Extensionality and Logicality*

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Abstract

Tarski characterized logical notions as invariant under permutations of the domain. The outcome, according to Tarski, is that our logic, which is commonly said to be a logic of extension rather than intension, is not even a logic of extension—it is a logic of cardinality (or, more accurately, of "isomorphism type"). In this paper, I make this idea precise. We look at a scale inspired by Ruth Barcan Marcus of various levels of meaning: extensions, intensions and hyperintensions. On this scale, the lower the level of meaning, the more coarse-grained and less "intensional" it is. I propose to extend this scale to accommodate a level of meaning appropriate for logic. Thus, below the level of extension, we will have a more coarse-grained level of *form*. I employ a semantic conception of form, adopted from Sher, where forms are features of things "in the world". Each expression in the language embodies a form, and by the definition we give, forms will be invariant under permutations and thus Tarskian logical notions. I then define the logical terms of a language as those terms whose extension can be determined by their form. Logicality will be shown to be a lower level analogue of rigidity. Using Barcan Marcus's principles of *explicit* and *implicit extensionality*, we are able to characterize purely logical languages as "sub-extensional", namely, as concerned only with form, and we thus obtain a wider perspective on both logicality and extensionality.

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1 Prelude

A key component of Tarski's contribution to the foundations of the developing field of mathematical logic is his characterization of logical notions—the subject matter with which mathematical logic is concerned. In continuation with Klein's characterization of geometric disciplines through invariance properties, Tarski set logical notions as the most general notions, and characterized them as the elements of the set-theoretic hierarchy that are invariant under all permutations of the given underlying domain. Logical notions do not distinguish between elements of the domain, and are not, in general, sensitive to difference in extension:

> It is customary to say that our logic is a logic of extensions and not of intensions, since two concepts with different intensions but identical extensions are logically indistinguishable. In the light of [our results] this assertion can be sharpened: our logic is not even a logic of extensions, but merely a logic of cardinality, since two concepts with different extensions are still logically indistinguishable if only the cardinal numbers of their extensions are equal and the cardinal numbers of the extensions of the complementary concepts are also equal.

(Lindenbaum & Tarski, 1935, pp. 387-8)

Philosophers have, throughout history, dealt withs different levels of meaning to which we may refer when interpreting a language: some languages require recourse to the level of intensions, and others only to the more coarse-grained level of extensions. In this paper I elaborate on the remark made by Tarski (here with Lindenbaum and later in (Tarski, 1986) on his own), and I characterize logical languages—languages whose vocabulary is purely logical—as "sub-extensional": as sensitive to a level of meaning even more coarse-grained than that of extension. Lindenbaum and Tarski mention cardinality as the only aspect of meaning that is relevant to logic, but to be more accurate, permutation invariance allows further distinctions—those that more generally concern set-theoretic structure. The idea I aim to explore is that there is a level of meaning which concerns only structure and which logical languages make recourse to, which I will label *form*. That is, a linguistic expression can be thought to have, besides an intension and an extension, also a form, the latter construed semantically as a type of meaning more coarse-grained than extension.

Throughout the paper I will motivate this idea of form building on Tarski, as well as Barcan Marcus and Sher, and I will present some of its consequences. However, before moving forward, let me first put forth a characterization of form that will serve as the basis of our inquiry. The initial characterization will have the structure of an abstraction principle; it will, in this respect, resemble the initial characterzation of "isomorphism type" offered in, *e.g.*, (Levy, 1979). This principle captures the essence of what shall be meant by *form*, and will serve as an adequacy condition for any definition of form. Later on we shall see an explicit definition of form that satisfies this principle. We thus state when two expressions have the same form in their respective domains, and this equality will depend on their extension. We assume that the language is interpreted, and use "$ext_D(t)$" to denote the extension of a term t in a domain D. In so-called *extensional contexts*, all that matters to the truth value of a sentence is the extensions of its components. The elements of the domain serve as the building blocks for extensions. We shall see that in some contexts the identity of the elements of the domain is immaterial, and only the domain's cardinality will matter—those will be the so-called *formal contexts*. In such cases it suffices to speak of the form of an expression in a domain-size.

Adequacy Condition (*form*) Let t and t' be primitive expressions in a given language L. $form_D(t) = form_{D'}(t')$ if and only if there is a bijection $f : D \to D'$ such that $f(ext_D(t)) = ext_{D'}(t')$.[1]

This condition will be explained in due course. We can already see, however, that form, as presented here, is not a syntactic feature of expressions, but a semantic one. This kind of approach to forms and to formality is inspired by Sher's *semantic conception of form*, as I shall explain in §4. That form can be thought of as a level of meaning will be the main concern of this paper. At this point, I would like to draw attention to two crucial points. First, that forms of expressions are more coarse-grained than their extensions in any given domain. Secondly, that the adequacy condition is borne out of Tarski's characterization of logical notions as permutation invariant. We shall attend to Tarski's account shortly, but for now, note that forms are insensitive to the particular identity of individuals in the domain and are invariant under permutations of the domains over which they are defined.

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[1] We extend a bijection f between two domains to a function between the set-theoretic hierarchies built over those domains by recursion in the natural way. By "domains" we mean sets of ur-elements. The restrictions of domains to ur-elements enables the recursion by which f is extended to work, by blocking a set from being both a member and a subset of the domain.

Note that on the left we have an identity statement, and that the condition on the right (there is a bijection $f : D \to D'$ such that $f(ext_D(t)) = ext_{D'}(t')$) is an equivalence relation on pairs of domains and expressions, and thereby the bi-conditional has the structure of an abstraction principle.

2 Introduction: Logical Notions and Logical Terms

In a lecture delivered in 1966, Tarski gave an account of *logical notions*, the entities he assumed to be the subject matter of logical inquiry. Tarski characterized logical notions as the set-theoretic entities invariant under permutations of the given underlying domain. More recent discussions have taken up Tarski's characterization in an effort to demarcate *logical terms*. The idea is that logical terms denote logical notions, as characterized by Tarski or in a similar modified way. The common contention is that logical notions are the *extensions* of logical terms.

While *form* in the context of language is frequently associated with syntactic structure, the semantic tradition in logic following Tarski treats formality in a semantic manner. That is, formality does not have to do only with the way expressions combine to form more complex structures, but it also has to do with the things in the world which expressions mean or denote and their properties. The main proponent of the semantic approach to formality is Gila Sher, who takes the semantic conception of formality a step further. As for Tarski, for Sher formality is not merely a syntactic phenomenon, and the logical terms—characterized as formal—are defined through semantic characteristics. But further, for Sher, formality itself is not only a feature of expressions (vis-à-vis their semantic properties), but can be found "in the world". Thus, on such an approach, logical terms refer to formal features (of extensions) in the world. Quantifiers, for instance, are treated in the Fregean manner as second-level predicates. They thus denote higher-order extensions, *i.e.*, things in the world, some of which are *formal*. The quantifier *There are exactly 3* denotes the higher order property of having three members (Sher, 1996, p. 674). Forms are thus higher-order extensions, and are structural, rigidified, features or properties of lower order extensions. And so, for Sher formality is not merely a linguistic phenomenon: it is rather grounded in non-linguistic, "objectual" properties (Sher, 1996, p. 670). The connection between forms on the objectual level and language is manifested by the logical vocabulary.

In this paper, I take up Tarski's characterization of logical notions and Sher's semantic approach to form, and shift them to another field. Not only shall forms be non-linguistic and in the world, I shall moreover consider *(semantic) form* as its own type of semantic value comparable to *extension*, *intension* and *hypertension*. I treat those types as occupying different levels of meaning lying on a scale of coarseness, starting with the most coarse-grained at the bottom, and going up to the more fine-grained at the top. The level of form will be coarser than that of extension and thus might be considered as "sub-extensional". I shall then employ

Barcan Marcus's principles of explicit and implicit extensionality to situate forms under extensions in the scale. The basic idea is therefore that starting with an interpreted language, each expression (not just the logical ones) has, in addition to its extension and intension, also a form that will be derived from its extension. Tarski's logical notions serve as inspiration for this definition. Logical terms will then be defined. While in the Tarski-Sher tradition, logical terms are defined as those terms denoting logical notions, here logical terms will be characterized as having a special kind of form.

