Chang's Conjecture and weak square

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Abstract

We investigate how weak square principles are denied by Chang's Conjecture and its generalizations. Among other things we prove that Chang's Conjecture does not imply the failure of $\square_{\omega_1,2}$, i.e. Chang's Conjecture is consistent with $\square_{\omega_1,2}$.

1 Introduction

So far many set theorists have investigated the tension between compactness phenomena, such as large cardinal axioms, stationary reflection principles and forcing axioms, and incompactness phenomena, such as square principles and the existence of Aronszajn trees. One of important researches along this line is to study what kind of weak square principles are denied by each compactness phenomenon.

First recall weak square principles. The original square principle $\square_\nu$ was introduced by Jensen [5], and he proved that $\square_\nu$ holds for every uncountable cardinal ν in L and that $\square_\nu$ implies the existence of a ν^+-Aronszajn tree. So far various kinds of weak square principles have been considered. Among them the following $\square(\kappa)$ and $\square_{\nu,\rho}$ are often discussed:

Notation. *Let δ be an ordinal. Suppose that $\langle c_\alpha \mid \alpha < \delta\rangle$ is a sequence such that each c_α is club in α. Then a club $c \subseteq \delta$ is said to* thread *$\langle c_\alpha \mid \alpha < \delta\rangle$ if $c \cap \alpha = c_\alpha$ for all $\alpha \in \mathrm{Lim}(c)$. Moreover suppose that $\langle \mathcal{C}_\alpha \mid \alpha < \delta\rangle$ is a sequence*

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such that each $\mathcal{C}_\alpha$ is a family of club subsets of α. Then a club $c \subseteq \delta$ is said to thread $\langle \mathcal{C}_\alpha \mid \alpha < \delta \rangle$ *if $c \cap \alpha \in \mathcal{C}_\alpha$ for all $\alpha \in \mathrm{Lim}(c)$.*

Definition 1.1 (Todorčević)**.** *Let κ be a regular cardinal $\geq \omega_2$.*

$\square(\kappa)$ $\equiv$ *There exists a sequence $\langle c_\alpha \mid \alpha < \kappa \rangle$ with the following properties:*

(i) c_α is a club subset of α for each $\alpha < \kappa$.

(ii) c_α threads $\langle c_\beta \mid \beta < \alpha \rangle$ for each $\alpha < \kappa$.

(iii) There are no club $C \subseteq \kappa$ threading $\langle c_\alpha \mid \alpha < \kappa \rangle$.

A sequence $\langle c_\alpha \mid \alpha < \kappa \rangle$ witnessing $\square(\kappa)$ is called a $\square(\kappa)$-sequence.

Definition 1.2 (Schimmerling [7])**.** *Let ν be an uncountable cardinal and ρ be a cardinal such that $1 \leq \rho \leq \nu$.*

$\square_{\nu,\rho}$ $\equiv$ *There exists a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \nu^+ \rangle$ such that the following hold for each $\alpha < \nu^+$:*

(i) $\mathcal{C}_\alpha$ is a set consisting of club subsets of α of order-type $\leq \nu$.

(ii) $1 \leq |\mathcal{C}_\alpha| \leq \rho$.

(iii) Every $c \in \mathcal{C}_\alpha$ threads $\langle \mathcal{C}_\beta \mid \beta < \alpha \rangle$.

A sequence $\langle \mathcal{C}_\alpha \mid \alpha < \nu^+ \rangle$ witnessing $\square_{\nu,\rho}$ is called a $\square_{\nu,\rho}$-sequence.

Note that if $\rho \leq \rho'$, then $\square_{\nu,\rho}$ implies $\square_{\nu,\rho'}$. The original square principle $\square_\nu$ is $\square_{\nu,1}$, and it is easy to see that $\square_\nu$ implies $\square(\nu^+)$.

As we mentioned before, it have been studied what kind of weak square principles are denied by each compactness phenomenon. For example, using the argument by Todorčević [9], Magidor proved that PFA implies the failure of $\square_{\nu,\omega_1}$ for all uncountable cardinal ν. On the other hand Magidor proved that PFA is consistent with $\square_{\nu,\omega_2}$ for any uncountable cardinal ν. For another example, it is well-known, essentially due to Magidor [6], that the stationary reflection for subsets of $\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ implies the failure of $\square_{\omega_1,\omega}$. On the other hand this stationary reflection principle is consistent with $\square_{\omega_1,\omega_1}$ because it is consistent with CH, and CH implies $\square_{\omega_1,\omega_1}$.

In this paper we study how weak square principles are denied by Chang's Conjecture and its generalizations. Recall Chang's Conjecture and its generalizations:

Definition 1.3. *Suppose that μ and ν are infinite cardinals with $\mu < \nu$. Then $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ denotes the statement that for any structure $\mathcal{X}$ of a countable language whose universe is ν^+ there exists $x \prec \mathcal{X}$ such that $|x| = \mu^+$ and such that $|x \cap \nu| = \mu$. Chang's Conjecture is the principle $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$.*

Todorčević proved that $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ implies the failure of $\square_\nu$. (See [10] or [3].) We discuss whether $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ implies the failure of $\square(\nu^+)$ and for what ρ, $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ implies the failure of $\square_{\nu,\rho}$.

As for $\square(\nu^+)$ we have the following result:

Notation. *Let ν be an ordinal and $\mathbb{P}$ be a poset. $\mathbb{P}$ is said to be $<\nu$-*Baire *if a forcing extension by $\mathbb{P}$ does not add any new sequences of ordinals of length $< \nu$.*

Theorem 1.4. *Suppose that μ and ν are infinite cardinals with $\mu < \nu$ and that $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ holds. Then there exists a $<\nu^+$-Baire poset $\mathbb{P}$ which forces both $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ and $\square(\nu^+)$.*

It follows from Thm.1.4 that all consistent variations of $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ are also consistent with $\square(\nu^+)$.

Next we turn our attention to $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ and $\square_{\nu,\rho}$. Results depend on ν and μ. The case when ν is a singular cardinal was investigated by Foreman [3]. In this paper we discuss the case when ν is regular.

First we have the following in the case when $\mu = \omega$:

Theorem 1.5. *Suppose that ν is a regular uncountable cardinal and that there is a measurable cardinal $> \nu$. Then there exists a $<\nu$-Baire poset $\mathbb{P}$ which forces both $(\nu^+, \nu) \twoheadrightarrow (\omega_1, \omega)$ and $\square_{\nu,2}$.*

Corollary 1.6. *Assume that* ZFC + Measurable Cardinal Axiom *is consistent. Then* ZFC + *Chang's Conjecture* + $\square_{\omega_1,2}$ *is consistent.*

As for the case when $\mu > \omega$, we have the following:

Theorem 1.7. *Let μ be an uncountable cardinal and ν be a cardinal $> \mu$. Assume $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$. Moreover suppose that either of the following holds:*

(I) $\nu^{<\nu} = \nu$.

(II) $|\{\rho \in \mathrm{Reg} \mid \mu < \rho \leq \nu\}| < |\{\rho \in \mathrm{Reg} \mid \omega \leq \rho \leq \mu\}| < \omega_1$, *where* Reg *denotes the class of all regular cardinals.*

Then $\square_{\nu,\mu}$ fails.

For example if $\mu = \omega_1$, and $\nu = \omega_2$, then the condition (II) in the above theorem holds. Thus we have the following corollary:

Corollary 1.8. $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ *implies the failure of* $\square_{\omega_2,\omega_1}$.

Here recall that $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ is consistent with $2^{\omega_1} = \omega_2$ and that $2^{\omega_1} = \omega_2$ implies $\square_{\omega_2,\omega_2}$. (The former is due to Kunen and Laver. See [2].) Thus $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ is consistent with $\square_{\omega_2,\omega_2}$. In this sense the corollary above is optimal.

2 Preliminaries

In this section we give our notation and facts used in this paper.

We begin with notation concerned with sets of ordinals: For regular cardinals μ, ν with $\mu < \nu$ let $E^\nu_\mu := \{\alpha \in \nu \mid \mathrm{cf}(\alpha) = \mu\}$. Next let A be a set of ordinals. Then $\mathrm{Lim}(A) := \{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}$, and o.t.$(A)$ denotes the order-type of A. Moreover, for a regular cardinal ρ, A is said to be *ρ-closed* if $\sup(B) \in A$ for any $B \subseteq A$ of order-type ρ.

Next we give statements equivalent to $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$. The proof of the following is standard and left to the reader:

Lemma 2.1. *The following are equivalent for infinite cardinals μ, ν with $\mu < \nu$:*

(1) $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$.

(2) For any function $F : {}^{<\omega}(\nu^+) \to \nu^+$ there exists $x \subseteq \nu^+$ such that x is closed under F, such that $|x| = \mu^+$ and such that $|x \cap \nu| = \mu$.

