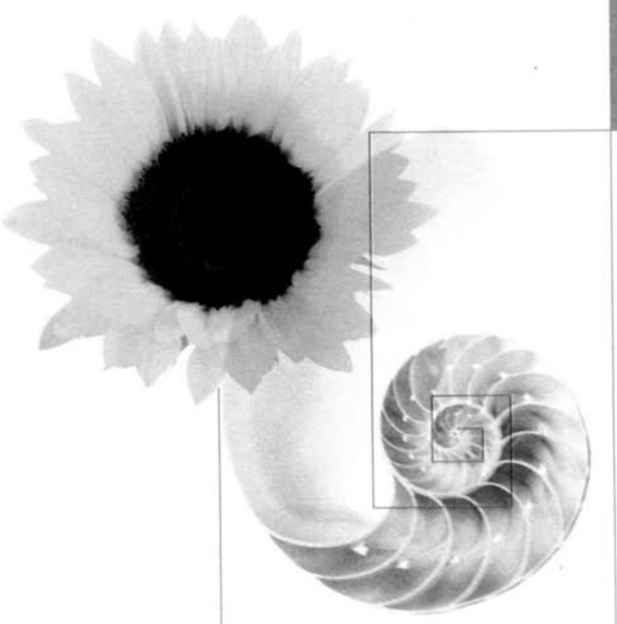
MATHEMATICAL

ART

IN MANILA



My warmest welcome to the exhibit "Mathematical Art in Manila" and our heartfelt gratitude to the Japan Foundation and Tokai University for making this wonderful exhibit possible.

The relationship between Mathematics and Art has always been very deep, from the symmetries of the ancient Greeks to Renaissance paintings and to the music of Bach.

I hope that this exhibit will bring to mathematicians a deeper sense of the relationship of our discipline with art and to artists some insight into the aesthetics and beauty in mathematics.

Once more, my congratulations and thanks to all who have made this possible, particularly to Dr. Jin Akiyama, to Dr. Mari-Jo P. Ruiz, to the Japan Foundation and to Tokai University.

Message

BIENVENIDO F. NEBRES, S.J.
President
Ateneo de Manila University

For some time now, I have expressed concern for a growing problem in Japan. Significant numbers of our students dislike science and mathematics. There are, of course, varied causes, but as a physicist, I believe that the problem can be alleviated if we could only engage students more in the creative process in our classrooms. If they can create something that works, they will experience the thrill of creating and discovery and they will fully understand that which they have created.

This exhibit makes me very proud because most of the objects here were created by the students and faculty of Tokai University. I would be very happy if some of the models here would find their way into mathematics classrooms in the Philippines.

Message

SONOSUKE KATORI
Director
Research Institute of Educational Development
Tokai University Educational System

The mathematician G.H. Hardy in an often quoted statement said "A mathematician, like a painter or a poet, is a maker of patterns....The mathematician's patterns, like the painter's or poet's, must be *beautiful*; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics."

This exhibit celebrates the beauty and the power of mathematics through artistic objects that demonstrate mathematical concepts and formulas, and through models and devices that offer the audience opportunities to experiment, to discover, to gain fresh insights into mathematical truths.

Most of the objects were made by the students, artists, engineers and technicians of Tokai University. The exhibit was first shown in Asahikawa, then in Seoul, and proceeds to Beijing from Manila. Funding to bring the exhibit to Manila was generously provided by The Japan Foundation. Other logistical support was provided by Ateneo de Manila University and Tokai University.

We thank The Japan Foundation, our universities, and our colleagues, who have worked long hours for this exhibit. We hope the exhibit will bring the audience a sense of wonder and a new appreciation for mathematics.

for the Exhibit Organizing Committee

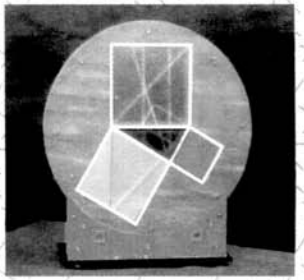
Message

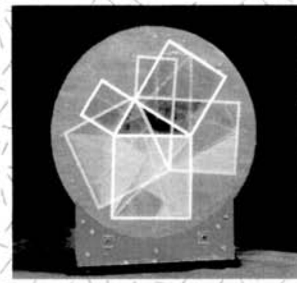
JIN AKIYAMA
Deputy Director
Research Institute of Educational Development
Tokai University

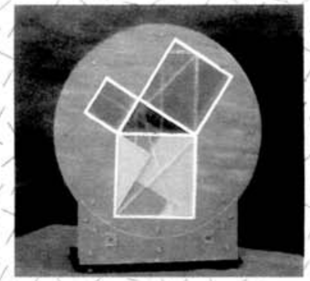
MARI-JO P. RUIZ
Dean
School of Arts and Sciences
Ateneo de Manila University

MATHEMATICAL ART IN MANILA

Text by Mari-Jo P. Ruiz and
the Brochure Committee of RIED
Tokai University



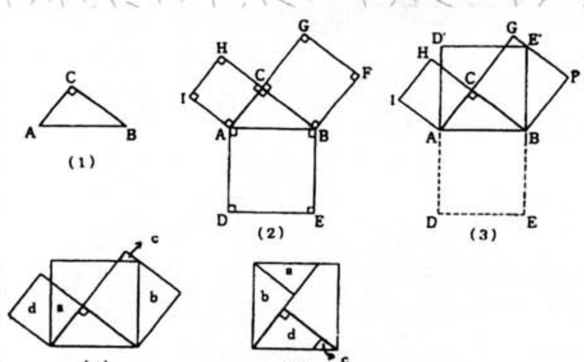




1-1. Rotary Model for the Pythagorean Theorem

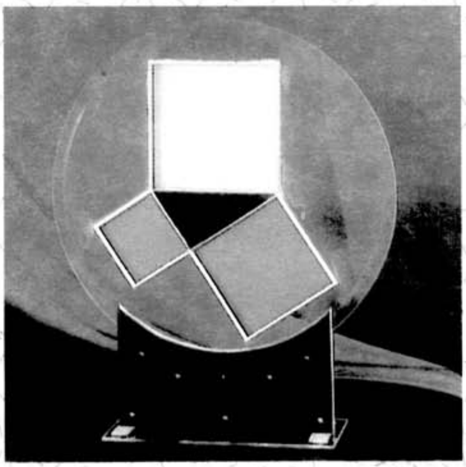
This model provides a visual illustration of the Pythagorean Theorem [AHKMNW]. It has three thin containers with square cross sections mounted on a circular base. The dimensions of the squares are determined by the Pythagorean Theorem. Initially, each of the smaller containers is filled with pieces of colored plastic. As the circular base rotates, the pieces of plastic from the smaller containers fall into the larger container and fill it up exactly.

