Hilary Putnam on the philosophy of logic and mathematics* [1]

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ABSTRACT: This paper focuses on Putnam's conception of logical truth as grounded in his picture of mathematical practice and ontology. Putnam's 1971 book *Philosophy of Logic* came one year later than Quine's homonymous volume. In the first section, I compare these two Philosophies of Logic which exemplify realist-nominalist viewpoints in a most conspicuous way. The next section examines Putnam's views on modality, moving from the modal qualification of his intuitive conception to his official generalized non-modal second-order set-theoretic concept of logical truth. In the third section, I emphasize how Putnam´s "mathematics as modal logic" departs from Quine's "reluctant Platonism". I also suggest a complementary view of Platonism and modalism showing them perhaps interchangeable but underlying different stages of research processes that make up a rich and dynamic mathematical practice. The final, more speculative section, argues for the pervasive platonistic conception enhancing the aims of inquiry in the practice of the working mathematician.

Keywords: Putnam, Quine, Logic, Indispensability, Ontology, Scientific Language, Mathematical Practice.

RESUMEN: Este artículo estudia la concepción de Putnam de verdad lógica que emana de su visión de la práctica de la matemática y de su ontología. *Philosophy of Logic*, el libro de 1971 de Putnam surge un año más tarde que el homónimo de Quine. En la primera sección, se comparan estas dos Filosofías de la Lógica que ejemplifican los puntos de vista del realismo y del nominalismo de modo conspicuo. La siguiente sección examina el enfoque de la modalidad de Putnam, que va desde la cualificación modal de su caracterización intuitiva de validez lógica a su concepción oficial generalizada no-modal conjuntista de segundo orden. La tercera sección subraya el modo en que «la matemática como lógica modal» de Putnam se distancia del «Platonism a regañadientes» de Quine. Aquí se sugiere una visión complementaria del Platonism y del modalismo, los cuales, aunque quizás intercambiables, se muestran subyaciendo a los diferentes estadios del proceso de investigación de una práctica de la matemática rica y dinámica. La sección final, más especulativa, conjetura algunas razones de la persistente concepción platónica implícita en la práctica del matemático.

Palabras clave: Putnam, Quine, Lógica, Indispensabilidad, Ontología, Lenguaje Científico, Práctica de la Matemática.

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Introduction

This paper focuses on Putnam's philosophy of logic and mathematics. Specifically, I examine his conception of logical truth, which is based on his view of mathematical practice and ontology. His conception can be seen as an endorsement of the mathematical paradigm of contemporary model theory framing a general picture of mathematics and the natural sciences. Even though his 1971 presentation is philosophical, in it Putnam emphasizes what can be thought of as an official view of logic as a mature scientific discipline. However, by considering Putnam's overall view of logic and science as expressed in his different writings, we can see that he never loses sight of the idea that mathematical logic, as part of mathematics, is also an applied science. For example, some form of a moderate scientific holism emanates from his remark that "the internal success and coherence of mathematics is evidence that it is true under *some* interpretation, and that its *indispensability for physics* is evidence that it is true under a realist interpretation" (Putnam 1979/85b, 74). On the other hand, he (Putnam 1994a, 464) qualifies his overall cooperative picture of science by paying attention to the diversity of sciences involving communal rational practices with no formalizable unique method. Moreover, Putnam (2012, 188) reminds the reader that in his 1979/85b mathematics was considered quasi-empirical "*within* mathematics itself". Furthermore, he also agrees with Parsons (1979-80, section III) in showing "concern that the obviousness of basic mathematical statements is left out unaccounted for by the Quinean picture". The same observation was made to include logical principles by the Corcoran (1979, 1237-8) review in connection with Quine's unexplained force of basic reasoning: "In the reviewer's opinion these views are attractive but more discussion of the *obviousness* of logical axioms and of the *visibly soundness* of rules of inference would be desirable". This suggests that one trait of Putnam's conception entertains an intended balance between the *local* and the *global* at the time of assessing the cooperative spirit among different scientific domains, ranging from mathematics and physics on the one hand, to more "extreme cases" (Putnam 1994a, 463), such as, Darwin's theory of natural selection, relativity theory, and quantum mechanics. Thus, Putnam's naturalism is not like Quine's, at least to the extent that he recognizes that the universe of discourse of a science may countenance things that neither can be properly discussed in a canonical first-order language nor do they require the whole machinery of science to be tested. This view also evolves towards Putnam's later modal proposal for the formulation of scientific principles avoiding Quine's commitment to abstract objects and disclosing a different view upon indispensability arguments (Putnam 2012, 182). He also recognizes that logic plays a crucial role in the way our interaction with our environment has enabled our survival and amassing of knowledge through experience, hypothesis-testing and prediction. Putnam (1968) thinks that logic is empirical in the sense that logical theory could turn out to be false for empirical reasons. Contrary to Quine, however, he thinks that the success of our logical theories is ultimately sustained by the existence of an objective reality that goes beyond sensory experience and the mere concern for the grammar of language. Logic and mathematics, according to Putnam, are about an objective reality, of which equivalent descriptions can be given within different conceptual frameworks. In the end, it is tempting to conjecture that his views constitute a sort of compromise between logic and mathematics on the one hand, and the best account of the role these play in general scientific knowledge on the other.

Putnam's Philosophy of Logic

Putnam's 1971 *Philosophy of Logic* came just one year later than Quine's book with the same title. Some aspects of the two philosophies of logic exemplify the nominalist and realist viewpoints in a most conspicuous way.