(3) For any structure $\mathcal{M}$ whose universe includes ν^+ there exists $M \prec \mathcal{M}$ such that $|M \cap \nu^+| = \mu^+$ and such that $|M \cap \nu| = \mu$.

Next we give a lemma and notation on the Skolem hull. For a structure $\mathcal{M}$ and $x \subseteq M$ let $\mathrm{Sk}^{\mathcal{M}}(x)$ denotes the smallest M with $x \subseteq M \prec \mathcal{M}$ if such M exists. (Otherwise, $\mathrm{Sk}^{\mathcal{M}}(x)$ is not defined.) We call $\mathrm{Sk}^{\mathcal{M}}(x)$ *the Skolem hull* of x in $\mathcal{M}$. We use the following lemma:

Lemma 2.2 (folklore)**.** *Let θ be a regular uncountable cardinal and $\mathcal{M}$ be a structure obtained by adding countable many constants, functions and predicates to $\langle \mathcal{H}_\theta, \in \rangle$. Suppose that $B \subseteq A \in M \prec \mathcal{M}$, and let*

$$N \;:=\; \{f(b) \mid f : {}^{<\omega}A \to \mathcal{H}_\theta \;\wedge\; f \in M \;\wedge\; b \in {}^{<\omega}B\}\;.$$

Then the following hold:

(1) $N = \mathrm{Sk}^{\mathcal{M}}(M \cup B)$.

(2) $\sup(N \cap \lambda) = \sup(M \cap \lambda)$ *for all regular cardinals* $\lambda \in M$ *such that* $\lambda > |A|$.

Proof. (1) First note that $M \cup B \subseteq N$ and that if $M \cup B \subseteq N' \prec \mathcal{M}$, then $N \subseteq N'$. So it suffices to show that $N \prec \mathcal{M}$. We use the Tarski-Vaught criterion.

Suppose that φ is a formula, that $c \in {}^{<\omega}N$ and that $\mathcal{M} \models \exists v \varphi[v, c]$. It suffices to find $d \in N$ such that $\mathcal{M} \models \varphi[d, c]$.

Because $c \in {}^{<\omega}N$, we can take a function $f : {}^{<\omega}A \to \mathcal{H}_\theta$ in M and $b \in {}^{<\omega}B$ such that $c = f(b)$. Then there exists a function $g : {}^{<\omega}A \to \mathcal{H}_\theta$ such that for any $a \in {}^{<\omega}A$ if $\mathcal{M} \models \exists v \varphi[v, f(a)]$, then $\mathcal{M} \models \varphi[g(a), f(a)]$. We can take such g in M by the elementarity of M.

Then $d := g(b) \in N$. Moreover $\mathcal{M} \models \varphi[d, c]$ by the choice of g and the assumption that $\mathcal{M} \models \exists v \varphi[v, c]$. Therefore d is as desired.

(2) Fix a regular cardinal $\lambda \in M$ with $|A| < \lambda$. Clearly $\sup(N \cap \lambda) \geq \sup(M \cap \lambda)$. On the other hand, for any $f : {}^{<\omega}A \to \mathcal{H}_\theta$ in M and any $b \in {}^{<\omega}B$,

$$f(b) \;\leq\; \sup\{f(a) \mid a \in {}^{<\omega}A \wedge f(a) \in \lambda\} \;\in\; M \cap \lambda\;.$$

Hence $\sup(N \cap \lambda) \leq \sup(M \cap \lambda)$. □

Below we give our notation and a fact on forcing:

Let ν be an ordinal. A poset $\mathbb{P}$ is said to be *ν-strategically closed* if Player II has a winning strategy in the followng game $\Gamma(\mathbb{P}, \nu)$ of length ν:

I	p_0	p_1	$\cdots$	$-$	$p_{\omega+1}$	$\cdots$	$-$	$p_{\omega+\omega+1}$	$\cdots$
II	q_0	q_1	$\cdots$	q_ω	$q_{\omega+1}$	$\cdots$	$q_{\omega+\omega}$	$q_{\omega+\omega+1}$	$\cdots$

Player I opens the game by choosing an arbitrary $p_0 \in \mathbb{P}$, and then Player II chooses $q_0 \leq p_0$. In the ξ-th stage Player I and II move as follows: If ξ is successor, then first Player I chooses $p_\xi \leq q_{\xi-1}$, and Player II chooses $q_\xi \leq p_\xi$

after that. If ξ is limit, then Player I does nothing. Player II chooses a lower bound q_ξ of $\{q_\eta \mid \eta < \xi\}$ if it exists. Otherwise, the game is over at this stage. Player II wins if the game continued for ν-stages, that is, he could choose q_ξ for all limit $\xi < \nu$. Otherwise, Player I wins.

It is easy to see that if $\mathbb{P}$ is ν-strategically closed, then $\mathbb{P}$ is $< \nu$-Baire.

Let $\mathbb{P}$ be a poset and M be a set. A sequence $\langle p_\xi \mid \xi < \zeta \rangle$ is said to be $(M, \mathbb{P})$-*generic* if it is a descending sequence in $\mathbb{P} \cap M$, and for any dense open $D \subseteq \mathbb{P}$ with $D \in M$ there exists $\xi < \zeta$ with $p_\xi \in D$. A condition $p \in \mathbb{P}$ is said to be *strongly* $(M, \mathbb{P})$-*generic* if p is a lower bound of some $(M, \mathbb{P})$-generic sequence. For a $\mathbb{P}$-generic filter G over V let $M[G] := \{\dot{a}^G \mid \dot{a} \text{ is a } \mathbb{P}\text{-name in } M\}$, where $\dot{a}^G$ denotes the evaluation of $\dot{a}$ by G.

Finally we give our notation on forcing notions for collapsing cardinals. Let ν be a regular cardinal, and let α and β be ordinals with $\alpha < \beta$. Then let $\mathrm{Col}(\nu, \alpha)$ be the poset ${}^{<\nu}\alpha$ ordered by reverse inclusions. Moreover let $\mathrm{Col}(\nu, {<}\alpha)$ be the $< \nu$-support product of $\langle \mathrm{Col}(\nu, \gamma) \mid \gamma < \alpha \rangle$, and let $\mathrm{Col}(\nu, [\alpha, \beta))$ be the $< \nu$-support product of $\langle \mathrm{Col}(\nu, \gamma) \mid \alpha \leq \gamma < \beta \rangle$. Thus if α is an inaccessible cardinal, then $\mathrm{Col}(\nu, {<}\alpha)$ is the Lévy collapse making α to be ν^+. We use the following lemma:

Lemma 2.3 (folklore)**.** *Let ν and κ be regular cardinals with $\nu \leq \kappa$. Suppose that $\mathbb{P}$ is a $< \nu$-closed poset of size κ which forces $|\kappa| = \nu$. Then $\mathbb{P}$ is forcing equivalent to* $\mathrm{Col}(\nu, \kappa)$.

Proof. Let $\dot{\pi}$ be a $\mathbb{P}$-name for a surjection from ν to $\mathbb{P}$. For each $q \in \mathrm{Col}(\nu, \kappa)$, by induction on $\mathrm{dom}\, q$, we define a descending function $s_q : \mathrm{dom}\, q \to \mathbb{P}$. We define s_q's so that if $q' \leq q$, then $s_{q'} \supseteq s_q$:

First suppose that $\mathrm{dom}\, q$ is a limit ordinal. Then let $f_q := \bigcup_{\xi < \mathrm{dom}\, q} f_{q \restriction \xi}$. Next suppose that f_q has been defined. We define $f_{q'}$ for each $q' \leq q$ with $\mathrm{dom}\, q' = (\mathrm{dom}\, q) + 1$. First take $A \subseteq \mathbb{P}$ with the followng properties:

(i) A is a maximal antichain in the suborder of $\mathbb{P}$ consisting of all lower bounds of $\{f_q(\xi) \mid \xi < \mathrm{dom}\, q\}$.

(ii) $|A| = \kappa$.

(iii) Every element of A decides $\dot{\pi}(\mathrm{dom}\, q)$.

(iv) If $p \in A$, and $p \Vdash$ "$\dot{\pi}(\mathrm{dom}\, q) = r$", then either $p \leq r$ or $p \perp r$.

We can take such A by the $<\nu$-closure of $\mathbb{P}$ and the fact that $\mathbb{P}$ forces $|\kappa| = \nu$. Let $\langle p_\alpha \mid \alpha < \kappa \rangle$ be a 1-1 enumeration of A. Then for each $q' \leq q$ with $\operatorname{dom} q' = (\operatorname{dom} q) + 1$ let $f_{q'}$ be the extension of f_q such that $f_{q'}(\operatorname{dom} q) = p_{q'(\operatorname{dom} q)}$.

