Mathematical Explanations in Euler's Königsberg

Tim Räz*

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Abstract

I examine Leonhard Euler's original solution to the Königsberg bridges problem. Euler's solution can be interpreted as both an explanation within mathematics and a scientific explanation using mathematics. At the level of pure mathematics, Euler proposes three different solutions to the Königsberg problem. The differences between these solutions can be fruitfully explicated in terms of explanatory power. In the scientific version of the explanation, mathematics aids by representing the explanatorily salient causal structure of Königsberg. Based on this analysis, I defend a version of the so-called "Transmission View" of scientific explanations using mathematics against objections by Alan Baker and Marc Lange, and I discuss Lange's notion of "distinctively mathematical explanations" and Christopher Pincock's notion of "abstract explanations".

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*e-mail: tim.raez@gmail.com

1 Introduction

In the present paper, I examine Leonhard Euler's original solution to the Königsberg bridges problem, a case of application of mathematics that has become standard in the pertinent philosophical debates. Based on a reconstruction of the case, we can interpret Euler's solution both as an explanation within mathematics and as a scientific explanation, and we can discern two different kinds of contributions of mathematics to these explanations.

First, mathematics contributes to explanatory power at the level of pure mathematics. Euler proposes not one, but at least three different solutions to the Königsberg problem. His discussion of the respective strengths and weaknesses of these solutions can be fruitfully explicated in terms of differences in explanatory power. Characteristics of explanatory power are providing relevant information and a reduction of complexity. I also propose that the application of a theorem to mathematical instances can constitute a purely mathematical explanation. Thus, explanations within mathematics need not be proofs.

Second, the mathematics aids the applied version of the explanation by representing aspects of the causal structure of the city of Königsberg. This is a causal explanation on a liberal notion of causal explanation, if we take the pragmatics of explanations into account. In the applied case, the purely mathematical explanation transmits to the applied explanation via a bridge principle. The picture resulting from this analysis is a version of the so-called "Transmission View", an account of mathematical explanations of scientific phenomena.

I then compare and contrast several recent accounts of mathematical explanations, and of the Königsberg case, with the present account. In particular, I defend the Transmission View against objections by Alan Baker (2012) and Marc Lange (2013), and I discuss Lange's "distinctively mathematical explanations", as well as the notions of "abstract explanations" and "abstract acausal representations" proposed by Christopher Pincock (2007, 2012, ????).

2 Euler's Königsberg

Philosophers have been interested in the Königsberg case for some time; see Penco (1994); Franklin (1994); Wilholt (2004) for some early discussions. One of the reasons is that Euler's solution can be interpreted as an explanation of a scientific phenomenon in which mathematics plays an important role, and it has been used to propose novel kinds of scientific explanations. The present account improves on previous reconstructions by basing the discussion on Euler's original paper on the case, as well as other historical sources.[1]

In this section, after a short look at the historical genesis of the Königsberg bridges problem, I scrutinize Euler's original solution to the problem in his paper "Solutio problematis ad geometriam situs pertinentis" ("The solution of a

[1] Molinini (2012) applies the same methodology to Euler's work in the context of mathematical explanations.

problem relating to the geometry of position"), and I compare Euler's approach to a modern approach.[2]

2.1 Prehistory

The story of how Euler learned of the Königsberg bridges problem is not completely known, see Sachs et al. (1988). Euler probably first heard about the problem from letters by Carl Leonhard Gottlieb Ehler, who acted as an intermediary between Euler and Heinrich Kühn, a professor of mathematics from Danzig. In a letter to Euler on March 9, 1736, Ehler alludes to an earlier formulation of the problem, asks for a solution, and adds a schematic map; see figure 1.[3] of the city of Königsberg, indicating the direction of flow of the river and stating the names of bridges, the island, and neighborhoods.

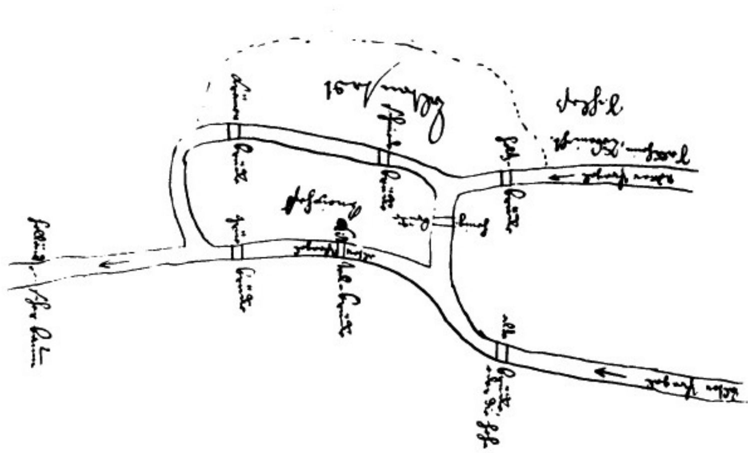

Figure 1: Ehler's Map of Königsberg (inverted)

2.2 The Königsberg Paper

We will now reconstruct Euler's line of thought in the Königsberg paper.

Euler begins by stating his systematic interest in the Königsberg bridges problem in paragraph 1. He takes it to be an example of a new, special kind of

[2] I use the widely available translation Euler (1956). See Hopkins and Wilson (2004) for a useful overview of Euler's paper. I thank an anonymous referee for his suggestion to consider Euler's original publication, and for pointing out several interesting aspects of Euler's approach.

[3] The figure is inverted in order to facilitate comparison with the corresponding picture from Euler's paper; see figure 2 below.

geometry, which does not involve quantities and measures, but only position – what we now call topology.

In paragraph 2, Euler distinguishes two problems. He illustrates the situation in Königsberg using a schematic map, reproduced as figure 2. He assigns capital letters A, B, C, D to the areas, and lower case letters $a, b, c, \ldots$ to the bridges connecting areas. The first problem is to find out whether it is possible to find a path that crosses every bridge of this system exactly once. Euler notes that there is no definite answer to this problem as yet. We will call this the *Königsberg Problem.* Euler then generalizes the problem and asks how one can determine the solution, not only for this particular configuration, but for any kind of system, i.e., any kind of branching of the river and any number of bridges. We will call this second problem the *General Problem.*

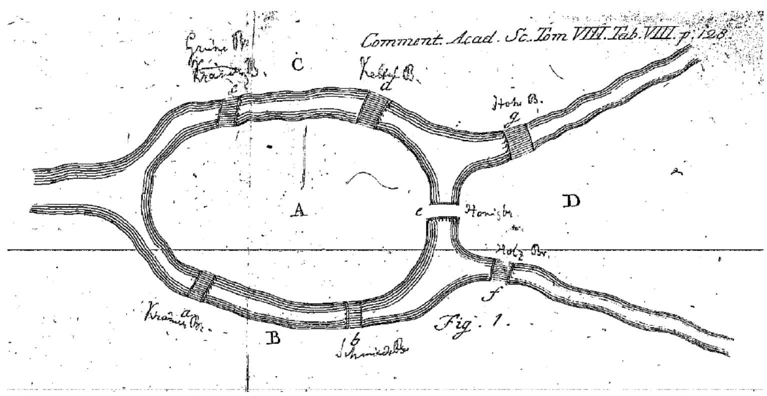

Figure 2: Euler's Map of Königsberg

Several aspects of this paragraph are noteworthy. Firstly, there is the introduction of two kinds of letters, one for places, or areas, the other for connections, or bridges. This distinction, as we will see in a moment, is key to a graph-theoretic approach to both problems. This notational innovation does not feature in Ehler's letter of March 9, 1736. Secondly, Euler suppressed some information contained in Ehler's map, such as the names of bridges and the direction of flow of the river, probably because he considered them to be irrelevant to the solution. Third, the immediate generalization of the problem shifts the focus from mere puzzle-solving to a real mathematical question – some solutions that are appropriate for the special case of Königsberg will not do in the general case, which requires a more elegant argument. The Königsberg Problem and the General Problem should be carefully distinguished.