The style of presentation of Putnam's 1971 *Philosophy of Logic* reveals a clear intention to argue in favor of the most plausible understanding of a realist conception of logic by showing the shortcomings of nominalist strategies. His rhetoric is compact, although endowed with some pedagogical license. On page 27, Putnam endorses the view that logic is concerned with *general* principles, such as:

> "For all classes S, M, P: if all S are M and all M are P, then all S are P."

The conception of logical validity that emerges from this core example favors the choice of dealing with logical truth rather than logical consequence. However, this decision has important philosophical ramifications. Briefly, in any *finite* universe of sentences, logical implication can be defined on the basis of logical truth, and viceversa. The option chosen is of no technical or philosophical significance: P logically implies c if and only if the conditional whose antecedent is the conjunction of the sentences in P and whose consequent is c, is logically true. However, the issue has import when considering *infinite* universes of sentences (perhaps not even closed under conjunctions and conditionals). Clearly, not every text expressing a first-order argument admits of a suitable one-sentence translation in a standard language. Specifically, no such text with an infinite set of premises allows a single-sentence translation. Of course, Quine bypasses this issue by calling on the compactness of first-order logic. Putnam's higher-order logic, however, is not compact; so one is inclined to consider his presentation focused on logical truth as a kind of pedagogical license.

The important point is that such a formulation leaves no doubt that expressing logical validity requires a universal quantifier ranging over classes or sets. Let us call this conception of validity "contemporary setism". We should note that setism presupposes ontology of sets provided by the underlying set-theory adopted in the semantic definition of the logical properties of our logic. Let me point out at the outset some interesting features of Putnam's conception.

Logical validity in Putnam's picture is extra-linguistic or language transcendent. To be precise, in his account, logical validity is predicated on general truths of set theory. It is not, for example, the interpreted sentence "If all men are rational beings and all rational beings are mortal, then all men are mortal." that is a logical truth in his conception, but rather the corresponding generalized set-theoretic (material) truth quoted above. Appealing to the contrast with nominalism, we may recall that Quine (1970/85) holds that in order for a sentence to be logically true, it is necessary and sufficient for it to be true and to remain true under any uniform *lexical substitution* of its content-terms. Since, intuitively speaking, the relation of a sentence to its lexical substitutions is a matter of grammar, then logical truth is thus a matter, as Quine emphasizes, of grammar and truth. Now, Putnam is quick to point out that the nominalist strategy makes logical truth *relative* to a previously specified class of logical constants in a given interpreted first-order language. Furthermore, at least in the case of Quine, the interpretation of the language is kept fixed and no changes to the extensions attached to the non-logical terms are allowed. So in this respect, the nominalist, and Quine in particular, thinks that logical truth is relative to the means of expression of a given language. In this syntactic or *intra-linguistic view*, as opposed to the Putnam's already mentioned extra-linguistic view, changing the language changes the concept of logical truth involved. In this sense, Quine's concept of validity is *immanent*, whereas Putnam's is transcendent (see Quine 1970/86, 19-29). Thus, according to Quine, as Putnam correctly criticizes, given two languages, we have two concepts of logical truth which share in common that each of their interpretations is kept fixed. Perhaps this seems ahistorical, to say the least. Tarski, in his seminal 1936 paper on consequence, cogently argued that his semantic or *extra-linguistic* conception was superior to the syntactic or intra-linguistic conception. In effect, it is at least conceptually possible for there to be a sentence in a given language that does not have an instance of a lexical substitution (or a *counter-variant*, following Quine's terminology) due to the limited means of expression in that language. Such a sentence would be rendered as a logical truth in the language, with the result of over-generation of the intuitively adequate validity class.

We should not overlook the fact that in Putnam's conception of validity, a conception which, as mentioned above, follows the main trend in contemporary model theory, it is irrelevant whether the language is interpreted or not. Specifically, interpretations are set-theoretic objects: elements of the universe of pure sets. Again, this view presupposes ontology of sets. Validity is predicated on a second-order *universal* set-theoretic sentence and it amounts to the nonexistence of a certain sort of set that provides for a counter-interpretation or counter-model. Similarly, invalidity amounts to the existence of a set that provides for such a counter-model. Of course, the natural question arising here is whether there are enough sets to supply suitable counter-interpretation domains for every invalid sentence.[2] Besides, as it is argued in the concluding remarks of my 1997 paper, this reductionist move of logical truth to set-theoretic (plain) truth involves a kind of ontological circularity, since logic is traditionally understood as the science underlying all the sciences, including set-theory. Putnam thus seems to endorse a debatable identification of a logical property (logically necessary truth) with material (plain truth) of the particular science of sets.

Nevertheless, one may be inclined to say that Putnam's proposal is superior to Quine's because the range of his variable in the validity principle appears to be wider since it appeals to *all classes*, whereas Quine appeals to *all lexical substitution instances in L*. However, this is not so, at least not for the case of a first-order language capable of expressing elementary arithmetic with identity as non-logical. To *see* that every sentence in a first-order language of arithmetic is logically true in the substitutional sense if and only if it is logically true in the model-theoretic sense, we can proceed as explained in what follows. The conditional from left to right can be established by contraposition. Suppose that a given sentence, Φ, of the language is not a logical truth in the model-theoretic sense. Then Φ has a counter-model; and so by the Löwenheim theorem, Φ has a counter-model in the universe of natural numbers. Now, by the Hilbert-Bernays theorem, the arithmetical predicates involved in this interpretation can be defined in the language.[3] Hence, every non-logically true sentence has a

[2] Kreisel 1967 is the classical locus discussing the tension between our pre-formal intuitions and our formal or set-theoretic models representing them. See specially section 2 in connection with logical validity. See also footnote 4 below.