Now we have defined f_q for all $q \in \mathrm{Col}(\nu, \kappa)$. Let $\mathbb{Q}$ be the suborder of $\mathrm{Col}(\nu, \kappa)$ consisting of all $q \in \mathrm{Col}(\nu, \kappa)$ whose domain is a successor ordinal. Moreover for each $q \in \mathbb{Q}$ let $d(q) = f_q((\operatorname{dom} q) - 1)$. It suffices to show that d is a dense embedding from $\mathbb{Q}$ to $\mathbb{P}$.

Clearly, $d(q_0) \leq d(q_1)$ in $\mathbb{P}$ if and only if $q_0 \leq q_1$. Thus it suffices to show that the range of d is dense in $\mathbb{P}$. Take an arbitrary $p \in \mathbb{P}$. We find $q \in \mathbb{Q}$ with $d(q) \leq p$. First we can take $p' \leq p$ and $\xi < \nu$ such that $p' \Vdash$ "$\dot{\pi}(\xi) = p$". Here note that the set $\{d(q) \mid \operatorname{dom} q = \xi + 1\}$ is predense in $\mathbb{P}$ by the construction of f_q's. Thus we can take $q \in \mathbb{Q}$ such that $\operatorname{dom} q = \xi + 1$ and such that $d(q)$ is compatible with p'. Then $d(q) \Vdash$ "$\dot{\pi}(\xi) = p$" by the property (iii) above and the compatibility of $d(q)$ and p'. Then $d(q) \leq p$ by the property (iv) above and the compatibility of $d(q)$ and p.

This completes the proof. □

3 $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ and $\square(\nu^+)$

In this section we prove Thm.1.4:

Theorem 1.4. *Suppose that μ and ν are infinite cardinals with $\mu < \nu$ and that $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ holds. Then there exists a $<\nu^+$-Baire poset $\mathbb{P}$ which forces both $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ and $\square(\nu^+)$.*

This follows from the standard facts below:

- A straight forward forcing notion for $\square(\nu^+)$ is ν^+-strategically closed.
- $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ is preserved by ν^+-strategically closed forcing extensions.

For the completeness of this paper we give the proof of these facts.

First we present a forcing notion for $\square(\kappa)$, which consists of initial segments of a $\square(\kappa)$-sequence:

Definition 3.1. *Suppose that κ is a regular cardinal $\geq \omega_2$. Then let $\mathbb{P}(\square(\kappa))$ be the poset of all $p = \langle c_\alpha \mid \alpha \leq \delta \rangle$ for some $\delta < \kappa$ such that the following hold for each $\alpha \leq \delta$:*

(i) c_α *is a club subset of* α.

(ii) c_α *threads* $\langle c_\beta \mid \beta < \alpha \rangle$.

$p_0 \leq p_1$ *in* $\mathbb{P}(\square(\kappa))$ *if and only if* p_0 *is an end-extension of* p_1.

If $p = \langle c_\alpha \mid \alpha \leq \delta \rangle \in \mathbb{P}(\square(\kappa))$, *then each* c_α *is denoted as* $p(\alpha)$, *and* δ *is denoted as* δ_p.

We show that $\mathbb{P}(\square(\kappa))$ is κ-strategically closed and forces $\square(\kappa)$.

Lemma 3.2. $\mathbb{P}(\square(\kappa))$ *is* κ*-strategically closed for any regular cardinal* $\kappa \geq \omega_2$.

Proof. Fix a regular cardinal $\kappa \geq \omega_2$. Consider the following strategy of Player II for $\Gamma(\mathbb{P}(\square(\kappa)), \kappa)$: First suppose that ξ is a successor ordinal $< \omega_1$ and that Player I has choosed p_ξ at the ξ-th stage. Then Player II chooses q_ξ which strictly extends p_ξ. Next suppose that ξ is a limit ordinal $< \omega_1$ and that $\langle q_\eta \mid \eta < \xi \rangle$ is a sequence of Player II's moves before the ξ-th stage. Let $c := \{\delta_{q_\eta} \mid \eta < \xi\}$. Then II plays

$$q_\xi \ := \ (\bigcup_{\eta<\xi} q_\eta)^\frown \langle c \rangle$$

at the ξ-th stage.

By induction on ξ we can easily prove that for each limit $\xi < \kappa$ the set $\{\delta_{q_\eta} \mid \eta < \xi\}$ is club in δ_{q_ξ} and threads $\bigcup_{\eta<\xi} q_\eta$. Thus $q_\xi \in \mathbb{P}(\square(\kappa))$, and q_ξ is a lower bound of $\{q_\eta \mid \eta < \xi\}$. Therefore the above is a winning strategy of Player II. $\square$

Lemma 3.3. *Let* κ *be a regular cardinal* $\geq \omega_2$. *Then* $\mathbb{P}(\square(\kappa))$ *forces* $\square(\kappa)$.

Proof. Let $\mathbb{P} := \mathbb{P}(\square(\kappa))$, and let $\dot{H}$ be the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter. We show that $\bigcup \dot{H}$ is a $\square(\kappa)$-sequence in $V^{\mathbb{P}}$.

First note that the set $\{p \in \mathbb{P} \mid \delta_p \geq \alpha\}$ is dense in $\mathbb{P}$ for any $\alpha < \kappa$. This can be easily proved using the κ-strategically closure of $\mathbb{P}$. Thus in $V^{\mathbb{P}}$, $\bigcup \dot{H}$ is a sequence of length κ satisfying the properties (i) and (ii) in the definition of $\square(\kappa)$ (Def.1.1). To check the property (iii), take an arbitrary $p \in \mathbb{P}$ and an arbitrary $\mathbb{P}$-name $\dot{C}$ for a club subset of κ. It suffices to find $p^* \leq p$ such that

$$p^* \Vdash_{\mathbb{P}} \text{“}\, \delta_{p^*} \in \mathrm{Lim}(\dot{C}) \wedge \dot{C} \cap \delta_{p^*} \neq p^*(\delta_{p^*}) \ \text{”} \ .$$

By induction on $n \in \omega$ take a descending sequence $\langle p_n \mid n \in \omega \rangle$ below p and an increasing sequence $\langle \alpha_n \mid n \in \omega \rangle$ in κ so that $p_{n+1} \Vdash$ “ $\min(\dot{C} \setminus \delta_{p_n}) = \alpha_n$ ”

and $\alpha_n < \delta_{p_{n+1}}$ for each $n < \omega$. Moreover let $c := \{\alpha_n \mid n < \omega \wedge n\text{: even}\}$. Note that any lower bound of $\{p_n \mid n \in \omega\}$ forces that $\dot{C} \cap \sup_{n<\omega} \delta_{p_n} \neq c$. Then $p^* := (\bigcup_{n<\omega} p_n)^\frown \langle c \rangle$ is as desired. □

Next we show that $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ is preserved by ν^+-strategically closed forcing extensions:

Lemma 3.4. *Suppose that μ and ν are infinite cardinals with $\mu < \nu$ and that $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$. Then $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ remains to hold in every ν^+-strategically closed forcing extension.*

Proof. Suppose that $\mathbb{P}$ is a ν^+-strategically closed poset. Take an arbitrary $p \in \mathbb{P}$ and an arbitrary $\mathbb{P}$-name $\dot{F}$ for a function from ${}^{<\omega}(\nu^+)$ to ν^+. It suffices to find $p^* \leq p$ and $x^* \subseteq \nu^+$ such that $|x^*| = \mu^+$, such that $|x^* \cap \nu| = \mu$ and such that $p^* \Vdash$ "x^* is closed under $\dot{F}$".

Take an enumeration $\langle a_\xi \mid \xi < \nu^+ \rangle$ of ${}^{<\omega}(\nu^+)$. Using the ν^+-strategically closure of $\mathbb{P}$ we can easily construct a descending sequence $\langle q_\xi \mid \xi < \nu^+ \rangle$ below p and a sequence $\langle \alpha_\xi \mid \xi < \nu^+ \rangle$ in ν^+ such that $q_\xi \Vdash$ "$\dot{F}(a_\xi) = \alpha_\xi$" for each $\xi < \nu^+$. Let $F : {}^{<\omega}(\nu^+) \to \nu^+$ be the function defined as $F(a_\xi) := \alpha_\xi$.

Because $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ holds in V, we can take $x^* \subseteq \nu^+$ such that $|x^*| = \mu^+$, such that $|x^* \cap \nu| = \mu$ and such that x^* is closed under F. Moreover take $\zeta < \nu^+$ such that ${}^{<\omega}x^* \subseteq \{a_\xi \mid \xi < \zeta\}$. Then $p^* := p_\zeta$ and x^* are as desired. □

It follows from Lem.3.2, 3.3 and 3.4 that $\mathbb{P}(\square(\nu^+))$ witnesses Thm.1.4.

