REPORTS ON MATHEMATICAL LOGIC
36 (2002), 55–61

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DECIDABLE FORMULAS OF INTUITIONISTIC PRIMITIVE RECURSIVE ARITHMETIC

A b s t r a c t. By formalizing some classical facts about provably total functions of intuitionistic primitive recursive arithmetic ($iPRA$), we prove that the set of decidable formulas of $iPRA$ and of $i\Sigma_1^+$ (intuitionistic Σ_1-induction in the language of PRA) coincides with the set of its provably Δ_1-formulas and coincides with the set of its provably atomic formulas. By the same methods, we shall give another proof of a theorem of Marković and De Jongh: the decidable formulas of HA are its provably Δ_1-formulas.

1. Notation

Following Wehmeier [5] let $iPRA$ be the intuitionistic theory in the language of PRA which is the first order language containing function symbol for each primitive recursive function, whose nonlogical axioms are the defining equations for all primitive recursive functions plus the axiom

Received 1 August 2000
Mathematics Subject Classification: Primary 03F55; Secondary 03F50, 03F30, 03B20.
Keywords: Primitive Recursive Arithmetic, Provably Total Function.

scheme of induction restricted to atomic formulas.

$i\Sigma_1^+$ is $iPRA$ plus induction over Σ_1 formulas.

PRA is $iPRA$ together with classical logic, and similarly is $I\Sigma_1^+$.

For a function f, let $\Gamma_f(\bar{x}, y)$ be (the formula of) its graph (intuitively, $\Gamma_f(\bar{x}, y) \equiv f(\bar{x}) = y$). $\mathbf{T}, \mathbf{U}$ are Kleene's functions, μ is the minimalization operator and $\langle\ ,\ \rangle$ is the pairing function with the projections π_1, π_2.

A function f is called provably total in a theory T, if $T \vdash \forall x \exists y \Gamma_f(x, y)$, and is called provably Δ_1, if moreover Γ_f is provably Δ_1 in T. (For an accurate definition of a Δ_1 formula see [3] or Definition 3.1 below.)

2. Provably Recursive Functions

In classical arithmetics provably Δ_1 functions are also called *provably recursive*, the following theorems show that provably Δ_1 functions are really provably recursive, in intuitionistic arithmetics as well.

Let S be any arithmetical theory containing $iPRA$.

Lemma 2.1. *Σ_1 provably total functions are Δ_1, that is, if*

$$S \vdash \forall x \exists ! y \exists t \varphi(x, y, t) \text{ for } \varphi \in \Delta_0$$

then

$$S \vdash \exists t \varphi(x, y, t) \ \leftrightarrow \forall z, u(\neg\varphi(x, z, u) \vee z = y).$$

The proof is straightforward, noting that the equality is decidable in S.

Lemma 2.2. *Provably (total) recursive functions are provably Δ_1, that is, if*

$$S \vdash \forall x \exists ! y A(x, y) \text{ and } S \vdash \exists c \forall x \exists ! z(\mathbf{T}(c, x, z) \wedge A(x, \mathbf{U}(z)))$$

then

$$S \vdash A(x, y) \leftrightarrow \exists z(\mathbf{T}(c, x, z) \wedge y = \mathbf{U}(z))$$

(and so by the previous lemma, A is Δ_1.)

Lemma 2.3. *Provably total Σ_1 functions are provably recursive, that is, if*

$$S \vdash \forall x \exists ! y A(x, y) \text{ and } A \in \Sigma_1$$

then

$$S \vdash \exists c \forall x \exists ! z(\mathbf{T}(c, x, z) \wedge A(x, \mathbf{U}(z))).$$

Proof. If $A = \exists t\varphi(t,x,y)$ and $\varphi \in \Delta_0$ then let

$$c = \overline{\lambda x.\pi_2(\mu\langle a,b\rangle\ \{\varphi(a,x,b)\})}$$

(= the code of the function which with input x gives the output: $\pi_2(\mu\langle a,b\rangle\ \{\varphi(a,x,b)\})$). ■

3. Decidable Formulas of $iPRA$, $i\Sigma_1^+$ and HA.

Definition 3.1. [4]

$$\Delta(\psi,\chi) = \forall\overline{x}(\exists y\psi(\overline{x},y) \leftrightarrow \forall z\chi(\overline{x},z))$$

$$P\Delta(\psi,\chi) = \forall\overline{x}\ \forall y\forall z\exists u\exists v((\psi(\overline{x},y) \to \chi(\overline{x},z)) \wedge (\chi(\overline{x},u) \to \psi(\overline{x},v)))$$

For any formula $\phi(\overline{x})$, we say $\phi \in \Delta_1(S)$ (provably Δ_1) if there are Δ_0-formulas $\psi(\overline{x},y), \chi(\overline{x},z)$ such that $S \vdash \phi(\overline{x}) \leftrightarrow \exists y\psi(\overline{x},y)$ and $S \vdash \Delta(\psi,\chi)$.