[3] Quine (1954) explains and proves the Hilbert-Bernays result, showing its role for my present purposes.

counter-variant obtained by suitable substitution of its content terms by arithmetical ones. The conditional from right to left follows from the (weak) completeness of first-order logic. Any sentence that is logically true in the model-theoretic sense is deducible by means of some standard calculus, which by virtue of its soundness only generates true sentences under all substitutions.[4] However, Putnam follows the tradition established by Frege, Russell, Bolzano and Tarski in which logic is higher-order and he would not hesitate in considering Quine's view restrictive since first-order logic is a proper sub-logic of virtually every logic used as an underlying logic in the classical mathematics and science literature. Of course, Quine (1950/82, 259-260) had to "borrow furniture" from the Platonic realm, since he recognizes that sciences require classes for their intelligibility. Nevertheless, for the case of first-order logic, Quine's parsimonious ontology welcomes the avoidance of sets.

Putnam (1971,5) emphasizes that the example of validity given above, namely: "For all classes S, M, P: if all S are M and all M are P, then all S are P." on its modern interpretation just expresses the transitivity of the relation of subclass. He emphatically adds that "this is a far cry from what traditional logicians thought they were doing when they talked about Laws of Thought and terms" Of course, Putnam (1971, 3) is well aware of the historical fact that "the methods used in logical research today are almost exclusively mathematical ones". Putnam clearly acknowledges that the principles of classical logic do not change even though a new domain of investigation, such as quantum mechanics, may suggest the interest of exploring a non-classical logic for that particular domain. He recognizes that what does change is our comprehension of the classical properties. In short, Putnam is aware that there is general agreement concerning whether a given statement is (classically) valid or invalid, but disagreement about the proper understanding of that validity.

As Corcoran (1973b) indicates, current mathematical logic can be taken to be an applied branch of mathematics that produces model-artifacts resembling logical principles which underlie mathematical practice. The "material" for the model is provided by string theory for the syntax and by set-theory for the semantics, thereby generating standards of well-formed formulas and logical validity. So far, so good. However, even though the mathematical character of logic is as well established as any other mathematical discipline such as physics or chemistry, the set-theoretic reduction suggested in the semantics raises two problems. The first is whether some versions of set-theory contain only true principles. Particularly, some philosophers and mathematicians have shown concern regarding less evident axioms that imply the existence of large cardinals (see Boolos 1998). Second, if the principles of set theory are true, then either they are materially true or their truth involves some kind of necessity. In the first case, as already mentioned, validity would be dependent of plain truth about sets. In the second, validity would be dependent on a kind of necessary (mathematical) truth, which I believe some logicians would take to be weaker than logical necessity.[5] In other words, *prima facie*, the realm of mathematical possibility seems to

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[4] The alleged co-extensionality of Quine's substitutional account with the model-theoretic is discussed and put into question in Eder (2016) following and expanding an argument developed in Boolos (1975).

[5] See, for example Corcoran's 1972 paper "Conceptual Structure of Classical Logic", where the author already cast doubts on the material adequacy of Tarski's model-theoretic concept of logical consequence. He suggests that Tarski might have considered a necessary condition for validity to be both necessary and sufficient: "What I am suggesting is that there might well be an argument (P, c) which is invalid but

be smaller than the realm of logical possibility. Perhaps the offshoot of this predicament is that a mathematical logic, or in other words, a mathematical analogue of a logic—whether real or putative—cannot tell us what the nature of validity *is* but rather only what validity *is like* by providing a proxy representation. Certainly, this suggestion could follow the lines of Putnam's defense of the empirical character of general mathematics, and logic in particular. However, Putnam does not tell us if the data for mathematical logic are general principles emerging from the structure of the world or rather examples of reasoning as exhibited in particular domains of investigation. On this score, he sometimes seems to favor a view of logic as formal ontology in the sense of Russell when he proposes quantum logic for the underlying logic of that particular empirical realm.[6] Standard principles studied by logic as formal ontology are excluded middle and non-contradiction. On the other hand, logic has been also understood as grounding formal epistemology by focusing on reasoning and the way we process information from initial premises or principles by means of deduction. It must be said that Putnam also refers to logic as the science of reasoning in this sense but in my opinion there are no definite grounds to settle this issue. Finally, it should also be emphasized that Putnam's official conception of validity is non-modal. There is no trace or hint of modality in the foregoing second-order formulation of validity.

Back and forth with modalities

Despite of the closing remark of the previous section we should not overlook the fact that Putnam's pre-formal understanding of validity shows explicit modal wording seemingly expressing an unofficial modal conception. For example, the reader finds in some of the 1971 Putnam's criticism of the nominalist reductive strategy evidence of his modal qualifications of validity, which according to him, cannot be accommodated by the nominalist. The intuitive or underlying sense of validity that he appeals to in these passages is explicitly modal; for, he says that the nominalist cannot afford the intended idea of validity as truth under all *possible* substitution instances in all *possible* formalized languages. However, the official option that Putnam endorses for logical validity is *generalized* non-modal second-order set-theoretic truth as pointed out above. In short, Putnam's pre-formal conception of validity is modal, whereas his favorite formal conception *de-modalizes* validity by means of a purely quantificational account. This suggests that even though Quine rejects modal notions *tout court* while Putnam does not, both understand—at least officially—logical properties as different forms of generalization.

