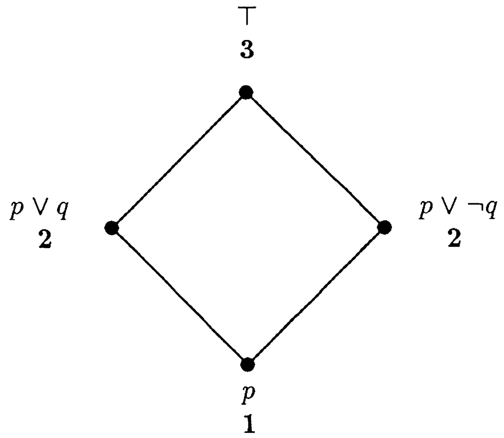

Figure 2: Example 1

of K is equivalent under Cn with one of the sentences mentioned, our information clearly determines a unique regular and virtually connected hierarchy $<_H$ over K. Applying definition 5, we get that $K\dot{-}p$ contains exactly those sentences in K which are equivalent to $p \vee q$, $p \vee \neg q$ or $\top$. But this means that $K\dot{-}p$ is not closed under Cn: it contains $p \vee q$ and $p \vee \neg q$ but does not contain p. And of course, closing under Cn would only make bad things worse, because we would then have p in $K\dot{-}p$ which is to say that the alleged contraction of K with respect to p does not remove p from K at all. (End of example.)

The rest of this paper is devoted to the demonstration that the concepts of safe contraction and epistemic entrenchment contraction are nevertheless equivalent in a very strong sense. Given an arbitrary hierarchy $<_H$ over K, the main problem is how to "conjure up" the conjunctiveness condition (E3) without disturbing anything else. The following definition will turn out appropriate.

Definition 7. Let K be a theory.

(i) If $<_E$ is an E-relation with respect to K, then the associated (regular and virtually connected) hierarchy $<_H = H(<_E)$ is just $<_E$ restricted to K.

(ii) If $<_H$ is a hierarchy over K and $<_{GH} = GH(<_H)$, then the associated E-relation $<_E = E(<_H)$ is given by

$$\phi <_E \psi \text{ iff } \begin{array}{l} \text{there is an } M \subseteq K \text{ such that } M \vdash \psi \text{ and} \\ \qquad \text{for every } N \subseteq K \text{ such that } N \vdash \phi,\ N <_{GH} M. \end{array}$$

Part (i) of definition 7 is trivial. Part (ii) says that ψ is epistemically more firmly entrenched than ϕ iff there is a "proof set" M for ψ in K which is safer (in terms of $<_{GH}$) than every "proof set" N for ϕ in K. This is, I think, a very perspicuous way of extracting a notion of epistemic entrenchment from a given hierarchy.

Lemma 2. If $<_E = E(<_H)$, then $\phi <_E \psi$ iff there is a finite $M \subseteq K$ such that $M \vdash \psi$ and for every finite $N \subseteq K$ such that $N \vdash \phi$, $N <_{GH} M$.

Proof. From left to right: Assume that there is an $M \subseteq K$ such that $M \vdash \psi$ and $N <_{GH} M$ for every $N \subseteq K$ such that $N \vdash \phi$. By compactness, there is some finite $M_0 \subseteq M$ with $M_0 \vdash \psi$, and by (GH1), $N <_{GH} M_0$ for every $N \subseteq K$ such that $N \vdash \phi$, so in particular for every finite such N.
From right to left: Assume that there is a finite $M_0 \subseteq K$ such that $M_0 \vdash \psi$ and $N_0 <_{GH} M_0$ for every finite $N_0 \subseteq K$ such that $N_0 \vdash \phi$. Let $N \subseteq K$ be such that $N \vdash \phi$. By compactness, there is some finite subset $N_0 \subseteq N$ such that $N_0 \vdash \phi$. So $N_0 <_{GH} M_0$, so by (GH1) $N <_{GH} M_0$. □

This lemma brings out the fact that we can restrict our attention to the finite subsets of K. The next lemma justifies part (i) of definition 7, and part (ii) for virtually connected hierarchies $<_H$. If $<_H$ is only transitive then definition 7(ii) leads to the generalized concept of epistemic entrenchment of Rott (1991c), and if $<_H$ is not even transitive then it involves a slight abuse of the term 'E-relation' which should not cause, however, any confusion.