Fact 3.2. [6] *$iPRA$ (resp. $i\Sigma_1^+$, resp. HA) is Π_2-conservative over PRA (resp. $I\Sigma_1^+$, resp. PA.)*

Lemma 3.3. *For $\psi,\chi \in \Delta_0$ and $T = iPRA$ or $i\Sigma_1^+$, we have:*

$$T \vdash \Delta(\psi,\chi) \text{ iff } T \vdash P\Delta(\psi,\chi) \text{ iff } T^c \vdash \Delta(\psi,\chi).$$

(T^c is the classical counterpart of the theory T.)

Proof. The proof of Lemma 2 in [4] works, using Fact 3.2. ■

Fact 3.4. [5] *If $HA \vdash \forall\overline{x}\ \exists yA(\overline{x},y)$ then $HA \vdash \exists c\ \forall\overline{x}\ \exists z(\mathbf{T}(c,\overline{x},z) \wedge A(\overline{x},\mathbf{U}(z)))$.*

Fact 3.5. [6] *If $T \vdash \forall\overline{x}\ \exists yA(\overline{x},y)$ then there is a (primitive recursive) function symbol f such that $T \vdash \forall\overline{x}\ A(\overline{x},f(\overline{x}))$, for $T = iPRA$ or $i\Sigma_1^+$.*

Theorem 3.6. *If $\phi \in \Delta_1(HA)$ then $HA \vdash \forall\overline{x}(\phi(\overline{x}) \vee \neg\phi(\overline{x}))$.*

Proof. Suppose for $\psi,\chi \in \Delta_0$, $HA \vdash \phi(\overline{x}) \leftrightarrow \exists y\psi(\overline{x},y) \leftrightarrow \forall z\chi(\overline{x},z)$, so by Lemma 3.3 (for HA, [4]), $HA \vdash \forall\overline{x}\ \exists u,v(\chi(\overline{x},u) \to \psi(\overline{x},v))$, so by Fact 3.4,

$$HA \vdash \exists c\ \forall\overline{x}\ \exists z(\mathbf{T}(c,\overline{x},z) \wedge [\chi(\overline{x},\pi_1(\mathbf{U}(z))) \to \psi(\overline{x},\pi_2(\mathbf{U}(z)))]).$$

On the other hand we have $HA \vdash \forall \overline{x}\ \forall y(\psi(\overline{x}, y) \vee \neg\psi(\overline{x}, y))$ so,

$$HA \vdash \exists c\ \forall \overline{x}\ \exists z(\mathbf{T}(c, \overline{x}, z) \wedge [\psi(\overline{x}, \pi_2(\mathbf{U}(z))) \vee \neg\psi(\overline{x}, \pi_2(\mathbf{U}(z)))]),$$

so,

$$HA \vdash \exists c\ \forall \overline{x}\ \exists z(\mathbf{T}(c, \overline{x}, z) \wedge [\exists y\psi(\overline{x}, y) \vee \neg\chi(\overline{x}, \pi_1(\mathbf{U}(z)))]),$$

so,

$$HA \vdash \exists c\ \forall \overline{x}\ \exists z(\mathbf{T}(c, \overline{x}, z) \wedge [\exists y\psi(\overline{x}, y) \vee \neg\forall z\chi(\overline{x}, z)]),$$

thus, $HA \vdash \forall \overline{x}(\phi(\overline{x}) \vee \neg\phi(\overline{x}))$. ■

Theorem 3.7. *If $HA \vdash \forall \overline{x}(\phi(\overline{x}) \vee \neg\phi(\overline{x}))$ then $\phi \in \Delta_1(HA)$.*