4 $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ and $\square_{\nu,\rho}$

4.1 $\mu = \omega$

Here we prove Thm.1.5:

Theorem 1.5. *Suppose that ν is a regular uncountable cardinal and that there is a measurable cardinal $> \nu$. Then there exists a $<\nu$-Baire poset $\mathbb{P}$ which forces both $(\nu^+, \nu) \twoheadrightarrow (\omega_1, \omega)$ and $\square_{\nu,2}$.*

Let κ be a measurable cardinal $> \nu$. The desired forcing extension will be the Lévy collapse $\mathrm{Col}(\nu, < \kappa)$ followed by a straight forward forcing notion for $\square_{\nu,2}$. First we present the forcing notion for $\square_{\nu,2}$:

Definition 4.1. *Let ν be an uncountable cardinal. Then let $\mathbb{P}(\square_{\nu,2})$ be the poset of all $p = \langle \mathcal{C}_\alpha \mid \alpha \leq \delta \rangle$ for some $\delta < \nu^+$ such that the following hold for all $\alpha \leq \delta$:*

(i) $\mathcal{C}_\alpha$ is a family of club subsets of α of order-type $\leq \nu$.

(ii) $1 \leq |\mathcal{C}_\alpha| \leq 2$.

(iii) Each $c \in \mathcal{C}_\alpha$ threads $\langle \mathcal{C}_\beta \mid \beta < \alpha \rangle$.

$p_0 \leq p_1$ in $\mathbb{P}(\square_{\nu,2})$ if and only if p_0 is an end-extension of p_1.

If $p = \langle \mathcal{C}_\alpha \mid \alpha \leq \delta \rangle \in \mathbb{P}(\square_{\nu,2})$, then each $\mathcal{C}_\alpha$ is denoted as $p(\alpha)$, and δ is denoted as δ_p.

We prove the following:

Proposition 4.2. *Suppose that ν is a regular uncountable cardinal and that κ is a measurable cardinal $> \nu$. Then $\mathrm{Col}(\nu, < \kappa) * \dot{\mathbb{P}}(\square_{\nu,2})$ is $< \nu$-Baire and forces both $(\nu^+, \nu) \twoheadrightarrow (\omega_1, \omega)$ and $\square_{\nu,2}$.*

First we present basic properties of $\mathbb{P}(\square_{\nu,2})$ (Lem.4.3, 4.4 and 4.5):

Lemma 4.3. *Suppose that ν is a regular uncountable cardinal. Then $\mathbb{P}(\square_{\nu,2})$ is $\nu + 1$-strategically closed.*

Proof. Consider the following strategy of Player II for $\Gamma(\mathbb{P}(\square_{\nu,2}), \nu + 1)$: First suppose that ξ is a successor ordinal $< \nu + 1$ and that Player I has choosed p_ξ at the ξ-th stage. Then Player II chooses q_ξ strictly extending p_ξ. Next suppose that ξ is a limit ordinal $< \nu + 1$ and that $\langle q_\eta \mid \eta < \xi \rangle$ is a sequence of Player II's moves before the ξ-th stage. Let $c := \{\delta_{q_\eta} \mid \eta < \xi\}$. Then II chooses

$$q_\xi \ := \ (\bigcup_{\eta < \xi} q_\eta)^\frown \langle \{c\} \rangle$$

at the ξ-th stage.

By induction on ξ we can easily prove that for each limit $\xi < \nu + 1$ the set $\{\delta_{q_\eta} \mid \eta < \xi\}$ is club in δ_{q_ξ} and threads $\bigcup_{\eta < \xi} q_\eta$. Thus $q_\xi \in \mathbb{P}(\square(\kappa))$, and q_ξ is a lower bound of $\{q_\eta \mid \eta < \xi\}$. Therefore the above is a winning strategy of Player II. $\square$

Using the $\nu + 1$-strategically closure of $\mathbb{P}(\square_{\nu,2})$, we can easily prove the following:

Lemma 4.4. *Let ν be an uncountable cardinal. Then for any $\delta < \nu^+$ the set $\{p \in \mathbb{P}(\square_{\nu,2}) \mid \delta_p \geq \delta\}$ is dense in $\mathbb{P}(\square_{\nu,2})$.*

Then we have the following by the construction of $\mathbb{P}(\square_{\nu,2})$:

Lemma 4.5. *Let ν be an uncountable cardinal. Suppose that H is a $\mathbb{P}(\square_{\nu,2})$-generic filter over V. Then $\bigcup H$ is a $\square_{\nu,2}$-sequence in $V[H]$.*

In Prop.4.2, $\mathrm{Col}(\nu, <\kappa) * \dot{\mathbb{P}}(\square_{\nu,2})$ is $< \nu$-Baire by Lem.4.3 and forces $\square_{\nu,2}$ by Lem.4.5. Thus it suffices for Prop.4.2 to prove the following:

Lemma 4.6. *Suppose that ν is a regular uncountable cardinal and that κ is a measurable cardinal $> \nu$. Then $\mathrm{Col}(\nu, <\kappa) * \dot{\mathbb{P}}(\square_{\nu,2})$ forces $(\nu^+, \nu) \twoheadrightarrow (\omega_1, \omega)$.*

To prove Lem.4.6 we need some preliminaries. Before starting preliminaries we present the outline of the proof of Lem.4.6:

Outline of proof of Lem.4.6. Let G be a $\mathrm{Col}(\nu, <\kappa)$-generic filter over V. In $V[G]$ suppose that $p \in \mathbb{P}(\square_{\nu,2})$, that $\dot{F}$ is a $\mathbb{P}(\square_{\nu,2})$-name for a function from ${}^{<\omega}\kappa$ to κ. In $V[G]$ it suffices to find $x^* \subseteq \kappa$ and $p^* \leq p$ such that $|x^*| = \omega_1$, such that $|x^* \cap \nu| = \omega$ and such that $p^* \Vdash$ "x^* is closed under $\dot{F}$".

For this first we prove that a variant of the Strong Chang's Conjecture holds in $V[G]$ (Lem.4.12). (For the Strong Chang's Conjecture see Shelah [8] Ch.XII §2 or Foreman-Magidor-Shelah [4].) Using this variant of the Strong Chang's Conjecture, in $V[G]$ we construct a $\subseteq$-increasing sequence $\langle M_\xi \mid \xi < \omega_1 \rangle$ of countable elementary submodels of $\langle \mathcal{H}_\theta, \in, \mathbb{P}(\square_{\nu,2}), p, \dot{F} \rangle$ for some sufficiently large regular cardinal θ and a descending sequence $\langle p_\xi \mid \xi < \omega_1 \rangle$ below p with the following properties:

(i) $M_\xi \cap \nu = M_0 \cap \nu$ for each $\xi < \omega_1$.

(ii) p_ξ is strongly $(M_\xi, \mathbb{P}(\square_{\nu,2}))$-generic and belongs to $M_{\xi+1}$ for each $\xi < \omega_1$.

(iii) $\{p_\xi \mid \xi < \omega_1\}$ has a lower bound.

Note that $M_{\xi+1} \cap \nu^+$ is strictly larger than $M_\xi \cap \nu^+$ by the property (ii).

Let $M^* := \bigcup_{\xi<\omega_1} M_\xi$ and $x^* := M^* \cap \nu^+$. Then $|x^*| = \omega_1$ by the remark above, and $|x^* \cap \nu| = \omega$ by the property (i). Moreover let p^* be a lower bound of $\{p_\xi \mid \xi < \omega_1\}$. Note that p^* is strongly $(M^*, \mathbb{P}(\square_{\nu,2}))$-generic. Therefore p^* forces that x^* is closed under $\dot{F}$. □

Now we start preliminaries for the proof of Lem.4.6. For our variant of the Strong Chang's Conjecture we use the following forcing notion which adds a club subset of ν^+ of order-type ν threading a $\square_{\nu,2}$-sequence.

Definition 4.7. *Let ν be a regular uncountable cardinal. Suppose that $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \nu^+ \rangle$ is a $\square_{\nu,2}$-sequence. Then let $\mathbb{C}(\vec{\mathcal{C}})$ be the poset of all c such that*

(i) c is a closed bounded subset of ν^+ of order-type $< \nu$,

(ii) c threads $\langle \mathcal{C}_\alpha \mid \alpha \leq \max c \rangle$. (This is equivalent to that $c \in \mathcal{C}_{\max c}$ if $\max c \in \mathrm{Lim}(c)$.)

$c_0 \leq c_1$ in $\mathbb{C}(\vec{\mathcal{C}})$ if and only if c_0 is an end-extension of c_1.