Lemma 3. Let K be a theory, $<_H$ a hierarchy over K and $<_E$ an E-relation with respect to K. Then

(i) $H(<_E)$ is a regular and virtually connected hierarchy over K.

(ii) $E(<_H)$ is a regular hierarchy over K, and it satisfies (E2) – (E5) and (E3$^{\uparrow}$); if $<_H$ is transitive then $E(<_H)$ satisfies (E3$^{\downarrow}$); if $<_H$ is virtually connected then $E(<_H)$ satisfies (E1).

(iii) $E(H(<_E)) = <_E$.

Proof. (i) We write $<'_H$ for $H(<_E)$. For (H1), first observe that (E1) - (E3) give us asymmetry over K: If $\phi <'_H \psi$, then by (E1) either $\phi <'_H \phi \wedge \psi$ or $\phi \wedge \psi <'_H \psi$. But as the former is excluded by (E2), the latter must hold. Hence, by (E3), $\phi \wedge \psi \not<'_H \phi$. Moreover, by (E2), $\psi \not<'_H \phi \wedge \psi$, so by (E1) $\psi \not<'_H \phi$. Asymmetry and virtual connectivity, (E1), imply transitivity and acyclicity. — For ($H2^{\downarrow}$), let $\phi \vdash \psi$ and $\psi <'_H \chi$. From the former and (E2), we get $\psi \not<'_H \phi$, so by (E1) $\phi <'_H \chi$. — ($H2^{\uparrow}$) is proven similarly. — (H3) is (E1) restricted to K.

(ii) We write $<'_E$ for $E(<_H)$. For (H1), suppose that $\phi_1 <'_E \phi_2 <'_E \ldots <'_E \phi_n <'_E \phi_1$. Letting $+$ denote addition modulo n and applying lemma 2, we have for every $i = 1, \ldots, n$: there is a finite $M_{i+1} \subseteq K$ such that $M_{i+1} \vdash \phi_{i+1}$ and $N_i <_{GH} M_{i+1}$ for every finite $N_i \subseteq K$ such that $N_i \vdash \phi_i$. In particular, we have $M_1 <_{GH} M_2 <_{GH} \ldots <_{GH} M_n <_{GH} M_1$ for these finite M_i's, contradicting (GH3).

($H2^{\uparrow}$) and ($H2^{\downarrow}$) follow immediately from the definition of $<'_E$.

For (E2), let $\psi \vdash \phi$ and suppose that $\phi <'_E \psi$. That is, by lemma 2, there exists a finite $M \subseteq K$ such that $M \vdash \psi$ and $N <_{GH} M$ for every finite $N \subseteq K$ such that $N \vdash \phi$. $M \not<_{GH} M$, by (GH9). But $M \vdash \phi$, contradicting our supposition.

For (E3), suppose for reductio that both $\phi \wedge \psi <'_E \phi$ and $\phi \wedge \psi <'_E \psi$. That is, by lemma 2, there is a finite $M \subseteq K$ such that $M \vdash \phi$ and $P <_{GH} M$ for every finite $P \subseteq K$ such that $P \vdash \phi \wedge \psi$, and there is a finite $N \subseteq K$ such that $N \vdash \psi$ and $P <_{GH} N$ for every finite $P \subseteq K$ such that $P \vdash \phi \wedge \psi$. By (GH4), $P <_{GH} M \cup N$ for every finite $P \subseteq K$ such that $P \vdash \phi \wedge \psi$. But $M \cup N$ is finite and $M \cup N \vdash \phi \wedge \psi$, so $M \cup N <_{GH} M \cup N$, contradicting (GH9).

For (E4), let $K \neq L$. First assume that there is a ψ such that $\psi <'_E \phi$. We need to show that $\phi \in K$. But $\psi <'_E \phi$ means in particular that there is a $M \subseteq K$ such that $M \vdash \phi$, so $\phi \in K$, because K is a theory. For the converse, assume that $\phi \in K$. Take any $\psi \notin K$. Such a ψ exists, since $K \neq L$, and $K \not\vdash \psi$, since K is a theory. We have $K \vdash \phi$, and trivially $N <_{GH} K$ for every $N \subseteq K$ such that $N \vdash \psi$, because there is no such N. So $\psi <'_E \phi$ by definition.

For (E5), let $\not\vdash \phi$. Then every N such that $N \vdash \phi$ is non-empty and, by (GH12), $N <_{GH} \emptyset$. Taking $\psi = \top$ and $M = \emptyset$, and we find that $\phi <'_E \top$ by definition.