Proof. Suppose $HA \vdash \forall \overline{x}(\phi(\overline{x}) \vee \neg\phi(\overline{x}))$, so

$$HA \vdash \forall \overline{x}\exists y[(y = 0 \to \phi(\overline{x})) \wedge (y = 1 \to \neg\phi(\overline{x}))],$$

then $HA \vdash \forall \overline{x}\ \exists! y\{y \leq 1 \wedge (y = 0 \leftrightarrow \phi(\overline{x}))\}$, so, by Fact 3.4,

$$HA \vdash \exists c\ \forall \overline{x}\ \exists z(\mathbf{T}(c, \overline{x}, z) \wedge \{\mathbf{U}(z) \leq 1 \wedge (\mathbf{U}(z) = 0 \leftrightarrow \phi(\overline{x}))\}),$$

and since $\mathbf{T}(c, \overline{x}, z_1) \wedge \mathbf{T}(c, \overline{x}, z_2) \Rightarrow z_1 = z_2$, we have

$$HA \vdash \exists c\ \forall \overline{x}\ \exists! z(\mathbf{T}(c, \overline{x}, z) \wedge \{\mathbf{U}(z) \leq 1 \wedge (\mathbf{U}(z) = 0 \leftrightarrow \phi(\overline{x}))\}).$$

So, by Lemma 2.2, there is a $\Psi(\overline{x}, y) \in \Delta_1(HA)$, such that

$$HA \vdash \Psi(\overline{x}, y) \leftrightarrow \{y \leq 1 \wedge (y = 0 \leftrightarrow \phi(\overline{x}))\},$$

so, $HA \vdash \phi(\overline{x}) \leftrightarrow \Psi(\overline{x}, 0)$, that is $\phi \in \Delta_1(HA)$. ■

Theorem 3.8. *If $\phi \in \Delta_1(T)$ then $T \vdash \forall \overline{x}(\phi(\overline{x}) \vee \neg\phi(\overline{x}))$, for $T = iPRA$ or $i\Sigma_1^+$.*

Proof. Suppose for $\psi, \chi \in \Delta_0$, $T \vdash \phi(\overline{x}) \leftrightarrow \exists y\psi(\overline{x}, y) \leftrightarrow \forall z\chi(\overline{x}, z)$, so by Lemma 3.3, $T \vdash \forall \overline{x}\ \exists u, v(\chi(\overline{x}, u) \to \psi(\overline{x}, v))$, so by Fact 3.5, there is an $f \in PRA$ such that

$$T \vdash \forall \overline{x}(\chi(\overline{x}, \pi_1(f(\bar{x}))) \to \psi(\overline{x}, \pi_2(f(\bar{x})))).$$

On the other hand, we have $T \vdash \forall \overline{x}\ \forall y(\psi(\overline{x}, y) \vee \neg\psi(\overline{x}, y))$, so,

$$T \vdash \forall \overline{x}(\psi(\overline{x}, \pi_2(f(\overline{x}))) \vee \neg\psi(\overline{x}, \pi_2(f(\overline{x})))),$$

so,

$$T \vdash \forall \overline{x}(\exists y \psi(\overline{x}, y) \vee \neg \chi(\overline{x}, \pi_1(f(\overline{x})))),$$

so,

$$T \vdash \forall \overline{x}(\exists y \psi(\overline{x}, y) \vee \neg \forall z \chi(\overline{x}, z)),$$

thus, $T \vdash \forall \overline{x}(\phi(\overline{x}) \vee \neg \phi(\overline{x}))$. ■

Theorem 3.9. *If* $T \vdash \forall \overline{x}(\phi(\overline{x}) \vee \neg \phi(\overline{x}))$ *then* $\phi \in Atoms(T)$, *for* $T = iPRA$ *or* $i\Sigma_1^+$.

Proof. Suppose $T \vdash \forall \overline{x}(\phi(\overline{x}) \vee \neg \phi(\overline{x}))$, so, $T \vdash \forall \overline{x}\ \exists y(y = 0 \leftrightarrow \phi(\overline{x}))$, so, by Fact 3.5 there is an $f \in PRA$, such that $T \vdash \forall \overline{x}(f(\overline{x}) = 0 \leftrightarrow \phi(\overline{x}))$. So, $\phi \in Atoms(T)$. ■

Corollary 3.10. *Decidables(iPRA)*$=\Delta_1(iPRA)=$*Atoms(iPRA), Decidables(*$i\Sigma_1^+$*)*$=\Delta_1(i\Sigma_1^+)=$*Atoms(*$i\Sigma_1^+$*).*

By the similar methods one can characterize the decidable formulas of $i\Sigma_1$, (see [5] for the definition of $i\Sigma_1$ and $I\Sigma_1$).