In paragraph 3, we learn of a first method for solving the Königsberg Problem: it consists of "tabulating all possible paths" and examining whether one of them uses every bridge exactly once – call this the *Brute Force Method.* Euler rejects this method because it is "too tedious and too difficult": there are too many possible paths, and for bigger systems, this method becomes intractable. The approach generates information that is irrelevant to the problem at hand.

This insight leads to a shift of focus: Euler restricts the task to establishing whether the required path exists, which does not require the specification of an actual path, i.e., a witness. Euler does not specify how to carry out the brute force search in detail.[4]

A crucial innovation is introduced in paragraph 4. It consists of the use of a particular notation for paths in the bridge system in terms of the crossing of bridges; see figure 2: The crossing of any one of the bridges a and b between A and B can be written as AB. A path from A over B to D is noted as ABD, using any one of the bridges connecting these areas.

This notational shift is characteristic of the graph-theoretic nature of Euler's approach to both problems and marks the invention of graph theory. In modern graph theory, a graph[5] is represented by a set of vertices V, and a set of edges E, represented by pairs of vertices, $E \subseteq V^2$. This is exactly what Euler's notation achieves: Two structurally related kinds of objects – bridges and areas in the present case – are brought into notational correspondence by writing edges (here: bridges) as pairs of vertices (here: areas). There are no longer two separate sets of labels for the two kinds of objects, but one is expressed in terms of the other.

As Euler himself notes, the bulk of the paper consists of putting this simple yet powerful idea to work in a sequence of methods for determining the existence of what we now know as "Euler path", i.e., a path that crosses every bridge exactly once, culminating in Euler's Theorem. The key idea is that if we write bridges as pairs of areas, we can compare the algebraic condition for the length of Euler paths with algebraic conditions for the areas.

First, Euler finds a method that is sufficient to solve the Königsberg Problem. We can represent a path on a bridge system consisting of n bridges by a string of $n + 1$ capital letters (areas). A bridge between areas A and B is written AB. The notation does not distinguish between different bridges that connect the same two areas, as this information is irrelevant for the existence of a path. However, we know that if we want to use every bridge in Königsberg exactly once, the string has to consist of 8 letters.

We can determine how many times a capital letter (area) has to occur in a string (path). Denote the number of bridges connected to an area X with m_X.[6] If m_P, i.e., the number of bridges leading to area P, is odd, then P will have to occur $\frac{m_P+1}{2}$ times in the string: If three bridges lead to area P, then the letter P will feature twice in the string, whether we start in P or not. For five bridges, the letter has to occur three times, and so on. We can now apply this result to

[4] One way of implementing it is to write down all (finitely many) paths of length seven starting from any one of the areas A, B, C, D, and see whether any one of these paths consists of seven different bridges. A distinctive feature of the brute force approach is that we do not need both kinds of letters introduced above – the lower-case bridge labels will do the job.

[5] More specifically, a multigraph, as there are some pairs of vertices that are connected by more than one edge.

[6] Euler does not employ the notation m_X for the number of bridges of area X; his argument hardly uses any algebraic expressions; it is stated in prose, which makes it somewhat hard to follow. The reconstruction given here follows Euler closely, but transforms some of his reasoning into algebra, in order to make it more accessible. Thanks to Antje Rumberg for pressing me on this issue.

the Königsberg system. The letter A has to occur three times, and B, C, D two times. This adds up to 9, which is bigger than 8. It is therefore impossible to find a path that crosses every bridge in Königsberg exactly once.

Euler's next step is to extend this method to systems with even areas. If an area has an even number of bridges, there are two possibilities. If area Q is the starting point of the trip and m_Q is even, then the letter Q will occur $\frac{m_Q}{2}+1$ times. If area R is not the starting point and m_R is even, the letter R will occur $\frac{m_R}{2}$ times. Putting these results together yields a method, which we will call the *Intermediate Method*, and which solves the General Problem.

Euler's argument for the Intermediate Method, and the subsequent deduction of Euler's Theorem from the Intermediate Method, can be restated in algebraic form as follows. Recall that n designates the total number of bridges in the system; the length of a string representing an Euler path on this system therefore must be of length $n+1$. We write P for areas with an odd number m_P of bridges, Q for areas with even m_Q where we start, and R for areas with even m_R where we do not start. The Intermediate Method can be explained best on the basis of the following equation:

$$\underbrace{\sum_P \frac{m_P+1}{2}}_{\text{odd areas}} + \underbrace{\sum_R \frac{m_R}{2}}_{\text{even nonstart areas}} + \underbrace{\frac{m_Q}{2}+1_Q}_{\text{even start area}} = n+1 \qquad (1)$$

On the LHS, we have inserted the rules of how many letters each area contributes to a string, depending on whether the areas are odd, even, and whether we start in an odd or an even area. The Intermediate Method consists in computing the sum on the LHS, and checking whether the result is equal to $n+1$; it gives us a necessary condition for the existence of an Euler path: no Euler path exists if equality is violated. The equation also shows that if there are odd areas, we can "bring down" the sum on the LHS if we start the path in an odd area, as the contribution of 1 to the even start area, 1_Q, drops out in this case.

Once we have established this relationship, we can, following Euler, deduce *Euler's Method*. Note, first, that the number of odd areas has to be even, because summing up the number of bridges connected to all areas counts every bridge in the system twice, and therefore has to yield an even number. Now we multiply equation (1) by 2 and replace the resulting $2n$ on the RHS by the sum of all areas:

$$\underbrace{\sum (m_P+1)}_{\text{odd areas}} + \underbrace{\sum m_R}_{\text{even nonstart areas}} + \underbrace{m_Q+2}_{\text{even start area}} = \underbrace{\sum_X m_X + 2}_{\text{all areas}} \qquad (2)$$

From equation (2), we can deduce that equality only holds in two cases. First, if there are no odd areas at all, the first summand on the LHS drops out, and equality follows. Second, if there are two odd areas, we have to start in one of them, because otherwise, both the first and the third summand contribute 2 to the LHS, exceeding the RHS by 2. If there are four, six, etc. odd areas, the equality is violated because the first summand contributes at least an additional 4, which exceeds the RHS by at least 2.

This is what I will call "Euler's Original Theorem". It gives two conditions that are jointly necessary for the existence of an Euler path: it exists only if a) the valence of all areas is even, or if b) the valence of exactly two areas is odd, and we start the journey in one of these areas.

2.3 Euler and Modern Graph Theory

It is important to point out several differences between Euler's account on the one hand, and modern accounts on the other.[7] First, many modern formulations of the Königsberg Problem and the General Problem heavily rely on graph diagrams, such as in the following figure, to explain the problem.

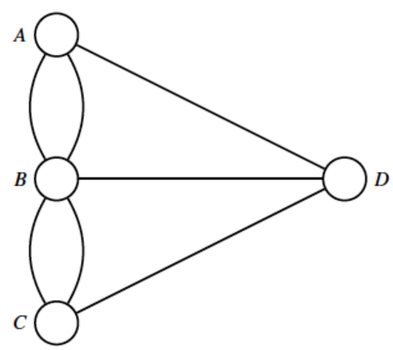

Figure 3: Königsberg Graph

However, no such diagram can be found in Euler's paper. He only uses various schematic maps of Königsberg, see figure 2, and of other, imaginary bridge systems. As Robin J. Wilson (1986, p. 272) points out, graph diagrams only appear 150 years later. The fact that Euler's reasoning does not rely on a graph diagram does not mean that it is not graph theoretical. I argued above that the introduction of graph-theoretic methods in the paper occurs in paragraph four with the identification of bridges (lower case letters) with pairs of areas (pairs of capital letters). This is the relevant innovation, not the use of graph diagrams.

Second, Euler's question is different from the modern formulation: he asks if it is possible to cross every bridge in Königsberg exactly once, without the assumption that starting and end point coincide, i.e., that the Euler path be closed. Based on his argument, we can see that a closed path is possible if, and only if, there are only even areas, because otherwise, we would only "use" an odd number of bridges of the start area. In the case of two odd areas, we have to start in one of the odd ares, and end in the other.

[7] See Diestel (2006, pp. 21) for a modern account, including a proof of Euler's Theorem.