In fact we use a poset $\mathbb{C}(\bigcup H)$ for a $\mathbb{P}(\square_{\nu,2})$-generic filter H. We give basic facts, Lem.4.8, 4.9 and 4.10, on this poset before proceeding to a variant of the Strong Chang's Conjecture:

Lemma 4.8. *Let ν be a regular uncountable cardinal and $\dot{H}$ be the canonical $\mathbb{P}(\square_{\nu,2})$-name for a $\mathbb{P}(\square_{\nu,2})$-generic filter. Then*

$$D \ := \ \{p * \check{c} \in \mathbb{P}(\square_{\nu,2}) * \mathbb{C}(\textstyle\bigcup \dot{H}) \mid c \in V \ \wedge \ \delta_p = \max c\}$$

is a $< \nu$-closed dense subset of $\mathbb{P}(\square_{\nu,2}) * \mathbb{C}(\bigcup \dot{H})$ of size 2^ν.

Proof. Clearly $|D| = 2^\nu$. We show that D is $< \nu$-closed and dense. Let $\mathbb{P} := \mathbb{P}(\square_{\nu,2})$ and $\dot{\mathbb{C}} := \mathbb{C}(\bigcup \dot{H})$.

First we prove the density of D. Suppose that $p * \dot{c} \in \mathbb{P} * \dot{\mathbb{C}}$. Because $\mathbb{P}$ is $\nu+1$-strategically closed, we can take $p^* \leq p$ and $c' \in V$ such that $p^* \Vdash$ "$\dot{c} = \check{c'}$". By extending p^* if necessary, we may assume that $\delta_{p^*} > \max c'$. Let $c^* := c' \cup \{\delta_{p^*}\}$. Then $p^* * \check{c^*} \leq p * \dot{c}$, and $p^* * \check{c^*} \in D$.

Next we prove the $< \nu$-closure of D. Suppose that ζ is a limit ordinal $< \nu$ and that $\langle p_\xi * \check{c_\xi} \mid \xi < \zeta \rangle$ is a strictly descending sequence in D. Let $c' := \bigcup_{\xi<\zeta} c_\xi$ and

$$p^* \ := \ (\textstyle\bigcup_{\xi<\zeta} p_\xi)^\frown \langle \{c'\} \rangle \ .$$

Note that c' threads $\bigcup_{\xi<\zeta} p_\xi$ because each c_ξ threads p_ξ. Thus $p^* \in \mathbb{P}(\square_{\nu,2})$. Let $c^* := c' \cup \{\sup c'\}$. Then $p^* * \check{c^*}$ belongs to D and is a lower bound of $\{p_\xi * \check{c_\xi} \mid \xi < \zeta\}$. $\square$

¿From Lem.4.8 we have the following easily:

Lemma 4.9. *Let ν be a regular uncountable cardinal. Suppose that H is a $\mathbb{P}(\square_{\nu,2})$-generic filter over V and that I is a $\mathbb{C}(\bigcup H)$-generic filter over $V[H]$. Then the following hold:*

(1) $\bigcup I$ is a club in $(\nu^+)^V$ of order-type ν. So $|(\nu^+)^V| = \nu$.

(2) $(\bigcup H)^\frown \langle \{\bigcup I\}\rangle \in \mathbb{P}(\square_{\nu,2})^{V[H*I]}$.

¿From Lem.2.3, 4.8 and 4.9 (1) we have the followng:

Lemma 4.10. *Assume that $2^\nu = \nu^+$. Let $\dot{H}$ be the canonical $\mathbb{P}(\square_{\nu,2})$-name for a $\mathbb{P}(\square_{\nu,2})$-generic filter. Then $\mathbb{P}(\square_{\nu,2}) * \mathbb{C}(\bigcup \dot{H})$ is forcing equivalent to $\mathrm{Col}(\nu, \nu^+)$.*

We proceed to a variant of the Strong Chang's Conjecture. The following is a key lemma. In the following note that if p is a strongly $(N, \mathbb{P}(\square_{\nu,2}))$-generic condition, then $\sup(N \cap \kappa) \leq \delta_p$ by Lem.4.4:

Lemma 4.11. *In V let ν be a regular uncountable cardinal and κ be a measurable cardinal $> \nu$. Suppose that G is a $\mathrm{Col}(\nu, <\kappa)$-generic filter over V. In $V[G]$ let $\dot{H}$ be the canonical $\mathbb{P}(\square_{\nu,2})$-name for a $\mathbb{P}(\square_{\nu,2})$-generic filter. Then in $V[G]$ there are club many $N \in [\mathcal{H}_{\kappa^+}]^\omega$ such that for any strongly $(N, \mathbb{P}(\square_{\nu,2}))$-generic condition p if there is $c \in p(\sup(N \cap \kappa))$ with*

$$p \Vdash \text{“}c \cup \{\sup(N \cap \kappa)\} \text{ is strongly } (N[\dot{H}], \mathbb{C}(\textstyle\bigcup \dot{H}))\text{-generic”},$$

then there exists a countable $N^* \prec \langle \mathcal{H}_{\kappa^+}, \in\rangle$ and $p^* \in \mathbb{P}(\square_{\nu,2}) \cap N^*$ such that

(i) $N^* \supseteq N$, and $N^* \cap \nu = N \cap \nu$,

(ii) $p^* \leq p$.

Proof. In V let U be a normal measure over κ, let W be the transitive collapse of the ultrapower of V by U, and let $j_0 : V \to W$ be the ultrapower map. Moreover let θ be a sufficiently large regular cardinal.

In $V[G]$ let $\mathcal{N} := \langle \mathcal{H}^{V[G]}_{\kappa^+}, \in\rangle$ and $\mathcal{M} := \langle \mathcal{H}^{V[G]}_{\theta}, \in, \kappa, U, \mathcal{H}^V_{\kappa^+}, G\rangle$. Moreover let $\mathbb{P}$ be $\mathbb{P}(\square_{\nu,2})$ defined in $V[G]$ and $\dot{\mathbb{C}}$ be $\mathbb{C}(\bigcup \dot{H})$. Here note that $\mathcal{N}$ is definable in $\mathcal{M}$ and that $\mathbb{P}$, $\dot{H}$ and $\dot{\mathbb{C}}$ are all definable in $\mathcal{N}$.

In $V[G]$ take an arbitrary countable $M \prec \mathcal{M}$ and let $N := M \cap \mathcal{N}$. Moreover suppose that p and c are as in the lemma for N. It suffices to prove the existence

of a countable $N^* \prec \mathcal{N}$ and $p^* \in \mathbb{P} \cap N^*$ with the properties (i) and (ii). For this we use a generic elementary embedding.

First let $c' := c \cup \{\sup(N \cap \kappa)\}$. Note that $p * \check{c'}$ is strongly $(N, \mathbb{P} * \dot{\mathbb{C}})$-generic and thus that it is strongly $(M, \mathbb{P} * \dot{\mathbb{C}})$-generic. Take a $\mathbb{P} * \dot{\mathbb{C}}$-generic filter $H * I$ over $V[G]$ containing $p * \check{c'}$. Moreover take a $\mathrm{Col}(\nu, [\kappa+1, j_0(\kappa)))$-generic filter J over $V[G * H * I]$ containing a strongly $(M[H * I], \mathrm{Col}(\nu, [\kappa+1, j_0(\kappa)))$-generic condition. We can take such J because $M[H * I]$ is countable, and $\mathrm{Col}(\nu, [\kappa+1, j_0(\kappa)))$ is $<\nu$-closed. Let $\bar{G} := G * H * I * J$. Below we work in $V[\bar{G}]$.

Note that

$$\mathrm{Col}(\nu, <\kappa) * \dot{\mathbb{P}}(\square_{\nu,2}) * \mathbb{C}(\bigcup \dot{H}) * \mathrm{Col}(\nu, [\kappa+1, j_0(\kappa)))$$

is absolute between V and W because ${}^{\kappa}W \cap V \subseteq W$. Moreover it is forcing equivalent to $\mathrm{Col}(\nu, < j_0(\kappa))$ in W by Lem.4.10. Define $j_1 : V[G] \to W[\bar{G}]$ as $j_1(\dot{a}^G) := j_0(\dot{a})^{\bar{G}}$ for each $\mathrm{Col}(\omega_1, <\kappa)$-name $\dot{a}$. Then j_1 is an elementary embedding which extends j_0.

By the elementarity of j_1 it suffices to show that in $W[\bar{G}]$ there exist a countable $N^* \prec j_1(\mathcal{N})$ and $p^* \in j_1(\mathbb{P}) \cap N^*$ such that

(i)′ $N^* \supseteq j_1(N)$, and $N^* \cap j_1(\nu) = j_1(N) \cap j_1(\nu)$,

(ii)′ $p^* \leq j_1(p)$.