For ($E3^{\uparrow}$), let $\phi <'_E \psi$ and $\phi <'_E \chi$. This means, by definition, that there is an $M \subseteq K$ and an $N \subseteq K$ such that $M \vdash \psi$, $N \vdash \chi$, and $P <_{GH} M$ and $P <_{GH} N$ for every $P \subseteq K$ such that $P \vdash \phi$. Thus $M \cup N \vdash \psi \wedge \chi$ and, by (GH2), $P <_{GH} M \cup N$ for every $P \subseteq K$ such that $P \vdash \phi$, so $\phi <'_E \psi \wedge \chi$ by definition.

Now let $<_H$ be transitive. Suppose that $\phi \wedge \psi <'_E \psi$. For ($E3^{\downarrow}$), we have to show that $\phi <'_E \psi$. By lemma 2, our supposition means that there is a finite $M \subseteq K$ such that $M \vdash \psi$ and $N <_{GH} M$ for every finite $N \subseteq K$ such that $N \vdash \phi \wedge \psi$. Now take this M. By lemma 2 again, we are ready if we can show that $P <_{GH} M$ for every finite $P \subseteq K$ such that $P \vdash \phi$. Take any such P. Clearly, $M \cup P \vdash \phi \wedge \psi$. So by supposition $M \cup P <_{GH} M$. Since M is finite, we get by (GH7) that $P <_{GH} M$, as desired.

Now let $<_H$ be virtually connected. Suppose that $\phi \not<'_E \chi$ and $\chi \not<'_E \psi$. For (E1), we have to show that $\phi \not<'_E \psi$. Our suppositions mean that, first, for every $M_1 \subseteq K$ such that $M_1 \vdash \chi$ there is an $N_1 \subseteq K$ such that $N_1 \vdash \phi$ and $N_1 \not<_{GH} M_1$, and, secondly, that for every $M_2 \subseteq K$ such that $M_2 \vdash \psi$ there is an $N_2 \subseteq K$ such that $N_2 \vdash \chi$ and $N_2 \not<_{GH} M_2$. Applying (GH8), we get that for every $M_2 \subseteq K$ such that $M_2 \vdash \psi$ there is an $N_1 \subseteq K$ such that $N_1 \vdash \phi$ and $N_1 \not<_{GH} M_2$. That is, by definition, $\phi \not<'_E \psi$.

(iii) Writing $<'_E$ for $E(H(<_E))$, we have to show that $\phi <'_E \psi$ iff $\phi <_E \psi$. By the definition of an E-relation and by parts (i) and (ii) of this lemma, both $<_E$ and $<'_E$ satisfy (E4) and (E5), so it suffices to consider the principal case where ϕ and ψ are from $K - Cn(\emptyset)$.

First suppose that $\phi <'_E \psi$. By definition, then, there is an $M \subseteq K$ such that $M \vdash \psi$ and for every $N \subseteq K$ such that $N \vdash \phi$ it holds that for every $\chi \in M$ there is a $\rho \in N$ such that $\rho <_E \chi$. Since $\phi \in K$, we can in particular take $N = \{\phi\}$. So there is an $M \subseteq K$ such that $M \vdash \psi$ and $\phi <_E \chi$ for every $\chi \in M$. By compactness, M can be chosen finite, so there are $\chi_1, \ldots, \chi_n$ such that $\chi_1 \wedge \ldots \wedge \chi_n \vdash \psi$ and $\phi <_E \chi_i$ for every $i = 1, \ldots, n$. By repeated application of (E3$^\uparrow$), we get $\phi <_E \chi_1 \wedge \ldots \wedge \chi_n$, so $\phi <_E \psi$ by continuing up, as desired.