Third, Euler does not assume that graphs are connected. This is a necessary assumption for the theorem: if a graph is not connected, we will not be able to find an Euler path even if all vertices have even valence, simply because one part of the graph is inaccessible from another. Euler probably omitted this condition because it is implicitly clear that there is no solution in these cases.

Fourth, Euler only proved one direction of what is now known as "Euler's Theorem", viz. the statement that a closed Euler path exists if, and only if, every area has even number of edges. He did not prove that if a closed Euler path exists, every vertex has an even number of edges; this was only proved 135 years later; see Wilson (1986, p. 270). Note that this direction of the theorem is not needed for the solution to the Königsberg Problem.

In the rest of the paper, I will refer to Euler's two conditions that are jointly necessary for the existence of Euler paths as "Euler's Original Theorem", and to the statement that, on connected graphs, a closed Euler path exists if, and only if, every vertex has even valence, as "Euler's Theorem". I will not restate the conditions that the graph be connected and that the path be closed in the latter case in order to avoid repetition.

3 Philosophical Analysis: Setting the Stage

We now turn to a philosophical analysis of Euler's work. In this section, we will introduce, and recall, some important distinctions that have been drawn in the philosophical literature, and, implicitly or explicitly, in Euler's paper. We will argue that Euler's work can be interpreted as explanatory in nature.

3.1 Intra-Mathematical Explanations (IME) and Scientific Explanations Using Mathematics (SEM)

We will analyze the Königsberg case in terms of mathematical explanations. The debate on mathematical explanations distinguishes between mathematical explanations of purely mathematical phenomena, so-called "Intra-Mathematical Explanation" (IME), and scientific explanations that make use of mathematics, so-called "Scientific Explanation using Mathematics" (SEM).[8] Both kinds of explanation are controversial and have been debated in recent years. While the debate on IME is in need of more good real-life examples, SEM is under debate mainly because it has been used to defend mathematical Platonism in the context of indispensability arguments.[9]

To draw the distinction between these two kinds of explanations, and apply it to Euler's work, we have to presuppose a clear separation between an empirical and a mathematical domain, which carries over to the distinction between IME

[8]The labels IME and SEM are due to Baker (2012); see Mancosu (2011) for a useful overview of the debate on explanations in pure and applied mathematics.

[9]See Colyvan (2001) for a book-length defense of indispensability and Baker (2009) for the now-standard formulation of an explanatory version of indispensability arguments. I will not discuss whether Euler's solution to the Königsberg bridges problem speaks in favor of mathematical Platonism.

and SEM. This should not be too problematic in the present case: We can think of Euler's work, be it explanatory or not, as only concerning pure graph theory and being unrelated to any particular domain of application. On the other hand, the arguments allow us to draw conclusions about the existence or non-existence of paths in cities, and Königsberg in particular.

3.2 Why Explanations?

Why should we interpret Euler's work as explanatory, be it as an IME or as an SEM? There are several.

The most important reasons for interpreting the Königsberg case as an explanation are systematic. Euler's solution does not merely show that there is no Euler path in the Königberg system, but it also establishes why this is the case. This view is widely shared; there seems to be a consensus in the philosophical literature that the Königsberg case is a case of an SEM, and there are various proposals for analyzing it as such. I will discuss these accounts below in section 6. However, if we accept an SEM version of Euler's explanation, we should also accept an IME version of the explanation. I will give an argument for this claim in section 6.1 below. In the following, I will presuppose that Euler's paper is at least partially geared towards answering why questions.

Is there any historical evidence that Euler himself thought of his work as being explanatory in nature? Euler does not use the word "explanation" in the Königsberg paper, but some of his methodological remarks can be fruitfully analyzed in terms of explanatory power. In paragraph 3, he writes:

> The particular problem of the seven bridges of Königsberg could be solved by carefully tabulating all possible paths, thereby ascertaining by inspection which of them, if any, met the requirement. This method of solution, however, is too tedious and too difficult because of the large number of possible combinations, and in other problems where many more bridges are involved it could not be used at all. When the analysis is undertaken in the manner just described it yields a great many details that are irrelevant to the problem; undoubtedly this is the reason the method is so onerous. Hence I discarded it and searched for another more restricted in its scope; namely, a method which would show only whether a journey satisfying the prescribed condition could in the first instance be discovered; such an approach, I believed, would be much simpler.

In this quote, he gives several reasons why the Brute Force Method is not satisfactory: It is inefficient, and it may not be applicable to more complicated systems. A second, even deeper reason is that the Brute Force Method produces irrelevant details, which is the reason for the inefficiency. To overcome these difficulties, he sets out to find a simpler method, which will only decide whether an Euler path exists, without producing a witness in the positive case.

If we interpret Euler's reasons for discarding the Brute Force Method as positive desiderata for solutions, we can conclude that he wants to get to the very heart of the matter – he wants relevant information, not irrelevant details. He is not after a solution that merely solves the problem, he also wants a solution that gives us insight into the issue. I will analyze the main desiderata mentioned in this quote, such as simplicity, reduction of complexity, and relevant information, in some detail in section 4.4 below.

A recent paper by Daniele Molinini (2012) shows that Euler was interested in explanatory proofs, generally speaking. Molinini points out that in the context of the proof of existence of an instantaneous axis of rotation of rigid bodies, Euler prefers a geometrical approach as opposed to an analytical treatment, and explicitly emphasizes the explanatory nature of the former approach (Ibid., p. 119). This makes it plausible that Euler was interested in mathematical explanation in other cases as well.

It could be thought that Euler's main goal is to find an efficient solution to the Königsberg problem – a fast solution, which need not be explanatory in a deeper sense.[10] This, however, is implausible in view of a remark later on. In paragraph 16, after discussing the Intermediate Method, he writes:

> By this method [i.e. the Intermediate Method] we can easily determine, even in cases of considerable complexity, whether a single crossing of each of the bridges in sequence is actually possible. But I should now like to give another and much simpler method, which follows quite easily from the preceding, after a few preliminary remarks.

This remark makes it plausible that Euler is not only after an efficient solution: the efficiency of the Intermediate Method is contrasted with the simplicity of the further, "much simpler method", i.e., Euler's Method. If efficiency were the only relevant criterion, Euler would probably point out the additional boost in efficiency provided by Euler's Method.

3.3 The Königsberg Problem and the General Problem

In paragraph 2, Euler distinguishes between the question whether there is a certain path in the city of Königsberg, the *Königsberg Problem*, and the general question of conditions for paths in similar kinds of system, the *General Problem*. Both problems give rise to explanations. In the case of the Königsberg Problem, the relevant why question is: why is it impossible to cross every bridge in Königsberg exactly once? In the case of the General Problem, the question is: why is it possible to cross every bridge in some systems (of a certain kind), and not in others? The two problems are not independent: a solution to the General Problem also solves the Königsberg Problem as a special case.

Do the explanations that answer these two questions belong to pure mathematics (IME), or are they scientific explanations that come with a portion of mathematics (SEM)? We can construe both explanations as belonging to both

[10] I thank an anonymous referee for raising this issue.

kinds of explanations: they can concern one real (or several actual and possible) bridge systems, or they can concern one (or a type of) graph-theoretic problem. On of the goals of the present paper is to get a clear picture of how the purely mathematical formulation relates to the scientific formulation of the explanations.

4 Königsberg as an Intra-Mathematical Explanation

In this section, I discuss the Königsberg case as an Intra-Mathematical Explanation (IME).

4.1 Euler's Three Methods

Euler notes early on, in paragraph 3, that we can tabulate all possible paths and examine whether one of these conforms to our specifications, and that this solves the Königsberg Problem.[11] He quickly dismisses this approach, the Brute Force Method, as not satisfactory. He is after a more telling method for solving the problem. He thus devises a method that also solves the General Problem, but in a more satisfactory manner: In the Intermediate Method, we sum up the number of times the letter has to occur in a string, depending on whether the number of bridges connected to an area is odd or even. As he is still not completely satisfied with the Intermediate Method, he proceeds to an even better method, Euler's Method, which establishes that an Euler path exists only if all vertices have an even number of edges, or if exactly two vertices have an odd number of edges, i.e., he establishes Euler's Original Theorem. All three methods provide answers to the why question of the Königsberg Problem, but in varying degrees.