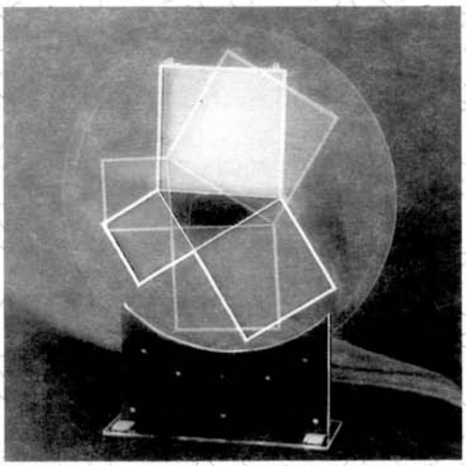
The process of obtaining the shapes of the pieces of plastic is shown below.

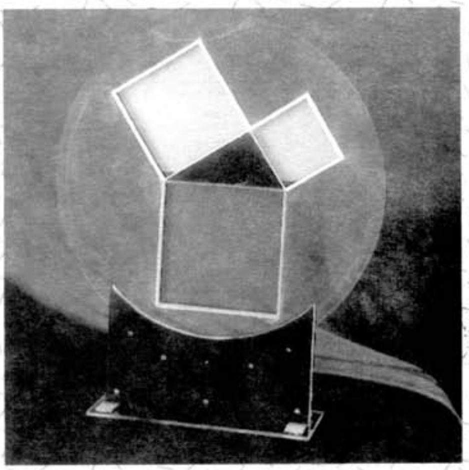


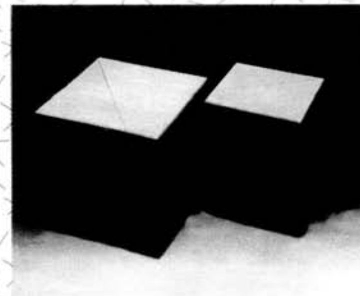
1-2. Another Rotary Model for the Pythagorean Theorem

This model has containers of the same size as the first model and the same set-up except that the two smaller containers are filled with colored water [Gu]. As the circular base rotates, the water flows into the larger container and fills it exactly when the two smaller ones empty completely.





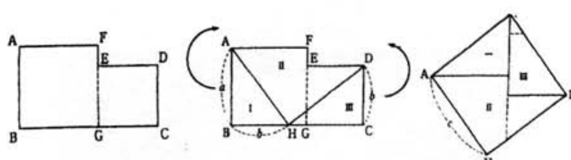


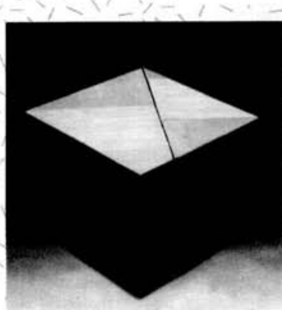


(a)

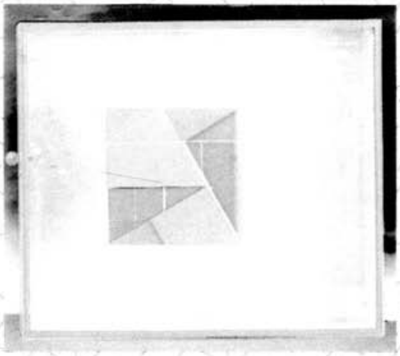
1-3. Pythagoras' Stool

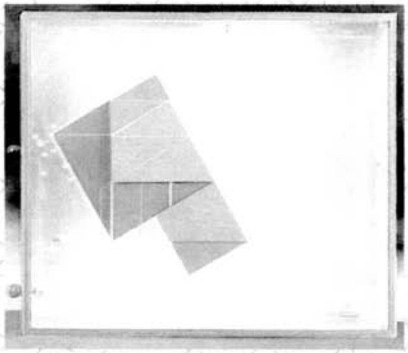
Any two squares of different sizes can be dissected, and the resulting parts rearranged to form a square which is larger than both [Mot]. This is the principle used in the Pythagoras' stool shown in (b). Start with two smaller stools (a) and assemble them so that their top side forms a hexagon. Partition the hexagon and reassemble the pieces as shown above.

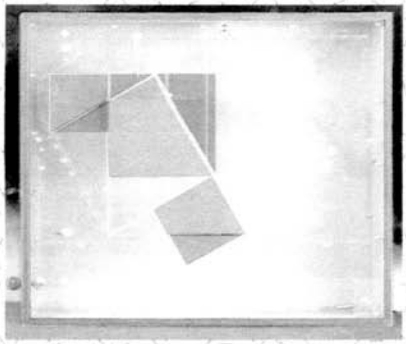




(b)



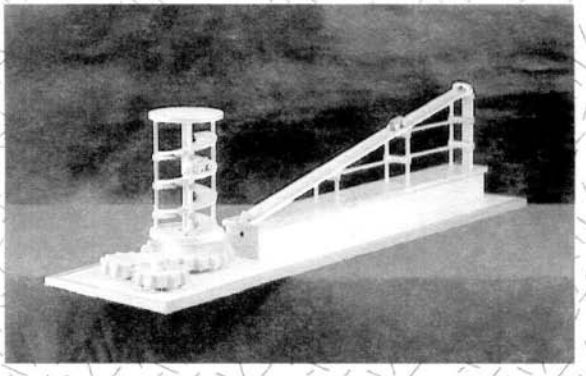




1-4. Sequentially *n*-Divisible Square

A square is said to be sequentially n-divisible if it has a dissection into convex polygons which combine to form sequentially 2, 3, ..., n-1, n unequal squares. This is a model of a sequentially 5-divisible square. It is partitioned into 15 convex polygons. The polygons can be reassembled as shown in the figure to first form two unequal squares then three, and eventually five unequal squares.