To show the converse, suppose that $\phi <_E \psi$. By continuing down, then, $\rho_1 \wedge \ldots \wedge \rho_m <_E \psi$ for all $\rho_1, \ldots, \rho_m$ such that $\rho_1 \wedge \ldots \wedge \rho_m \vdash \phi$. (Since $\not\vdash \phi$ by hypothesis, $m \geq 1$.) Hence $\rho_j <_E \psi$ for some $j = 1, \ldots, m$. For otherwise, if $\rho_j \not<_E \psi$ for every j, we would get, since either $\rho_1 \wedge \rho_2 \not<_E \rho_1$ or $\rho_1 \wedge \rho_2 \not<_E \rho_2$ by (E3), $\rho_1 \wedge \rho_2 \not<_E \psi$, by (E1), and by repeated application of (E3) and (E1) again, $\rho_1 \wedge \ldots \wedge \rho_m \not<_E \psi$, contradicting the above. By compactness, we then get that for every $N \subseteq K$ such that $N \vdash \phi$, it holds that $N \neq \emptyset$ and there is a $\rho \in N$ such that $\rho <_E \psi$. Now $\psi \in K$. So taking $M = \{\psi\}$, we find that there is an $M \subseteq K$ such that $M \vdash \psi$ and for every $N \subseteq K$ such that $N \vdash \phi$ it holds that $N \neq \emptyset$ and for every $\chi \in M$ there is a $\rho \in N$ such that $\rho <_E \chi$. That is, by definition, $\phi <'_E \psi$. □

Part (i) of lemma 3 demonstrates that Alchourrón and Makinson's concept of a hierarchy is a weakening of the concept of a (strict) E-relation, even if the hierarchy is supposed to be regular and virtually connected. A similar weakening for non-strict relations of epistemic entrenchment is proposed in Schlechta (1991). Alternative weakenings which are closer to the spirit of the original ideas of Gärdenfors are discussed in Lindström and Rabinowicz (1991) and Rott (1991c). The "cheap" method of contraction construction as described in definition 5 cannot sensibly be based on

Schlechta's preference relations (as example 1 makes clear), nor — as far as I can see — on Lindström and Rabinowicz's epistemic entrenchment orderings, but it can be based on the generalized epistemic entrenchment relations introduced by myself.

Of course, we do not in general have $<_H \subseteq H(E(<_H))$, because $H(E(<_H))$ satisfies the conjunctiveness condition (E3) (within K) which is not required for $<_H$. More interestingly, we do not even get $H(E(<_H)) \subseteq <_H$, i.e., $E(<_H)$ restricted to K is not just the result of cancelling certain pairs from $<_H$ until it satisfies (E3). All this is true even when $<_H$ is regular and virtually connected:

Example 2. Consider the propositional language L with two atoms p and q, let Cn be classical propositional logic and $K = Cn(\{p, q\})$. Let $<_H$ be the regular and virtually connected hierarchy over K which is characterized by $p \wedge q \sim_H p <_H q \sim_H p \leftrightarrow q \sim_H \neg p \vee q <_H p \vee q \sim_H p \vee \neg q <_H \top$. See figure 3. As every element of K