4.2 Degrees of Explanatory Power

The Brute Force Method does not meet Euler's expectations for a solution to the Königsberg Problem. It answers the why question – none of the paths in a complete list is an Euler path – but the answer is not very telling. Euler writes, "it yields a great many details that are irrelevant to the problem". We could interpret the Brute Force Method as not being explanatory at all. However, I think this would be going too far. The method does give us a reason – there simply is no Euler path among all possible candidates – but it is not very insightful. I therefore prefer to interpret it as a *bad* explanation. On the other end of the spectrum, the Euler Method provides an answer with high explanatory power. It tells us that the reason for the non-existence of an Euler path is

[11]We do not need graph theory to solve the Königsberg Problem, because we can tabulate paths in terms of bridge labels alone; there is no need for the area letters. The fact that we can express one set (bridge labels) in terms of the other (area labels), which marks the invention of graph theory, is irrelevant to the Brute Force Method.

that the four areas have odd valence, which violates both conditions of Euler's Original Theorem. Finally, the Intermediate Method is somewhere in between.

This suggests that we think of the difference between the explanatory power of the three methods not in terms of categorical differences – explanatory vs. non-explanatory methods, or proofs – but in terms of (comparative) degrees of explanatory power: We can rank the three methods relative to each other. This has several advantages. First, it does not force us to claim that the Brute Force Method is not explanatory *tout court*, i.e. that it does not give us any insight at all. *Some* information is provided by the complete list of possible paths.[12] Second, employing degrees of explanatory power provides a more fine-grained picture, allowing us to rank the three methods.

It has to be emphasized that an analysis along these lines will yield a relative and local notion of explanation: we get degrees of explanatory power, and comparison is possible with respect to one particular theorem. It may be impossible to rank two proofs (or explanations) of two theorems (or mathematical facts) in terms of their explanatory power. Therefore, the account defended will not yield necessary and sufficient conditions for explanations in mathematics that apply absolutely and globally.

4.3 IME by Proof and by Instantiation

For all their differences, the three methods share one important feature: all three can be used to solve the Königsberg Problem, and, in principle, they can also be used to solve the General Problem.[13] The two problems yield two kinds of explanations: One concerning the General Problem, the other concerning the Königsberg Problem. The first kind is an *IME by Proof*, the second kind an *IME by Instantiation*.

The former kind consists of a method explaining why a class of systems has a certain property, while the latter kind explains by applying this method to a particular system. Euler shifts his attention from the Königsberg Problem to the General Problem early on, in paragraph 2. This underlines the different status of the two kinds of explanations. While both are deductive, and thereby constitute proofs in a wide sense of the notion, only the solution to the General Problem is the proof of a *theorem*, i.e., constitutes a proof on a more narrow conception of proof. The solution to the Königsberg Problem is merely an application of that theorem.[14]

The distinction between the proof of a theorem, and the instantiation of the same theorem, carries over to all three methods. If we apply the methods to

[12] There are proofs, so-called *zero-knowledge proofs*, that only show that a result is true, without giving us *any* knowledge as to why this is so, in a strict, cryptographic sense of knowledge; see Aaronson (2013, Sec. 9.1). However, these are non-classical, probabilistic proofs, and deductive analogues will convey *some* information. I thank Scott Aaronson for correspondence on this point.

[13] The qualification is necessary because the Brute Force Method has a high complexity and may be practically inapplicable to large systems.

[14] The idea that we can draw this distinction can be found in Baker (2012). I will discuss the ramifications of the distinction for Baker's ideas in section 6.1 below.

the Königsberg Problem, we use general facts, such as theorems, but the proofs of these facts are not part of the explanation. For example, the mere appeal to Euler's Theorem, and the fact that the Königsberg system does not satisfy the theorem's conditions, is a legitimate and satisfactory explanation. The same is true, to a certain extent, for the Brute Force explanation. The explanation is satisfactory if we accept that the Brute Force Method works, and the proof that this is so need not be part of the explanation.[15]

This is not to say that we have to accept the methods on faith. We can request proof of these results. However, this is a request for different explanations, namely the explanations of the validity of methods or theorems. We can keep apart the explanation of a theorem, which can consist in a proof, and an application of the same theorem in the explanation of a particular *mathematical* fact. We should not be too puzzled about this, as the same is common in scientific explanations: we use a regularity to explain an event, and we can explain the regularity by appealing to a different, more general regularity. These are just two different explanations. This is not to say that the explanatory power of an IME by Proof, and an associated IME by Instantiation, are independent. We will explore the relation between the two kinds of explanations after identifying the explanatory virtues of methods, to which we will now turn.

4.4 Explanatory Differences

We now turn to the (explanatory) differences between Euler's three methods. Euler notes several differences. In paragraph 3, he writes that he is after a method that is simpler than the Brute Force method. Later, after explaining the Intermediate Method in paragraph 16, he writes that he wishes to introduce an even simpler solution; see the last quote in section 3.2 above. Simplicity is a well-known candidate contributor to explanatory power.[16] However, it is a notoriously difficult and vague notion. We will thus try to dissolve simplicity into other, clearer concepts that Euler could have had in mind.

4.4.1 Complexity

First, in paragraph 3, Euler elaborates that the Brute Force Method yields a "large number of possible combinations", which makes it difficult to carry out for more complicated systems, or even inapplicable. The three methods differ in how simple they are to execute. The Brute Force Method is more tedious than both other methods. Then, Euler's Method is easier to carry out than the Intermediate Method, as for the latter, we have to count the number of bridges for each area, determine the number of times the area letter has to occur in a path string, sum the results up, and compare the number with the length

[15] The instantiation of the Brute Force Method could consist in an (exhaustive) list of paths, together with an appeal to the fact that the list has been compiled according to a certain method. A proof of the Brute Force Method would have to establish that the method is indeed reliable.

[16] See Baker (2010) for a useful discussion.

of Euler paths of the system. On Euler's Method, we only need to determine whether zero or two areas have an odd number of bridges. This is easier to determine than the calculation of the Intermediate Method. I propose that this variety of simplicity can be spelled out in terms of *Complexity*.

It is plausible that a reduction of complexity can improve explanatory power. A method that reduces complexity can enhance our understanding by making the reason for the existence or non-existence of a path accessible and concise. I think that Euler's pre-theoretic notion of complexity could be spelled out in terms of computational complexity, and that it would be helpful to carry out such an analysis. I will not explore this possibility here.[17]

However, Euler is not only after a reduction of complexity. He notes that while the Intermediate Method is already quite successful at reducing complexity, he prefers a "much simpler method", Euler's Method; see the last quote in section 3.2 above. This suggests that while complexity can contribute to explanatory power, it is not sufficient: the Intermediate Method already achieves a sufficient reduction of complexity, but it is wanting in a different sense of simplicity.

4.4.2 Relevant Information

In paragraph 3, Euler goes on to write that the Brute Force Method gives us "a great many details that are irrelevant to the problem". Therefore, providing *Relevant Information* yields methods with more explanatory power. Euler thinks that the lack of relevant information is responsible for the complexity of the Brute Force Method.

Euler's Method provides us with a very transparent reason as to why an Euler path does or does not exist. We get an immediate grasp on the relevant property, the valence of vertices. The Intermediate Method provides us with a reason that is *somewhat* transparent – the method of determining the numbers for odd and even areas and that we have to sum them up makes intuitive sense – but it is not as clear as Euler's Theorem. Still, it is clearer than the reason provided by the Brute Force Method, which is nontransparent. Let us spell out this idea in more detail.

Euler writes that the Brute Force Method gives us details that are irrelevant to the problem at hand. In one sense, this is wrong: the method does not give us any irrelevant details at all, because we need a complete list of possible paths of a certain length to establish that there really is no Euler path on the system. Yet, in another sense, Euler is right. Going through all these possibilities to figure out whether there is a path or not seems like overkill. Why is that so?