The partitioning algorithm is based on the Pythagorean Theorem and is described in [AN1]. In general there exists an n-divisible square consisting of 2n + 5 pieces.



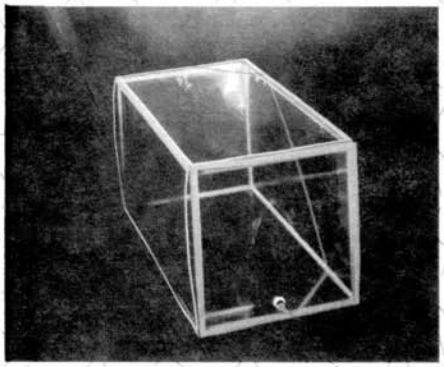
1-5. The Length of a Spiral that Winds Around a Cylinder

If a spiral winds around a circular cylinder with a constant slope then its length can be obtained as follows. Think of unwinding the side of the cylinder, then the spiral is unraveled and becomes a straight line segment. Hence the length ℓ of a spiral wound n times around a circular cylinder with a base of radius r and of height h coincides with the length of the diagonal of a rectangle of size $2nr\pi\,^{2}\mathrm{x} + h$ and is

$$\ell = \sqrt{(2nr\pi)^2 + h^2}$$

by the Pythagorean Theorem. A similar analysis will give the length of a vine wound around a tree trunk [Ni].

These models demonstrate the concept of unwinding a spiral. A spiral ramp winds around a cylindrical structure and an inclined plane represents the unwound spiral. A gear triggers the ascent of both the car on the spiral ramp and the car on the inclined plane. Since they traverse the same distance, the cars reach the top of structures at the same time.



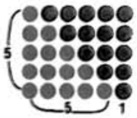
1-6. Shortest Paths Along the Surface of a Rectangular Parallelpiped

An ant and a grain of sugar are on the surface of a rectangular parallelopiped, the ant at the point P and the grain of sugar at the point Q. What is the shortest path that the ant can take, along the surface of the parallelopiped, to get to the grain of sugar? This problem is adapted from a puzzle posed by H.E. Dudeney in 1903 [Ga1].

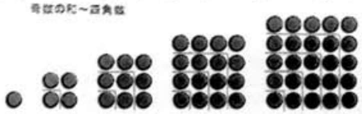
This problem is solved by determining all possible developments of the parallelopiped in which the points P and Q can be connected by a straight line segment and choosing the one for which $\overline{PQ}$ is shortest.

A development of a polyhedron π a plane figure obtained by cutting along edges of π and laying out the surface of π on the plane. This model shows the development that results in the shortest path.

(a)



(b)



(c)

2-1. Pythagorean Models

The configurations of stones shown in (a) represent the successive sums 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, 1 + 2 + 3 + 4 + 5

and were used by the Pythagoreans to demonstrate the formula

$$\sum_{k=1}^{n} k = \frac{1}{2} n(n = 1)$$

as follows [D]. Use a second copy of the last configuration in (a) to create the 5 by 6 rectangle in (b). This demonstrates that

$$2(1 + 2 + 3 + 4 + 5) = 5(6)$$

and can be extended to show in general, that

$$2\sum_{k=1}^{n} k = n(n + 1).$$

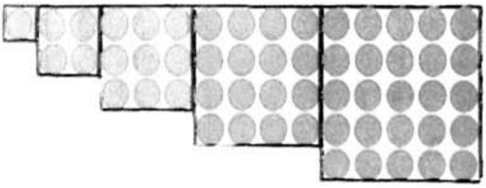
The configurations in (c) are made up of ℓ-shaped partitions called gnomon, whose base and height are equal [D]. The configurations are used to demonstrate the formula

$$\sum_{k=1}^{n} (2k - 1) = n^2.$$

The gnomons in the last configuration have consecutively 1, 3, 5, 7 and 9 stones and comprise a 5 x 5 square showing that

$$1 + 3 + 5 + 7 + 9 = 5^2,$$

which extends to the general formula.



2-2. Another Pythagorean Model

This model illustrates

the formula

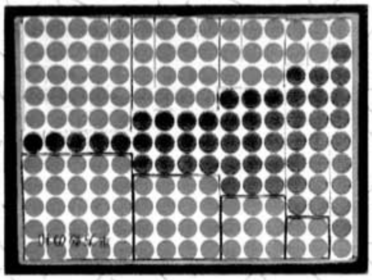
$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

for n = 5. The number 1^2, 2^2, 3^2, 4^2 and 5^2 are represented as squares filled with the appropriate number of stones as shown in (a). A rectangular configuration of size (1 + 2 + 3 +4 + 5) by [2(5) + 1] is created as shown in (b) by using two copies of (a) and by filling up the empty spaces with blue stones. For the general case, the size of the rectangle is

$\frac{1}{2} n(n+1)$ by $(2n+1)$,

using the standard formula for 1 + 2 + ... + n.

The number of blue stones is equal to the number of red stones in (a) as follows. The blue stones are arranged in 5 blocks of columns. Each column in a block has the same number of stones. The first block has 9 which is equal to the number of red stones in the largest gnomon of the largest square. The second block has 7 which is equal to the number of red stones in the second largest gnomon of the largest square. The number of stones in the first columns of the remaining blocks correspond to the number of red stones in the remaining gnomons of the largest square.



Similarly, the number stones in the second columns of the successive blocks of blue stones correspond to the number of red stones in the successive gnomons of the second largest square, and so on. Hence the number of blue stones is equal to the number of red stones in (a).

This leads to

$$3(1^2+2^2+3^2+4^2+5^2) = \frac{1}{2}(5)(6)(11)$$

and in general to

$$3\sum_{k=1}^{n} k^2 = \frac{1}{2} n(n+1)(2n+1),$$

which gives the formula [CG, Go].