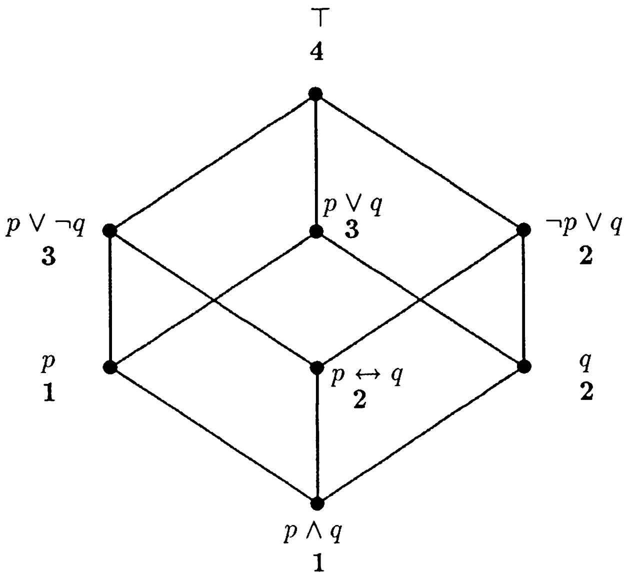

Figure 3: Example 2

is equivalent under Cn with one of the sentences mentioned, it is easy to check that our information in fact determines a unique regular and virtually connected hierarchy over K. We find that $M = \{p \vee q, p \vee \neg q\} \vdash p$ and that every N such that $N \vdash q$ includes a sentence ϕ with $\phi <_H p \vee q$ and $\phi <_H p \vee \neg q$, so that $N <_{GH} M$ for every proof set N of q in K. But this just means that $q <_E p$, although $p <_H q$ and thus $q \not<_H p$. Note that the only E-relation $<_E$ with $<_E \subseteq <_H$ is the trivial one which has $\phi \sim_E \psi$ for every ϕ and ψ in $K - Cn(\emptyset)$: by the definition of $<_H$, $\phi \not<_H p$, and, by (E3), either $p \sim_E (p \vee q) \wedge (p \vee \neg q) \not<_E p \vee q$ or $p \sim_E (p \vee q) \wedge (p \vee \neg q) \not<_E p \vee \neg q$, and again by the definition of $<_H$, $p \vee q \not<_H \psi$ and $p \vee \neg q \not<_H \psi$, so by repeated application of (E1), $\phi \not<_E \psi$. The proof of $\psi \not<_E \phi$ is analogous. This seems to indicate that there are interesting hierarchies $<_H$ which cannot be narrowed down to interesting E-relations by just cancelling pairs from $<_H$. (A method of this kind is employed in Schlechta (1991).) (End of example.)

What we do get, however, is that the safe contraction based on a hierarchy $<_H$ which continues down $\vdash$ over K and the epistemic entrenchment contraction based on an E-relation $<_E$ are equivalent, if $<_H$ and $<_E$ are related by either part (i) or part (ii) of definition 7. In fact, this is the main result of the present paper.

Theorem 4. Let K be a theory, $<_H$ a hierarchy which continues down $\vdash$ over K, and let $<_E$ an E-relation with respect to K. Then

(i) $\quad C(E(<_H)) = C(<_H)$.

(ii) $\quad C(H(<_E)) = C(<_E)$.

Proof. Let $<_{GH} = GH(<_H)$.

(i) If $\vdash \phi$, then both $C(E(<_H))$ and $C(<_H)$ yield $K \dot{-} \phi = K$. So let $\not\vdash \phi$. According to the contraction function $C(E(<_H))$, then, a sentence ψ is in $K \dot{-} \phi$ iff ψ is in K and there is an $M \subseteq K$ such that $M \vdash \phi \vee \psi$ and for every $N \subseteq K$ such that $N \vdash \phi$, $N <_{GH} M$. According to the contraction function $C(<_H)$, on the other hand, a sentence ψ from K is in $K \dot{-} \phi$ iff it is implied by a set M' of sentences which are $<_H$-safe with respect to ϕ. That is, iff there is an $M' \subseteq K$ such that $M' \vdash \psi$ and for any minimal set $N' \subseteq K$ such that $N' \vdash \phi$, it holds that $N' <_{GH} M' \cap N'$. What we have to show is that for every ψ in K the following two conditions are equivalent:

$(*)$ $\quad$ there is an $M \subseteq K$ such that $M \vdash \phi \vee \psi$ and for every $N \subseteq K$ such that $N \vdash \phi$, $N <_{GH} M$,

and

$(**)$ $\quad$ there is an $M' \subseteq K$ such that $M' \vdash \psi$ and for every $\subseteq$-minimal $N' \subseteq K$ such that $N' \vdash \phi$, $N' <_{GH} M' \cap N'$.

To show that $(*)$ implies $(**)$, take an M from $(*)$ and set $M' = M \cup \{\neg\phi \vee \psi\}$. Since K is a theory containing ψ, $M' \subseteq K$ and since $M \vdash \phi \vee \psi$, $M' \vdash \psi$. If $N' \subseteq K$ is a $\subseteq$-minimal set such that $N' \vdash \phi$, N' does not contain $\neg\phi \vee \psi$. For otherwise the deduction theorem for Cn tells us that $N' - \{\neg\phi \vee \psi\} \vdash (\neg\phi \vee \psi) \rightarrow \phi$, thus $N' - \{\neg\phi \vee \psi\} \vdash \phi$, contradicting the minimality of N'. Hence $N' \cap M' = N' \cap M$. It remains to show that $N' <_{GH} N' \cap M$. But $(*)$ tells us that $N' <_{GH} M$, so by (GH1) $N' <_{GH} N' \cap M$, as desired.

Conversely, to show that $(**)$ implies $(*)$, we take an M' from $(**)$. Obviously,

$M' \vdash \phi \vee \psi$. Take a subset M of M' such that M minimally implies $\phi \vee \psi$. Since Cn is assumed to be compact, M is finite. If $M = \emptyset$, then $(*)$ follows from the hypothesis $\nvdash \phi$ and (GH12). So let M be non-empty. Now take an $N \subseteq K$ such that $N \vdash \phi$. For $(*)$, we have to show that for every $\chi \in M$ there is a $\rho \in N$ such that $\rho <_H \chi$. Suppose for reductio that there is some $\chi \in M$ with $\rho \not<_H \chi$ for every $\rho \in N$.