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for which there is no (re)interpretation making the premises true and the conclusion false. In any case, according to the Tarskian definition of validity, the invalidity of an argument depends on *the existence* of a suitable domain and there might not be enough domains to provide counter interpretations for all invalid arguments". So Corcoran, anticipated today's debate on what he first called "Tarski's thesis" in clear reference to the analogous way in which people discuss Church's thesis. In addition, the class of set-theoretic domains of interpretation clearly determines the range of modal possibility built on that class. A fortiori, this suggests that logical necessity may surpass the kind of necessity provided by the set-theoretic machinery. This line of argument is also taken up by Etchemendy in his 1990 book.

6 Russell (1919/93, 169) says that "logic is concerned with the real world just as truly as zoology, though with its more abstract and general features".

During the 1970s, Putnam's explicit views on modalities change. However, as I have indicated, this modal standpoint was not new to Putnam's thinking, since it was already clearly present in 1967b. In his *Philosophy of Logic* (1971), Putnam shows a pre-formal modal conception of logical truth; whereas for his formal presentation, he removes modalities in favor of a pure extensional set-theoretic conception. As in many other areas of philosophy, the possible influence of Quine's rejection of a modal notion is one to consider as background to this. As Shapiro (2000a, 238) indicates: "The general program is to demur from talk of necessity and possibility, replacing it with talk of abstracts objects like sets and numbers". Putnam (1979/85b, 70) describes this move by saying that "mathematics got rid of *possibility* by simply assuming that up to isomorphism anyway, all possibilities are simultaneously *actual*—actual, that is, in the universe of sets". I have said that for the case of model theory, in his 1971 book, this move amounts to de-modalizing Putnam's own intuitive view of logical truth. His next explicit position (1979/85b, 70) was to recover a modal discourse for mathematics in general, in order to propound a realist conception by emphasizing its objectivity but avoiding commitments to the existence of mathematical objects. His main thrust seems "to bring some comfort to the Platonist", who is usually in some trouble when it comes to explaining our epistemic access to abstract entities.

The new modal standpoint now embraced is that "mathematics has *no* objects of its own at all". As Shapiro (2000a, 243-244) emphasizes, modal mathematics makes "a subject with no object" in clear reference to the title of the 1997 Burgess and Rosen's influential book. The idea is that mathematics lacks a proper or specific universe of (actual) objects to count as its own subject-matter. The proposal is to substitute existential assertions in favor of modal assertions, such as: "certain things are possible and certain things are impossible". To back up this anti-Quinean mathematics-as-modal-logic account, Putnam points to the development of modal logic and possible-worlds semantics: "It seems to us that those philosophers who object to the notion of possibility may, in some cases at least, simply be ill-acquainted with physical theory, and not appreciate the extent to which an apparatus has been developed for describing 'possible worlds'." Presumably, Putnam is referring here to the power of possible-worlds semantics.

Now, if mathematics lacks objects of its own, then what is the difference between mathematics and logic? Logic has traditional been said to be the science preceding other sciences which lack a specific domain, or a neutral topic. If we consider possible worlds to furnish an account of the modal conception of validity, then we have that in order for a sentence to be logically true, it is necessary and sufficient for it to be true in every possible world. If this view of logical truth is explained by reference to possible-worlds semantics, then we end up relying on set-theoretic structures of a distinct mathematical nature. Needless to say, possible worlds raise fundamental questions concerning their ontological nature that make them controversial.[7] Ultimately they rely on some substantive background ontology that provides the material from which to build up "logical spaces", namely, set-theoretic structures. I suggest calling Putnam's view so sketched "structural modalism".

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7 However, in other papers, Putnam offers different interpretations of possibilities in terms of the state spaces adopted in probability theory or the phase spaces of physics. I thank an anonymous referee for pointing out Putnam's alternative choices for depicting possibilities.

Certainly the idea that a science such as mathematics lacks objects of its own was not new. Putnam's general picture of his 1967a was influenced by the structural approach to geometry found in Hilbert's 1899 *Grundlagen* and Dedekind's 1888 structural account of arithmetic.[8] On this view, mathematics is not about any particular object, but about objects of any sort which structured in some appropriate way satisfy the given axioms. This approach became highly influential in the early twentieth century, even though it departed from the more traditional view that mathematicians divide their labor and typically work in specific domains of research. In general, the domain of a given science is its subject-matter or *genus* in the Aristotelian sense of the word. In the *Posterior Analytics* 76b10, Aristotle says that each science requires three things: its genus, its basic concepts and its basic principles. The idea that each science has a subject-matter or domain is so entrenched that virtually every basic textbook says that the domain of elementary arithmetic is the class of natural numbers; the domain of geometry is the class of points; the domain of string theory is the class of all strings; the domain of set theory is the so-called universe of sets; etc.

Now, what price does structural modalism have to pay in order to avoid Platonism with respect to actually existent abstract objects? First of all, Putnam (1979/85b, 71) clearly indicates that "the notion of possibility does not have to be taken as a primitive in science" since we can "define a structure to be possible (mathematically speaking) just in case a model exists for a certain theory, where the notion of a model is the standard set theoretic one", thus taking the notion of set as basic and the notion of possibility as derived. He also supports the reverse direction, treating "the notion of possibility as basic and the notion of set existence as the derived one". Here the slogan states: "Sets are permanent possibilities of selection" (*Ibidem*). Clearly Putnam subscribes to the view that the existential and the modal expression of mathematical propositions are just equivalent descriptions. He indicates that the structural modal account is useful to understand that one can express the same mathematical fact (whatever that is) without the need of appealing to abstract objects. In addition, he indicates in this connection that there are "real puzzles, especially if one holds a causal theory of reference in some form, as to how one can refer to mathematical objects at all. I think, that these puzzles can be clarified with the aid of modal notions". In my opinion, what Putman accomplishes with the modal turn is certainly bypassing the Platonist's difficulty in explaining the issue of accessibility to abstract entities.