(a)

(b)

2-3. Pyramidal Model

The blocks provide another demonstration of the formula

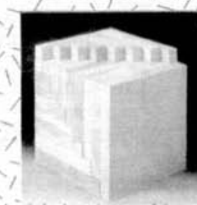
$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

Start with the pyramid shown in (a). Although n = 8 in this model, the procedure will hold for any n . The top block represents 1^2 , the blocks on the second level represent 2^2 , the blocks on the next level represent 3^2 , and so on up to n^2 . To prove the formula, construct the block shown in (d), whose dimensions are n(n + 1)(2n + 1), by combining 6 pyramids of the kind shown in (a), as follows. First move the blocks in the pyramid to obtain the configuration on the right-hand-side of (b). Then put together two such configurations to form the configuration on the left-hand-side of (b). Compose the configurations in (b) to obtain the configuration shown in (c). A reflected copy of the configuration in (c) is then combined with the first to form (d). Hence

$$6\left(1^2 + 2^2 + 3^2 + \ldots + n^2\right) = n(n+1)(2n+1)$$

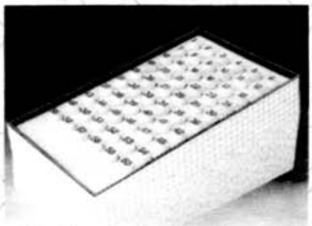
and so the formula holds.

This construction was first introduced by a Chinese mathematician, Yang Hui, in the 13th century [FT, Ho].



(c)

(d)

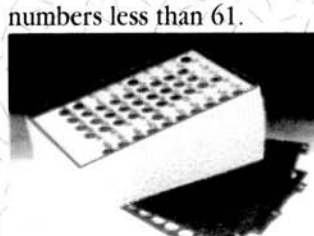


(a)

2-4. Sieve of Eratosthenes

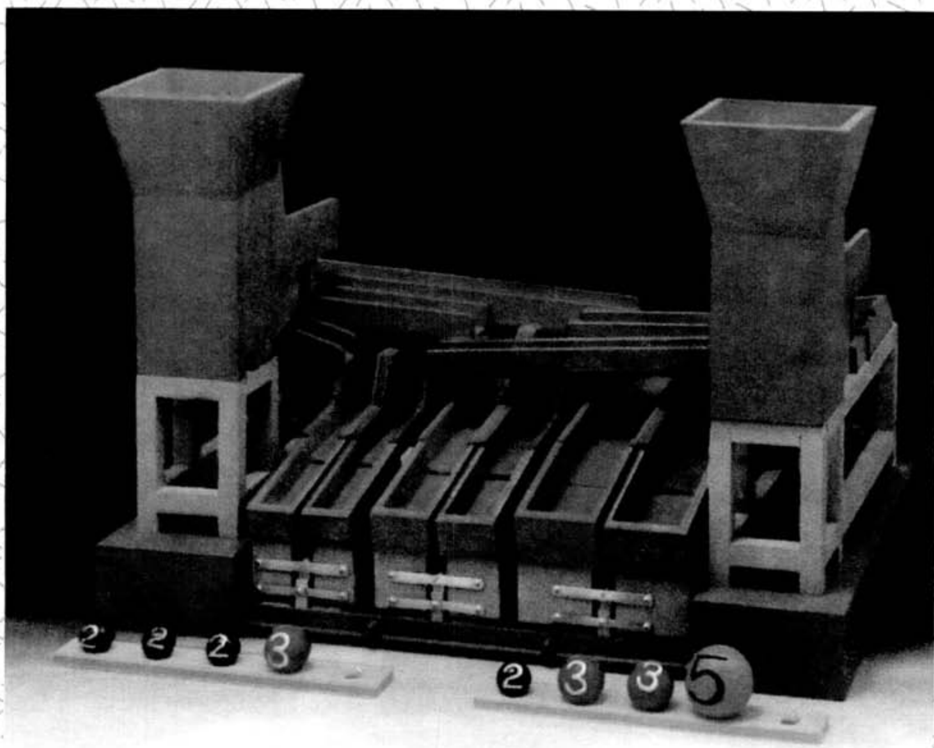
The sieve of Eratosthenes is a well-known method for determining all the prime numbers less than a given number. Eratosthenes was a Greek mathematician who lived from around 275 to 194 B.C. The method is as follows: Given the set of integers {1, 2, ..., n}, remove 1, retain 2 but remove all other multiples of 2, the next remaining number is 3, retain it but remove all of its other multiples, the next remaining number is 5, retain it but remove all of its multiples, and so on.

The device shown in (a) is an implementation of the sieve of Eratosthenes which will identify all prime numbers less than 61. The device is covered on top by a transparent board on which are written the numbers 1 to 60. Under this board are 60 balls corresponding to the 60 numbers. Under the balls are five boards, each with a chosen configuration of holes as shown in (b). These boards will be pulled out one after the other. When the first board from the top is pulled out, the ball corresponding to the number 1 falls inside the box. When the second board is pulled, all numbers which are multiples of 2, except 2 itself, fall in, and so on. When the fifth board is pulled, all numbers which are multiples of 7, except 7 itself, fall in. Since a composite integer n has a prime factor at most $\sqrt{n}$, it suffices to drop the balls which correspond to the multiples of the prime numbers less than $\sqrt{61}$, i.e., prime numbers less than seven. The remaining balls are the prime numbers less than 61.



(b)

2-5. Greatest Common Factor and Least Common Multiple

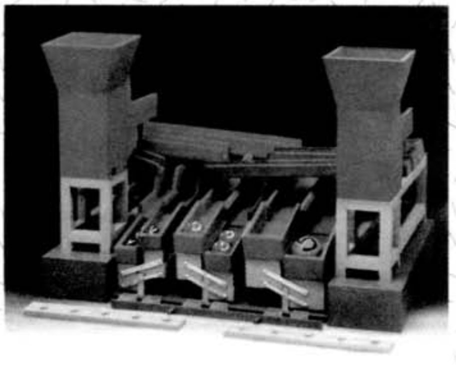


The device shown obtains the greatest common factor (gcf) and least common multiple (lcm) of two natural numbers automatically. To keep the device portable, we limit the natural numbers to those whose prime factors are 2, 3, and 5 but the principles used in the construction of the device can be extended.