In the proposed theory we thereby obtain a wider perspective on both logicality and extensionality. The formality of logical languages is characterized as a low level of intensionality. Namely, we are able to characterize purely logical languages as "sub-extensional"—as concerned only with *form*.

We shall see that as Tarskian logical notions, forms are invariant under permutations. An added benefit we obtain from the proposed theory is that Tarski's proposal becomes an inherent part of a theory about language. Tarski first formulated his characterization in the context of Klein's Erlangen Program, where various geometric disciplines were characterized through different invariance criteria. Tarski then construed logical notions—the subject matter of logical inquiry—as the most general notions, compared to other notions in the Program. However, when employed in a logical-linguistic inquiry, the comparison to geometric disciplines seems out of place. Here, I adopt Tarski's idea of logical notions as general and coarse-grained, and import it to a linguistic scale, where the granularity of various levels of meaning is compared. Logic is commonly said to be concerned with forms of expressions, and here this saying will be understood in a new way, as forms will constitute now their own, *sui generis*, semantic role.

The plan of the paper is as follows. In §3, I present Tarski's characterization of logical notions in the context of Klein's Erlangen Program. In §4, I define forms of primitive expressions. I compare forms to Tarskian logical notions, and invoke Sher's semantic conception of form. In §5, I discuss logicality: I define logical terms by a special property of their form. I show that logicality is analogous to rigid designation on a lower level of meaning. In §6, I relate forms to other levels of meaning through Barcan Marcus's *principles of explicit extensionality*, the basic idea of which is that different levels of meaning can be loosely ordered by the granularity of their associated semantic values, from the more extensional to the more intensional. I then discuss Barcan Marcus's *principles of implicit extensionality*, where various levels of meaning are compared in terms of the contexts in which they enable substitution *salva veritate*. In both cases, the logical vocabulary defined in §5 is used in characterizing the languages and contexts appropriate for the level of form. Further, I show, using McGee's theorem on the definability of permutation invariant operations, that the first order infinitary language $L_{\infty\infty}$ can serve to define "formal contexts", those contexts where expressions of the same form can be substituted *salva veritate*. In §7, I define forms of complex expressions, and discuss the logicality of sentences. Logical sentences in this context, as other logical expressions, are sensitive to the cardinality of the domain, and thus constitute a wider class than that of logical truths and falsehoods.

3 Tarski and Klein's Erlangen Program

Take a geometric space, or manifold, $\mathcal{S}$. For simplicity, assume that $\mathcal{S}$ is two-dimensional, *i.e.*, a plane, and that $\mathcal{S}$ consists of points that are specified by two coordinates from the real field. By taking sets of points in this space we can form lines and geometric figures. Take a *notion* in this context to be any entity in the set-theoretic hierarchy built over a given domain, in this case the geometric space $\mathcal{S}$. Now, we can consider the notion of *triangle*. A triangle is a set of points consisting of three intervals adjoining at the ends. The notion of triangle is thus the set of all such sets.

Now, it is a basic feature of notions of Euclidean geometry that they are indifferent to *motions*: transformations of the space that preserve distances (also termed *isometric transformations*).[2] That is, if you take an isometric transformation, a one-one and onto function of the plane which preserves distance, the image of a triangle under that transformation will be a triangle. If you extend the transformation in a natural way to apply to sets built over the plane, then the image of the set of triangles under a motion is just the set of triangles: and thus we say that the notion of triangle is invariant under motions. And more generally, every property referred to in Euclidean geometry is invariant under motions. In a way, then, there is some arbitrariness in the system of coordinates. Euclidean geometry at its base is not concerned with the particular identity of points in space, but rather with higher-order structures, such as properties and relations between figures, that are unaffected by distance-preserving mappings.

This modern conception of geometry is most prominent in the work of Felix Klein (1893). In the so-called *Erlangen Program*, Klein showed how various geometric disciplines can be characterized by groups of transformations over a geometrical space. Indeed, we can take the previous observation on Euclidean

[2] We use "transformation", as Tarski, to mean a one-one function from the space onto itself.

geometry a step further and use a group of transformations to *define* the notions relevant to the discipline. In the case of Eucledean geometry, we can use isometric transformations as defining the discipline.

A space with a group of transformations gives us an identity criterion of sorts for the entities of the relevant discipline, which is more coarse-grained than extensional identity of sets (Marquis, 2008). We can have two different sets of points which can yet be identified if they do not differ with respect to any notion of Euclidean geometry. In any such case, one of the sets will be the image of the other under some isometric transformation. We can thus define an equivalence relation $\equiv_{Euc}$ which holds between two sets if one is the image of another under an isometric transformation. Note that we obtain such an equivalence relation from any class of transformations as long as they form a group.[3] The relation $\equiv_{Euc}$ can be thought of as "identity with respect to the notions of Euclidean geometry."

Klein observed that various geometrical disciplines can be characterized by a space and a group of transformations. If we look at affine transformations, transformations which preserve betweeness and co-linearity, we obtain the notions of affine geometry. Affine transformations include isometric transformations and much more, and so the notions of affine geometry are a subset of the notions of Euclidean geometry. The more transformations we include, the more general the notions we obtain. Topology is yet a more general discipline, allowing only notions invariant under homeomorphisms (continuous transformations with a continuous inverse).

Klein's Erlangen Program has been employed and extended in various ways (see *e.g.* (Marquis, 2008)). An extension to logic was most famously carried out by Tarski ((Tarski, 1986), but see also (Mautner, 1946)). How does logic fit into this scheme, which classifies geometric disciplines and concerns geometric spaces at its base? The idea is that if we take Klein's method to the limit, and consider *all* transformations on a given space, we thus obtain the most general notions. Tarski's logical notions are those notions that are invariant under all transformations (or *permutations*, to use the expression prevalent in contemporary literature). In this case we can conceive of an equivalence relation $\equiv_{log}$ which holds between two notions if one is the image of the other under some permutation, coding "identity with respect to the notions of logic." The generality of logical notions strips them from any content specific to some geometric discipline: no notion that could be reasonably conceived as distinctive of some geometric disci-

[3] The group's associated operation is composition of functions. Reflexivity is obtained from the identity element which is the identity function, symmetry is obtained from the existence of an inverse, and transitivity is obtained by closure under composition.

pline, as abstract as it may be, will be logical.

Indeed, which notions are permutation-invariant? On the first, base level, that of points in space, no notion is invariant: logic does not distinguish between particular points. But so far, there is no departure from Eulidean geometry, which, as we have seen, does not concern the particular identity of points, but rather only higher-order properties of figures in geometric space. On the next level, that of sets of points, only the empty set and the whole space are invariant under permutations. Again, that is the case also in Euclidean geometry: only those two sets are invariant under isometric transformations. The difference in generality between the two disciplines becomes apparent when we move further, to relations over the given space or to sets of subsets of the domain. Consider the quadric relation $R(x,y,z,w)$ which holds if the distance between x and y is equal to the distance between z and w. This relation is invariant under isometric transformations, but not under all transformations. Alternatively, consider the second level property of being a triangle, which holds of a set of points only if they form a triangle. Surely, the notion of triangle is invariant under isometric transformations, but not under *all* transformations. When it comes to relations, the only logical notions we have are those that have to do with equality or distinctness (so, the only logical binary relations are the empty and universal relations, the relation of being equal, and the relation of being non-equal). As for second-level properties and relations, we only have notions that are set-theoretic or have to do with size: being non-empty, having three elements, being a subset of, being bigger than, etc.