Putnam (2012) points out that his modal view is worked out in detailed in Hellman's 1989 book, where the project of a modal structuralist conception of mathematics is developed. Hellman's project makes mathematics *logical* to the extent that the realm of possibility in which his program is nested is neither physical nor metaphysical, but logical. Under this modal proposal, a mathematical statement, A, has to be translated into or rewritten in a modal second-order language stating that for every possible structure of the appropriate kind, A would hold in that structure. Clearly, this modal paraphrasing and rewriting avoids direct quantification over mathematical entities. Thus, Hellman's intended project is eliminative providing a non-Platonist approach avoiding set-theoretical formulations. Putnam, on the other hand, simply indicates that the modal formulation has now a proper development which was only sketched in his 1971 book. This is consistent with taking the modal structuralist description as one equivalent description of the objectual (contrary to Hell-

[8] Again, my thanks go to an anonymous referee for bringing up this important historical point.

man's intentions) but with no epistemic or ontological priority.[9] In the end Putnam's philosophy appears to be inclusive of both types of discourse in science, objectual and modal, whereas Hellman's philosophy is eliminative and thus exclusive.

The natural question arising concerns the nature of logical modalities. Putnam's support of the development of modal logics leads us back to the set-theoretic nature of possible worlds. Clearly, under what I have already termed contemporary "setism", a possible world is a set-theoretic structure; a set, for that matter. Hence, in order for a sentence to be logically possible, it is necessary and sufficient for there to be a set which satisfies the given sentence. In turn, according to the present program, the existential clause that there is such a set is to be modalized accordingly. The threat of circularity is evident. If the idea is to avoid or bypass sets, modalizing existential statements via possible worlds leads back to the set-theoretic ontology. Shapiro (2000a, 275) reports that Hellman accepts this circularity objection and "he demurs from the standard model-theoretic accounts of the logical modalities". Hellman's way out is to take logical modal notions as primitive: not reducible to set-theory. Of course, we may feel that we are left here with a mystery surrounding a very rich primitive logical notion, to say the least.

The search for complementary views of mathematical practice

It is difficult to offer sensible arguments that resolve the present debate between Platonism and modalism. Quine and Putnam agree that it would be dishonest to deny the existence of values for the quantificational variables. However, while Quine appears as a reluctant Platonist, admitting that even a soft science such as zoology requires the existence of classes, Putnam, appears to be a non-Platonist who is somehow comforted by his modal logic turn. Quine's ontological commitment welcomes abstract entities as values of the bound variables of scientific discourse while his web of beliefs does not harbor modalities at all. In contrast, Putnam's philosophical choice entertains an overall model of science that involves modalities that release pressure on commitments to objects. The present discussion makes evident that these two authors, Quine and Putnam, held different conceptions of logic with implications for the status of their respective views on the so-called indispensability arguments. In the received view, Quine and Putnam were identified in defending this influential argument which holds that successful science amply employs mathematics and to the extent that we consider science as true, mathematics as part of it, must also be true. This in turn leads to the conception that true mathematics requires the existence of mathematical entities. Contrary to Colyvan's (2004) reduction of these two views to the one he calls the Quine-*dash*-Putnam indispensability argument, which is explicitly rejected in Putnam (2012: 181-83), we have suggested above that there is already enough evidence in their respective philosophies of logic to sustain two conceptions with emergent implications for their assessment of indispensability arguments: that of Quine and that of Putnam.[10] Putnam's modal turn reduces the weight that the indispensability argument is supposed to

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9 I am indebted to my reviewer for framing this interpretative point.

10 See also D. Liggins (2008), who arrives at the need to distinguish the two views on indispensability based on a careful interpretation of Quine and Putnam preventing an oversimplified discussion of a unique indispensability view.

carry in favor of Platonism. The intention is advancing a form of new compromise realism that follows the lines of the objectivity of mathematics, as we encounter in Kreisel (1967), as opposed to the role of the existence of objects. This is such that —according to Putnam (1971: 57)— it would allow intellectual honesty to be recovered by those defenders of the indispensability of quantification over abstract entities who live ill at ease with Platonism.

Historically, Platonism appears to be the older tendency within the philosophy of mathematics whereas modalism appears to have a shorter and more recent development. To be more precise, the present debate seems to be between setism—as the present realization of the Platonist tendency—and modalism. Moreover, Putnam (1979/85, xii) qualifies Platonism as "a research program; not something fixed once and for all but something to be modified and improved by trial and error", a traditional philosophy of mathematics endowed with an epistemological theory and an ontology.[11] Parsons (1990, 289) also suggests Putnam as the inventor of modalism clearly indicating its recent arrival in the philosophy of mathematics often in connection with some form of structuralism.

From a sociological standpoint of the working mathematician Platonism appears more influential in the current philosophy of mathematics than modalism. Along these lines, Parsons (1967/72, 201) affirmed that Platonism "is the dominant attitude in the practice of modern mathematicians". However, Putnam's modalism also leads an important tendency within contemporary mathematics that can be traced as far as back as Zermelo's influential paper of 1930.