Suppose the numbers are a = 24 and b = 90. Their prime factorizations are first determined: $a = 24 = 2^3 \times 3$

$b = 90 = 2 \times 3^2 \times 5$.

Then these factorizations are represented by balls of graduated sizes, based on the sizes of the primes, and marked accordingly (a).



The device has two funnels. Place all the balls in the representation of a in one and the balls in the representation of b in the other. Within the device are sorting and balancing mechanisms which make use of the sizes and weights of the balls. The process sends the balls of the same size from a and b into adjacent compartments (b). Of the two adjacent compartments, the one with fewer balls rises while the one with more balls falls. To take care of cases where there are an equal number of balls in each compartment, a small weight has been placed to favor the right compartment; this weight is too small to affect the general case.

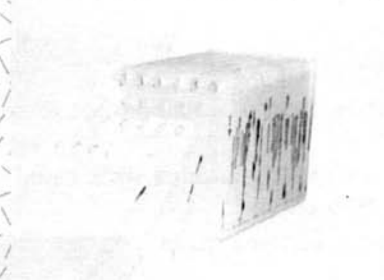
Take the balls in the higher level compartments and multiply the numbers appearing on them to obtain the gcf. Do the same for the balls appearing on the lower compartments to obtain the lcm.

2-6. Binary Sorter

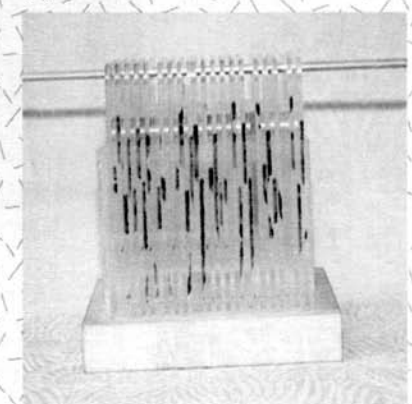
This device consists of forty square plastic boards that are bundled together to form a cube. On one of the lateral sides of the cube is a picture (a). Each board has six holes or indentations cut out from its top side (b). The holes and indentations are obtained as follows [CR, pp. 258-260]. Label the boards 1 to 40, starting from the leftmost side of the picture and associate the number n, $1 \leq n \leq 40$, with the binary notation for n - 1. In each board, the holes stand for the zeros in its representation while the indentations stand for the ones.

When the boards are randomly disarranged, the picture becomes jumbled (c) but can be returned to its original state by the following procedure. Starting on the side nearest the picture, take a rod, put it through the first column of holes, and lift the boards (d). Those boards which have holes in the first column will be lifted while those which

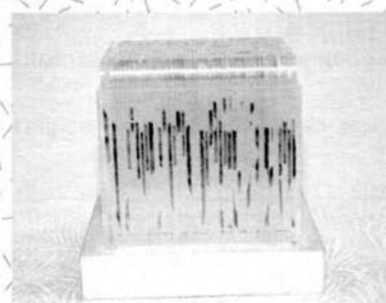
(a)



(b)



(c)

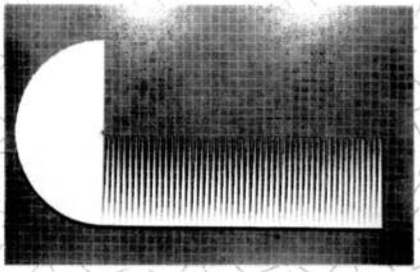


(d)

have indentations will remain. Push the remaining boards to the back and put the lifted boards down in front. Repeat the procedure for the second through sixth columns of holes. After all this, the picture will have reappeared in its original form on the side of the cube.

The operation of the rod on the first column of holes chooses those boards whose rightmost digits are zeros and moves them to the front (top of the list). This is shown below in (3). The operation of the rod on the second column of holes chooses those boards which have zeros as the second digit from the right and moves them to the front as shown below in (4). Proceeding in this manner, after the sixth operation, the boards will have been arranged in the original order shown on the right in (1).

Original order	Disarrangement	After first lifting operation	After second lifting operation
1000000	* * * * * *	* * * * * 0	* * * * 0 0
2000001	* * * * * *	* * * * * 0	* * * * 0 0
•	•	•	•
•	•	•	•
•	•	•	•
16001111	* * * * * *	* * * * * 0	* * * * 0 0
17010000	* * * * * *	* * * * * 0	* * * * 0 1
18010001	* * * * * *	* * * * * 0	* * * * 0 1
•	•	•	•
•	•	•	•
•	•	•	•
32011111	* * * * * *	* * * * * 0	* * * * 0 1
33100000	* * * * * *	* * * * * 1	* * * * 0 1
34100001	* * * * * *	* * * * * 1	* * * * 1 0
•	•	•	•
•	•	•	•
•	•	•	•
40100111			
(1)	(2)	(3)	(4)



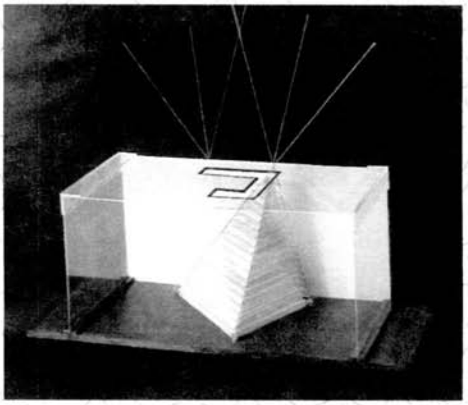
3-1. Area of a Circle

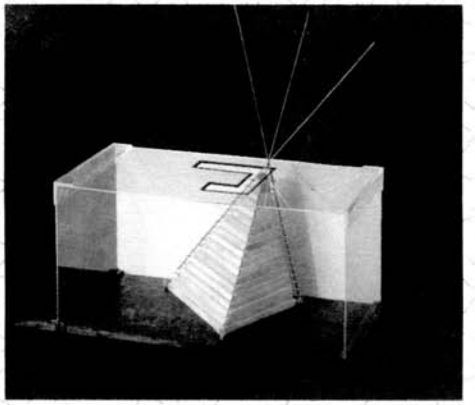
The two models shown provide convincing demonstrations of the formula for the area of a circle: $A = \pi r^2$.