We see that there is nothing particularly geometric about the logical notions. It is as if by accident we used as the base a geometric space, as the dimension or coordinates associated with the points in space come nowhere into play: we could have used any domain as our base, with all its permutations. And indeed, considering a domain and its associated transformations is an idea of great generality which transcends the realm of geometry, as is made manifest in contemporary category theory.[4]

Notwithstanding Tarski's characterization, logic is traditionally conceived of from a very different perspective, as concerned with validity of arguments. Logical validity is commonly said to be a matter of the form of the argument, and as arguments are often taken to be linguistic entities, logic has a distinctive connection with language. Tarski himself made a central contribution to our current understanding of validity in his (Tarski, 1936). Thirty years later, Tarski's lec-

[4]Though, through the influence of Klein, the mere idea of characterizing a discipline by a space and a group of transformations is thought of as "geometric," referring to the method of classification rather than the distinctive content of geometric disciplines.

ture connecting logic to a program so influential in mathematics represents, in contrast, the modern conception of logic as a bona fide mathematical discipline. Tarski's focus is on the subject-matter of this rising discipline, and so here he turns away from the traditional philosophical concerns. The turn to the geometric comparison class in Klein's Erlangen Program is foreign to the traditional view of logic as having a distinctive connection with validity and language.

Nevertheless, Tarski's characterization of logical notions has been deployed widely in current debates on *logical terms*: the part of the vocabulary of a language which determines its relation of logical consequence. According to the criterion of permutation-invariance, logical terms are the terms that denote Tarskian logical notions, and those are the terms that are to be held fixed when checking for logical validity. Now it is important to stress that Tarski himself did not present his characterizarion of logical notions as relating to the question of logical terms, at least not as those are connected to the concept of logical consequence. Quite the contrary: in his lecture (Tarski, 1986) Tarski expressly distinguishes the matter of logical notions and the matter of logical truth. Logical consequence, or validity, is not even mentioned. It is thus rash to interpret Tarski's logical notions as filling a gap left open in his paper on logical consequence (Tarski, 1936), where Tarski expresses skepticism about the existence of a sharp boundary between logical and nonlogical terms.[5] The later employment of permutation-invariance in the literature on logical consequence thus extends the initial project of the 1966 lecture which simply seems to address mathematical logic as a mathematical discipline with its own subject-matter, on a par with other mathematical disciplines, including the geometric ones.[6] In sum, it appears that logical notions were de-

[5]Especially as we may note that the idea of permutation invariance is prominent in (Lindenbaum & Tarski, 1935) and so was salient for Tarski at the time of writing the paper on logical consequence (Tarski, 1936)—and yet Tarski does not appeal to it. Tarski notes that the results in (Lindenbaum & Tarski, 1935) were presented as early as 1932-3. In a letter to Morton White, as late as 1944 (White & Tarski, 1987), Tarski remains skeptical regarding a principled criterion for logicality, and again, does not even mention the results from the paper with Lindenbaum published in 1935.

Another interpretive option is to view Tarski as attending to the 1936 lacuna in his 1966 lecture, but not seeing himself as providing an absolute criterion. The characterization of logical notions as invariant under permutations still leaves room for relativity: what comes out as logical will depend on the background set theory (see (Gómez-Torrente, 2015)). This interpretation sits well with the fact that Tarski does not declare that he had filled the gap of the 1936 paper, but it does not explain why Tarski, in the later lecture, avoids drawing any connection to that paper or to the concept of logical consequence.

[6]But see the later (Tarski & Givant, 1987) where an explicit connection between logical terms in a given formal language and logical notions is made. Note, however, that when the link is finally made, it is in the context of a mathematical text on languages for set theory. The definition there of logical

fined independently from the motivations guiding the model-theoretic definition of logical consequence in (Tarski, 1936).

There is, however, a connection between logical notions and language and linguistic expressions that Tarski makes in his 1966 lecture. Tarski notes that the notions definable in conventional systems such as that of *Principia Mathematica* are all invariant under permutations, referring to a result from (Lindenbaum & Tarski, 1935). But note that this remark does not set logical notions apart from geometric notions in an essential way. In the same article referred to, the definability of geometric notions is described as well, with respect to various axiomatic systems for geometry. Thus, the notions definable in a system for Euclidean geometry are all found to be invariant under isometric transformations. By these lights, Tarskian logical notions appear to be of a most general nature through a comparison to geometric notions. Even if the end-result appears to be independent of all geometric content, logical notions have no *special* connection to language, let alone to truth or logical consequence. Logical notions are connected to language as much as any set of geometric notions: each can be associated with a subset of the vocabulary used to refer to them—we use the geometric vocabulary to talk about geometry and we use the logical vocabulary to talk about logic.

Now, one can retain this Tarskian attitude and extend the analogy between logic and other disciplines to the context of consequence relations as well. Gila Sher takes a step further and makes an explicit connection between permutation invariance (or, to be accurate, isomorphism invariance) through logical terms to logical consequence. As with Tarski, logic is the most general discipline, though Sher does not limit herself to the geometrical context. Thus, according to Sher, as there is logical consequence, so there is biological or legal consequence (Sher, 1996, p. 670). For each kind of consequence relation one fixes an appropriate vocabulary, where logical consequence is distinguished as being general and formal. Indeed, the denotations of logical terms (which are isomorphism invariant, and are thus Tarskian logical notions) are, according to Sher, formal features of the world, whereas the biological vocabulary concerns biological features of the world, and so on. We shall return to this point in the next section.

In what follows I will employ some of the fundamental ideas of Tarski and Sher on logical notions. However, I will make their linguistic role more significant. The relevant comparison class will not be that of mathematical disciplines (or

symbols (p. 57) serves for proving a mathematical theorem, and is not used elsewhere in the book. The definition is not backed by any philosophical analysis, and no connection is made to Tarski's work on logical consequence (though other relevant works: Tarski's lecture and Tarski's earlier paper with Lindenbaum are mentioned).

other scientific disciplines), but will rather be of a linguistic nature, and more specifically, will consist of levels of meaning. Sher's semantic conception of form will facilitate the new connection.

Recall the quote from Lindenbaum and Tarski with which the paper starts. At the end of the 1966 lecture, Tarski reiterates:

> This result seems to me rather interesting because in the nineteenth century there were discussions about whether our logic is the logic of extensions or the logic of intensions. It was said many times, especially by mathematical logicians, that our logic is really a logic of extensions.[7] This means that two notions cannot be logically distinguished if they have the same extension, even if their intensions are different. As it is usually put, we cannot logically distinguish properties from classes. Now in the light of our suggestion it turns out that our logic is even less than a logic of extension, it is a logic of number, of numerical relations. We cannot logically distinguish two classes from each other if each of them has exactly two individuals, because if you have two classes, each of which consists of two individuals, you can always find a transformation of the universe under which one of these classes is transformed into the other. Every logical property which belongs to one class of two individuals belongs to every class containing exactly two individuals.

(Tarski, 1986, p. 151).

It is the idea gestured at by Tarski in this quote that I would like to develop in the following sections. From a linguistic perspective, logic is concerned neither with intension nor with extension, but with an even more course-grained level of meaning which I will call *form*. Tarski's quote would have us associate this level of meaning with cardinality. Indeed, cardinality plays a major role, but a more accurate description of what is preserved by permutations is *isomorphism type*. Strictly speaking, Tarski's quote is applicable only to sets of the first level: if A and B are sets of individuals of the same cardinality (here: "number"), and their complement in the domain are also of the same cardinality, then $A \equiv_{log} B$. However, when it comes to relations, cardinality is insufficient for identifying logical notions, as further set-theoretic structure can make a difference (to take a simple example, by any acceptable set-theoretic definition of the ordered pair, $\{\langle a, a\rangle\} \not\equiv_{log} \{\langle a, b\rangle\}$ if $a \neq b$: both sets are singletons, but none can be obtained by a

[7] Tarski refers here in a footnote to (Whitehead & Russell, 1910, *III* (2)).

permutation from the other). The notion of form we shall explore in what follows will be associated with isomorphism type. However, as we shall see, cardinality plays an essential role in logical languages, as the cardinality of the domain is all that matters for the truth values of their sentences.

4 The Semantic Notion of Form

On the picture I propose, as there are intensional and extensional languages, there are also *formal* languages: these languages do not mind differences in extension, but only differences in *form*. In this section I shall define forms, and in later sections I shall characterize formal languages in this new sense.