Every time we write down one possible path, we draw on the structure of the bridge system, because this structure dictates what sequences are possible. However, we use the same structural information more than once, for example

[17] Note that while the Intermediate Method fares better than the Brute Force Method in terms of execution, the Brute Force Method is simpler to state than the Intermediate Method, i.e., the program-size complexity of the algorithms may increase, and thus not be correlated with explanatory power in all cases. I thank Hannes Leitgeb for bringing up this issue.

if two paths share an initial segment. The Brute Force Method extracts the structural information of the bridge system in a very redundant manner.[18] The Intermediate Method fares better in this respect than the Brute Force Method. We compute the length of potential Euler paths in a compositional manner, by exploiting a structural property of areas, their valence. As we use every area only once, redundancy is reduced. The property of the system we are interested in depends on the property of the parts of the system in a clear manner, which makes the Intermediate Method more transparent. Finally, if we turn to Euler's Method, we notice that the Intermediate Method still exhibits some redundancy: It requires us to determine the exact number of bridges connected to each area, compute another number from this, and sum up the results. In Euler's Method, most of this information is irrelevant. All that matters is whether the number of bridges connected to an area is even or odd. In some sense, Euler's Theorem is maximally informative in that it uses no irrelevant information at all: it is necessary and sufficient to know whether the numbers are even or odd to answer the why question.[19]

This suggests that the specification of a property in the *explanans* contributes to explanatory power insofar as it provides relevant information about the *explanandum* – i.e., to the degree that it is a "relevant property".[20] Not all of the structural information of graphs matters for the *explanandum* we are after. A highly relevant property squeezes exactly the right amount of information out of the structure, while a not-so-relevant property squeezes out too much, or not enough. One measure of relevance is the logical strength of a property. The less (irrelevant) information it contains, the better it gets, the ideal case being a relevant property that is logically equivalent to the *explanandum*.[21]

In the Brute Force Method, there is no property of graphs that we use in particular, we go directly for the possible paths, and the graph structure

[18] This does not speak against all kinds of algorithms that compile lists of paths; the redundancy could be overcome by using a clever search algorithm. This would also lower the (computational) complexity of the method. I thank Thomas Müller for pointing this out.

[19] Note, again, that Euler's Original Theorem does not prove that if all valences are even, a closed Euler path exists.

[20] The idea that explanatory power can be measured in terms of relevance of properties is inspired by the analysis of explanatory proofs in terms of "characterizing properties" proposed by Mark Steiner (1978a). He writes: "My view exploits the idea that to explain the behavior of an entity, one deduces the behavior from the essence or nature of the entity. [...] Instead of 'essence', I shall speak of 'characterizing properties' ..." (Ibid., p. 143) In an IME, the explanatory work is done by a characterizing property, a property used in the *explanans* that characterizes the *explanandum*. While I have sympathy for Steiner's proposal, I will not defend it here, as it has several problems. The proposal has been criticized on several occasions; see Resnik and Kushner (1987), and Hafner and Mancosu (2005). It should be noted that my account deviates from Steiner's in several respects. Most importantly, I do not suggest to interpret relevant properties as providing necessary or sufficient conditions for a categorical distinction between explanatory and non-explanatory proofs, as pointed out above in section 4.2.

[21] The idea that explanatory power is related to the specification of a sufficient and necessary condition, as opposed to a merely sufficient one, has recently been discussed by Christopher Pincock (????): his conception of "abstract explanation" requires an equivalence. I will discuss Pincocks' proposal below in section 6.5.

is "distributed" over the paths. In the Intermediate Method, we identify a relevant property of graphs, the valence of vertices. Summing up the valences as in equation (1) yields a necessary condition for the existence of an Euler path that is violated and thus accounts for our *explanandum*. Finally, Euler's Method improves on the relevant property of the Intermediate Method: We only have to know whether all valences are even or exactly two are odd. Euler's Theorem yields a necessary and sufficient condition for our *explanandum*; it is weaker than the relevant property used in the Intermediate Method.[22]

The concept of relevant properties providing sufficient conditions of varying degrees, or even sufficient and necessary conditions, is related to complexity as discussed above: If we have a good relevant property, this can lead to a reduction of the complexity of the corresponding method or proof. Still, these are two different measures of "informativeness": One is a measure of efficiency, the other one of the logical strength of conditions given by a theorem.

Summing up, we propose that the specification of a more or less relevant property contributes to more or less explanatory power of a method, or proof. One measure of relevance is the logical strength of properties. Weakening a sufficient property yields a more explanatory method; ideally, we get a property that is not merely sufficient, but also necessary for the *explanandum*. Other measures of relevance could be considered.

4.5 Explanatory Differences: From Theorem to Instantiation

So far, we have discussed the different features of Euler's three methods in terms of explanatory power, and we have ranked the three methods based on these differences. The discussion only concerned the notion of IME by Proof, but we are also interested in the ramifications of the preceding discussion for the corresponding cases of IME by Instantiation. Is the explanatory power of the general cases inherited by the application of the methods to mathematical instances?[23] Let us examine how differences in complexity and relevant information in the general case might affect the explanatory virtues of their instances.

First, if a certain method has low complexity, the application of this method to a particular case will not always lead to a faster solution. The notion of computational complexity only applies to the general case; it is usually a statement about how difficult it is to solve problems in the worst case. In instances where the worst case scenario is overly pessimistic, it may be easier to solve a problem based on a method that is of high complexity. Take, for example, a system with a circular graph. If we apply the Brute Force Method to this graph, a list of the

[22]It could be objected that Euler's Original Theorem only established that the two conditions found by Euler were merely shown to be sufficient for the non-existence of an Euler path. However, what is relevant here is that these conditions improve on the condition given by the Intermediate Method. Euler's Theorem, which establishes, additionally, that one of the conditions is also necessary for the non-existence of a closed Euler path, can be interpreted as a further refinement, but also specialization, of Euler's Original Theorem.

[23]I thank an anonymous referee for raising this issue.

possible paths will comprise one item, which is an Euler path, while on Euler's Method, we have to check the valence of each vertex.

However, in many cases, a method with low complexity will lead to a simpler solution in the particular case, and if this is so, the explanatory virtue of the general method carries over to the instances: Solution a of a particular problem is superior to solution b because solution a was found by applying method A, which is more efficient than method B. Compare the solutions to the Königsberg problem by the Brute Force Method and by Euler's Method: The former is more tedious at least partially because the Brute Force Method has high complexity.[24]

Turning, secondly, to a comparison of relevant information in instances, an instance may inherit the virtue of providing relevant information from a general method. Take the application of the Intermediate Method and Euler's Method to the Königsberg case. On the Intermediate Method, we calculate the number on the LHS of equation (1) above. As it is bigger than $n + 1$, no Euler path exists. On Euler's Method, we merely have to point out that more than two vertices have odd valence. It seems that here, the virtue of the latter explanation is partly due to the fact that Euler's Method picks out a more salient property of systems. On the other hand, it may also happen that an instance does not inherit this explanatory virtue of a general method. For instance, if a method provides merely a sufficient condition that is not also necessary, the method can simply be inapplicable to a particular instance, such that there is no explanation of the particular fact based on one of the methods.

In sum, both criteria of explanatory power do not carry over to particular instances in a straightforward manner. The relation between IME by Proof and by Instantiation is more delicate than one might have thought, and it certainly needs further scrutiny.

5 Königsberg as a Scientific Explanation using Mathematics

In this section, we turn to the question of how to conceive of the Königsberg case as an Scientific Explanation using Mathematics (SEM).

5.1 The Transmission View Revisited

Above, we discussed Euler's three methods on the level of pure mathematics. The advantage of the Intermediate Method over the Brute Force Method, and that of the Euler Method over the Intermediate Method, is based on the introduction of notation, which helps to extract information from a mathematical structure in an efficient manner.[25] The resulting increase in explanatory power

[24] Of course, much hinges on what "many" in the preceding sentence means. Questions about the *typical* complexity of the methods become relevant.

[25] The importance of notation, in particular in explanatory contexts, has recently been pointed out by Mark Colyvan (2012, ch. 8). It would be worthwhile to explore the ramifications of the present case for Colyvan's ideas.

is purely mathematical and has nothing to do with empirical application. This suggests that we can analyze the SEM versions of these explanations into two components. Some aspects of the explanations have to do with the relation between some structure and the world – the SEM component – while other parts, such as the three methods, operate on the purely mathematical level and are independent of this relation – the IME component. What aspects are to be located in which component?

