Say My Name.
An Objection to Ante Rem Structuralism

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Abstract

In this paper I raise an objection to *ante rem* structuralism, proposed by Stewart Shapiro: I show that it is in conflict with mathematical practice. Shapiro introduced so-called "finite cardinal structures" to illustrate features of *ante rem* structuralism. I establish that these structures have a well-known counterpart in mathematics, but this counterpart is incompatible with *ante rem* structuralism. Furthermore, there is a good reason why, according to mathematical practice, these structures do not behave as conceived by *ante rem* structuralism.

1 Introduction

When it comes to the nature of mathematical objects, many philosophers and mathematicians embrace a form of structuralism. Philosophers of mathematics have tried to formulate a metaphysics of mathematics in structuralist terms for quite some time. One kind of structuralism is particularly popular: Stewart Shapiro's *ante rem* structuralism, first proposed in Shapiro (1997).

In this paper, I critically assess *ante rem* structuralism. After a short introduction to *ante rem* structuralism, I raise my principal objection to this position by showing that it is in conflict with mathematical practice. Shapiro introduced so-called "finite cardinal structures" to illustrate features of *ante rem* structuralism. I establish that these structures have a well-known counterpart in group theory, but this counterpart is incompatible with *ante rem* structuralism: It has an *in re* character. Furthermore, there is a good reason why, according to mathematical practice, these structures do not behave as conceived by *ante rem* structuralism: We want to be able to establish connections between different representations of abstract structures, and in order to do this, we rely on "coordinates", non-structural properties of structures.

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2 Ante Rem Structuralism: The Story So Far

The idea behind *ante rem* structuralism is that we should not think of mathematical structures in terms of their instantiations (*in re*), but in terms of the structural features that the structures have independently of, or "before", instantiations (*ante rem*). For example, *the* structure of natural numbers is independent of its instantiations as, say, some ordinal structure. It is exhaustively characterized by the axioms of natural numbers. The natural numbers are the places in this structure, characterized in terms of the structural relations such as the successor function. This is the so-called places-are-objects perspective of *ante rem* structuralism.

Ante rem structuralism is an attractive position because each mathematical structure is taken seriously in itself. We work exclusively with the properties and relations that are naturally available in a structure; it is not necessary to interpret structures in terms of, say, set theory. This meshes well with many mathematician's conception of the autonomy of mathematical subdisciplines: A graph theorist is working with graphs, he is not doing some version of set theory.

A serious objection, however, has been raised against *ante rem* structuralism; see Burgess (1999) and Keränen (2001). The objection is based on two facts. According to *ante rem* structuralism, we can characterize mathematical objects exclusively in terms of the structural properties (including relations) of the structure to which the objects belong. Secondly, Shapiro can be read as endorsing a form of the Principle of the Identity of Indiscernibles (PII): if two objects of a structure share all structural properties, then they should be identified; see Shapiro (2008, p. 286). This leads to the objection that structures with certain symmetries are not adequately captured by *ante rem* structuralism.

The concept of structures exhibiting more or less symmetry can be made more precise using the concept of *automorphism*. An automorphism is a bijective, structure-preserving function (isomorphism) from the structure to itself. In the case of natural numbers, there is only one automorphism, the identity function. Structures on which only this (trivial) automorphism can be defined are called *rigid*. Structures admitting of non-trivial automorphisms are called *non-rigid*. Places of structures linked by a non-trivial automorphism are called *structurally indiscernible*.