The idea of form to be employed here is a purely semantic one. As the other notions of meaning mentioned, forms will be nonlinguistic entities. This might defy an intuition by which form has to do with syntax. The idea is not to replace the syntactic idea of form, but to focus on the semantic perspective.

We begin with an interpreted language, understood in a very wide sense. Thus, we assume that each well-formed expression of the language may be assigned at least one of an extension, an intension and a hyperintension, via some intended model appropriate for the language. We view those different types of semantic values as inhabiting different *levels of meaning*. Normally, an interpretation or model for a language takes into account just some of these levels. Thus, we might consider extensions, but not intensions as the relevant meanings for some discussion—and make do with extensional models, or we might consider extensions and intensions, but not hyperintensions—and work with possible worlds models. Here we assume that all the levels exist, though perhaps disregarded in certain contexts.

Moreover, the levels of meaning are distinguished by their granularity, where the most coarse-grained meanings are extensions, and the finer the distinctions we make, the more intensional the relevant notion of meaning is. In the formalism, extensions are set-theoretic constructions over a given domain: elements, sets, relations etc. are the extensions of individual constants, monadic predicates, polyadic predicates etc.[8] We adopt here the standard technical definition of intensions as functions from possible worlds to appropriate extensions therein. The

[8]Extensions for connectives can be defined as operators from sets of assignments satisfying subformulas to a set of assignments satisfying the complex formula. See §7, and in particular fn. 18, for more details. I shall not be concerned with modal operators and other expressions that belong solely to intensional languages.

extension of *Dog* in a given domain is the set of dogs in the domain, and the intension is a function picking out in each possible world in a given range of possible worlds the set of dogs there.[9] Intensions provide us with the modal profile of an expression, but have been argued to be still too coarse-grained to account for synonymy. All necessarily true sentences, for instance, have the same intension. And so a more fine-grained notion is required. Hyperintensions serve to account for synonymy and can be construed in different ways. We can follow Tichý's rendering of Fregean senses in *Transparent Intensional Logic* and identify hyperintensions with constructions: each expression will encode the procedure by which its extension is to be computed in each possible world; see (Tichý, 1988; Duží, 2010, 2012). Other options include identifying hyperintensions with structured intensions (Carnap, 1947; Cresswell, 1975) or algorithms (Moschovakis, 2006).

Now, extensions, as the most coarse-grained type of semantic value, are commonly perceived as lying at the bottom of the induced scale. I would like to propose that: a) we can think of *forms* of expressions as constituting a level of meaning lying below extensions (as "sub-extensions"), and that b) permutation invariance can guide us to a definition of form through the adequacy condition we have formulated in the very first section. The two parts of the proposal will be dealt with in conjunction, supporting each other.

Let us then turn to forms, elaborating on the adequacy condition stated in the first section. The form of an expression will be defined based on its extension. Note that the extension of an expression can be derived from its intension by applying the intension to a distinguished actual world. The intension can normally be derived from the hyperintension by "forgetting" the extra structure. Indeed, the different levels of meaning mentioned here are linearly ordered in terms of granularity.

For greater generality, let us assume that while the language is interpreted, the domain is not settled. This means that we can consider the extension of an expression in different domains, but as the language is interpreted, the extension of the expression is determined by the domain. Recall our notation: for any expression α, $ext_D(\alpha)$ is the extension of an expression α in a domain D.

Forms will be invariant under permutations, as are Tarskian logical notions. However, instead of looking at permutations on a given domain, we shall widen our perspective to include bijections in general—allowing for variable domains. Note that Tarski's characterization of logical notions assumes one given domain

[9] There are reasons to keep models mathematically pure—consisting of only mathematical objects—in which case the extension of *Dog* in a domain will be a set of mathematical objects used as a surrogate for a set of dogs. I will leave these issues aside.

(indeed, a geometric space). However, contemporary model-theoretic semantics employs models with different domains for greater generality. And so, the mappings relevant to invariance conditions should take this into account. Thus, Sher's criterion for logicality is that of invariance under isomorphisms of structures, induced by bijections between underlying domains (see (Sher, 1991, pp. 49ff)). The move from permutations to bijections is a natural one, and is accepted in the literature on logical terms, and so Tarski's permutation invariance and Sher's isomorphism invariance are normally grouped together as the *Tarski-Sher criterion* for logicality. For us, this move allows for a domain-independent notion of form: the form of an expression will depend on the domain in which it is interpreted, and yet two expressions can have the same form over different domains.

Something like the "isomorphism type" of the extension of an expression would do the job of form. An "isomorphism type" of a structure can be defined by abstraction, as in Levy (1979, p. 49), who in turn generalizes Cantor's method of abstraction in the definition of ordinal type. Levy states an abstraction principle as a temporary assumption, and later on gives an explicit set-theoretic definition satisfying the assumption. We follow a similar method, using the adequacy condition presented in the prelude. We thus state a condition on forms that has the structure of an abstraction principle, *i.e.*, we give an identity criterion for forms on the basis an equivalence relation interpreted as "having the same form". At this point we shall discuss forms for primitive expressions only. To reiterate:

Adequacy Condition (*form*) Let t and t' be primitive expressions in a given language L, and let D and D' be domains. We say that $form_D(t) = form_{D'}(t')$ if and only if there is a bijection $f : D \to D'$ such that $f(ext_D(t)) = ext_{D'}(t')$.[10]

First, we note that *forms*, presented thus, are more coarse-grained than extensions: if $ext_D(t) = ext_D(t')$ then $form_D(t) = form_D(t')$, but not necessarily vice versa. The level of meaning of form will be associated with logic—with logical languages and logical contexts—as will be made clear in what follows.

By the above adequacy condition, all individual constants have the same form over any given domain, and moreover—over all domains of the same size. For two monadic predicates to have the same form, both their extension and their anti-extension (the complement of their extension in the domain) must be of the same cardinality. The predicates *Even* and *Odd* by their standard interpretation over the domain of the natural numbers have the same form, which in turn is distinct

[10]Recall that we extend a bijection f between two domains to a function between the set-theoretic hierarchies built over those domains by recursion in the natural way.

from the form of the predicate *Greater than* 2 over the natural numbers. The latter, however, has the same form as *Different from* 7, 9 *and* 12 over the natural numbers. With relations, we need more than cardinality to determine form—we need something along the lines of "isomorphism type": the predicates '>' (*Greater than*) and '<' (*Lesser than*) have distinct forms over the natural numbers (though over the integers they do have equal forms). Equal form is possible only over domains of the same size. This is a consequence of building on the Tarski-Sher approach: only domains of the same size can have bijections between them.[11]

The adequacy condition captures the essence of what we shall refer to as "form". But it would be good to have an explicit definition—if only to show that the adequacy condition is consistent, and to have a workable mathematical concept at hand. I provide here an explicit, set-theoretic definition, based on the cardinality of the domain. I therefore assume, using the axiom of choice, that every set is equinumerous with an initial ordinal. We define cardinal numbers to be initial ordinals, and use $|D|$ to designate the cardinality of D for a set D. The following definition seems to capture the idea of "isomorphism type" relevant to our needs on the backdrop of ZFC.

Definition 1 (*form*) Let t be a primitive expression in a given language L, then:

$$form_D(t) = \{f(ext_D(t)) : f \in |D|^D, \ f \text{ is a bijection}\}$$

By this definition, the form of an expression is the set of all images of its extension mapped to the cardinality of the domain. The form of individual constants is simply the cardinality of the domain: if $ext_D(t) \in D$ then every member of $|D|$ is reached by some bijection from D, and $form_D(t) = |D|$. The form of a monadic predicate relative to D is a set of subsets of $|D|$, and the form of a monadic quantifier (taken as a second level predicate) relative to D is a set of sets of subsets of $|D|$. Note that the form of the universal and existential quantifiers is a singleton. We shall discuss this feature in §5.