The core of the explanation is the structure of the Königsberg bridge system, the mathematical object represented in figure 3. This structure has to stand in a certain relation to the world. This relation is the SEM component of the explanation. The IME component consists in a mathematical analysis of the structure with respect to a mathematical property, the existence of an Euler path. The application of any one of the three methods to the structure provides such an analysis.

The idea that we can distinguish two components of an SEM is reminiscent of a certain picture of SEM, the so-called "Transmission View", first proposed by Mark Steiner (1978b).[26] According to the Transmission View, SEM work via a transmission of an IME to some physical *explanandum*. When, in an SEM, we use a mathematical *explanans* M in the explanation of a physical *explanandum* P, this can be recast as the combination of an IME, with *explanans* M and a mathematical *explanandum* M^*, and a bridge principle connecting M^* and P. Thus, the IME is transmitted to the world via the bridge principle, and thus transformed into an SEM, and if we lose the bridge principle, we are left with an IME.

The Transmission View has attractive features that are worth pointing out. First, it accounts for the fact that mathematics can have explanatory benefits both within mathematics due to the IME component, and in application to the world due to the SEM component, and that these can be analyzed and assessed independently, at least to a certain degree. In the Königsberg case, this is descriptively adequate. Second, the division of labor between an SEM and an IME component mirrors the methodological differences between mathematics and empirical science, while maintaining that both are in the business of providing explanations. Finally, from the point of view of philosophical methodology, it is useful to break a concept down into components, and to reconstruct it from there, using a "divide and conquer" strategy, i.e., consider the IME and the SEM components separately, and then think about how they interact. Even if this picture breaks down, it will be instructive to analyze why it does not work.

It is for these reasons that I want to defend the Transmission View. My goal is to defend the Transmission View as one important kind of SEM; however, I do not claim that it is the only way to understand SEM. One particular claim by Steiner's version of the Transmission View, pointed out by Alan Baker (2012) in his reconstruction of Steiner's account, that I do not wish to endorse, is that in an SEM, all the explanatory work is done by the IME component. I think

[26] The label "Transmission View" was introduced by Alan Baker (2012).

this is wrong: The bridge principle connecting the mathematics and the world is far from trivial; it is an important contribution to the SEM. We should think hard about what this connection is, and what mathematics contributes – we will turn to this in the next section. I will discuss recent objections against the Transmission View below in section 6.

5.2 Presuppositions ...

We begin our analysis with the question of what explicit and implicit assumptions are necessary to formulate the Königsberg Problem as a mathematical problem. What do we assume when we choose the map, and the labels, as the starting point of our mathematical investigation? An analysis, and comparison, of the historical sources suggests that we can distinguish at least three kinds of assumptions.

First, in his letter, Ehler relied on the map in figure 1 to formulate the Königsberg Problem. This provided Euler with a preselection of what counts as an acceptable solution to the Königsberg Problem: Ehler was interested in "structural" answers, i.e. answers related to facts about how the parts of the city hang together. This presupposes that the system of paths is stable over time. Second, most details about the city of Königsberg are irrelevant and therefore left out. I have already pointed out one example: the information about the direction of flow of the river is contained in Ehler's map, but omitted in Euler's map. On the other hand, relevant aspects are highlighted in Euler's map by adding labels. By assigning lower-case letters to bridges, the crossing of a bridge is turned into a fundamental process, the details of which are irrelevant: the crossing of several bridges can be written as a sequence of lower-case letters. Third, both map and labels represent structural constraints on possible paths. We presuppose that we can cross the bridges in certain sequences only, that areas with distinct letters, say C and D, are disjunct, and that the map is complete, i.e., that there are no bridges outside the map.

How should we interpret these presuppositions? What do they tell us about the kind of explanation we are dealing with? I propose to interpret the presuppositions as causal and pragmatical in nature.

5.3 ... Causal ...

When it comes to scientific explanations, causal explanations are something like the gold standard. Traditionally, causal explanations explain by specifying a cause of the *explanandum*. Do the three methods, applied to the Königsberg system, provide causal explanations? Not according to the traditional conception of causal explanations. Euler's explanations not provide a single cause that explain the impossibility to travel the system in a certain way. Also, if the above analysis of these explanations is correct, the explanations are not *purely* causal, because the IME component is an explanatory contribution of pure mathematics to these explanations, and mathematics is commonly taken to be non-causal.

However, the explanations are not non-causal either. We can interpret the mathematical structure causally, in the following manner. The lower case letters stand for basic causal processes: the label x stands for the crossing of bridge x, where we do not care about the exact path that is taken, or the direction in which the bridge is crossed. The structure of the bridge system restricts the possibilities of how these causal processes can be combined: not any succession of lower case letters is a path; the basic causal processes can be combined in certain ways, but not in others. Thus, the explanations are of the form: given that certain fundamental causal processes (crossing bridges) can only be combined in certain ways, we can decide mathematically whether it is possible to find a sequence such that every fundamental causal process occurs exactly once. The explanations rely on the causal network of the Königsberg bridge system and provide relevant causal information.[27]

If the Königsberg SEM can be reconstructed within the framework of the Transmission View, such an outcome has to be expected: The explanation is not causal, as it has a mathematical component; however, it is not non-causal either, because the bridge principle adds a causal component.

5.4 ... and Pragmatical

How should the causal presuppositions of the Königsberg SEM be interpreted? One possibility is to view them as conditions of applicability: If the presuppositions are true of the system, then the Königsberg SEM will work, if not, then we cannot apply the mathematical structure to the bridge system, and the explanation breaks down. However, I think that this perspective is too narrow. If someone were to point out that there will obviously be other ways of crossing the rivers besides the bridges represented in the map, we would not concede that this invalidates the explanation, but that the explanation still applies, because we have taken a certain pragmatic perspective on the problem.[28] Euler's formulation of the Königsberg Problem is natural only after we have already accepted this perspective. Is it not obvious that we have to answer the question as to why it is impossible to cross all the bridges exactly once based on the map and the structural constraints. The context, and the way in which the question is formulated, go a long way towards suggesting what kind of answer we will consider to be acceptable, and the mathematical solution removes all ambiguity. These issues are related to the pragmatic aspects of explanations.[29]

[27] Marc Lange (2013) has argued against the view that the Königsberg case should be classified as a causal explanation, even on a wide reading of that notion. I will discuss his reservations below in section 6.4.

[28] The pragmatic aspect of mathematical explanations, and of the present case in particular, have been emphasized by Marc Lange (2013), see section 6.4 below.

[29] The *locus claussicus* for the pragmatics of scientific explanation is Bas van Fraassen's pragmatic theory of explanations, as proposed in van Fraassen (1980, ch. 5). See, however, Jakob (2007, section 2.3) for criticism of van Fraassen's account in general, and Sandborg (1998) for a critical analysis of van Fraassen's theory, in the context of mathematics. I will not reconstruct the Königsberg case in van Fraassen's theory here. All that matters is that we can think of explanations as answers to why questions, that we ask these questions with

The starting point of the Königsberg case is a request for an explanation, as we saw in section 5.2. The why question prompting the explanation is: why is it impossible to cross all bridges in Königsberg exactly once and return to the starting point? This question was asked with certain presuppositions, and the way in which the question is asked determines what is explanatorily relevant. For example, relevance is expressed by the maps and the notation. Ehler, who asked the question in his letter, was interested in *structural* reasons for the failure of finding a certain path, by providing a sketch of the situation in Königsberg. Euler further narrowed down the relevant aspects of the structure, by omitting irrelevant details and by introducing the letters for areas and bridges. Euler's search for an appropriate mathematical representation was, at the same time, a search for a representation that captures the explanatorily relevant factors, as requested in the formulation, and in the context, of the initial why question. Thus, the transition, from the initial request for an explanation to the reformulation in a mathematical framework, also plays an explanatory role in that the reformulation makes the pragmatic relevance relation explicit.