The circle is dissected into congruent sectors of small area. The sectors are extended for half of the circle over a length πr, half of the circumference of the circle, to form the comb-like figure shown in (a). The sectors for the other half of the circle are similarly extended to obtain a second comb. Then the two combs are meshed to approximate a rectangle. Since the length of the rectangle is πr and its width is r, its area is πr^2 and coincides with the area of the circle.

The model shown in (b) is said to have been introduced by an Italian, Torricelli, in the 17th century [HN]. It is useful to think of the circle as a cross section of a piece of cake popularly called baumkuchen. Let 0 denote the center of the circle. As in the baumkuchen, the circle is partitioned into annular rings centered at 0. Then the circle is cut along a radius 0H and each annular ring extended along an appropriate line parallel to the tangent to the circle at H. The extended annular rings approximate a right triangle, of height r and base $2\pi r$, whose area $2\pi r \times r/2 = \pi r^2$ coincides with the area of the circle.

3 Formulas on Area and Volume



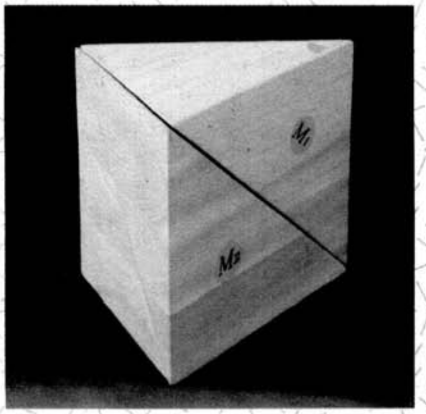


3-2. Pyramids with Congruent Bases and the Same Height

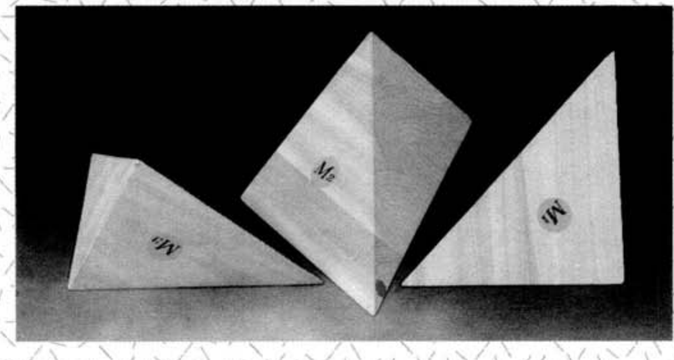
This model demonstrates that the volume of all triangular pyramids with congruent bases and the same height are equal. It consists of similar triangular boards of graduated sizes piled on top of one another. Three sticks, going through small metal rings attached to the vertices of the triangular boards meet in a ring at the apex of the pyramid.

When the ring at the apex is moved horizontally or vertically along the opening provided, the shape of the pyramid changes but the base and height remain the same. Since the volume of any of these pyramids is approximated by the sum of the volumes of the triangular boards, it is clear that the volumes of the pyramids are the same.

Similar devices can be used to demonstrate the validity of the assertion for other types of pyramids.

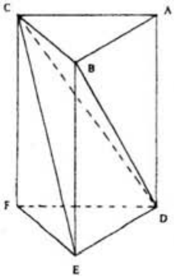


3-3. The Relation Between Volumes of Prisms and Pyramids



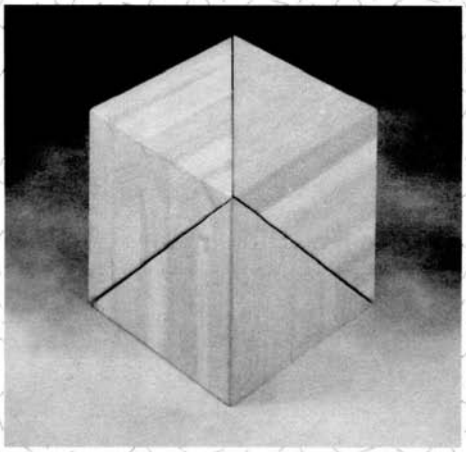
The volume of a triangular pyramid is one third that of a triangular prism with a congruent base and the same height. This fact is demonstrated on the prism shown in (a) which is dissected into three pyramids M_1, M_2, and M_3 shown in (b), which have the same volume.

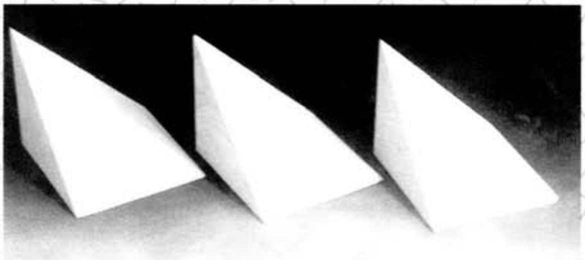
The dissection process is carried out as in (1) below where M_1 = ABCD, M_2 = BCDE, and M_3= CDEF. It can be shown that the volume of M_1 = the volume of M_2 because they are triangular pyramids with the same base and height. Similarly the volume of M_2 = the volume of M_3 for the same reason.