The explicit definition of *form* satisfies the adequacy condition: defined explicitly, we still have that $form_D(t) = form_{D'}(t')$ if and only if there is a bijection $f : D \to D'$ such that $f(ext_D(t)) = ext_{D'}(t')$. From now on Definition 1 will be assumed throughout, but much of what follows does not refer to the added detail

[11] We shall set aside the issue of whether this is a welcome consequence. I do not rule out modifications to the adequacy condition that relieve this dependency on the cardinality of the domain (along the lines of proposed modifications to the Tarski-Sher criterion proposed in the literature (Feferman, 1999; Bonnay, 2008)).

there, and will be consistent with any definition of form satisfying the adequacy condition.

We shall see later that when focusing on logical languages, all that will matter for meaning is the size of the domain, so Tarski's remarks fit in neatly. As a precursory explanation, we can make an analogy with the higher levels of meaning. I remarked that in extensional contexts, we consider only one world at a time, "forgetting" the array of possible worlds required for intensional contexts. In formal settings, alluding to the new notion of form, even the domain is redundant: all that matters is its size. We shall see that the languages concerned only with form are those conventionally regarded as purely logical. Note, for instance, that the identity predicate has the same form in all domains of the same size. For such languages, a domain-size and interpretation function suffice for determining the truth values of all sentences.

We note that the notion of form defined here accords with Tarski's definition of logicality. Forms are invariant under permutations in the sense that when permuting the domain, while the extension of a term might shift, its form will remain the same. More generally, the adequacy condition entails that forms are invariant under isomorphisms. Moreover, by the explicit definition we have given, forms are Tarskian logical notions over cardinality domains. Thus, the level of form makes fewer distinctions than that of extension, the form of an expression disregards the identity of particular objects forming the extension—it only concerns the structure of the extension—and so as an aspect of the meaning of an expression, it can be said to be truly formal and general along the lines of (Tarski, 1936, 1986) and (Sher, 1996).

Both forms and Tarskian logical notions are introduced through an invariance condition, each characterized by the way it is situated in a different comparison class: that of types of meaning and that of geometrical notions. There are two main differences between forms and Tarskian logical notions. One is technical, that we have just mentioned: the present definition of form allows for variable domains, and the forms themselves are constructed over a cardinality domain rather than a given geometric space.

There is also a conceptual difference between forms and Tarskian logical notions, which concerns the relevant item's role in semantics. In the Tarski-Sher tradition, an invariance condition is used to distinguish a particular kind of extensions—the so-called "formal" ones: those are the logical notions, and they are denoted by the logical vocabulary. Here we use an invariance condition to define a distinct level of meaning, as *derived* from extensions: each extension determines a form, and both levels relate to one and the same linguistic expression. We thus have two different semantic relations at play. We say that an expression *denotes* its extension. Additionally, we can say that an expression *embodies* a form. We take logic to be concerned with this very coarse-grained level of meaning, abstracting from the particularities of extensions or meaning that is more fine-grained—remaining with only the rough features pertaining to form.

What is offered here can be viewed as a modification of Sher's semantic conception of form, one which puts her ideas in a new light. As mentioned earlier, Sher views forms as features of the world. Logical terms, for Sher, are formal in the sense that they denote formal features of the world:

> These terms denote properties that are formal and general roughly in the sense of being structural or mathematical and applying to objects (extensions of predicates) in general (Sher, 1996, p. 668).

Sher defines logical terms as invariant under isomorphisms. As mentioned earlier, Sher employs a more comprehensive class of transformations than Tarski (isomorphisms rather than permutations—allowing for varying domains), but on each domain the value given to a logical term (its extension) is a logical notion in Tarski's sense.

Here I have adopted Sher's general approach to form as a nonlinguistic entity capable of serving as a semantic value, but I have modified the idea that forms are the subclass of extensions that are denoted by logical terms. Here, forms constitute their own level of meaning—a separate type of semantic value—so that *all* expressions are assigned a form which they embody. Logical terms will be characterized later as embodying a special kind of form.

We have removed permutation invariance from the geometric comparison class to that of levels of meaning. The mode of comparison has been modified, as we do not use invariance under various kinds of transformations on a base space to distinguish levels of meaning.[12] Still, as we shall see in §6, we use identity criteria as conceived in the previous section to fit forms into the new scale. First, however, we should say how logicality fits into this picture.

5 Logicality

Now we turn to the definition of logical *terms*, which will be based on the new notion of form. This definition will be in use when we characterize *formal languages*

[12]Technically, we could opt for defining extensions, intensions and hyperintensions through invariance under transformations, but the result would be, it seems to me, unnatural and unenlightening.

and *formal contexts* in §6.

It should be noted that the aim here is not to argue for a criterion for logicality, but rather to set a connection between logicality and levels of meaning. The definition of logical terms below is equivalent to the definition of logical terms as denoting extensions that are invariant under isomorphisms, and is in accord with the Tarski-Sher approach to logicality. This criterion, as widely accepted as it is, has received criticism for both overgenerating (McCarthy, 1981; Feferman, 1999; Bonnay, 2008) and for undergenerating (Dutilh Novaes, 2014; Woods, 2014). Yet, the definition of form as "isomorphism type" can be viewed as a starting point open to modification. Indeed, some of the alternative accounts for logicality that have been proposed in the literature might still fit in the picture we have here, as long as relevant adjustments are applied to the definition of form.

Now, we shall define *logical terms* to be formal terms, terms whose meaning at the lower level is exhausted by their form: their form determines their extension in every domain. Given our explicit definition of form, we can define logical terms to have a singleton form—and so their extension can be retrieved from their form. In addition we require that the form of a logical term is as stable as can be—that it does not vary across domains of the same size.

Definition 3 (*logical term*) A primitive expression t is a *logical term* if the following conditions hold:

1. $|form_D(t)| = 1$ for every domain D.
2. If there is a bijection between two domains D and D', then $form_D(t) = form_{D'}(t)$.

First, let us consider some examples. Take the monadic predicate '< 2' (*Less than 2*) over the domain $\mathbb{N}$ of the natural numbers. $ext_{\mathbb{N}}(< 2) = \{0, 1\}$ and so $form_{\mathbb{N}}(< 2) = \{A \subseteq \aleph_0 : |A| = 2\}$ (the form of '< 2' is the set of all subsets of $\aleph_0$ with two elements). So $|form_{\mathbb{N}}(< 2)| = \aleph_0$ and so '< 2' is not a logical term. On the other hand, consider the existential quantifier. Treated, as is customary, as a second level predicate, we have for any domain D, $ext_D(\exists) = \{A \subseteq D : A \neq \emptyset\}$. And so, applying the definition of form, we obtain $form_D(\exists) = \{\{A \subseteq |D| : A \neq \emptyset\}\}$, which is indeed a singleton (the images of the extension under all bijections are equal), and thus the existential quantifier satisfies the first condition in the definition of logicality. The second condition holds as well: the form of the existential quantifier does not vary between domains of the same size, so the existential quantifier is a logical term.

More generally, from this definition follow the following facts:

Proposition 1 *Let t be a logical term. Then:*

1. *for every primitive expression t' and every domain D, if $form_D(t) = form_D(t')$ then $ext_D(t) = ext_D(t')$.*

2. *A primitive expression t is a logical term iff the extension of t is invariant under isomorphisms (i.e., for every bijection f between two domains D and D', $f(ext_D(t)) = ext_{D'}(t)$).*

The first fact is a reiteration of the idea that the extension of a logical term is determined by its form. The second fact shows that in the case of logical terms, invariance is projected up to the level of extension. Note that the first condition in Definition 3 is equivalent to invariance under permutations of the extension of the term, and that the further, independent, condition forces uniformity in the interpretation of the term across equinumeroud domains, and strengthens the first condition to invariance under isomorphisms.

The logicality of a term can be viewed as lower level rigidity. A term is a *rigid designator* if it has the same extension in all worlds (in which it denotes). For rigid terms, we can say that their intensions are completely determined by their extension. Analogously, a term is logical if and only if its extension in any given domain is determined by its form. Rigidity and logicality are independent properties: a term can be rigid but not logical and vice versa.[13] But rigidity and logicality are parallel on this approach. Normally, the higher level of meaning determines the lower ones, but not vice versa. In the special cases of rigidity and logicality, the higher level of meaning is also determined by the lower level, as it happens to be just as coarse-grained.