We first note that $\neg\chi \nvdash \phi \vee \psi$. For $\neg\chi \vdash \phi \vee \psi$, taken together with $M \vdash \phi \vee \psi$, i.e. $(M - \{\chi\}) \cup \{\chi\} \vdash \phi \vee \psi$, would give us $M - \{\chi\} \vdash \phi \vee \psi$, by our assumption that Cn satisfies disjunction in the antecedent. But the latter condition contradicts the hypothesis that M minimally implies $\phi \vee \psi$. (The point of this paragraph is proven as lemma 3.1 in Alchourrón and Makinson 1985.)

Now consider the set $\neg\chi \vee N =_{\mathrm{df}} \{\neg\chi \vee \rho : \rho \in N\}$. Since $N \subseteq K$ and K is a theory, $\neg\chi \vee N \subseteq K$. Since $N \vdash \phi$, clearly $\{\chi\} \cup (\neg\chi \vee N) \vdash \phi$. Now take some $N' \subseteq \{\chi\} \cup (\neg\chi \vee N) \subseteq K$ such that N' minimally implies ϕ. We find that $\chi \in N'$. For suppose for reductio that $\chi \notin N'$. Then $N' \subseteq \neg\chi \vee N$, and as $N' \vdash \phi$, we get that $\neg\chi \vee N \vdash \phi$. Thus $\neg\chi \vdash \phi$, contradicting the above observation that $\neg\chi \nvdash \phi \vee \psi$.

Considering the fact that N' minimally implies ϕ and that $\chi \in M' \cap N'$, we can apply $(**)$ in order to see that there is a $\sigma \in N'$ such that $\sigma <_H \chi$.

By the irreflexivity of $<_H$, which follows from the acyclicity condition (H1), $\sigma \neq \chi$. So σ is of the form $\neg\chi \vee \rho$ for some $\rho \in N$. Now $\rho \vdash \sigma$, so, by $(\mathrm{H2}^{\downarrow})$, $\rho <_H \chi$, contradicting our supposition.

(ii) By lemma 3(i), $H(<_E)$ is a regular hierarchy over K, hence, by part (i) of this theorem, $C(H(<_E)) = C(E(H(<_E)))$. But as $<_E$ is an E-relation with respect to K, lemma 3(iii) gives us that $E(H(<_E)) = <_E$, so $C(E(H(<_E))) = C(<_E)$. In sum, we get $C(H(<_E)) = C(<_E)$. □

It is remarkable that part (i) of the theorem uses only the fact that $<_H$ is irreflexive and continues down $\vdash$. Granted that safe contractions make sense in this case, it shows that "the cheap method" of contracting theories according to definition 5 is applicable even if $<_E =_{\mathrm{df}} E(<_H)$ is no full E-relation. For in general, when $<_H$ is not virtually connected, $E(<_H)$ does not satisfy (E1). It is shown in Rott (1991c) that relations of "generalized epistemic entrenchment" are fit to be used for contractions constructed according to definition 5. However, $E(<_H)$ need not even satisfy the milder requirements of generalized epistemic entrenchment, unless $<_H$ is transitive.

As a consequence of theorem 4, we get the following *representation theorem* that generalizes a result implicit in Alchourrón and Makinson (1986) to the infinite case:

Corollary. Every contraction function $\dot{-}$ over K satisfying the Gärdenfors postulates $(K\dot{-}1)$ – $(K\dot{-}8)$ can be represented as a safe contraction function generated by a regular and virtually connected hierarchy $<_H$ over K, i.e., there is such a $<_H$ with

$\dot{-} = C(<_H)$.

Proof. Let $\dot{-}$ over K satisfy (K$\dot{-}$1) - (K$\dot{-}$8), and define $<_E$ by putting

$$\phi <_E \psi \text{ iff } \psi \in K\dot{-}\phi \wedge \psi \text{ and } \nvdash \phi \wedge \psi.$$

It is shown in Gärdenfors and Makinson (1988, theorem 5 and corollary 6) that $<_E$ is an E-relation with respect to K and that $C(<_E) = \dot{-}$. Hence, by theorem 4(ii), $C(H(<_E)) = \dot{-}$, so the restriction $H(<_E)$ of $<_E$ to K is a suitable hierarchy. By Lemma 3(i), $H(<_E)$ is regular and virtually connected. □

As another corollary, we get that safe contractions based on regular and transitive hierarchies satisfy a weaker form of (K$\dot{-}$8) which plays a central role in Rott (1991c).