Within this reconstruction, the initial why question is a question about the real bridge system; it is not a purely mathematical question about graphs. The determination of the explanatory relevant factors is at least partially located in the translation of the question about the real system into the question about the graph. The determination of explanatory relevant causal factors becomes obsolete once we conceive of the question as purely mathematical. These two ways of framing the explanation should not be conflated. Pragmatic aspects are salient in the coordination of the real bridge system, and the mathematical structure.

Summing up, the SEM component of the explanation we considered, the bridge principle connecting the Königsberg graph with the bridge system, provides the parts of the graph with a causal interpretation, which, in turn, can be justified pragmatically, i.e., the bridge principle captures those aspects of the real bridge system we consider to be explanatorily relevant in the present context.

6 Königsberg in the Philosophical Discussion

The above discussion sheds light on some issues in the recent philosophical debate on scientific explanations using mathematics.

6.1 Objections to the Transmission View

Alan Baker (2012) and Mark Lange (2013) have raised objections against the Transmission View. I will argue here that these are difficult to maintain in view of the above discussion.

certain implicit presuppositions in mind, which are closely related to the context, and that the context helps in determining which explanations, or answers to the why question, are acceptable.

Baker identifies two problems with the Transmission View. He first proposes a counterexample, the explanation of the structure of the bee's honeycomb based on the mathematical honeycomb conjecture. This counterexample has recently been criticized in Räz (2013). I will not comment on this problem here. Secondly, Baker argues that it is a problem for the Transmission View that we can use a theorem in an SEM without citing the proof of this theorem in the SEM.

Baker supports his argument with a well-known case of SEM, the cicada case.[30] The *explanandum* is that certain cicadas have live cycles with periods of 13 and 17 years. The *explanans* consists of biological and mathematical premisses. The biological premisses are that it is evolutionary advantageous to minimize intersection with other periods (to avoid predators emerging in periods, or to avoid hybridization), and that there are ecological constraints on the possible length of life cycles (they have to lie in a certain interval). The mathematical premiss, a number-theoretic theorem, is that prime periods minimize intersection. From this we can deduce that cicadas will have life cycles of 13 or 17 years. Baker argues that in this explanation, a number-theoretic theorem is used, but the proof of said theorem does not feature in the explanation. From this he concludes that the Transmission View is flawed, as the standard explanations of the cicada case do not use the proof of the number-theoretic theorem. According to him, it is sufficient for the explanation *that* the theorem has been proved.

I agree with Baker that in the cicada case, the proof of the number-theoretic theorem need not be part of the explanation. However, this only undermines the Transmission View if we presuppose that all IMEs have to involve a proof. If this is not so, we can accept the cicada case as a good explanation without using the proof of the number-theoretic theorem. In the cicada case, the number-theoretic theorem also explains why the numbers 13 and 17 maximize the lowest common multiple in a certain interval. This is a – maybe not very interesting – purely mathematical explanation of a mathematical fact. I argued above that the application of Euler's Theorem to the Königsberg graph is an IME by Instantiation, such that no (substantive) proof features in the explanation. If we accept the distinction between IME by Proof and by Instantiation, then Baker's argument is in jeopardy.

Marc Lange (2013) is also critical of the Transmission View:

> [N]one of the mathematical explanations in science that I have mentioned [including the Königsberg case, TR] incorporates a mathematical explanation in mathematics. These mathematical explanations in science include mathematical facts, of course, but not their proofs – much less proofs that explain why those mathematical facts hold. (Ibid., pp. 507)

I think this is wrong, for the following reason. I agree that the explanation of why there is no Euler path in Königsberg need not contain a proof of Euler's

[30]The cicada case was introduced in Baker (2005); see also the discussion in Baker (2009).

Theorem. However, Euler's Theorem, without proof, also explains the *mathematical* fact that there is no Euler path on the Königsberg graph. At least in the present case, the SEM version of the explanation does incorporate an IME, and a reconstruction in terms of the Transmission View is unproblematic (of course, this does not answer the question as to how widely the Transmission View is applicable).

Baker and Lange could deny that the example of an IME by Instantiation that I just described is in fact an acceptable explanation. However, compare the following IME with its SEM "twin":

1. **IME** *Explanandum:* There is no Euler path on the Königsberg graph. *Explanans:* In the Königsberg graph, all four vertices have odd valence. According to Euler's Theorem, a graph has an Euler path if and only if all, or all but two, vertices have even valence.

2. **SEM** *Explanandum:* There is no path in the bridge system of Königsberg that uses every bridge exactly once, i.e. there is no Euler path in Königsberg. *Explanans:* Given a set of reasonable presuppositions, the bridge system of Königsberg has the structure of a graph in which all four vertices have odd valence. According to Euler's Theorem, a graph has an Euler path if and only if all, or all but two, vertices have even valence.

Both Baker and Lange accept the SEM version as a (good) explanation. Then, however, I find it hard to deny that the IME version is a good explanation as well. Baker and Lange face the challenge of explaining how we can accept one of the above explanations, and not the other.

6.2 Abstract Explanations I (Pincock 2007)

Christopher Pincock (2007) proposes interpreting the Königsberg Problem as an instance of "abstract explanations". These are explanations that pick out certain relations of a physical system, while other aspects of the system are ignored. Abstract explanations can rely on mathematics, by using a structure-preserving mapping between the physical system and a mathematical domain, but this mapping does not depend on an arbitrary choice of units, or a coordinate system – it captures an intrinsic feature of the system. When we give an abstract description of a situation, we tell the truth, and nothing but the truth, but not the whole truth.[31]

I think that abstraction is an important part of the process, which we can discern in the historical path to the formulation of the explanation. However, Pincock's account of the Königsberg case is incomplete in two respects.

First, if my claim, that the mathematical formulation of the problem has a pragmatic aspect, is correct, then the relation between the real bridge system

[31]The "nothing but the truth" part distinguishes abstraction from idealization: in abstraction, we leave out certain properties in our description of a system, while in idealization, our description of the system comprises claims that are, strictly speaking. wrong; see e.g. Batterman (2010) for a recent discussion.

and the mathematical graph potentially violates the "nothing but the truth" clause. For example, it may be possible to find an Euler path in the system, if we took alternative paths in the real bridge system into account, but this is simply excluded for the explanation's sake. If we take the pragmatic aspect of the explanation into account, then there is no direct mapping between the mathematical structure and the world. The relation between the mathematics and the world is more complicated: the mathematical structure helps expressing our explanatory presuppositions.

Secondly, abstract explanations fail to account for the explanatory contributions of pure mathematics. If all a mathematical structure does is to represent some structure in the world, how can mathematics be explanatorily helpful? Why don't we base the explanation directly on the structure in the world, which, on this account, is the same as the mathematical structure?[32] On my account, one answer is that we can extract the relevant information from the abstract structure in a more or less informative manner.

6.3 Abstract Acausal Representations (Pincock 2012)

In his recent book, Christopher Pincock (2012) classifies the Königsberg case as an "abstract acausal representation" – acausal because it does not represent change over time. Pincock writes:

> The mathematics here is not tracking genuine causal relations, but is only reflecting a certain kind of formal structure whose features in the physical system have some scientific significance (Ibid., p. 53)

I agree with Pincock that the graph does not represent change over time. However, I think it is wrong to classify the representation as acausal for this reason. I proposed above that we can give the components of the structure, i.e. the edges, a straightforward causal interpretation. In this sense, the mathematical structure *is* tracking genuine causal relations.

6.4 Distinctively Mathematical Explanations (Lange 2013)

Marc Lange (2013) categorizes the Königsberg case as a so-called "distinctively mathematical explanation". These are scientific explanations in which mathematics plays a distinctive role. They provide information that is modally stronger than causal information, and do not even fall under a broad notion of causal explanation. They work "by showing how the explanandum arises from the framework that any possible causal structure must inhabit, where the 'possible' causal structures extend well beyond those that are logically consistent with all of the actual natural laws there happen to be" (Ibid., p. 505).