We show that $N^* := M[H * I * J] \cap j_1(\mathcal{N})$ and $p^* := (\bigcup H)^\frown \langle \{\bigcup I\} \rangle$ witness this.

Let $\bar{\mathcal{M}} := \langle \mathcal{H}_\theta^{V[\bar{G}]}, \in, \kappa, U, \mathcal{H}_{\kappa^+}^V, \bar{G} \rangle$. Note that $M[H * I * J] \prec \bar{\mathcal{M}}$.

First we show that p^* is as desired. Note that p^* is definable in $\bar{\mathcal{M}}$ and belongs to $j_1(\mathcal{N})$. So $p^* \in N^*$. Moreover $p^* \in j_1(\mathbb{P})$ by Lem.4.9 (2). Furthermore $p^* \leq p$ because $p \in H$, and $j_1(p) = p$ because the rank of p is below the critical point κ of j_1. Thus $p^* \leq j_1(p)$.

Next we show that N^* is as desired. First note that $j_1(\mathcal{N})$ is definable in $\bar{\mathcal{M}}$. So $N^* = M[H*I*J] \cap j_1(\mathcal{N}) \prec j_1(\mathcal{N})$ by the elementarity of $M[H*I*J]$. Below we check the property (i)′. Note that $j_1 \restriction \mathcal{N}$ is also definable in $\bar{\mathcal{M}}$ and thus that $M[H * I * J]$ is closed under $j_1 \restriction \mathcal{N}$. Recall that $N = M \cap \mathcal{N}$, and note that $j_1(N) = j_1[N]$ because N is countable. So $j_1(N) \subseteq M[H * I * J] \cap j_1(\mathcal{N}) = N^*$. For the latter statement note that $M[H*I*J] \cap \mathrm{On} = M \cap \mathrm{On}$ because $H * I * J$

contains a strongly M-generic condition. Hence $N^* \cap \nu = N \cap \nu$. But ν is below the critical point κ of j_1. So $N^* \cap j_1(\nu) = j_1[N] \cap j_1(\nu) = j_1(N) \cap j_1(\nu)$.

This completes the proof. □

Using the previous lemma we can prove that a variant of the Strong Chang's Conjecture holds in $V^{\mathrm{Col}(\nu,<\kappa)}$:

Lemma 4.12. *In V let ν be a regular uncountable cardinal and κ be a measurable cardinal $> \nu$. Suppose that G is a $\mathrm{Col}(\nu,<\kappa)$-generic filter over V. Then in $V[G]$ the following holds for any regular cardinal $\theta > 2^\kappa$: For any countable $M \prec \langle \mathcal{H}_\theta, \in, \nu \rangle$, any $(M, \mathbb{P}(\square_{\nu,2}))$-generic sequence $\langle p_n \mid n \in \omega \rangle$ and any club $d \subseteq \sup(M \cap \kappa)$ threading $\bigcup_{n<\omega} p_n$ there exists a countable $M^* \prec \langle \mathcal{H}_\theta, \in, \nu \rangle$ and $p^* \in \mathbb{P}(\square_{\nu,2}) \cap M^*$ such that*

(i) $M^* \supseteq M$, and $M^* \cap \nu = M \cap \nu$.

(ii) $p^* \leq p_n$ for all $n < \omega$,

(iii) $d \in p^*(\sup(M \cap \kappa))$.

Proof. We work in $V[G]$. Fix a sufficiently large regular cardinal θ, and suppose that M, $\langle p_n \mid n < \omega \rangle$ and d are as in the lemma. We find M^* and p^* as in the lemma.

First note that $\mathcal{H}_{\kappa^+} \in M$ because $\kappa = \nu^+$ and $M \prec \langle \mathcal{H}_\theta, \in, \nu \rangle$. Let $Z \in M$ be a club subset of $[\mathcal{H}_{\kappa^+}]^\omega$ witnessing Lem.4.11. Then $N := M \cap \mathcal{H}_{\kappa^+} \in Z$.

Next let $\mathbb{P} := \mathbb{P}(\square_{\nu,2})$, $\dot{H}$ be the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter and $\dot{\mathbb{C}} := \mathbb{C}(\bigcup \dot{H})$. Moreover let $\langle \dot{D}_n \mid n < \omega \rangle$ be an enumeration of all $\mathbb{P}$-names for dense open subsets of $\dot{\mathbb{C}}$ which belong to M.

Then we can take a sequence $\langle \dot{c}_n \mid n < \omega \rangle$ of $\mathbb{P}$-names in M such that $\Vdash$ "$\dot{c}_n \in \dot{D}_n \wedge \dot{c}_{n+1} \leq \dot{c}_n$" for all $n < \omega$. Moreover, by Lem.4.3, for each $n < \omega$ we can take a bounded closed subset $c_n \in M$ of κ such that $p_m \Vdash$ "$\dot{c}_n = \check{c_n}$" for some $m < \omega$. Let $c := \bigcup_{n<\omega} c_n$. Note that c is club in $\sup(M \cap \kappa)$. Let

$$p \;:=\; (\textstyle\bigcup_{n<\omega} p_n)^\frown \langle \{c, d\} \rangle \;.$$

Then p is strongly $(N, \mathbb{P})$-generic and forces that $c \cup \{\sup(N \cap \kappa)\}$ is strongly $(N[\dot{H}], \dot{\mathbb{C}})$-generic. Here recall that $N \in Z$. Let N^* and p^* be those obtained by Lem.4.11 for N and p. Moreover let

$$M^* \;:=\; \{f(p^*) \mid f : \mathbb{P} \to \mathcal{H}_\theta \;\wedge\; f \in M\} \;=\; \mathrm{Sk}^{\langle \mathcal{H}_\theta, \in, \nu \rangle}(M \cup \{p^*\}) \;.$$

(See Lem.2.2.) We claim that M^* and p^* are as desired.

Clearly $p^* \in \mathbb{P} \cap M^*$, and p^* satisfies the properties (ii) and (iii) in the lemma by the choice of p^*. Furthermore $M^* \prec \langle \mathcal{H}_\theta, \in, \nu \rangle$, and $M^* \supseteq M$ clearly.

What remains to prove is that $M^* \cap \nu \subseteq M \cap \nu$. For this first note that $M^* \cap \mathcal{H}_{\kappa^+} \subseteq N^*$ because $N \cup \{p^*\} \subseteq N^* \prec \langle \mathcal{H}_{\kappa^+}, \in \rangle$, and $N = M \cap \mathcal{H}_{\kappa^+}$. Moreover $N^* \cap \nu = N \cap \nu$ by the choice of N^*. Therefore $M^* \cap \nu \subseteq N^* \cap \nu = N \cap \nu = M \cap \nu$. □

Now we prove Lem.4.6 using Lem.4.12:

Proof of Lem.4.6. Let G be a $\mathrm{Col}(\nu, <\kappa)$-generic filter and $\mathbb{P}$ be $\mathbb{P}(\square_{\nu,2})$ in $V[G]$. Working in $V[G]$, we show that $\mathbb{P}$ forces $(\kappa, \nu) \twoheadrightarrow (\omega_1, \omega)$.

Take an arbitrary $p \in \mathbb{P}$ and an arbitrary $\mathbb{P}$-name $\dot{F}$ of a function from ${}^{<\omega}\kappa$ to κ. It suffices to find $p^* \leq p$ and $x^* \subseteq \kappa$ such that $|x^*| = \omega_1$, such that $|x^* \cap \nu| = \omega$ and such that $p^* \Vdash$ "x^* is closed under $\dot{F}$".

Take a sufficiently large regular cardinal θ. By induction on $\xi < \omega_1$ we construct a $\subseteq$-increasing continuous sequence $\langle M_\xi \mid \xi < \omega_1 \rangle$ of countable elementary submodels of $\langle \mathcal{H}_\theta, \in, \nu \rangle$ and a descending sequence $\langle p_\xi \mid \xi < \omega_1 \rangle$ below p with the following properties:

(i) $p, \dot{F} \in M_0$.

(ii) $M_\xi \cap \nu = M_0 \cap \nu$ for each $\xi < \omega_1$.

(iii) p_ξ is strongly $(M_\xi, \mathbb{P})$-generic and belongs to $M_{\xi+1}$ for each $\xi < \omega_1$.

(iv) $\{\sup(M_\eta \cap \kappa) \mid \eta < \xi\} \in p_\xi(\sup(M_\xi \cap \kappa))$ for each limit $\xi \in \omega_1 \setminus \{0\}$.

Below let $p_{-1} := p$.

First take a countable $M_0 \prec \langle \mathcal{H}_\theta, \in, \nu \rangle$ with $p, \dot{F} \in M_0$.