Non-rigid structures, such as the complex numbers, do have places, e.g. i and $-i$, that are structurally indiscernible – but which are nonetheless not identical: the additive inverse of i is $-i$, not i itself. As 0 is the only complex number additively inverting itself, and i is not 0, i and $-i$ have to be different. But, according to (IND), i and $-i$ should be identified.[1] *Ante rem* structuralism

[1] Note that i and $-i$ need only be identified according to (IND), which is just one possible formulation of PII. There are several notions of indiscernibility on the market; see Ketland (2011) and Ladyman et al. (2010) for a discussion of these notions and their interrelations). According to *weak discernibility*, i and $-i$ are discernible by a formula $\phi(x, y)$ expressing the fact that x and y are additive inverses: $\phi(i, i)$ is false in the complex number structure, while $\phi(i, -i)$ is true. If (IND) were based on weak discernibility, then i and $-i$ would be discernible (but see Ketland (2006) for criticism of weak discernibiliy).

appears not to adequately capture mathematics, which is unacceptable for a nonrevisionist position, such as Shapiro's.

In reaction to this objection, Shapiro (2008) agrees that it would be fatal if *ante rem* structuralism were committed to the above form of PII. However, he denies that this is the case. He thinks that in mathematics, identity cannot be defined in a non-circular way, and that mathematics presupposes identity.[2] *Ante rem* structuralism can thus be amended in the following way: we use only structural properties that are naturally available in a mathematical structure to characterize the objects belonging to that structure, and identity is one of these structural properties.

If we accept identity as a primitive relation, then Shapiro has successfully averted attacks based on PII. For the sake of the argument, I accept Shapiro's solution and assume that identity is available as a primitive relation. What follows has nothing to do with metaphysically motivated principles, such as PII.

3 The Problem: No-Name Places

The feature of *ante rem* structuralism that I consider to be problematic concerns reference in mathematics. To see the problem more clearly, I will underline an implicit distinction made by Shapiro.

In Shapiro's opinion, one attractive feature of *ante rem* structuralism is that "in most cases, reference is straightforward" (Shapiro, 2008, p. 290). One straightforward case is the structure of natural numbers with unique, structurally characterized places interpreted as objects: the numeral "4" refers to the fifth place in this structure. While Shapiro accepts the idea that "singular terms in true sentences [... suggest] that there are objects denoted by those terms", he denies the converse: "It is simply false that to be an object is to be the sort of thing that can be picked out uniquely with a singular term." (Ibid.) How can this be the case?

Shapiro gives several examples where reference to mathematical objects fails. For big structures, such as the real numbers, at least one problem of reference is well-known: Given a countable supply of names, we cannot name or describe all real numbers at once, as they are uncountable. We can "diagonalize out" of any list of members of these structures. Therefore, the countable supply of names cannot be in a one-one-correspondence with the members of these structures. There are probably further problems with reference to members of big or random structures, but I will not discuss them further, as the claim about failure of reference due to uncountability is uncontroversial.

My focus will be on a different type of example: structures that are *too homogeneous* for reference. Shapiro thinks that certain mathematical structures with symmetries have the property that we cannot name or refer to the objects, or places, in these structures because they are *too homogeneous*. He writes:

[2]See Shapiro (2008, p. 292) and Leitgeb and Ladyman (2008) on this point. The proposal that identity is presupposed in mathematical practice has been made in Ketland (2006), as Shapiro notes.

> There simply is no naming *any* point in Euclidean space, nor any place in a finite cardinal structure and in some graph, no matter how much we idealize on our abilities to pick things out. The objects are too homogeneous for there to be a mechanism, even in principle, for singling out one such place, as required for reference, as that relation is usually understood.(Shapiro, 2008, p. 291)

I take it that the reason why we cannot name the objects in these structures is that there are no structural properties to pick them out, or discern them. Identity is of no help, as structurally indiscernible places can be nonidentical. I will call this the "no-naming constraint" of *ante rem* structuralism.

I think that the no-naming constraint is an undesirable feature of *ante rem* structuralism, because it is in conflict with mathematical practice. This I will show by examining the paradigm of homogeneous structures, the finite cardinal structures mentioned in the above quote. Finite cardinal structures comply with the no-naming constraint to the extreme: none of their places can be named, because they are too homogeneous. I will show that a correlate of finite cardinal structures in mathematics does not comply with the no-naming constraint.