Corollary. If $<_H$ is a regular and transitive hierarchy over K then $C(<_H)$ satisfies

(K$\dot{-}$8c) if $\psi \in K\dot{-}\phi \wedge \psi$ then $K\dot{-}\phi \wedge \psi \subseteq K\dot{-}\phi$.

Proof. Immediate from theorem 4(i) and lemma 3(ii) above and theorem 2(i) of Rott (1991c). □

It is interesting to have a closer look at the finite case. A set of sentences K is *finite* (modulo Cn) iff the consequence relation Cn partitions it into finitely many equivalence classes. If K is a finite theory then it can be viewed as a finite Boolean algebra. Of special interest are the *top elements* or *co-atoms* of K, that is, the elements ϕ of K such that $\nvdash \phi$ and for every $\psi \in Cn(\phi) \cap K$ either $\vdash \psi$ or $\vdash \phi \leftrightarrow \psi$. Let T_K be the set of all top elements of K and $T_K(\phi)$ be the set of all top elements of K that "cover" ϕ, i.e., $T_K(\phi) = T_K \cap Cn(\phi)$. When K is a finite theory, we can simplify the principal case in the construction of an E-relation out of a hierarchy (definition 7(ii)):

Lemma 5. Let K be a finite theory and $<_H$ be a regular hierarchy over K. Then the following two conditions are equivalent for any ϕ and ψ in $K - Cn(\emptyset)$:

(i) there is an $M \subseteq K$ such that $M \vdash \psi$ and for every $N \subseteq K$ such that $N \vdash \phi$, $N <_{GH} M$

(ii) $T_K(\phi) <_{GH} T_K(\psi)$.

Proof. Clearly, $T_K(\phi) \subseteq K$ and, by Boolean algebra, $T_K(\phi) \vdash \phi$, and likewise for ψ. — To show that (i) implies (ii), we note that (i) implies that there is an $M \subseteq K$ such that $M \vdash \psi$ and $T_K(\phi) <_{GH} M$. Now take an arbitrary $\chi \in T_K(\psi)$. We have to show that there is a $\rho \in T_K(\phi)$ such that $\rho <_H \chi$. Since $M \vdash \psi$ and χ is a top element of K implied by ψ, there must be a $\sigma \in M$ such that $\sigma \vdash \chi$. But by $T_K(\phi) <_{GH} M$, there is a $\rho \in T_K(\phi)$ such that $\rho <_H \sigma$, so by (H2$^\uparrow$), $\rho <_H \chi$, as desired. — To show that (ii) implies (i), we show that for every $N \subseteq K$ such that $N \vdash \phi$ it holds that

$N <_{GH} T_K(\psi)$. Let $N \subseteq K$ be such that $N \vdash \phi$ and let $\chi \in T_K(\psi)$. We have to show that there is a $\rho \in N$ such that $\rho <_H \chi$. By $T_K(\phi) <_{GH} T_K(\psi)$, there is a $\sigma \in T_K(\phi)$ such that $\sigma <_H \chi$. Since $N \vdash \phi$ and σ is a top element of K implied by ϕ, there is a $\rho \in N$ such that $\rho \vdash \sigma$. So by (H2$^{\downarrow}$), $\rho <_H \chi$, as desired. □

The results connecting hierarchies with E-relations enable us to derive some properties of safe contractions quite easily. Here is an example.

Corollary. Let K be a finite theory and $<_H$ and $<'_H$ be two regular hierarchies over K. Then $C(<_H) = C(<'_H)$ if and only if $<_H$ and $<'_H$ agree within T_K.