Next suppose that ξ is successor or equals to 0 and that $p_{\xi-1}$ and M_ξ have been taken. Then take p_ξ and $M_{\xi+1}$ as follows: First take an $(M_\xi, \mathbb{P})$-generic sequence $\langle q_n \mid n < \omega \rangle$ below $p_{\xi-1}$. Then by Lem.4.12 we can take a countable $M_{\xi+1} \prec \langle \mathcal{H}_\theta, \in, \nu \rangle$ and $p_\xi \in \mathbb{P} \cap M_{\xi+1}$ such that $M_{\xi+1} \supseteq M_\xi$, such that $M_{\xi+1} \cap \nu = M_\xi \cap \nu$ and such that $p_\xi \leq q_n$ for all $n < \omega$. Then p_ξ and $M_{\xi+1}$ satisfy the induction hypotheses.

Finally suppose that ξ is a limit ordinal with $0 < \xi < \omega_1$ and that $\langle M_\eta \mid \eta < \xi \rangle$ and $\langle p_\eta \mid \eta < \xi \rangle$ have been taken. Then take M_ξ, p_ξ and $M_{\xi+1}$ as follows: First let $M_\xi := \bigcup_{\eta<\xi} M_\eta$. Note that $\langle p_\eta \mid \eta < \xi \rangle$ is an $(M_\xi, \mathbb{P})$-generic sequence.

Note also that $d := \{\sup(M_\eta \cap \kappa) \mid \eta < \xi\}$ threads $\bigcup_{\eta<\xi} p_\eta$ by the induction hypothesis (iv). Then by Lem.4.12 we can take a countable $M_{\xi+1} \prec \langle \mathcal{H}_\theta, \in, \nu\rangle$ and $p_\xi \in \mathbb{P} \cap M_{\xi+1}$ such that $M_{\xi+1} \supseteq M_\xi$, such that $M_{\xi+1} \cap \nu = M_\xi \cap \nu$, such that $p_\xi \leq p_\eta$ for all $\eta < \xi$ and such that $d \in p_\xi(\sup(M_\xi \cap \kappa))$. Then M_ξ, p_ξ and $M_{\xi+1}$ satisfy the induction hypotheses.

Now we have constructed $\langle M_\xi \mid \xi < \omega_1\rangle$ and $\langle p_\xi \mid \xi < \omega_1\rangle$. Let $M^* := \bigcup_{\xi<\omega_1} M_\xi$ and $x^* := M^* \cap \kappa$. Then $|x^* \cap \nu| = \omega$ by the property (ii) above. Note also that $M_{\xi+1} \cap \kappa$ is strictrly larger than $M_\xi \cap \kappa$ by the property (iii). So $|x^*| = \omega_1$.

Next let $d^* := \{\sup(M_\xi \cap \kappa) \mid \xi < \omega_1\}$ and

$$p^* \ := \ (\textstyle\bigcup_{\xi<\omega_1} p_\xi)^\frown \langle\{d^*\}\rangle \ .$$

Then $p^* \in \mathbb{P}$, and p^* is a lower bound of $\{p_\xi \mid \xi < \omega_1\}$. So $p^* \leq p$, and p^* is strongly $(M^*, \mathbb{P})$-generic. But $\mathbb{P}, \dot{F} \in M^* \prec \langle \mathcal{H}_\theta, \in\rangle$. So p^* forces that $x^* = M^* \cap \kappa$ is closed under $\dot{F}$. □

This completes the proof of Thm.1.5.

4.2 $\mu > \omega$

Here we prove Thm.1.7:

Theorem 1.7. *Let μ be an uncountable cardinal and ν be a cardinal $> \mu$. Assume $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$. Moreover suppose that either of the following holds:*

(I) $\nu^{<\nu} = \nu$.

(II) $|\{\rho \in \mathrm{Reg} \mid \mu < \rho \leq \nu\}| < |\{\rho \in \mathrm{Reg} \mid \omega \leq \rho \leq \mu\}| < \omega_1$, *where* Reg *denotes the class of all regular cardinals.*

Then $\square_{\nu,\mu}$ fails.

The following is a key lemma:

Lemma 4.13. *Assume that μ and ν are as in Thm.1.7. Let θ be a regular cardinal $\geq \nu^+$ and $\mathcal{M}$ be a structure obtained by adding countable many constants, functions and predicates to $\langle \mathcal{H}_\theta, \in\rangle$. Then there exists $M \prec \mathcal{M}$ with the following properties:*

(i) $\mathrm{o.t.}(M \cap \nu^+) = \mu^+$.

(ii) $|M \cap \nu| = \mu \subseteq M$.

(iii) $M \cap \nu^+$ *is stationary in* $\sup(M \cap \nu^+)$.

First we prove Thm.1.7 using Lem.4.13:

Proof of Thm.1.7 using Lem.4.13. For the contradiction assume that μ and ν are as in Thm.1.7 and that $\square_{\nu,\mu}$ holds. Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha \leq \nu^+ \rangle$ be a $\square_{\nu,\mu}$-sequence, and let $\mathcal{M}$ be the structure $\langle \mathcal{H}_{\nu^{++}}, \in, \vec{\mathcal{C}} \rangle$.

Then we can take $M \prec \mathcal{M}$ with the properties (i)–(iii) in Lem.4.13. Take an arbitrary $c \in \mathcal{C}_{\sup(M \cap \nu^+)}$. Then o.t.$(c) \leq \nu$, and cf(o.t.$(c)$) $= \mu^+$ because o.t.$(M \cap \nu^+) = \mu^+$. But $|M \cap \nu| = \mu$. Hence $M \cap$ o.t.(c) is bounded in o.t.(c). By (iii) in Lem.4.13 we can take $\alpha \in \mathrm{Lim}(c) \cap M$ such that o.t.$(c \cap \alpha) \notin M$. But $c \cap \alpha \in \mathcal{C}_\alpha$, and $\mathcal{C}_\alpha \subseteq M$ because $\mathcal{C}_\alpha \in M$ and $|\mathcal{C}_\alpha| \leq \mu \subseteq M$. Thus $c \cap \alpha \in M$, and so o.t.$(c \cap \alpha) \in M$. This is a contradiction. □

The rest of this subsection is devoted to the proof of Lem.4.13. For this we need preliminaries. First we give a lemma which assures the property (i) in Lem.4.13:

Lemma 4.14 (folklore)**.** *Let μ and ν be infinite cardinals with $\mu < \nu$, and let θ be a regular cardinal $\geq \nu^+$. Assume that $M \prec \langle \mathcal{H}_\theta, \in \rangle$, that $|M \cap \nu^+| \geq \mu^+$ and that $|M \cap \nu| = \mu$. Then* o.t.$(M \cap \nu^+) = \mu^+$.

Proof. For the contradiction assume that o.t.$(M \cap \nu^+) \neq \mu^+$. Then o.t.$(M \cap \nu^+) > \mu^+$ because $|M \cap \nu^+| \geq \mu^+$. Let α be the μ^+-th element of $M \cap \nu^+$. Here note that $\nu \in M$ because M contains ordinals between ν and ν^+. By the elementarity of M we can take an injection $\sigma : \alpha \to \nu$ in M. Note that $\sigma[M \cap \alpha] \subseteq M \cap \nu$. Then $\nu^+ = |M \cap \alpha| = |\sigma[M \cap \alpha]| \leq |M \cap \nu| = \nu$. This is a contradiction. □

Next we give a lemma relevant to the property (ii) in Lem.4.13. In the following lemma note that $\mathrm{Sk}^{\mathcal{M}}(M \cup B)$ and $\mathrm{Sk}^{\mathcal{M}}(M \cup \mu)$ exist by Lem.2.2 (1) and the fact that $B, \mu \subseteq \nu \in M \prec \mathcal{M}$:

Lemma 4.15. *Let μ and ν be infinite cardinals with $\mu < \nu$, and let θ be a regular cardinal $\geq \nu^+$. Suppose that $\mathcal{M}$ is a structure obtained by adding countable many constants, functions and predicates to $\langle \mathcal{H}_\theta, \in, \mu, \nu \rangle$. Moreover assume that M is an elementary submodel of $\mathcal{M}$ such that $|M \cap \nu| = \mu$.*

(1) Assume also (I) in Thm.1.7. Then $|\mathrm{Sk}^{\mathcal{M}}(M \cup B) \cap \nu| = \mu$ *for any bounded subset* B *of* $\sup(M \cap \nu^+)$ *with* $|B| \leq \mu$.

(2) Assume also (II) in Thm.1.7. Then $|\mathrm{Sk}^{\mathcal{M}}(M \cup \mu) \cap \nu| = \mu$.