This creates a problem for Shapiro, because he also endorses the so-called *faithfulness constraint*. This is the "*desideratum* [...] to provide an interpretation that takes as much as possible of what mathematicians say about their subject as literally true, understood at or near face value" (Shapiro, 2008, p. 289, emphasis in original).[3] Shapiro wants his position to be in agreement with mathematical practice as much as possible. If naming the objects of finite cardinal structures is no problem in practice, then either the no-naming constraint or the faithfulness constraint has got to go.

4 Finite Cardinal Structures in Mathematics

Shapiro characterizes the cardinal-four structure, one kind of finite cardinal structures, as follows:

> The cardinal-four structure [...] has four places, and *no relations*. [...] Since there are no relations to preserve, every bijection of the domain is an automorphism. Each of the four places is structurally indiscernible from the others and yet, by definition, there are four such places, and so not just one. (Shapiro, 2008, p. 287)

The cardinal-four structure has four objects, or places, and no relation between these objects; therefore, every bijection between the objects is an automorphism. Technically speaking, the places are pairwise structurally indiscernible.

We will now locate the cardinal-four structure in mathematical practice. Initially, it is unclear how to interpret the cardinal-four structure in ordinary

[3] See Shapiro (1997, ch. 1) for more on the faithfulness constraint.

mathematical terms, because if we cannot name the objects of a structure, it is not clear how to define a function on the structure.[4] We will therefore choose a familiar starting point, and work our way from there. We will use the familiar idea that structures can be characterized via structure-preserving functions.

Usually, a structure is defined by giving some domain, say $C = \{1, 2, 3, 4\}$, on which we can define functions in the usual way. There are no relations on C, so every $f : C \to C$, with f bijective, is an automorphism. In mathematics, a bijection on a (finite) domain which is not required to respect any relations, is called a permutation of C. Mathematicians are interested in permutations because the set of permutations of a (finite) domain, equipped with composition of functions, forms an important group called *symmetric group*, written S_n if the size of C is n. The members of the group S_4 are the permutations of C.[5]

Clearly, C is not the cardinal-four structure: the elements of C are natural numbers, thus we can name them. This carries over to the permutation group on C: According to the *ante rem* structuralist, some of the permutations of C should be indistinguishable. Take the functions f, defined as $f(1) = 2, f(2) = 1, f(3) = 3, f(4) = 4$, and g, defined as $g(1) = 3, g(2) = 2, g(3) = 1, g(4) = 4$. They are different members of S_4. However, the only difference between f and g is that f permutes 1 and 2, while g permutes 1 and 3.

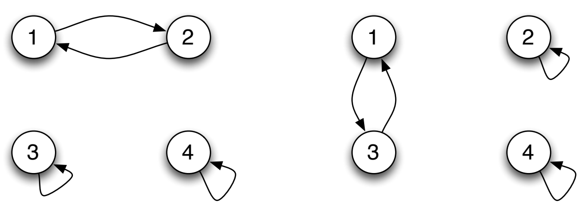

Figure 1: Functions f (left) and g (right)

Thus we cannot use the permutation group S_4 to characterize the cardinal-four structure: f and g are distinguishable, which should not be the case, as 2 and 3 play the same structural role. We have to "identify" f, g, and any other permutation of C that only swap two places of C and leaves all other places untouched.

In mathematics, the result of this identification is well-known and has many different, but equivalent descriptions. A particularly intuitive approach is based on the notion of cycles. To understand how this works, we need the cycle notation of permutations. A cycle of length $r \leq n$, written $(i_1 i_2 ... i_r)$, is a permutation $a \in S_n$ such that $a(i_1) = i_2, a(i_2) = i_3, ..., a(i_{r-1}) = i_r, a(i_r) = i_1, a(i_k) = i_k$ for $k \neq 1...r$, i.e. it sends r places of the domain around in a cycle and leaves the other places alone. For example, f above is the cycle (12).