The idea of having equivalent descriptions must surely prompt us to uncover the ellipsis: description equivalent of what? The problem is to clarify what they are descriptions of if this can be only given in one of those descriptions. An interesting analogy can be found in Tarski (1929, 27), when he indicates that the science of Euclidean geometry can be formalized by taking the domain to be either the class of points or the class of solids. The fact of the matter is that in either case, there are appropriate primitive concepts in terms of which the usual geometrical concepts can be constructed. For example, Pieri (1908) provides a system of geometry whose primitive terms are "point" and the three-place relation "equidistance"; and Tarski himself gives a set of postulates whose primitive terms are "sphere" and the two-place relation "being a part of". For practical reasons, we could agree that these are just different formulations of one and the same science. However, if we take ontology seriously, the domains of these formalizations are different and hence there is a theoretical sense in which these two sciences are different.

Putnam (1967b, 72) also indicates that his view embraces that "the chief characteristic of mathematical propositions is the very wide variety of equivalent formulations that they possess [...] in mathematics the number of ways of expressing what is in some sense the same fact (if the proposition is true) while apparently not talking about the same objects is especially striking". Unfortunately, I could not find a clear sense of the word 'proposition' in Putnam's discussions. He often predicates truth of statements instead of propositions. A statement is usually understood as an ordered pair composed of a sentence and its intended interpretation. On different occasions he predicates truth of assertions, which are usually taken to be the result of a pragmatic act. A proposition on the other hand, has traditionally been understood as being abstract and providing the meaning or the interpreta-

[11] See also Putnam (1979/85b, 72).

tion of a sentence. What does exactly mean for a proposition in this sense to have different equivalent formulations? Consider by way of tentative illustration the case of the 1931 Gödel's axiomatization of arithmetic, whose universe of discourse is restricted to the class of natural numbers. In this framework the proposition that one is a number is a tautology to the effect that one is an existing [number]. On the other hand, in the 1889 Peano's presentation of arithmetic the universe of discourse is the unrestricted class of all individuals. In this framework the proposition that one is a number is an atomic non-tautological proposition to the effect that one is a numerical individual. Finally, take the Whitehead-Russell formalization in the hierarchy of types based on the universe of all [logical] individuals. In this framework the proposition that one is a number is a proposition to the effect that the class of all of the singletons belongs to the numerical class. It is not easy to see in what sense there is one and the same proposition here instead of three different propositions expressed in the languages of different frameworks. Putnam does not provide for a workable identity criterion for propositions to elucidate this issue. At any rate, the point is that *prima facie* the propositional content of these three arithmetics is not the same. Likewise, the aboutness provided by their respective different universes of discourse is also different displaying different underlying ontologies: a Gödel number is specific, a Peano number is an individual and a Whitehead-Russell number is a logical class of equivalent classes. The message here is that the universe of discourse (of a given discourse) is important in determining which propositions are expressed by which sentences and every such a discourse has an underlying ontology to be considered. This situation calls for a basic equilibrium between amplifying the scope to attain the global and lessening the depth to attain the local within our aims of inquiry.[12] The trouble with these cases is how to transfer the lessons of the equivalent formulations of objectual mathematics to the realm of the present philosophical debate between Platonism and modalism.

In this connection I would also like to suggest that mere attribution of pragmatic equivalence of the thing and the modal languages in ordinary scientific *parlance* may also disregard the historical, and perhaps even the epistemic priority of a thing-language over a modal language. This is not to deny the complementariness of these two ways of presenting mathematical results but to point out that the sense in which scientists discuss equivalent formulations may sometimes fail to consider the onto-epistemic genesis of our alleged equivalent theories. Some so-called "applications" are prior to others, and they are indicators of the time and the epistemic dependent goals of a given science. The aims behind such priority should not be overlooked and may need to be brought to light if the modal structural philosophy becomes prevalent. In short, possible equivalent formulations are identified from previous realizations of—in some unspecified sense—the very same theory and these are distinctively referential. The suggestion I am making here is not to disregard the fact that a given science is often identified not only with its true propositions, but also with its dynamic epistemic goals and the pragmatic considerations involved in its development

[12] The scenario shown in these arithmetical examples opens up the contemporary debate of the so-called incompleteness of mathematical objects; i.e., the issue of whether there is a fact of the matter to properly answer the question for the identity of the "ones" in these three arithmetics. The topic deserves a paper on its own but the reader can follow the present debate in MacBride (2005) which surveys a detailed and clear discussion of the influential views of Resnik and Shapiro on the matter, along the lines of their respective structuralist conceptions of mathematics.

which seem to be entangled with certain background ontology as it was illustrated above. In some cases at least, it seems rather unnatural to detach these considerations from the characteristic subject-matter or aboutness of particular mathematical theories and its associated metaphysics. It seems also natural to think of abstracting form particular concrete arithmetical systems so as to be able to discuss possible structures whose realizations are those prior particular systems. As Shapiro (2000a, ch. 10) also discusses and illustrates, cognitive psychology sustains an analogous process of pattern recognition responsible for the learning process of the identification of possible structures and the abstraction process underlying the type-token distinction. Be that as it may, history of mathematics and ontogenesis seem —*prima facie* and roughly— to support an analogous evolving process of modal and structuralist abstraction from prior concrete realizations.[13] Hellman's eliminative modal structuralism wants to avoid the classical epistemological problem of existence of abstract objects by discussion of systems which are logically possible. To a certain extent it may be arguable the sense in which Putnam's modalism is structural but certainly it is not eliminative. Holding the objectual or the modal picture, he says, is often determined by which of the equivalent formulations of the mathematical propositions the mathematician takes to be as primary. He also insists (1971, 75) that "none of these approaches (objectual or modal) should be regarded as more true than any other; the realm of mathematical fact admits of many 'equivalent descriptions'".[14]