(1)

(c)

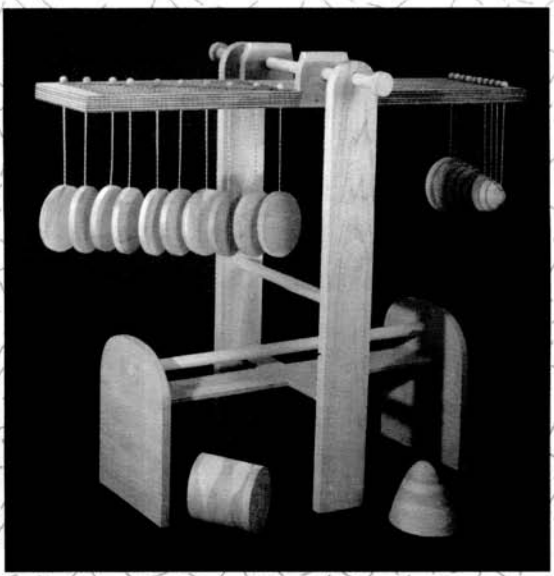




(d)

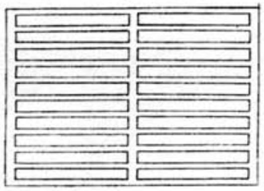
Similar relationships hold for the volumes of any of the following pairs: a cone and a cylinder, a quadrangular pyramid and a quadrangular prism, and a circular cone and a circular cylinder, provided each pair has congruent bases and the same height.

A cube can be dissected into three congruent quadrangular pyramids as shown in (c) and (d) [Ni].



(a)

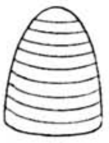
3-4. Archimedes' Balance - The Volume of a Paraboloid



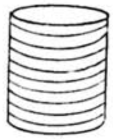
(1)

Archimedes determined the volume of a paraboloid by comparing slices of a circular cylinder of known volume with slices of the paraboloid, using a specially designed balance. A device was constructed on the basis of Archimedes' idea with some modification [A]. Instead of a bar, the balance uses a board that is divided into two parts, left and right, with 10 slits cut out from each part (1).

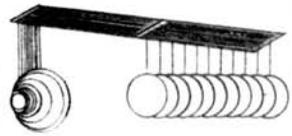
The circular cylinder is chosen so that it has the same height and base as the paraboloid (2). The paraboloid A and the cylinder B are sliced into ten parts of the same thickness. The topmost slice of A is hung at the leftmost end of the first slit on the left side, while one slice of B is hung on the first slit on the right side at the point which will create equilibrium. The same procedure is repeated for the

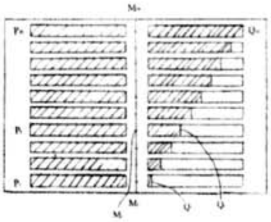


A



B





next slices from A and B using the second slits on the left and right side of the boards, respectively. The process is repeated until all slices are hung (3). We observe that the position taken by the slices of B form a straight line.

Along the i^{th} slits, label the point at the mid-line of the board M_i and the points at which the slices of A and B hang P_i and Q_i respectively. Then by the laws of Physics, if a_i is the mass of the slice of A and b_i is the mass of the slice of B, then

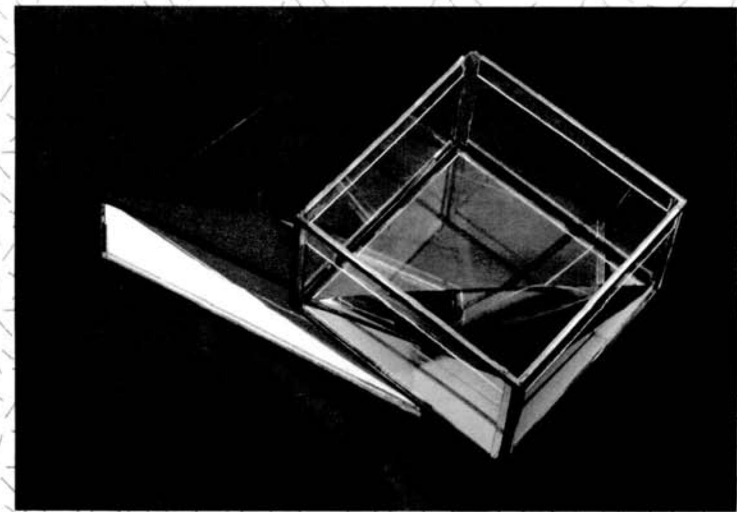
$$a_i \times \overline{P_iM_i} = b_i \times \overline{Q_iM_i}$$

$$\text{or } a_i : b_i = \overline{Q_iM_i} : \overline{P_iM_i},$$

i.e., the ratio between the masses of the i^{th} slices of A and B is inversely proportional to the ratio between the distances $\overline{P_iM_i}$ and $\overline{Q_iM_i}$. Since the paraboloid and the cylinder are made of the same material, this relation holds for the volumes of the i^{th} slices of A and B. Considering all the slices together, the ratio between the volumes of A and B is approximately inversely proportional to the ratio of the shaded area on the left side of the board and the shaded area on the right side of the board (4), i.e.,

$$A : B \approx \text{area of triangle } M_1M_{10}Q_{10} : \text{area of rectangle } M_1M_{10}P_{10}P_1 = 1 : 2$$

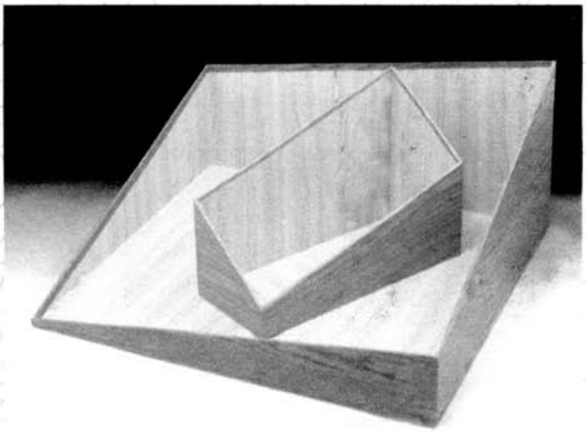
Hence the volume of the paraboloid is approximately $\frac{1}{2}$ the volume of the associated circular cylinder.



3-5. Universal Measuring Devices Without Gradations

The measuring devices shown have no gradations but can measure any integral amount of liquid up to their total capacity, assuming that we allow liquid to be scooped up only once from the original container. We refer to such devices as universal measuring devices.