Proof. (1) Suppose that B is a bounded subset of $\sup(M \cap \nu^+)$ with $|B| \leq \mu$, and let $N := \mathrm{Sk}^{\mathcal{M}}(M \cup B)$. We show that $|N \cap \nu| \leq \mu$. Take $\alpha \in M \cap \nu^+$ such that $\alpha > \sup(B)$. Note that

$$N \cap \nu \ = \ \{f(b) \mid f : {}^{<\omega}\alpha \to \nu \ \wedge \ f \in M \ \wedge \ b \in {}^{<\omega}B\} \ .$$

Moreover $|\{f \mid f : {}^{<\omega}\alpha \to \nu\}| = \nu$ because $\nu^{<\nu} = \nu$. Then

$$|\{f \mid f : {}^{<\omega}\alpha \to \nu \ \wedge \ f \in M\}| \ = \ |M \cap \nu| \ = \ \mu$$

by the elementarity of M. Then

$$|N \cap \nu| \ \leq \ |\{f \mid f : {}^{<\omega}\alpha \to \nu \ \wedge \ f \in M\}| \cdot |{}^{<\omega}B| = \mu \ .$$

(2) Let $N := \mathrm{Sk}^{\mathcal{M}}(M \cup \mu)$. For the contradiction assume that $|N \cap \nu| \neq \mu$. Then $|N \cap \nu| > \mu$. Let λ be the least cardinal such that $|N \cap \lambda| \geq \mu^+$.

First note that $\mu^+ \leq \lambda \leq \nu$. Note also that there are only countable many cardinals between μ^+ and ν by (II) in Thm.1.7. Thus all cardinals between μ^+ and ν belong to M by the elementarity of M. In particular, $\lambda \in M$. Moreover all limit cardinal between μ^+ and ν have countable cofinality. Thus λ must be a successor cardinal by the choice of λ.

Then o.t.$(N \cap \lambda) = \mu^+$ by Lem.4.14. In particular, $\sup(M \cap \lambda) < \sup(N \cap \lambda)$. This contradicts Lem.2.2 (2). □

Next we present a lemma relevant to the property (iii) in Lem.4.13:

Lemma 4.16. *Let* κ *and* θ *be regular cardinals with* $\kappa < \theta$. *Suppose that* $M \prec \langle \mathcal{H}_\theta, \in, \kappa \rangle$. *Moreover suppose that* ρ *is a regular cardinal* $< \kappa$ *and that* $\mathrm{cf}(\sup(M \cap \lambda)) \neq \rho$ *for any regular cardinal* $\lambda \in M$ *with* $\rho < \lambda \leq \kappa$. *Then* $M \cap \kappa$ *is* ρ*-closed.*

Proof. For the contradiction assume that $x \subseteq M \cap \kappa$, that o.t.$(x) = \rho$ and that $\sup(x) \notin M \cap \kappa$. Note that $\sup(x) < \sup(M \cap \kappa)$ because $\mathrm{cf}(\sup(M \cap \kappa)) \neq \rho$. Let $\alpha := \min(M \setminus \sup(x)) > \sup(x)$, and take an increasing continuous sequence $\langle \alpha_\beta \mid \beta < \mathrm{cf}(\alpha) \rangle \in M$ cofinal in α. Then $\sup(x) = \alpha_\beta$ for some limit

$\beta < \mathrm{cf}(\alpha)$ by the elementarity of M. Moreover $\beta = \sup(M \cap \mathrm{cf}(\alpha))$ again by the elementarity of M. Note that $\mathrm{cf}(\beta) = \mathrm{cf}(x) = \rho$. Then $\mathrm{cf}(\sup(M \cap \mathrm{cf}(\alpha))) = \rho < \mathrm{cf}(\alpha)$. Furthermore $\mathrm{cf}(\alpha) \in M$. This contradicts the assumption on ρ. $\square$

Now we prove Lem.4.13:

Proof of Lem.4.13. By expanding $\mathcal{M}$ if necessary, we may assume that μ and ν are constants of $\mathcal{M}$.

First we prove Lem.4.13 assuming (I) in Thm.1.7. By $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ take $N \prec \mathcal{M}$ such that $|N \cap \nu^+| = \mu^+$ and such that $|N \cap \nu| = \mu$. Let $\rho_0 := \mathrm{cf}(\sup(N \cap \nu))$, and take an increasing cofinal sequence $\langle \gamma_\xi \mid \xi < \rho_0 \rangle$ in $\sup(N \cap \nu)$. Note that $\rho_0 \leq \mu$.

Then by induction on $\xi \leq \rho_0$ take M_ξ as follows: Let $M_0 := \mathrm{Sk}^{\mathcal{M}}(N \cup \mu)$. If ξ is a limit ordinal, then let $M_\xi := \bigcup_{\eta<\xi} M_\eta$. Suppose that $\xi < \rho_0$ and that M_ξ has been taken. Then let B_ξ be the closure of $M_\xi \cap \gamma_\xi$ (with respect to the ordinal topology), and let $M_{\xi+1} := \mathrm{Sk}^{\mathcal{M}}(M_\xi \cup B_\xi)$.

Let $M := M_{\rho_0}$. We claim that M satisfies the properties (i)–(iii) in Lem.4.13.

Clearly $\mu \subseteq M$. Moreover, using Lem.4.15 (1), by induction on $\xi \leq \rho_0$ we can easily prove that $|M_\xi \cap \nu| = \mu$. Thus M satisfies the property (ii). Moreover $|M \cap \nu^+| \geq |N \cap \nu^+| = \mu^+$, and $M \prec \mathcal{M}$. So M also satisfies the property (i) by Lem.4.14.

To check (iii), take a regular cardinal $\rho_1 \leq \mu$ which is different from ρ_0. We can take such ρ_1 because μ is uncountable. Then it easily follows from the construction of M that $M \cap \sup(N \cap \nu)$ is ρ_1-closed. Moreover $\sup(M \cap \nu) = \sup(N \cap \nu)$ by Lem.2.2 (2), and so $\mathrm{cf}(\sup(M \cap \nu)) = \rho_0 \neq \rho_1$. Furthermore $\mathrm{cf}(\sup(M \cap \nu^+)) = \mu^+ \neq \rho_1$. In summary, $\mathrm{cf}(\sup(M \cap \lambda)) \neq \rho_1$ for every regular cardinal $\lambda \in M$ such that $\rho_1 < \lambda \leq \nu^+$. Then $M \cap \nu^+$ is ρ_1-closed by Lem.4.16. Because $\mathrm{cf}(\sup(M \cap \nu^+)) = \mu > \rho_1$, it follows that $M \cap \nu^+$ is stationary in $\sup(M \cap \nu^+)$.

Next we prove Lem.4.13 assuming (II) in Thm.1.7. By $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ take $N \prec \mathcal{M}$ such that $|N \cap \nu^+| = \mu^+$ and such that $|N \cap \nu| = \mu$. We show that $M := \mathrm{Sk}^{\mathcal{M}}(N \cup \mu)$ satisfies the properties (i)–(iii) of Lem.4.13.

Clearly $\mu \subseteq M$, and $|M \cap \nu| = \mu$ by Lem.4.15 (2). So M satisfies (ii). Then M also satisfies (i) by Lem.4.14.

To check (iii), take a regular cardinal $\rho \leq \mu$ such that $\mathrm{cf}(\sup(M \cap \lambda)) \neq \rho$ for all regular cardinals λ with $\rho < \lambda \leq \nu$. By the assumption (II) in Thm.1.7 we

can take such ρ. Note that $\mathrm{cf}(\sup(M \cap \nu^+)) = \mu^+ \neq \rho$. So $M \cap \nu^+$ is ρ-closed by Lem.4.16. Because $\mathrm{cf}(\sup(M \cap \nu^+) = \mu^+ > \rho$, it follows that $M \cap \nu^+$ is stationary in $\sup(M \cap \nu^+)$. $\square$

5 Questions

We end this paper with two questions.

First question is on the large cardinal assumption in Thm.1.5. It is known, due to Silver and Donder [1], that Chang's Conjecture is equi-consistent with the existence of an ω_1-Erdös cardinal. But in Thm.1.5 we assumed the existence of measurable cardinal which has stronger consistency than that of an ω_1-Erdös cardinal. We do not know whether we need a measurable cardinal for Chang's Conjecture together with $\square_{\omega_1,2}$:

Question 5.1. *What is the consistency strength of Chang's Conjecture together with* $\square_{\omega_1,2}$*?*

The second question is whether we can drop the assumption (I) and (II) from Thm.1.7:

Question 5.2. *Let* μ *be an uncountable cardinal and* ν *be a cardinal* $> \mu$*. Does* $(\nu^+, \nu) \twoheadrightarrow (\mu^+, \mu)$ *imply the failure of* $\square_{\nu,2}$*?*

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