[4]This worry has been formulated before, see Hellman (2005, p. 545, fn. 10).

[5]The portion of elementary group theory used in the following can be found in any introduction to group theory, see e.g. Rotman (1995).

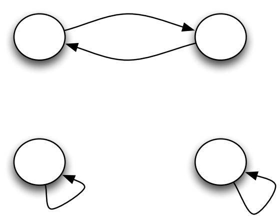

Figure 2: Function on Cardinal-Four Structure

Alternatively, we can also write f as (12)(3)(4), i.e. 3 and 4 are cycles of length one.

It is a theorem of group theory that all permutations can be written as a product of disjoint cycles. Thus we can think of permutations as cycles. This is very useful for our purposes, because we can use the cycle notation to classify all permutations into *cycle types*, also known as cycle structure. The cycle type of a permutation only depends on the number of cycles of length one, two, etc. of the permutation. The cycle type of a permutation $a \in S_n$ is written $(1^{m_1}, 2^{m_2}, ..., n^{m_n})$, meaning that the permutation a has m_1 cycles of length 1, m_2 cycles of length 2, and so on. For example, the permutation $(12)(3)(4) \in S_4$ above is of type $(1^2, 2^1, 3^0, 4^0)$. Some thought reveals that the permutations in S_4 fall into five cycle types. Here is one instance of each type: (1)(2)(3)(4), (12)(3)(4), (123)(4), (1234), (12)(34).

Cycle types are one way of representing finite cardinal structures. A cardinal-four structure is characterized by the fact that we can define five "essentially different bijections" on its places: All permutations are automorphisms, but some permutations have to be be identified. Now, the five essentially different bijections on the cardinal-four structure coincide with the five cycle types of S_4, see figure 3 below. These cycle types represent kinds of permutations by abstracting from the particular numbers (or places) that are permuted. They only appeal to facts such as the number of places mapped to themselves, the number of places mapped to each other, the number of places mapped in three-cycles, and so on – the structure of cycles.

It is not necessary to capture finite cardinal structures in terms of cycle types; the idea can be restated in many different forms. For example, there is a natural correspondence between cycle types and certain subgroups of S_n called conjugacy classes: Two permutations are of the same cycle type if and only if they are in the same conjugacy class.[6] Another perspective is in terms of partitions of natural numbers. A partition of n is a (non-strictly) increasing sequence of natural numbers $i_1, i_2, ...i_r$ such that $i_1 + i_2 + ... + i_r = n$.[7] There

[6] If G is a group and a a member of G, the conjugacy class of a is the set of b such that $b = xax^{-1}$ for some x in G.

[7] Note, incidentally, that a closed form expression for $p(n)$, the number of partitions of n is not known, see Simon (1996, p. 96). By extension, the same is true for the number of "essentially different" bijections on finite cardinal structures.

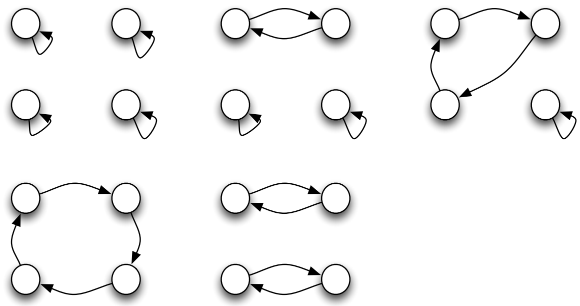

Figure 3: Cycle Types of S_4

is a one-one correspondence between partitions of n, cycle types of length n, conjugacy classes of S_n, and "essentially different" bijections on C, and we can use any one of these concepts to capture finite cardinal structures.[8]

5 Ante Rem Structuralism vs. Mathematical Practice

After this detour into mathematics, we are ready for our philosophical problem. Are cycle types *ante rem* structures in Shapiro's sense or not? More specifically, is it possible to name their places? Mathematical textbooks do not give a direct answer to this question, because naming is not a mathematical notion. However, they give an indirect answer.