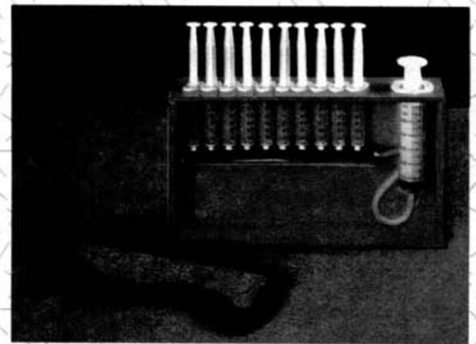
The device in which is on top in (a) is a rectangular parallelopiped which has a capacity of 6 liters. It can measure 1, 2, ..., or 6 liters of liquid by simply being tilted appropriately [Na]. (1) and (2) show 6 liters and 3 liters, respectively. In (3), the liquid takes the shape of a triangular pyramid.



Since its base is half the base of the device, the amount of liquid is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ of the capacity of the cup, or 1 liter.

To pour 5 liters, scoop 6 liters and tilt the device as in (3). Since 1 liter remains, 5 liters have been poured. To pour 2 liters, scoop 3 liters by tilting the device as shown in (2), then tilt it again as in (3) so that one liter remains in the device. To pour 4 liters, first scoop 6 liters, then tilt as in (2) to pour 3 liters into the second container, then tilt as in (3) to pour back 1 liter into the original container; finally, pour the remaining 1 liter into the second container.

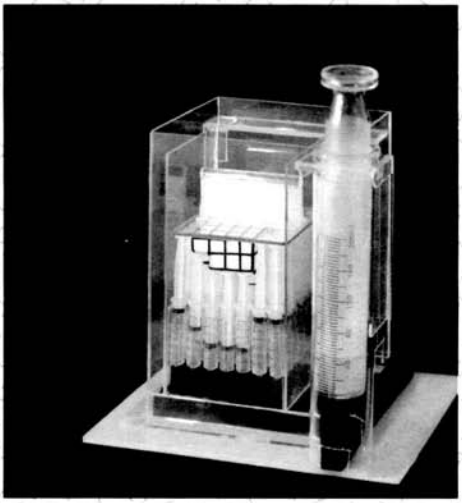
The second device shown in (a) has a capacity of 10 liters [Na] while those shown in (b) have capacities of 20 and 114 liters [WAN].



3-6. Area by Integration

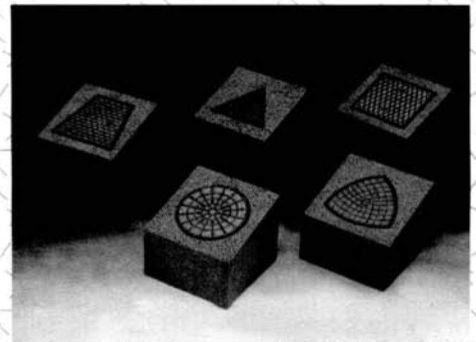
This device is used to illustrate how areas are obtained by integration. In particular, it will be used to approximate the area between a curve and the x-axis. It consists of ten syringes of diameter 1 arranged in a row (a). Each is connected to the large syringe so that any displacement of air in each is channeled to the large syringe and the amount measured there.

The area under consideration is represented in reverse by a frame and 10 plungers of the appropriate height. The plungers are aligned with the syringes (b). Since each syringe is of diameter 1, the total displacement measured in the large syringe is an approximation of the desired area.



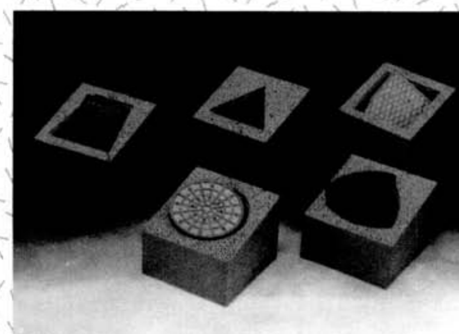
3-7. Volume by Integration

This device approximates the volumes of a solid that lies between a surface in 3-dimensional space and the fist quadrant of the xy-place. It extends the previous concept and relates volume to a double integral. It consists of a set of syringes arranged in consecutive rows. Each syringe is again connected to a large syringe as in the previous device. The large syringe measures any displacement of air in any of the small syringes. The solid is represented in reverse by a frame with appropriate plungers. The plungers are aligned with the syringes to displace the appropriate amount of air so that the total displacement, measured in the large syringe, approximates the volume of the solid.



(a)

4-1. Why are Manhole Covers Round?

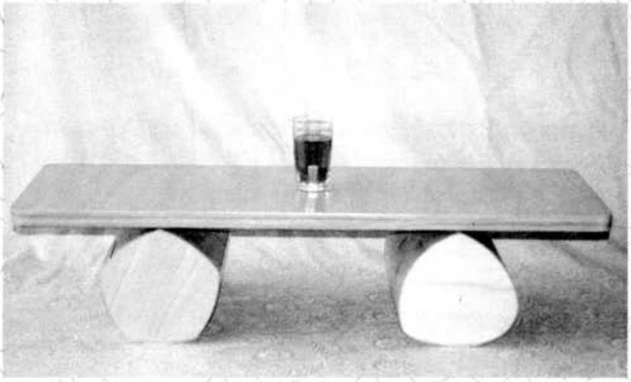


(b)

A square cover can fall into a square manhole of the same size (b). This is because the diagonal of a square is longer than the length of its sides. On the other hand, a round cover cannot fall into a round manhole of the same size because a circle has constant width, i.e., the distance between any two parallel lines which are tangent to the circle is constant.

Other figures of constant width can also be used successfully as manhole covers. One such figure is the Reuleaux triangle (the lower right manhole cover in (a) and (b)). To draw a Reuleaux triangle, start with the vertices of an equilateral triangle. Then, using one vertex of the triangle as a center and an edge of the triangle as a radius, draw a circular arc of 60 degrees from the edge. Do the same using each vertex in turn. There are infinitely many figures of constant width, among them are Reuleaux pentagons, Reuleaux heptagons, and so on.

4 Figures with Constant Width



4-2. Rollers of Constant Width

A board placed on top of two logs is sometimes used to move heavy loads. The circular cross-sections of the logs allow them to roll smoothly. The same can be said about rollers whose cross-sections are other figures with constant width as is demonstrated by these devices.