Mathematics, its knowledge and practice

At first sight, knowledge of mathematics and applied logic seems to presuppose different kinds of knowledge involving not just propositional knowledge or knowing-that, but also operational knowledge or knowing-how. These two also seem to presuppose essential knowledge *of*. For example, in order to know *that* no square number is a double square number in the universe of positives, it is necessary to know *how* to square: how to raise a number to its second power. Likewise, knowing how to square seems to require knowledge *of* numbers, presumably some kind of experiential acquaintance with whatever it is we call numbers. The Platonist makes the most straight forward interpretation of the previous description of mathematical experience taken knowledge of to be primary and knowing how to be in some sense operational or manipulative.[15] We can even picture a non-logicist Platonist describing her mathematical experience involving heuristics of hypothesis discovery as well as apodictic of proofs. For such a Platonist having a pre-formal proof is as good as having a formal one. The former is epistemic while the latter reconstructs in a formal system what has already been attained. A successful realization of the axiomatic method is in

[13] Contemporary structuralism do not seem to wave the need for objects in its ontology, namely structures, whether in their *ante rem* conception of characterizing a pre-existing structure or in their *in re conception*, characterized as an equivalent class of actual or possible systems. See Shapiro (2000a, chapter 10) for a detailed and perspicuous discussion of this distinction.

[14] Putnam (1967b, 45) credits Reichenbach with the happy expression 'equivalent descriptions'. Putnam often uses the expression 'equivalent formulations' instead.

[15] See Corcoran and Sagüillo (2018, sec. III.3, especially n. 24).

this sense one way (among several) of organizing already available knowledge.[16] To an important degree some of these features are found in Putnam's (1979/85b) quasi-empirical characterization of mathematics where the emphasis is made on hypothesis discovery and quasi-empirical testing of hypothesis by means of the hypothetic-deductive method, a venerable tradition that Putnam identifies in Polya's elaborated methodology of plausible reasoning. Here Putnam (*ibidem* 61) attempts to show "that even the elementary theory of non-negative integers is not *a priori*" without denying of course that mathematics is more *a priori* than physics and making thus a case for the relativity of the *a priori/a posteriori* epistemic dichotomy. One important conclusion Putnam draws in this connection (*ibidem* 76) is the essential complementariness of the method of proof and what he calls quasi-empirical inference.

Putnam (1971) was also clear that the nominalist option of stripping our scientific language of all reference to non-physical entities was not available, perhaps hardly even conceivable. Adding the dilemma of causal interaction with numbers and sets leads to Putnam's modal proposal.[17] It should be recalled that the transition from the thing-language to the modal-language is mediated with an intermediate stage of abstraction introducing structures. Putnam (1967b, 48-49) would insist that there are different systems of arithmetic, for example those mentioned in the previous section above, all three exhibiting the same structure of an omega sequence. In this connection, Corcoran (1992, sec. 5) discusses the historical antecedents of the 1931 Gödel axiomatization of arithmetic which he depicts as a modern characterization of a structural deductive system which was based on the previous underlying logic of *Principia*, which itself, was reflecting the deductive practice developed by Peano and Dedekind in the same field of number theory and by Veblen in geometry. Shapiro (1997, ch. 5) also traces a historical account of mathematics casting doubts that the changes brought about by structuralism in the 19th century could support the hypothesis that previous research strategies of ancient mathematicians going back as far as Thales were searching for knowledge of abstract structures instead of abstract things. He also discusses the case of geometry which exemplifies a transition from the mathematician's perception of it as a theory of real space to its contemporary structural model-theoretic variant interpretation. In the face of this well-known episode of the development of fundamental branches of mathematics it seems hard not to acknowledge some form of an objectual commitment in the ordinary talk and practice of the founders of the initial systems of arithmetic and geometry.

It also appears that the issue of not taking language at face value and consequently the need to rewrite scientific language, allegations Putnam makes against the nominalist, could also by applied to Putnam's modalism itself. After all, his stance seems to propose a non-trivial rewriting of scientific language taking explicitly and seriously modal notions and bypassing direct or literal existential commitments. Indeed, the issue is not merely rhetorical. Taking the language of science literally seems in many cases to be a matter of interpretation. We find the Platonist looking at scientific practice where the modalist aims to be

16 *Ibidem* sec. III.6.

17 Incidentally, Parsons (1990, n. 36) makes it clear that Putnam was not convinced that the modal option of eliminating mathematical objects was in "all respects clearer or more fundamental than taking mathematical language at face value as referring to mathematical objects."