In mathematics, different permutations such as f and g that belong to the same cycle type are always distinguishable. This follows from the way in which the permutations belonging to a certain cycle type are counted. A theorem tells us that the number of permutations of type $(1^{m_1}, 2^{m_2}, ..., n^{m_n})$ is $n!/\big(\Pi_{j=1}^{n}(m_j)!j^{m_j}\big)$ see Simon (1996, Theorem VI. 1.2.). Applied to the cycle type $(1^2, 2^1, 3^0, 4^0)$ of f and g, we find that the number of permutations of this type is $4!/\big(2!1^2 \cdot 1!2^1 \cdot 0!3^0 \cdot 0!4^0\big) = 6$; these are the 6 permutations of the set C that swap two places and leave two places untouched. This means that we can recover all the cycles belonging to a cycle type, all the different bijections on C, and especially f and g. We can move freely between cycle types as in figure 2 and cycles as in figure 1.

This is not so according to *ante rem* structuralism. In figure 2, the places 2 and 3, while nonidentical, are structurally indiscernible. There are no properties or relations to discern them, and we cannot name them. From the perspective of *ante rem* structuralism, there is exactly one function in figure 2. But there

[8]Shapiro (2008) points out a suggestion in Leitgeb and Ladyman (2008) according to which finite cardinal structures are (isomorphic to) certain graphs. This is yet another way to conceive of finite cardinal structures.

is no way to recover, or count, different permutations such as f and g in figure 1 that instantiate the function in figure 2.

The reason for this is that the *ante rem* structuralist can only use structural differences and identity to distinguish between f and g. However, they have the same structural role: they swap two places, and leave two places alone. In particular, the *ante rem* structuralist cannot use the fact that f and g are different because f swaps 1 and 2 while g swaps 1 and 3. All that can possibly matter for the *ante rem* structuralist is that two (nonidentical) places are swapped, while two further places, not identical to the former two, are left alone. There is one such situation, not two, or six.

Now, the *ante rem* structuralist could maintain that it is a primitive fact of identity that f and g are different permutations. However, this is not a fact that can be grounded in the identity and structural discernibility of the *places* that are permuted. Both f and g swap two nonidentical, structurally indiscernible places, so the nonidentity of places is of no help in distinguishing the two. The *ante rem* structuralist would need *additional* facts about the identity and discernibility of *functions*; more specifically, he would have to assume that there are exactly six different permutations with the same cycle type as f – and so for all other cycle types of all permutation groups. I think this is not an attractive option.

The easy way out would be to state the obvious: f and g are different because, well, 2 and 3 are different. This, however, the *ante rem* structuralist cannot do, as he would have to label the places of the cardinal-four structure as 1, 2, 3, 4, and then describe the different permutations between these numbers, which, arguably, amounts to naming the places. Once he has adopted the *ante rem* perspective, the *ante rem* structuralist cannot move freely from figure 2 to figure 1.

Why is the situation different for the mathematician? The mathematician simply uses non-structural properties to discern the places of cycle types; for example by defining the permutations on a set of natural numbers. He can then use the non-structural properties of the places of permutations to calculate how many permutations belong to each cycle type.[9] It appears that mathematicians adopt an *in re* perspective for circle types: They are not considered in isolation from their instantiations, but in close correspondence.

It could be asked[10] whether the problem of distinguishing the functions f and g could be solved by treating the places of the cardinal-four structure as parameters. We could skolemize the axiom of the cardinal-four structure

$$\exists x_1, x_2, x_3, x_4(x_1 \neq x_2 \wedge ... \wedge x_3 \neq x_4 \wedge \forall y(y = x_1 \vee y = x_2 \vee y = x_3 \vee y = x_4))$$

by eliminating the outermost existential quantifiers by introducing a new

[9]The point that in mathematics, we use non-structural properties to discern places in structures has been made before, see e.g. Hellman (2001). Hellman's criticism of *ante rem* structuralism is more general and severe than the one advanced here, as he considers the position to be incoherent.