In his discussion of the Königsberg case, Lange grants that we can give the parts of the Königsberg graph a causal interpretation, and that we presuppose

[32]See Bueno and Colyvan (2011, p. 351) for a prior formulation of this problem.

causal stability of the bridge system for the explanation. Distinctively mathematical explanations can cite causes, or rely on causal structure. However, he maintains that the Königsberg case is a distinctively mathematical explanation:

> [T]he fixity of the arrangement of bridges and islands, for example, is presupposed by the why question that the explanation answers: why did this attempt (or every attempt) to cross this particular arrangement of bridges – the bridges of Königsberg in 1735 – end in failure? [...] [T]he why question itself takes the arrangement as remaining unchanged over the course of any eligible attempt. If, during an attempt, one of the bridges collapsed before it had been crossed, then that journey would simply be disqualified from counting as having crossed the intended arrangement of bridges. The laws giving the conditions under which the bridges' arrangement would change thus do not figure in the explanans. (Ibid., p. 497)

Lange thinks that the explanatory work is done by the mathematics, while all the causal presuppositions (laws) are fixed by the why question, and do not figure in the *explanans*.

Some aspects of Lange's account of the Königsberg case are compatible with my account. He emphasizes the distinctive role of pragmatics in mathematical explanations, and notes that the why question, and the context, help in determining a subset of acceptable answers. Finally, he stresses the contribution of pure mathematics to scientific explanations.

The main point of contention is that while Lange grants that causal presuppositions about the empirical system are necessary in order to make the explanation applicable, this does not turn the explanation into a causal explanation, even on a broad reading of the notion. This is so because the explanatory power is independent of the causal interpretation of the mathematical structure: Even if we were not talking about the crossing of bridges, the explanation would still stand.

I agree that the mathematics is explanatory in its own right, by accounting for the purely mathematical fact that no Euler path exists in the Königsberg graph. However, I think that this is due to the fact that we can analyze the explanation into two components, one of which is an *intra*-mathematical explanation. This accounts for the modal nature of Euler's explanation. Adding the SEM component, the pragmatically motivated bridge principle, turns the IME into an explanation that explains *qua* interpretation of the structure in terms of causal processes. If the SEM component is removed, the remaining explanation does no longer convey causal information – but it also is no longer about the bridge system. Once we settle for this particular application, the explanation becomes irrevocably causal – on a broad reading of the notion.

6.5 Abstract Explanations II (Pincock forthcoming)

Pincock (????) proposes a substantially revised notion of "abstract explanation". He bases his proposal on a detailed analysis of a case study, the explanation of

Plateau's laws by the mathematician Jean Taylor in 1976. In this case, the *explanandum* is why soap bubbles and soap films obey Plateau's laws of the local structure of surfaces. Here is a sketch of Pincock's account, using his case study as an illustration.

The *explanandum* is: Why do systems of type X (soap bubbles and soap films) have property ϕ (obey Plateau's laws)? The abstract explanation of this fact is characterized by three features; see Pincock (????, p. 9): (1.) It provides a classification of systems with property ϕ by giving a necessary and sufficient condition: $\psi \leftrightarrow \phi$. In the case study, the necessary and sufficient condition ψ is the class of so-called "almost minimal sets"; these are two-dimensional surfaces that obey a technical notion of surface minimization. Taylor showed that almost minimal sets obey Plateau's laws. (2.) This classification draws on abstract entities, the almost minimal sets, such that the systems with property ϕ instantiate abstract entities of type ψ. (3.) There is a suitable link between the classification ψ and the *explanandum*, that systems of type X have property ϕ. This is necessary because a physical structure could obey, say, Plateau's laws, without any link to surface minimization, for example if the physical structure were built by hand. Taylor's result is applicable to soap bubbles and films only because we know that physical surface minimization is at play.

After the exposition of his account, Pincock goes on to compare abstract explanations to other recent accounts of non-causal and mathematical explanations, such as so-called "program explanations", first proposed in Jackson and Pettit (1990) and recently defended in the context of mathematical explanations in Lyon (2012), and "distinctively mathematical explanations" proposed by Lange (2013), see the discussion above. Here I will restrict myself to some comments of Pincock's discussion of Lange's position, as it is most relevant in the present context.

He raises two objections to Lange's account. First, distinctively mathematical explanations (just as program explanations) only require sufficient conditions. This, however, is too weak a requirement, as the *explanans* could contain redundant requirements. The second problem with distinctively mathematical explanations has to do with their pragmatic aspect. Pincock thinks that the account is too permissive:

> Lange must provide some further story of how the context shapes or constrains genuine explanatory questions. Otherwise, the danger is that every phenomenon will have a distinctively mathematical explanation and the special sort of case that Lange is after will be obscured. (Ibid., p. 17)

Introducing a pragmatic aspect into a conception of mathematical explanation à la Lange threatens to trivialize the account.

Let me briefly comment on Pincock's objections. First, Pincock's point, that specifying a sufficient *and* necessary condition in the *explanans* can be explanatorily beneficial, seems fair to me. Gaining knowledge about what systems do *not* have the property specified in the *explanandum* can help us understand the phenomenon in question. However, the distinction between mere sufficiency and equivalence does not justify a distinct kind of explanation. On my account, we can locate the distinction between program explanations and abstract explanations on the level of pure mathematics, and the distinction between the two cases is one in degrees of explanatory power: A theorem entering into an SEM can give a sufficient condition, or it can specify an equivalence. It can be explanatory in both cases, but it will provide us with more relevant information, i.e., be more explanatory, in the case of equivalence.

Second, let me address Pincock's objection to the pragmatic aspect of Lange's account. I don't know if a completely general characterization of the pragmatics of explanation is to be had, and pragmatics are probably not an equally important issue in all cases of mathematical explanation; for example, in the case of Plateau's laws, the question of how mathematical and physical optimization are related, seems much more important than pragmatics. I propose that, in the Königsberg case, the explanation using graph theory is good, because the causal interpretation of the mathematical structure yields reasonable causal restrictions that are pragmatic in nature. For example, we can accept that bridges outside the map are simply neglected for the purposes of the explanation, and that the system is stable over time. We could also reject Euler's explanation if we decided that such presuppositions are *ad hoc*, or otherwise unacceptable. Thus, the pragmatic aspect does not threaten to trivialize my and Lange's account, at least not in the Königsberg case.

7 Conclusions

The present paper has three parts. In the first part, we analyzed Euler's paper on the Königsberg bridges problem. We witnessed the first steps in the invention of graph theory, and the emergence of three methods for solving the Königsberg problem.

Second, we used the historical case for a systematic discussion; here are the most important systematic lessons from this discussion. First, we can interpret Euler's solutions both as explanations in pure mathematics (IME), and as scientific explanations using mathematics (SEM). Second, at least two kinds of explanations in pure mathematics are at work: a) methods that solve a general problem (IME by Proof), and b) instantiations of these methods to particular (mathematical) cases (IME by Instantiation). Third, we identified a local, gradual notion of explanatory power in the case of IME by Proof, which allows us to compare the explanatory power of a methods that solve the same problem. The criteria we identified are complexity and relevant information. Fourth, we found that the criteria for explanatory power just identified can, but need not, transfer to the corresponding IME by Instantiation. Fifth, turning to the SEM variety, we found that we can analyze this kind of explanation into a bridge principle and an IME component. This, and methodological considerations, lead us to endorse a version of the Transmission View of SEM; pure mathematics contributes to the explanatory power of SEM. Sixth, we proposed that mathematics contributes to the SEM in representing causal structure, which is justified pragmatically.

Third, we put these systematic lessons to work in the recent discussion on mathematical explanations in general, and of the Königsberg case in particular. We defended a version of the Transmission View against recent objections by Alan Baker and Marc Lange, and we compared the present proposal to accounts of mathematical explanations by Christopher Pincock and Marc Lange.

What's next? The account should be applied to other cases. One obvious, interesting example is the explanation of Plateau's laws, briefly discussed above. It has to be expected that the role of the bridge principle might be different in such cases. Then, he notion of IME should also be investigated further, using the characteristics of explanatory power proposed here. In particular, the proposal that explanatory power is correlated with computational complexity needs further scrutiny. There are probably further contributors to explanatory power of IMEs that have not been identified here. Finally, the notion of explanatory proof, and its interaction with the picture sketched here, should be substantiated.

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