more sophisticated about the way we read and write what mathematicians "really" do. The fact of the matter is that many philosophers and mathematicians of a classical persuasion regularly talk of and quantify over abstract entities without any sense of being dishonest, or of being particularly eccentric, or in discomfort.[18] Consider for example, the (honest) intellectual testimony of George Boolos in his 1998 paper "Must We Believe in Set Theory?" In that paper, Boolos restates a basic trait of rationality in the need for evidence in belief formation. Contrary to Gödel, Boolos, cast doubts on some of the axioms of set-theory. He clearly points out that some basic axioms are reasonably obvious, but that some others are far from being so evident. Boolos claims that axioms implying the existence of large cardinal numbers definitely do not "impose themselves upon us" showing clear disagreement with Gödel's (1947/89, 483-84) famous assertion to the contrary. Is Boolos dishonest for not committing himself to the existence of such large sets, despite the fact that his competent discussion of set-theory presupposes an underlying classical logic with objectual quantification? Of course not; much less does his responsible competence with set-theory imply that there is the need to eliminate certain mathematical objects in favor of modalities, even though modal paraphrasing is always a choice? Working mathematicians may honestly cast doubts on certain existential statements, but this does not necessarily make them modalists (neither *tout court* fictionalists for that matter). Granted, there are different degrees of deflating or inflating scientific language, but again, linguistic practice—including canonical formalization—seems supervenient on other material scientific practices that are typical of a dynamic context of discovery. Prior to speaking and writing properly the mature language of science—whether objectual or modal—the need arises for a good metaphysics for the intellectual enterprise that has formed the goals and the historical progress of a science. In the end, honest scientific practice is the practice of a rational believer. Dynamic research indicates that a believer, whether an individual or a community of thinkers, holds not just different metaphysical beliefs, but also entertains them with different epistemic intensities, according to the available evidence. Accordingly, the Quinean web of belief contracts and expands when we drop an old belief or add a new one. In short, we can change or reinforce our beliefs depending on the evidence and our web of belief is dynamic and constantly subjected to change and adjustment to recover equilibrium. In this sense—contrary to Quine—it is perhaps time to acknowledge that our web of belief is not "uniform and seamless".[19] The ideal of having different areas of research with different domains working together supplying an overall picture of knowledge and prediction should then be accommodated to particular tasks in specific areas of research. We philosophers clearly witness how different metaphysical tensions pull at the seams in the web. Here, the compromise needed is to be tolerant in the *local* if we want to allow for unavoidable licenses and simplifications in the *global*; and this seems to be both the payoff from and the price of Putnam's objectual and modal alleged equivalent readings of scientific statements.

[18] Likewise, it is acknowledged by many Platonists that our present conceptual and experiential background does not yet provide us with an adequate accessibility account of the borders and the relations between *concreta* and *abstracta*. On this account Putnam (1979/85b, 60) grants the referentiality of mathematical statements; *i.e.* that there is *something* denoted by expressions as 'set' and 'function' but denying that "reality is somehow bifurcated".

[19] See especially Shapiro (2000a, 337) and Maddy (2005, 456-58) on ways in which the scientific enterprise does not seem to adjust to Quine's holistic picture.

Perhaps there is something after all that may work in the philosophy of mathematics—*pace* Putnam's 1994 honest but pessimistic title. Metaphysical pluralism would do the job if we think that different philosophies of mathematics—Platonism and modalism for instance emphasize different aspects and research processes within a rich and historically framed mathematical practice. The suggestion here is to concoct an heterogeneous and dynamic conception of mathematics (mathematical logic included). Detailed analysis requires philosophical compromises. Perhaps the issue between Platonism and modalism should not be presented as a debate of antagonistic philosophies defended by the very same philosopher at different times and perhaps in different domains of research. By way of analogy, Tarski, held an ideal logicist picture of arithmetic while being an empiricist with respect to string theory. *Mutatis mutandis* there should not be any objection for a mathematician entertaining the object picture in some field while holding the modal picture in another modulo their context and historical development.

If mathematical practice grants this scenario then definitional equivalence of Platonism and modalism about the same domain of research does not seem to be the only condition that really matters for the history and philosophy of mathematics. Modalism requires prior contentual non-modal results if Aristotle's *dictum* mentioned above is to be made good. In other words, modalism may mask the historical fact that specific domain applications are onto-epistemically distinguishable by the Platonist.[20] In a way, modalism, despite Putnam's insistence on mere truth-value equivalence, appears *supervenient* on Platonism, perhaps it is a Wittgensteinian ladder that a modalist must discard. Putnam himself recognizes, for example, that: "Many of the physicist's methods (variational methods, Lagrangian formulations of physics) *depend* on describing the *actual* path of a system as that path of all the *possible* ones for which a certain quantity is a minimum or maximum".

Granting that "possible" is a fully legitimate notion in many fields, we tend to see these two philosophies, Platonism and modalism, as complementary (not just compatible neither necessarily antagonistic) in the practice of many (realist) working mathematicians whenever the onto-epistemic transition from a thing-language to a modal-language has been possible. Historical practice suggests that assigning successive roles to these two types of mathematical discourse could in the end show them to be compatible and equivalently interchangeable. However, definitional equivalence should not mask the fact that many contemporary mathematicians feel at home thinking derivatively of modalities thanks to the primitive robust acknowledgment of the semantic paradigm of set-theory. It is also worth recalling that by dint of avoiding talk of specific objects, Putnam modalism erases any trace of ontological costs characteristically linked to the range of the variables in a given language. He qualifies his view saying that "Of course, one can strain after objects if one wants" by looking at different interpretations. But again this overlooks the historical fact that an intended model of a theory has—in addition to a specific domain of objects—an epistemological privileged role in the dynamics of a science. Of course, this is not to deny but to supplement Tarski's (1941/94, 117) *dictum* that "as far as the construction of our

[20] Of course, I am not contemplating here any massive error that would justify a radical historical revision. Mathematical practice has change to a certain extent introducing new standards of rigor and clarity. But I tend to see this move as an evolving process within a more or less implicit but not neatly defined platonistic philosophical framework underlying the practice of the working mathematician.

theory is concerned, this [intended] model has no distinguished place with the totality of models". The final remark is that overlooking the historical development of concrete realizations of a theory can enhance modalism with a sense of possibility which may well erase any trace of the role of time and experience in the development of a science. Again, some models are prior to others and the intended interpretation of a theory carries a characteristic subject-matter or domain of objects pointing out the platonistic underpinnings of a primary mathematical practice.

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