[10]I thank an anonymous referee for this question.

parameter for each quantifier. This procedure is akin to the rule of existential instantiation. If we now conceive of the functions as being defined on these parameters, we can very well distinguish the functions f and g.[11] Drawing on second-order logic, it is even possible to formally deduce that there are exactly six different permutations on the cardinal-four structure, i.e., on the structure with exactly four objects.[12]

However, using parameters does not solve the problem, for the following reason. I certainly agree that the parameters can be used to *represent* the places of the cardinal-four structure. However, we can only distinguish the functions f and g as functions between parameters, which represent the places. If we want to establish, additionally, that we the functions f and g between the *places* are distinguishable as well, we would need a stable relation, a one-one correspondence between parameters and places (an interpretation of the parameters). This, however, would essentially amount to naming the places using parameters, which is impossible according to *ante rem* structuralism. The use of parameters also affects the possibility of deducing the intended result within second-order logic: such an argument presumably relies on skolemization, and therefore yields the result that there are exactly six permutations of four parameters, not four places.

6 Ante Rem vs. In Re

My objection against *ante rem* structuralism is not *a priori* or metaphysical. The problem is Shapiro's faithfulness constraint, which it is in tension with the no-naming constraint; I argued that the no-naming constraint has consequences that contradict mathematical practice. Of course, the *ante rem* structuralist can claim that cycle types and the other structures above do not really capture his idea of finite cardinal structures. However, these structures are as close as mathematics gets to Shapiro's finite cardinal structures. If he does not think that cycle types adequately capture his idea, we can reasonably question the relevance of these structures for mathematics – unless he comes up with a mathematical structure that captures finite cardinal structures even better.

According to Shapiro, the faithfulness constraint is relative and has to be weighted against other desiderata. If the feature of mathematics that is not faithfully mirrored by *ante rem* structuralism is only of minor importance, we could still dismiss it; after all, *ante rem* structuralism is able to capture some aspects of mathematical practice. Are there good reasons for conceiving of structures as *in re* rather than *ante rem*? Why is it important to count cycles of a certain type in a certain way?

[11] See Shapiro (2008) and Pettigrew (2008) on this issue.

[12] Such an argument would begin with the introduction of the above axiom; we would then eliminate the existential quantifiers using parameters. After constructing the six different permutations between four parameters, we would conclude, by *reductio*, that all permutations have to be identical to one of the six permutations just constructed; see Shapiro (1991) for the necessary background. I thank an anonymous referee for drawing my attention to this possibility.

There are good reasons for adopting an *in re* perspective. One reason is that it is an important part of mathematics to explore different perspectives, or representations, of one and the same abstract, *ante rem* structure. We saw an example of this practice above: We can think of finite cardinal structures in terms of cycle types, but also in terms of conjugacy classes or partitions of natural numbers. One advantage of these different representations is that we can use our knowledge of one of the representations for all the others.

However, in order to do this, we have to be able to prove that the different representations are equivalent, and in these proofs, we often use instances of abstract structures ("Let π be a permutation of type x ..."), and structure-preserving mappings between these instances. This is why it is important that we can move freely between an abstract structure and its instances. This is impossible if we adopt an *ante rem* perspective, as we saw in the case of the cardinal-four structure.

In conclusion, *ante rem* structuralism is right in emphasizing that we should take abstract mathematical structures seriously – they are more than their instantiations. However, I argued in this paper that we should not take abstraction too far. If we start to think of abstract mathematical structures as completely freestanding and independent of their instantiations, we lose sight of the fact that mathematics is also about the different representations of structures. If we want to make use of these representations, we have to be able to move back and forth between abstract structures and their instantiations, or an *ante rem* and an *in re* perspective.

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