Theorem 3.11. *Decidables*$(i\Sigma_1) = \Delta_1(i\Sigma_1)$.

Proof. By Theorem 3.8, it is straightforward that

$$\Delta_1(i\Sigma_1) \subseteq Decidables(i\Sigma_1).$$

Conversely if $i\Sigma_1 \vdash \forall \bar{x}(\phi(\bar{x}) \vee \neg \phi(\bar{x}))$, then $i\Sigma_1 \vdash \forall \bar{x} \exists! y\{y \leq 1 \wedge (y = 0 \leftrightarrow \phi(\bar{x}))\}$; so the formula $\Gamma(\bar{x}, y) \equiv \{y \leq 1 \wedge (y = 0 \leftrightarrow \phi(\bar{x}))\}$ defines a total function in $i\Sigma_1$, and so it should be primitive recursive, thus provably total in $i\Sigma_1$ by a Δ_1 formula. So there is a $\psi(\bar{x}, y) \in \Delta_1(i\Sigma_1)$ such that $i\Sigma_1 \vdash \{y \leq 1 \wedge (y = 0 \leftrightarrow \phi(\bar{x}))\} \leftrightarrow \Gamma(\bar{x}, y) \leftrightarrow \psi(\bar{x}, y)$, so $i\Sigma_1 \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x}, 0)$ i.e., $\phi \in \Delta_1(i\Sigma_1)$. ■

4. Provably Total Functions of HA

Wehmeier [6] showed that provably *total* functions of $i\Sigma_1$ are precisely primitive recursive functions, that is provably *recursive* functions of $I\Sigma_1$. A corresponding theorem for HA was proved by Damnjanovic [3]. Here we give a much simpler proof:

Theorem 4.1. *Provably* total *functions of HA are precisely the provably* recursive *functions of PA.*

Proof. It is well-known that provably recursive functions of PA are $< \epsilon_0$- primitive recursive functions (see [2]).

Suppose f is a $< \epsilon_0$-primitive recursive function, so $PA \vdash \forall x \exists ! y \Gamma_f(x,y)$ for a $\Gamma_f \in \Sigma_1$. Since PA is Π_2-conservative on HA, then $HA \vdash \forall x \exists y \Gamma_f(x,y)$. By the trivial property of the defining formula of f, we have

$$\forall y_1, y_2(\Gamma_f(x,y_1) \wedge \Gamma_f(x,y_2) \rightarrow y_1 = y_2).$$

So $HA \vdash \forall x \exists ! y \Gamma_f(x,y)$.

Conversely suppose $HA \vdash \forall x \exists y A(x,y)$, so there is an $n \in \mathbf{N}$ such that $HA \vdash \forall x \exists z (\mathbf{T}(n,x,z) \wedge A(x, \mathbf{U}(z)))$, so $PA \vdash \forall x \exists ! y \mathbf{T}(n,x,y)$ (since $\mathbf{T}(m,x,z_1) \wedge \mathbf{T}(m,x,z_2) \rightarrow z_1 = z_2$.)

Thus there is an $< \epsilon_0$-primitive recursive function g, such that $PA \vdash \Gamma_g(x,y) \leftrightarrow \mathbf{T}(n,x,y)$. Since $\Gamma_g, \mathbf{T} \in \Sigma_1$ so $HA \vdash \Gamma_g(x,y) \leftrightarrow \mathbf{T}(n,x,y)$. So $HA \vdash \forall x \exists z (\Gamma_g(x,z) \wedge A(x, \mathbf{U}(z)))$, and then

$$HA \vdash \forall x \exists y (\Gamma_{\mathbf{U}g}(x,y) \wedge A(x,y)).$$

(If $HA \vdash \forall x \exists ! y A(x,y)$ then also $HA \vdash \forall x, y (A(x,y) \leftrightarrow \Gamma_{\mathbf{U}g}(x,y))$.)

Since $\mathbf{U}$ is primitive recursive, then $\mathbf{U}g$ is $< \epsilon_0$- primitive recursive too. ■

It might be expected that provably total functions of $i\Sigma_n$ are the provably recursive functions of $I\Sigma_n$, but for $n = 2$ it is false! Burr [1] has shown that provably total functions of $i\Sigma_2$ are primitive recursive, although it is well-known that the Ackermann's function is provably recursive in $I\Sigma_2$.

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