Consequently, in both rigidity and logicality we can find a simplicity of meaning, which allows a certain parsimony in the semantics: in both cases, going up a level does not add any richness to the meaning of the term. *Mount Everest* is a

[13]The independence of logicality and rigidity would of course depend on our construal of possible world models and their relation to the domains alluded to here. To account for rigidity, we need to re-inflate the semantic apparatus. Recall that we have suggested considering extensions as derived from intensions applied to a distinguished actual world in each possible world model. Now note that rigidity will thus be relative to a possible world model, while logicality takes into account all models. However we re-inflate the semantics, if there are at least two distinct domains from which we draw extensions, we can provide examples for logical non-rigid expressions: a logical term defined over those domains whose extension is not empty will not be rigid. If, for instance, we treat the universal quantifier as a second-level predicate and take its extension in a domain D to be $\{D\}$, its extension will differ in possible worlds with different domains, and so despite being logical, it will not be rigid. An example for a non-logical rigid expression would be an individual constant that denotes the same object in each possible world—no individual constant is logical, even if it is rigid.

rigid designator, and as such, its modal profile in a set of possible worlds is degenerate: its extension in alternative possible worlds is always the same object, Mount Everest. Likewise with a logical term such as the existential quantifier, its "extensional profile" is degenerate: in each domain, the extension of a logical term is completely determined by its form. This analogy serves as a further step in situating form at a lower level of meaning. In the next section, I characterize the level of form using Barcan Marcus's principles of explicit and implicit extensionality, where logicality plays a further role.

6 Principles of Explicit and Implicit Extensionality

Barcan Marcus offered two perspectives on extensionality, expressed by principles of *explicit extensionality* and *implicit extensionality*. The aim of this section is to see how the new semantic notion of form fares with respect to Barcan Marcus's principles. The principles are used for characterizing where in the scale of the levels of meaning an interpreted language lies. Principles of *explicit extensionality* look at how coarse-grained the meanings assigned are at each level, and principles of *implicit extensionality* consider for each level of meaning the linguistic contexts in which equi-meaning expressions can be substituted *salva veritate*. Let us discuss these in turn.

Barcan Marcus explains that

> Our notion of intensionality does not divide languages into mutually exclusive classes, but rather orders them loosely as strongly or weakly intensional. A language is explicitly intensional to the degree to which it does not equate the identity relation with some weaker form of equivalence (Marcus, 1961, p. 304).

How should we read this quote? '$a = b$' can be said to hold with respect to a certain level of meaning, but perhaps not with respect to another. Note that the equality sign should be typed appropriately according to the expressions flanking it (whether these are singular terms, predicates, etc.) Normally, we only use identity as a relation between objects. By allowing predicates or other expressions to flank an equality sign we are able to state how coarse-grained we take meanings to be. Consider, for example, *Creature with a heart* and *Creature with a kidney* for a and b respectively. The extensions of these expressions might be the same, while their intensions differ, and so the equality holds in extensional languages, but not in intensional languages. In an extensional language, all possible distinctions appear at the level of extension (recall the quote from Lindenbaum and Tarski), and the relevant equivalence relation is that of having the same extension. Analogously with intensional and hyperintensional languages. Having the same extension is a weaker relation than having the same intension (the latter entails the former)—and thus extensional languages "equate the identity relation with some weaker form of equivalence".

The reader is invited to study the details of Barcan Marcus's formal formulations (laid out in a type-theoretic framework, see also (Marcus, 1960)). Here we shall take the core idea and employ it with respect to the newly constructed level of form in our model-theoretic framework.

We observe that principles of explicit extensionality are in fact identity criteria for meanings of linguistic expressions. In a "formal" language in the relevant sense, two expressions can be equated on the basis of having the same form. Now, on the present account, two primitive expressions have the same form in some domain D if and only if one is the result of the permutation of the domain applied to the other: $form_D(t) = form_D(t')$ if and only if $\pi(ext_D(t)) = ext_D(t')$ for some permutation π on D (this is a special case of the adequacy condition in §4). Thus the equivalence relation relevant to form in a given domain D is "$\pi(ext_D(t)) = ext_D(t')$ for some permutation π," i.e. $\equiv_{log}$ as defined in §3. For equality in form over all domains we require the condition to hold for t and t' over all domains.

A clear example of sub-extensional languages that are only sensitive to form is that of the so-called "purely logical" languages. Those are languages that have only a logical vocabulary, and do not make use of a signature. For the current purpose we can make use of the definition of logicality from §5, from which it follows that the logical vocabulary of a language consists of a selection of isomorphism-invariant terms. Of course, the purely logical version of standard first order logic is included. Again, criteria for logicality are up for debate, but are not the main concern of this paper: I focus rather on the relation between logicality and form, and the notion of form here is perfectly in line with one, widely accepted, notion of logicality. Now, if all the terms in a language are invariant under isomorphisms, whether an equality holds can be determined solely on the basis of the form of the expressions occurring in it. Such a language is formal by the principles of explicit extensionality (extended to the level of form).

Note, as remarked earlier, that assignments of truth values to sentences in purely logical languages depend only on the size of the domain and are independent of what particular elements it contains. That is, for each purely logical formula φ and models $M = \langle D, I \rangle$ and $M' = \langle D', I' \rangle$ such that D and D' have the same cardinality, $M \models \varphi$ if and only if $M' \models \varphi$. In such contexts, therefore, we can refer to a very thin notion of model, consisting only of a cardinality and an interpretation function. As here we are interested in interpreted languages, the interpretation function is assumed at the outset, and we simply refer to the truth value of a formula in a given domain-size. As a simple example, $\varphi = \exists x \exists y(x \neq y)$ is true in domain-size 2, but not in domain-size 1.

What other sub-extensional languages do we have? Recall the identity criteria for various kinds of geometrical notions we have seen in the previous section. Those were more fine-grained than the identity criterion for logical notions, and indeed, geometrical languages are not "formal". A language for each one of the geometrical disciplines might make any distinction the corresponding identity criterion would allow. However, since the identity criteria of geometrical and perhaps other disciplines are still more coarse-grained than identification by extension (their notions are invariant under certain permutations of the domain), the appropriate languages can still be thought of as "sub-extensional."

From a structuralist viewpoint of mathematics, it would appear that in general, languages suitable for mathematics should be sub-extensional, and even formal. Mathematical theories on a structuralist approach are concerned with structures, and the notion of structure is akin to our notion of form. Indeed, that may be true if the notions involved are invariant under a non-trivial class of permutations. A counterexample would be a mathematical language that includes individual constants. For example, if we include in a language for arithmetic 0 and 1 as individual constants, the result will not be sub-extensional.[14] But numbers can be construed as sets of sets denoted by second-level predicates.[15] Much will depend on the way the relevant notions are construed: whether as elements of the domain or as higher-order entities. If all notions are construed as invariant under isomorphisms, the relevant language is not only sub-extensional—it is, in particular, formal.

Before moving on, we note that even below the level of form, we can define the level of *semantic category* that is even coarser in the distinctions it makes.[16] The semantic categories of a language are defined by the type of object referred to by an expression in the category, and are normally assumed to be in correspon-

[14]Another example is that of i and $-i$. i and $-i$ share many properties, and in particular they have the same form, but they shouldn't be equated. This issue touches on the so-called *identity problem* in structuralism in mathematics, see e.g. (Button, 2006).

[15]See Sher's discussion in (Sher, 1996, p. 677) and in (Sher, 2013, §4).

[16]One exception is the degenerate case where two expressions are of different categories both have an empty extension and therefore the same form.

dence with the syntactic categories. For example, in a first order language we have the semantic categories: individual constant, n-ary relation, connective and quantifier.

Principles of *implicit extensionality* tell us when expressions can be substituted *salva veritate* in a given language or a language fragment (a set of contexts). Again, we look at various equivalence relations between expressions: having the same extension, having the same intension, etc. The more extensional a language is, the weaker the equivalence it requires for substitution of expressions without loss of truth. We thus say that extensional contexts (or extensional languages) allow substitution of extensionally equal expressions, while intensional contexts (or languages) exhibit failures of such substitution. Further, taking intensions to be functions from possible worlds to extensions, intensional contexts will be such that intensionally equal expressions can be substituted *salva veritate*: such are, *e.g.*, contexts of possibility and necessity. However, such substitution arguably fails for contexts of belief, which are considered to be *hyperintensional* contexts. Hyperintensions are defined in fine-grained manner so that expressions with equal hyperintensions could be intersubstitutable in such contexts.