Proof. If $<_H$ and $<'_H$ agree within T_K then $GH(<_H)$ and $GH(<'_H)$ agree within the power set of T_K. But then, by definition 7(ii) and lemma 5, $E(<_H) = E(<'_H)$. Therefore, by theorem 4(i), $C(<_H) = C(E(<_H)) = C(E(<'_H)) = C(<'_H)$. — Conversely, suppose that $C(<_H) = C(<'_H)$. Hence, by theorem 4(i), $C(E(<_H)) = C(E(<'_H))$, and also, by corollary 6 of Gärdenfors and Makinson (1988), $E(<_H) = E(C(E(<_H))) = E(C(E(<'_H))) = E(<'_H)$. In particular, $E(<_H)$ and $E(<'_H)$ agree within T_K. But as for any ϕ and ψ in T_K, $T_K(\phi) = \{\phi\}$ and $T_K(\psi) = \{\psi\}$, we can see from definition 7(ii) and lemma 5 that within T_K, $<_H$ agrees with $E(<_H)$ and $<'_H$ agrees with $E(<'_H)$. Hence $<_H$ and $<'_H$ agree within T_K. □

As David Makinson (personal communication) has pointed out, this corollary is also an immediate consequence of a lemma of Alchourrón and Makinson (1986, p. 192).

5 CONCLUSION

The foregoing arguments show, I believe, that safe contractions and epistemic entrenchment contractions are equivalent in a very strict sense. We can transform every virtually connected hierarchy $<_H$ into an E-relation by a very perspicuous construction, and conversely we already have a (regular and virtually connected) hierarchy if we have an E-relation $<_E$. The contraction functions that ensue are identical, if $<_H$ continues down $\vdash$ over K. It may also be worthwhile to remark that the interpretations of hierarchies and E-relations are similar. The notion of 'epistemic entrenchment', formerly called 'epistemic importance', must not be taken for a relation reflecting some kind of 'informational content' or 'inferential fruitfulness'.

Given a theory K and a contraction function $\dot{-}$ over K which satisfies the Gärdenfors postulates, there are in general many (regular and virtually connected) hierarchies $<_H$ over K that lead to this contraction function in the sense that $\dot{-} = C(<_H)$. But there is only one E-relation $<_E$ suitable for $\dot{-}$, and $<_E$ can be read off directly from the contraction behaviour: $\phi <_E \psi$ if and only if $\psi \in K \dot{-} \phi \wedge \psi$ and $\phi \notin K \dot{-} \phi \wedge \psi$ (cf. Gärdenfors and Makinson 1988, Rott 1991c). It is plausible to consider $H(<_E)$

as the canonical hierarchy for a given Gärdenfors contraction function $\dot{-}$, namely the hierarchy $<_H$ for which either $\phi \wedge \psi \not<_H \phi$ or $\phi \wedge \psi \not<_H \psi$, for all sentences ϕ and ψ. There is no reason to object to postulating epistemic entrenchment relations that satisfy the conjunctiveness condition (E3), because all the regular and virtually connected hierarchies $<_H$ for which $\dot{-} = C(<_H)$ give rise to one and the same E-relation $<_E = E(<_H) = E(C(<_H))$.

In the finite case, the only information needed is, both for safe and epistemic entrenchment contractions, the ordering over the set T_K of top elements of K. And again, given such an ordering, we can in general find many hierarchies but only one single E-relation conforming to this ordering. That E-relation is, so to speak, the most well-behaved hierarchy one can think of, and as such it might be called the canonical hierarchy corresponding to a prefixed ordering of T_K.

It should be noted that the gain from the results of this paper is philosophical rather than computational. Philosophically, people applying epistemic entrenchment are freed from the need of justifying (E3) (and (E4) and (E5)). But in order to perform theory contractions, the "cheap" method of definition 5 does not save us any work if all we are given is a hierarchy. This is because establishing an appropriate E-relation via definition 7(ii) requires at least as much computational effort as the contraction construction used in definition 2. In the case where K is finite and we have information about the relationships in T_K, both methods, safe contraction and epistemic entrenchment contraction, are equally simple.

Summing up: If we are interested in Gärdenfors contraction functions, there is no reason why contraction functions based on relations of epistemic entrenchment should be epistemologically more questionable than safe contraction functions based on hierarchies. Up to now, it is not entirely clear how to weaken or strengthen (E1) – (E5) in such a way that certain interesting properties of contraction functions get lost or added. First steps in this direction are taken in Rott (1991c). Furthermore, it would be desirable to describe the application of E-relations to sets of sentences that are not closed under logical consequence. About that nothing is known as yet. But *if* we are concerned with theory change along the Gärdenfors lines, then postulating the existence of epistemic entrenchment relations appears to be just as safe as postulating hierarchies. Everybody who is willing to accept safe contractions should be willing to accept epistemic entrenchment contractions as well. In my opinion, the maps proposed in this paper remove some of the more fundamental reservations against any undertaking which is, like e.g. Rott (1991b), based on relations of epistemic entrenchment.

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