Now we can ask what would be the *formal* contexts, those where the truth value of sentences is determined purely by the form of its constituent expressions. Those would be very stringent contexts, since uniform substitution (substituting everywhere the same expression for the same expression) is not required (*cf.* Barcan Marcus's quote above)—that is, our test allows substituting equi-form expressions for others in just some of their occurrences.

We can consider again mathematical contexts. And again, much will depend on the way those are construed. As an initial candidate, let us look at mathematical contexts where expressions are flanked with "the number of" or, indeed, "the form of", understood in the relevant manner. So, for example, in the mathematical sentence '$|\{x : Ax\}| \geq |\{x : Bx\}|$' where straight parentheses indicate cardinality, A and B are substitutable with any other respective equi-form predicates *salva veritate*. These specific cases may form formal contexts, but other set-theoretic notions may not. If the language includes a predicate A that is not invariant under isomorphisms, '$\{x : Ax\} \subseteq \{x : Bx\}$' is not a formal context: A cannot be substituted with every equi-form expression *salva veritate*.

Here, again, we can appeal to the purely logical languages. If all the terms in a language are logical, and so are invariant under isomorphisms, terms having the same form have the same extension (as was shown by Proposition 1 in the previous section) and are thus substitutable *salva veritate*.

Nonetheless, we may still wish to consider a fragment of a non-purely logical language as providing formal contexts, so that nonlogical terms can be considered for substitution. With some limitations on the sentences acting as contexts, we can obtain a precise identification of the formal contexts, using the theorem from McGee stated below. The formal contexts are formulas of $L_{\infty\infty}$ (a first order language allowing for any infinite disjunction and existential quantification over infinite sets of variables) where all but the substituted expression are logical terms. In such sentences, equi-form terms can be substituted *salva veritate*.

More specifically, $L_{\infty\infty}$ is defined thus:[17]

- The *predicates* of the language are $\{P_i : i < n\}$ for some natural n, each P_i is of arity k_i for some non-zero natural number k_i.
- The *atomic formulas* are expressions of the form $P_i\bar{x}$, where P_i is a predicate and $\bar{x}$ a variable sequence of P_i's arity, and of the form $x = y$ where x and y are variables.

Complex formulas:

- if φ is a formula, $\neg\varphi$ is a formula.
- If Φ is a set of formulas, its disjunction $\bigvee \Phi$ is a formula.
- If φ is a formula and U a set of variables, $(\exists U)\varphi$ is a formula.
- Conjunction and universal quantification are non-primitive, and are defined in the usual way.

Satisfaction is defined standardly. Given a domain D, an interpretation $\bar{R} = (R_1, ..., R_n)$ in D of the predicates P_1, ..., P_n, resp., a formula φ of $L_{\infty\infty}$, and an assignment σ to the free variables of φ in D, $(D, R) \models \varphi[\sigma]$ is defined by induction as usual. An operation Q_D is a function from interpretations $\bar{R} = (R_1, ..., R_n)$ to truth values. φ is said to define Q_D over D if for any $\bar{R} = (R_1, ..., R_n)$, $Q_D(\bar{R}) = T$ iff $(D, \bar{R}) \models \varphi$.

Theorem 2 *(McGee, 1996). Q_D is invariant under arbitrary permutations of the domain D if and only if Q_D is definable in $L_{\infty\infty}$.*

We shall modify some parameters in the definitions in order to obtain the desired formal contexts. As we are dealing with an interpreted language, we shall fix an interpretation $\bar{R}_* = (R_1^*, ..., R_n^*)$. So, for all $i \leq n$, $ext_D(P_i) = R_i^*$. Next, we

[17] We define $L_{\infty\infty}$ a bit differently from McGee, and instead follow (Feferman, 2010).

shall consider formulas where only one nonlogical predicate occurs (identity is considered logical).

Proposition 3 *Let $\varphi(P_i)$ be a formula of $L_{\infty\infty}$ which, perhaps, contains the predicate P_i for some $i \leq n$ and no other nonlogical predicate. Then for every domain D and every $j \leq n$ such that P_i and P_j are of the same arity and such that $form_D(P_i) = form_D(P_j)$ (and so in particular are of the same arity), $(D, \bar{R}_*) \models \varphi(P_i)$ if and only if $(D, \bar{R}_*) \models \varphi(P_j)$.*

Proof. Let Q_D be the operation described by $\varphi(P_i)$. $form_D(P_i) = form_D(P_j)$, so there is a permutation π of the domain such that $\pi(R_i^*) = R_j^*$. Q_D is invariant under permutations by McGee's theorem, and so $(D, \bar{R}_*) \models \varphi(P_i)$ iff $(D, \pi(\bar{R}_*)) \models \varphi(P_i)$, where $\pi(\bar{R}_*)$ is the interpretation induced by permuting the domain D as underlying $\bar{R}_*$. Since P_i is the only nonlogical predicate in $\varphi(P_i)$, for any interpretation $\bar{R} = (R_1, ..., R_n)$ such that $R_j = \pi(R_i^*)$, $(D, \bar{R}) \models \varphi(P_j)$ if and only if $(D, \pi(\bar{R}_*)) \models \varphi(P_i)$, and specifically $(D, \bar{R}_*) \models \varphi(P_j)$ if and only if $(D, \pi(\bar{R}_*)) \models \varphi(P_i)$, and from invariance, $(D, \bar{R}_*) \models \varphi(P_i)$ if and only if $(D, \bar{R}_*) \models \varphi(P_j)$. □

The result, in other words, is that in $L_{\infty\infty}$ contexts containing one nonlogical predicate, predicates with equal form are intersubstitutable *salva veritate*. In fact, we should restrict the relevant contexts to contain only one *occurrence* of a nonlogical predicate, if we are to allow, as Barcan Marcus does, for substitution that is not uniform. Now, there may be other contexts in which equi-form predicates are intersubstitutable *salva veritate*—this will, in part, depend on the richness of the language (the more impoverished the language, the less substitutions we can make, and so we potentially have more formal contexts). However, since all permutation invariant operations are definable in $L_{\infty\infty}$, the fragment we specified is the most comprehensive one we can specify without restrictions on the richness of the language. The result can be generalized to other types of expressions given appropriate modification of the language and of McGee's theorem.

7 Forms of Complex Expressions

So far, we have only considered the forms of primitive expressions. We can define the forms of complex expressions by generalizing the definitions from §4, as long as their extensions are set-theoretic constructs over the domain. However, we might wish to consider the case of sentences as well, where the extension is normally thought to be not a set-theoretic construct over the domain, but a truth value. The solution is to consider the set of assignments—sequences of members of the domain—that satisfy a formula as its extension, so the extension of a sentence is either a set of all infinite sequences over the domain (identified with truth) or the empty set (identified with falsity). The form of a formula is then a set of sets of sequences over the cardinality of the domain.

We first define the *extension* of a complex expression. The extension of a complex individual term is, as usual, a member of the domain. For a formula φ we define:

$$ext_D(\varphi) = \{\sigma : \ \sigma \in D^\omega \text{ s.t. } \sigma \text{ satisfies } \varphi \text{ in } D\}$$

In the case of a sentence φ, the set above either consists of all $\sigma \in D^\omega$ or is empty.[18]

We now extend the range of Definition 1 to apply to all expressions:

Definition 2 (*form*) Let t be an expression in a language L. Then:

$$form_D(t) = \{f(ext_D(t)) : \ f \in |D|^D, \ f \text{ is a bijection}\}$$

For a sentence φ, $form_D(\varphi) = \{\{\sigma : \ \sigma \in |D|^\omega\}\}$ or $form_D(\varphi) = \{\emptyset\}$: all true sentences have the same form and all false sentences have the same form on any given domain (and on any domain of the same size in which they have the same truth value). This result is in line with the characterization of form as more coarse-grained than extension. As is customary, we view all true sentences as sharing the same extension, and all false sentences sharing the same extension. On the level of form we could either keep this distinction, or treat all sentences as having the same form. Distinguishing between truth and falsity is something we should be able to do even in logical contexts, which we have characterized as "formal"—and so the form of a true sentence will be different from that of a false one.

We might now wish to consider extending the definition of logicality from §5 to apply to complex expressions in general, and to sentences in particular. Now, whether true or false, the form of a sentence is a singleton, and so all sentences satisfy the first condition in our definition for logicality in §5. Therefore, a sentence is logical if and only if it has constant form over equinumerous domains, as is required by the second condition in the definition of logicality. Among those sentences are the logical truths and the logical falsehoods: those sentences that

[18]We are now able to give the forms of connectives, as promised in f.n. 8. In this setting, the extension of a connective is an operation on sequences. For example, the extension of conjunction is the intersection operation on sets of sequences over the domain: $ext_D(\wedge) = \{\langle\Sigma_1,\Sigma_2,\Sigma_3\rangle : \Sigma_1,\Sigma_2,\Sigma_3 \in \mathcal{P}(D^\omega), \Sigma_3 = \Sigma_1 \cap \Sigma_2\}$.

are true (or false, respectively) in every domain and every re-interpretation of the nonlogical terms. However, other sentences satisfy the definition of logicality as well. A sentence is logical if its truth value is determined by the size of the domain, and this includes, in the case of a first-order language with identity, sentences such as '$\exists x \exists y (x \neq y)$'. A result that will soon be generalized is that every sentence in a first-order language composed of purely logical vocabulary will be logical. But this characterization is not exhaustive: sentences whose truth value is determined by the size of the domain might nonetheless contain nonlogical vocabulary, as, in particular, logical truths may contain nonlogical vocabulary.

Now, an open formula has more possible extensions than a closed one (namely, a sentence). Some open formulas are like sentences in that their extension includes all assignments or none in every domain, such as '$Px \vee \neg Px$' and '$Px \wedge \neg Px$'. In such cases the form will obviously be a singleton, and as in the case of sentences, their logicality will then depend on the second condition of having constant form across domains of the same size. Some formulas are satisfied by just some assignments over a given domain. Their form might not be a singleton in some domains, as in '$P(x)$' in domains where some, but not all, elements in the domain fall under P.

Open formulas are akin to relations. While the extension of a formula consists of assignments and a relation is a set of tuples of the arity of the relation's arity, an open formula which contains n unbound variables defines an operation from domains to n-ary relations therein. It's easy to see that the formula is logical if and only if the operation it defines is invariant under isomorphisms. For example, '$x = y$' is logical, as it defines the relation of equality in every domain, and thus an operation which is invariant under isomorphisms.

We can now state the general claim that in a first-order language, every complex expression which is composed of purely logical vocabulary as defined in §5 is logical by our definition—this can be proved by induction for each type of complex expression.[19] In particular, if a sentence is composed of purely logical vocabulary, its truth value will be constant across models of the same size.

There are, of course, logical formulas that contain nonlogical vocabulary, such as '$Px \vee \neg Px$'. We might like to say that in such formulas, the nonlogical terms occur "inessentially". However, whether every logical formula is equivalent to a

[19]This is a variant of what is known in model theory as the *isomorphism property* that holds of a logic if for every sentence φ and any models M and M' that are isomorphic vis-à-vis the nonlogical vocabulary in φ, $M \models \varphi$ if and only if $M' \models \varphi$. This property holds in every logic in which the terms fixed as logical are invariant under isomorphisms (see (Shapiro, 1998; Sagi, 2014)), so we may also consider going beyond standard first-order languages.

formula composed of purely logical terms will depend on features of the language that we have not specified—in particular, its size and the logical vocabulary that it contains.[20] To sum up, the logical sentences are all those whose truth value is determined by the cardinality of the domain. The logical formulas, more generally, are those whose extension is invariant under isomorphisms.

It is worth explaining the difference between our notion of logicality as applied to sentences and open formulas and the Tarskian notions of logical truth and logical falsehood.[21] The Tarskian notion of logical truth (or falsehood) requires truth (or falsehood) in *all models*. The vocabulary is divided into the logical and the nonlogical, and the nonlogical vocabulary gets reinterpreted in every model according to semantic category, while the logical vocabulary remains fixed.

The class of all models makes an appeal to two distinctions that go beyond the means we have used to define logicality. First, in the Tarskian definition, the vocabulary is divided into semantic categories (or types), and for each semantic category there is a range of possible extensions. The notion of semantic category, as I have briefly mentioned in §6, is sub-extensional and even more coarse-grained than that of form. It has no role in our definition of logicality.

Second, in the Tarskian definition, a distinction is made between the role of the logical vocabulary as the fixed vocabulary and the nonlogical vocabulary as not fixed: in the former case we do not consider alternative interpretations and in the latter we do. A model for a language assigns values to all non-fixed primitive expressions in the language according to their semantic category, while the interpretation of the fixed primitive expressions is fixed by a semantic rule. Here, as the language is interpreted, no such distinction is made, and all terms are fixed to the same extent (all have a fixed extension in every domain). We have defined the logical vocabulary, but it doesn't get anymore fixed than the nonlogical vocabulary.

The present definition of logicality makes no appeal to models in the above sense: we do not divide the language into two sets of expressions that are treated differently, some get fixed while others vary according to semantic category. Instead, we take an interpreted language, and we look at the behaviour of extensions under isomorphisms. Ours is an extension of the Tarski-Sher criterion for logicality, which applies to interpreted expressions. This criterion yields a notion of log-

[20]So, for instance, if the language is of first-order and includes the truth-functional connectives and identity, every logical formula is equivalent over equinumerous domains to a formula containing purely logical vocabulary.

[21]I appeal to the now accepted notion of logical truth which is derived from, but not identical to the notion of analytic sentence defined in (Tarski, 1936).

icality for sentences that is wider than Tarskian logical truth and logical falsehood combined, as it allows sensitivity to the size of the domain. For the same reason, the notion of *logical sentence* might not be a good candidate for the explication of the philosophical notion of *logical truth*. Nonetheless, if the recurrent theme of this paper is that our logic is a logic of cardinality or of number, as repeated in the quotes from Tarski and Lindenbaum, then here we have another manifestation of this phenomenon.

8 Concluding remarks

The characterization of logic has many facets, concerning consequence and truth, linguistic expressions and mathematical notions. The starting point of this paper is Tarski's characterization of logical notions. Tarski's logical notions are general in the sense of being invariant under the maximal group of transformations over a given domain, and are thus more coarse-grained than the geometrical notions Tarski compares them with. Through Sher's semantic conception of form we brought logical notions closer to the linguistic realm by devising for them a particular level of meaning, more coarse-grained than those of extension, intension and hyperintension.

Whereas in standard accounts logical notions are viewed as the extensions of logical terms, here we use logical notions as the *(semantic) forms* of all terms. Purely logical languages prove to be "sub-extensional" and provide the appropriate contexts for the level of form by both of Barcan Marcus's explicit and implicit principles of extensionality. We have also seen that formal contexts can be provided by a restricted fragment of $L_{\infty\infty}$. The definition of logical terms as those terms whose forms are singletons (and so their extensions are determined by their form) that are stable across equinumerous domains is equivalent to the widely accepted Tarski-Sher criterion of invariance under isomorphism, and at the same time treats logicality as rigidity at a low level.

In this framework, logical truth and validity can be determined by the logical vocabulary in the standard way, where the logical terms are those terms whose interpretations are held fixed. However, we offer a notion of a logical sentence that is a natural extension of the Tarski-Sher criterion for logicality to complex expressions. This notion does not rely on a distinction between the fixed and the non-fixed vocabulary, nor on the division of the language into semantic categories. The result we obtain is that a logical sentence is a sentence whose truth value depends only on the cardinality of the domain—a result we should expect if we agree with Tarski that "our logic is a logic of number".

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